Special Issue in honor of John W. Neuberger
Electronic Journal of Differential Equations, Special Issue 02 (2023), pp. 1–10.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

A NONEXISTENCE RESULT FOR p-LAPLACIAN SYSTEMS IN A BALL

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In memory of Professor John W. Neuberger

Abstract. We consider the p-Laplacian system

$$\begin{gathered} -\Delta_p u = \lambda f(v) \quad \text{in } \Omega; \\ -\Delta_p v = \lambda g(u) \quad \text{in } \Omega; \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\lambda > 0$ is a parameter, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $p > 1$ and Ω is a unit ball in $\mathbb{R}^N$ $(N \geq 2)$. The nonlinearities $f, g : [0, \infty) \to \mathbb{R}$ are assumed to be C^1 non-decreasing semipositone functions ($f(0) < 0$ and $g(0) < 0$) that are p-superlinear at infinity. By analyzing the solution in the interior of the unit ball as well as near the boundary, we prove that the system has no positive radially symmetric and radially decreasing solution for λ large.

1. Introduction

In this article, we consider the p-Laplacian coupled system

$$\begin{gathered} -\Delta_p u = \lambda f(v) \quad \text{in } \Omega; \\ -\Delta_p v = \lambda g(u) \quad \text{in } \Omega; \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $p > 1$ and $\Omega := \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$ $(N \geq 2)$ centered at the origin. The nonlinearities f and g satisfy:

- (H1) $f, g : [0, \infty) \to \mathbb{R}$ are nondecreasing C^1 functions with unique zeros v_0 and u_0, respectively, with $f(0) < 0$, $g(0) < 0$;
- (H2) there exist $\alpha, \beta \in (p-1, p*)$ such that

$$0 < \lim_{s\to\infty} \frac{f(s)}{s^\alpha} < \infty \quad \text{and} \quad 0 < \lim_{s\to\infty} \frac{g(s)}{s^\beta} < \infty,$$

2020 *Mathematics Subject Classification.* 34B18, 35B09, 35J92.
Key words and phrases. Positive radial solution; ball; p-Laplacian system; semipositone; p-superlinear at infinity.

Published March 27, 2023.

where

$$p^* = \begin{cases} \frac{Np}{N-p}; & p < N \\ +\infty; & p \geq N \end{cases}$$

is the critical Sobolev exponent.

Our main result reads as follows.

Theorem 1.1. *Under assumptions* (H1) *and* (H2)*, there exists* $\lambda^* > 0$ *such that* (1.1) *has no positive radial (radially symmetric and radially decreasing) solution for* $\lambda > \lambda^*$*.*

If f and g are p-sublinear at infinity, it was proved in [5] that there is no positive solution for λ small on a general bounded domain. This result was obtained by utilizing the variational characterization of the first eigenvalue λ_1 of $-\Delta_p$. Same technique cannot be applied to the p-superlinear case and thus turns out to be more challenging. Thus nonexistence result in general bounded domain, even for convex or strictly convex domain, remains open for both scalar and systems cases.

For the scalar case, non-existence of nonnegative solutions for λ large was established in [3, 9] in a ball when the nonlinearity is semipositone and p-superlinear at infinity. Using result of [1], it turns out that every nonnegative solution of a quasilinear equation in a ball with nonlinearity f satisfying the semipositone structure is positive, radially symmetric and radially decreasing which enabled the use of ODE techniques. See also [10], where nonexistence of positive solutions is established when a weight function is large for a semipositone superlinearproblem in a ball.

For $p = 2$ case, nonexistence result of nonnegative solutions, for coupled system, was established in [7]. For $p = 2$, it is known that all nonnegative solutions of a cooperative semipositone system in a ball in $\mathbb{R}^N$ $(N \geq 2)$, are componentwise positive (see [2]), and hence radially symmetric and radially decreasing by [8] and [11]. This enabled the use of ODE techniques in [7] to prove their result for nonnegative solutions. In this case, however, the nonexistence result for positive solution for λ large has been extended to the case when Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ in [4].

However, when $p \neq 2$, for the case of coupled system, we are not aware of any result in radial domains that will either imply nonnegative solutions are componentwise positive, or positive solutions are radially symmetric and radially decreasing. Thus, in this paper, we focused our attention to positive radial (radially symmetric and radially decreasing) solutions. To the best of our knowledge, Theorem 1.1 is the first nonexistence result for $p \neq 2$, when f and g are p-superlinear at infinity.

In Section 2, we discuss some preliminaries regarding an ODE whose solutions will be radial solutions of (1.1). We also establish several lemmas regarding the behavior of positive solutions in the interior and near the boundary of the ball that will be crucial in the sequel. In Section 3, we prove Theorem 1.1 by constructing an appropriate energy function, and compare the energy function considered in this paper with those used for the Laplacian system and p-Laplacian scalar case.

2. Preliminaries

Studying positive radial solutions of (1.1) is equivalent to studying the positive solutions of

$$\begin{gathered} -\left(r^{N-1}\phi_p(u')\right)' = \lambda r^{N-1} f(v) \quad \text{for } 0<r<1; \\ -\left(r^{N-1}\phi_p(v')\right)' = \lambda r^{N-1} g(u) \quad \text{for } 0<r<1; \\ u'(0)=u(1)=0; \\ v'(0)=v(1)=0, \end{gathered} \tag{2.1}$$

where $\phi_p(s) := |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0)=0$. Clearly ϕ_p is an odd increasing homeomorphism of $\mathbb{R}$ onto itself. The inverse mapping of ϕ_p, denoted by $(\phi_p)_{-1}$, is given by $(\phi_p)_{-1} = \phi_{p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$. Moreover ϕ_p is differentiable and its derivative, denoted by ϕ_p', is given by $\phi_p'(s) = (p-1)|s|^{p-2}$ for $s \neq 0$ and, $\phi_p'(0)=0$ provided $p>2$.

We define $F(t) := \int_0^t f(s)ds$ and $G(t) := \int_0^t g(s)ds$, and let V_0 and U_0 be the unique positive zeros of F and G respectively. We observe that $0<v_0<V_0$ and $0<u_0<U_0$.

We remark that since we study positive solution (u,v) of (2.1) that is radially decreasing, we have

$$u(0)>0 \text{ and } v(0)>0 \text{ with } u'(r)<0 \text{ and } v'(r)<0 \quad \text{on } (0,1]\,.$$

Letting

$$U_* := \min\{U_0, u(0)\} \quad \text{and} \quad V_* := \min\{V_0, v(0)\}\,,$$

it follows from (H2) that there exists $K>0$ such that

$$f(s) \geq K s^\alpha \text{ for } s > \frac{v_0+V_*}{2} \quad \text{and} \quad g(s) \geq K s^\beta \text{ for } s > \frac{u_0+U_*}{2}. \tag{2.2}$$

Now we establish several lemmas which will be crucial in proving our result. The first result below establishes relationship between positive solution and the zeros of the nonlinearities.

Lemma 2.1. *For any positive solution (u,v) of (2.1), we have $u(0)>u_0$ and $v(0)>v_0$.*

Proof. Assume to the contrary that $u(0) \leq u_0$ or $v(0) \leq v_0$. Without loss of generality, suppose that $v(0) \leq v_0$. Then, since v is radially decreasing, $v(r) < v(0) \leq v_0$ for $r \in (0,1)$ and thus u satisfies

$$\begin{gathered} -(r^{N-1}\phi_p(u'(r))' = \lambda r^{N-1} f(v(r)) < 0 \quad \text{for } 0<r<1; \\ u'(0)=u(1)=0\,. \end{gathered}$$

Then the maximum principle for the scalar equation, see [6], yields $u(r) \leq 0$ on $(0,1)$, a contradiction since (u,v) is a positive solution. Therefore $u(0)>u_0$ and $v(0)>v_0$, as desired. □

The result below characterizes each component of a positive solution and their derivatives, for λ large, in the interior of the ball.

Lemma 2.2. *Let (u,v) be a positive solution of (2.1). Then, there exists $\lambda^*>0$ such that for each $\lambda>\lambda^*$, there exist $r_1 = r_1(\lambda)$, $\widetilde{r_1} = \widetilde{r}_1(\lambda) \in \left[0, \frac{1}{2}\right]$ satisfying*

$$u(r_1) = \frac{u_0+U_*}{2} \quad \textit{and} \quad v(\widetilde{r}_1) = \frac{v_0+V_*}{2}.$$

Moreover, $|u'(r_1)|, |v'(\widetilde{r_1})| \to \infty$ *as* $\lambda \to \infty$.

Proof. First observe that since (u, v) is a positive solutions of (2.1), we have that $u(0) \geq U_* \geq \frac{u_0+U_*}{2}$ and $v(0) \geq V_* \geq \frac{v_0+V_*}{2}$. Now, without loss of generality, assume to the contrary that $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Then u must satisfy:

(i) $u(r) > \frac{u_0+U_*}{2}$ for all $r \in [0, 1/2]$, or
(ii) $u(r_1) = \frac{u_0+U_*}{2}$ for some $r_1 \in [0, 1/2]$.

We will show that both of these cases lead to contradictions.

Case 1. Suppose (i) holds. Then $u(r) > \frac{u_0+U_*}{2}$ and $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Integrating the first equation of (2.1) from 0 to $r \in (0, 1/2]$, using (2.2) and the fact that $v' < 0$, we obtain

$$\begin{aligned} r^{N-1}\phi_p(u'(r)) &= -\lambda \int_0^r t^{N-1} f(v(t))dt \\ &\leq -\lambda K \int_0^r v^{\alpha}(t) t^{N-1} dt \\ &= -\frac{\lambda K}{N} \left[t^N v^{\alpha}(t)\right]_0^r + \frac{\lambda K \alpha}{N} \int_0^r v^{\alpha-1}(t) v'(t) t^N \,\mathrm{d}t \\ &= -\frac{\lambda K}{N} r^N v^{\alpha}(r) + \frac{\lambda K \alpha}{N} \int_0^r v^{\alpha-1}(t) v'(t) t^N \,\mathrm{d}t \\ &< -\frac{\lambda K}{N} r^N v^{\alpha}(r). \end{aligned} \tag{2.3}$$

Simplifying, applying the inverse of ϕ_p to the previous inequality and using that ϕ_p is odd and $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned} u'(r) &< \phi_{p'}\Big(-\frac{\lambda r K v^{\alpha}(r)}{N}\Big) \\ &= -\phi_{p'}\Big(\frac{\lambda r K v^{\alpha}(r)}{N}\Big) \\ &= -\Big(\frac{\lambda r K}{N}\Big)^{p'-1} v(r)^{\alpha(p'-1)} \\ &= -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} v(r)^{\frac{\alpha}{p-1}}. \end{aligned} \tag{2.4}$$

Set $Q_1 := \min\big\{\inf_{[0,1/2]} u^{\frac{\alpha-p+1}{p-1}}, \inf_{[0,1/2]} v^{\frac{\alpha-p+1}{p-1}}\big\} > 0$. Then, since $\frac{\alpha}{p-1} > 1$, we obtain

$$\begin{aligned} u'(r) &< -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} v(r)^{\frac{\alpha}{p-1}} \\ &= -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} v(r)^{\frac{\alpha-p+1}{p-1}} v(r) \\ &\leq -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} Q_1 v(r)\,. \end{aligned} \tag{2.5}$$

Similarly, using the second equation of (2.1), we obtain

$$\begin{aligned} v'(r) &< -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} u(r)^{\frac{\alpha}{p-1}} \\ &= -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} u(r)^{\frac{\alpha-p+1}{p-1}} u(r) \\ &\le -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} Q_1 u(r). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\frac{(u+v)'(r)}{(u+v)(r)} < -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} Q_1\,, \quad r \in [0, 1/2].$$

Integrating the above inequality from 0 to 1/4 yields

$$\ln\Big(\frac{u(\frac14)+v(\frac14)}{u(0)+v(0)}\Big) = \int_0^{\frac14} \frac{(u+v)'(r)}{(u+v)(r)}\,\mathrm{d}r < -\Big(\frac{\lambda K}{N}\Big)^{\frac{1}{p-1}} Q_1 \int_0^{\frac14} r^{\frac{1}{p-1}}\,\mathrm{d}r = -\lambda^{\frac{1}{p-1}} C_0$$

where $C_0 := (\frac{K}{N})^{\frac{1}{p-1}} Q_1 \int_0^{\frac14} r^{\frac{1}{p-1}}\,\mathrm{d}r > 0$, and hence

$$u(1/4) + v(1/4) \le [u(0)+v(0)] e^{-\lambda^{\frac{1}{p-1}} C_0}.$$

Then there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, one has

$$v(1/4) < u(1/4) + v(1/4) < \frac{v_0 + V_*}{2}\,,$$

a contradiction to the fact that $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$.

Case 2. Suppose (ii) holds. Then $u(r_1) = \frac{u_0+U_*}{2}$ for some $r_1 \in [0, 1/2]$ and $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Using the inequality (2.4), for $r \in [0, 1/2]$, we obtain

$$u'(r) < -\Big(\frac{\lambda r K}{N}\Big)^{\frac{1}{p-1}} v(r)^{\frac{\alpha}{p-1}} < -\lambda^{\frac{1}{p-1}} \left(\frac{K}{N}\right)^{\frac{1}{p-1}} Q_2 r^{\frac{1}{p-1}}\,,$$

where $Q_2 := (\frac{v_0+V_*}{2})^{\frac{\alpha}{p-1}}$. Now integrating again from 0 to r_1, above inequality yields

$$u(r_1) - u(0) < -\lambda^{\frac{1}{p-1}} \Big(\frac{K}{N}\Big)^{\frac{1}{p-1}} Q_2 \int_0^{r_1} r^{1/(p-1)}\,\mathrm{d}r\,.$$

Therefore, there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, one has

$$u(r_1) < u(0) - \lambda^{\frac{1}{p-1}} \Big(\frac{K}{N}\Big)^{\frac{1}{p-1}} Q_2 \int_0^{r_1} r^{\frac{1}{p-1}}\,\mathrm{d}r < \frac{u_0 + U_*}{4},$$

a contradiction since $u(r_1) = \frac{u_0+U_*}{2}$. This concludes the proof of first part of the lemma.

Finally, since $0 < r_1 \le 1/2$, it follows from (2.5) that

$$|u'(r_1)| \ge \lambda^{\frac{1}{p-1}} \Big|\frac{K v^{\alpha}(r_1)}{N r_1}\Big|^{\frac{1}{p-1}} \to \infty \quad \text{as } \lambda \to \infty\,.$$

Similarly $|v'(\widetilde{r_1})| \to \infty$ as $\lambda \to \infty$. This completes the proof. □

The next lemma guarantees that each component of the positive solution will achieve any prescribed value below the zeros of the corresponding nonlinearities near the boundary of the ball.

Lemma 2.3. *Let (u,v) be a positive solution of (2.1) and $c,\widetilde{c}>2$ be any fixed constants. Then there exists $\lambda^{**}>0$ such that for all $\lambda>\lambda^{**}$, there exist $r_2=r_2(\lambda)$, $\widetilde{r_2}=\widetilde{r_2}(\lambda)\in\big[\frac{3}{4},1\big)$ satisfying*

$$u(r_2)=\frac{u_0}{c}\quad \textit{and}\quad v(\widetilde{r}_2)=\frac{v_0}{\widetilde{c}}.$$

Proof. Let $c,\widetilde{c}>2$, and assume that the lemma is false. Then, there exists a sequence $\{\lambda_n\}_n$ with $\lambda_n\to\infty$ as $n\to\infty$ and a corresponding sequence of positive solutions $\{(u_{\lambda_n},v_{\lambda_n})\}_n$ of (2.1) such that for all $n\in\mathbb{N}$, either $u_{\lambda_n}(r)\neq\frac{u_0}{c}$ for all $r\in[3/4,1)$ or $v_{\lambda_n}(r)\neq\frac{v_0}{\widetilde{c}}$ for all $r\in[3/4,1)$. Without loss of generality, assume that for all $n\in\mathbb{N}$, we have

$$u_{\lambda_n}(r)\neq\frac{u_0}{c}\quad\text{for all } r\in[3/4,1)\,.$$

Then, we need to analyze the following two cases:

Case 1. $v_{\lambda_n}(r)\neq\frac{v_0}{\widetilde{c}}$ for all $r\in[3/4,1)$ and for all $n\in\mathbb{N}$. Since u_{λ_n} is continuous, we observe that either $u_{\lambda_n}(r)>u_0/c$ for $r\in[3/4,1)$ or $u_{\lambda_n}(r)<u_0/c$ for $r\in[3/4,1)$. But the boundary condition $u_{\lambda_n}(1)=0$ implies that we must have $u_{\lambda_n}(r)<\frac{u_0}{c}$ on $[3/4,1)$. Similar argument yields $v_{\lambda_n}(r)<\frac{v_0}{\widetilde{c}}$ on $[3/4,1)$.

Integrating the first equation of (2.1) from $r\in(3/4,1)$ to 1, we obtain

$$r^{N-1}\phi_p(u'_{\lambda_n}(r))=\phi_p(u'_{\lambda_n}(1))+\lambda_n\int_r^1 s^{N-1}f(v_{\lambda_n}(s))\mathrm{d}s\,.$$

Then, using the facts that $v_{\lambda_n}<\frac{v_0}{\widetilde{c}}<\frac{v_0}{2}$, f is nondecreasing, $u'_{\lambda_n}(1)\le 0$, and ϕ_p is odd and increasing, the above equation yields

$$r^{N-1}\phi_p(u'_{\lambda_n}(r))\le\lambda_n f(v_0/2)\int_r^1 s^{N-1}\mathrm{d}s\le\frac{\lambda_n f(v_0/2)}{N}.$$

Using the properties of ϕ_p and the facts that $f(v_0/2)<0$ and $1/r^{N-1}>1$, we obtain

$$u'_{\lambda_n}(r)\le\phi_{p'}\Big(\frac{\lambda_n f(v_0/2)}{Nr^{N-1}}\Big)=-\Big|\frac{f(v_0/2)}{Nr^{N-1}}\Big|^{p'-1}\lambda_n^{p'-1}<-L\lambda_n^{\frac{1}{p-1}},$$

where $L:=|\frac{f(v_0/2)}{N}|^{p'-1}>0$. This gives $-u'_{\lambda_n}(r)>L\lambda_n^{\frac{1}{p-1}}$ and hence for $r\in[3/4,1)$, we have

$$u_{\lambda_n}(r)=-\int_r^1 u'_{\lambda_n}(s)\mathrm{d}s>L\lambda_n^{\frac{1}{p-1}}\int_r^1\mathrm{d}s=L\lambda_n^{\frac{1}{p-1}}(1-r).$$

In particular, for $r=4/5\in[3/4,1)$, we have

$$u_{\lambda_n}(4/5)\ge\lambda_n^{\frac{1}{p-1}}\frac{L}{5}\,.$$

Taking λ_n large enough, say for $\lambda_n\ge\big(\frac{5u_0}{2L}\big)^{p-1}$, we arrive at the contradiction $u_{\lambda_n}\big(\frac{4}{5}\big)\ge\frac{u_0}{2}$.

Case 2. There exist $n_0\in\mathbb{N}$ and $r_0\in\big[\frac{3}{4},1\big)$ such that $v_{\lambda_{n_0}}(r_0)=\frac{v_0}{\widetilde{c}}<\frac{v_0}{2}$. Proceeding as in Case 1 with $n\ge n_0$ and $r\ge r_0$, we arrive at the same contradiction as in Case 1. $\square$

By the mean value theorem, there exist $r_3 \in (r_1, r_2)$ and $\widetilde{r}_3 \in (\widetilde{r}_1, \widetilde{r}_2)$ such that

$$|u'(r_3)| = \Big|\frac{u(r_2) - u(r_1)}{r_2 - r_1}\Big| \leq \frac{\frac{U_*}{2}}{\frac{1}{4}} = 2U_* \leq 2U_0, \tag{2.7}$$

$$|v'(\widetilde{r}_3)| = \Big|\frac{v(\widetilde{r}_2) - v(\widetilde{r}_1)}{\widetilde{r}_2 - \widetilde{r}_1}\Big| \leq \frac{\frac{V_*}{2}}{\frac{1}{4}} = 2V_* \leq 2V_0. \tag{2.8}$$

Now we are ready to show that u' and v' are bounded for r close to 1.

Lemma 2.4. *There exist positive constants K_1 and K_2 (both independent of λ) such that*

$$|u'(r)| \leq K_1 \text{ for all } r \in [r_3, 1) \quad \text{and} \quad |v'(r)| \leq K_2 \text{ for all } r \in [\tilde{r}_3, 1).$$

Proof. Let $r_f, r_g \in (0,1)$ be such that $u(r_g) = u_0$ and $v(r_f) = v_0$. We claim that (a) $r_3 \in [r_f, 1)$ and (b) $\widetilde{r}_3 \in [r_g, 1)$. We will establish (a), then the proof of (b) follows similarly.

If $r_f \leq r_1$, then we are done since $r_3 > r_1$. Suppose $r_f > r_1$ and assume to the contrary that $r_1 < r_3 < r_f$. Then, by (2.7), $|u'(r_3)| \leq 2U_0$. On the other hand, it follows from Lemma 2.2 that $|u'(r_1)| \to \infty$ as $\lambda \to \infty$. Now, since $f(v(r)) > 0$ for all $r \in (0, r_f)$, it follows from the first equation of (2.1) that

$$(r^{N-1}\phi_p(u'(r)))' = -\lambda r^{N-1} f(v(r)) < 0\,.$$

Then using the facts that $u' < 0$ on $(0,1)$ and ϕ_p is an odd, increasing homeomorphism, we conclude that u' is decreasing and hence $|u'|$ is increasing on $(0, r_f)$ and thus $|u'(r_1)| \leq |u'(r_3)|$, a contradiction. Hence $r_3 \in [r_f, 1)$.

Now, since $f(v(r)) \leq 0$ for $r \in [r_3, 1) \subset [r_f, 1]$, repeating the argument above, $|u'|$ is decreasing on $[r_3, 1]$ and thus $|u'(r)| \leq |u'(r_3)| \leq 2U_0 =: K_1$ for all $r \in [r_3, 1]$.

Similarly, using (2.8), we can establish that $|v'(r)| \leq |v'(\widetilde{r}_3)| \leq 2V_0 =: K_2$ for all $r \in [\widetilde{r}_3, 1]$. This completes the proof. $\square$

3. Proof of Theorem 1.1

To reach a contradiction, suppose (u, v) is a positive solution of (2.1) for $\lambda > \max\{\lambda^*, \lambda^{**}\}$, where λ^* and λ^{**} are as given in Lemma 2.2 and Lemma 2.3, respectively. Define $E : [0,1] \to \mathbb{R}$ by

$$\begin{aligned} E(r) := &-\int_r^1 (\phi_p(u'(s)))'v'(s)\mathrm{d}s \\ &-\int_r^1 (\phi_p(v'(s)))'u'(s)\mathrm{d}s + \lambda F(v(r)) + \lambda G(u(r)). \end{aligned} \tag{3.1}$$

It follows from the boundary condition $u(1) = 0 = v(1)$ and $F(0) = 0 = G(0)$ that $E(1) = 0$. Moreover, it is easy to see that $E \in C^1(0,1) \cap C[0,1]$ and that

$$E'(r) = (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) + \lambda f(v(r))v'(r) + \lambda g(u(r))u'(r)\,.$$

First, we will analyze $E'(r)$ to determine the sign of $E(r)$ on $[0,1]$. To do so, observe that (2.1) can be rewritten as

$$\begin{gathered}
-(\phi_p(u'(r)))' - \frac{N-1}{r}\phi_p(u'(r)) = \lambda f(v(r)) \quad \text{for } 0<r<1; \\
-(\phi_p(v'(r)))' - \frac{N-1}{r}\phi_p(v'(r)) = \lambda g(u(r)) \quad \text{for } \ 0<r<1; \\
u'(0) = u(1) = 0; \\
v'(0) = v(1) = 0.
\end{gathered} \tag{3.2}$$

Then using (3.2) and the facts that $u'<0$, $v'<0$ and $\phi_p(\cdot)$ is an odd homeomorphism, we obtain

$$\begin{aligned}
E'(r) &= (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) + \lambda f(v(r))v'(r) + \lambda g(u(r))u'(r) \\
&= (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) - (\phi_p(u'(r)))'v'(r) \\
&\quad - \frac{N-1}{r}\phi_p(u'(r))v'(r) - (\phi_p(v'(r)))'u'(r) - \frac{N-1}{r}\phi_p(v'(r))u'(r) \\
&= -\frac{N-1}{r}\phi_p(u'(r))v'(r) - \frac{N-1}{r}\phi_p(v'(r))u'(r) < 0.
\end{aligned}$$

Then, $E(1)=0$ implies that

$$E(r) \ge 0 \quad \text{for } r \in [0,1]. \tag{3.3}$$

We define $r^* := \max\{r_3, \tilde{r_3}\}$, where r_3 and $\tilde{r_3}$ are as defined in (2.7) and (2.8), respectively. Since $u'(r), v'(r) < 0$ for $r \in (0,1]$, $E(r^*)$ can be expressed as

$$E(r^*) = \int_{r^*}^1 (\phi_p(u'(s)))'|v'(s)|\mathrm{d}s + \int_{r^*}^1 (\phi_p(v'(s)))'|u'(s)|\mathrm{d}s + \lambda F(v(r^*)) + \lambda G(u(r^*)).$$

We will analyze $E(r^*)$ below to arrive at a contradiction. We note that, by Lemma 2.4,

$$\int_{r^*}^1 (\phi_p(u'(s)))'|v'(s)|\mathrm{d}s + \int_{r^*}^1 (\phi_p(v'(s)))'|u'(s)|\mathrm{d}s$$

is bounded since $|u'(r)| \le K_1$ and $|v'(r)| \le K_2$ for all $r \in [r^*, 1]$.

Further, since $u(r_3) > u(r_2) = u_0/c \ne 0$ and $v(\widetilde{r_3}) > v(\widetilde{r_2}) = v_0/\widetilde{c} \ne 0$ for fixed $c, \widetilde{c} > 2$ and, $r_3 < r_2$ and $\widetilde{r_3} < \widetilde{r_2}$, we see that $r^* \not\to 1$ for λ large. This implies that $u(r^*)$ and $v(r^*)$ are bounded away from zero for λ large.

On the other hand, for $\lambda > \max\{\lambda^*, \lambda^{**}\}$, $u(r^*) \le u(r_3) < u(r_1) = \frac{U_*+u_0}{2} < U_0$ and $v(r^*) \le v(\widetilde{r}_3) < u(\widetilde{r}_1) = \frac{V_*+v_0}{2} < V_0$. Hence $F(v(r^*)) < 0$ and $G(u(r^*)) < 0$ and bounded away from zero. Thus for λ sufficiently large $E(r^*) < 0$, a contradiction to (3.3). Therefore, there is no positive radial (radially symmetric and radially decreasing) solution of (2.1), and hence of (1.1), for λ large. This completes the proof.

Remark 3.1. For $p=2$, the energy functional used in [7] is given by $J(r) = u'(r)v'(r) + \lambda F(v(r)) + \mu G(u(r))$ for $r \in [0,1]$. On the other hand, for $p=2$, $E(r)$ given by (3.1) becomes

$$E(r) = -\int_r^1 u''(s)'v'(s)\mathrm{d}s - \int_r^1 v''(s)u'(s)\mathrm{d}s + \lambda F(v(r)) + \lambda G(u(r))\,.$$

Integrating by parts on the first integral yields

$$\begin{aligned} E(r) &= -u'(1)v'(1) + u'(r)v'(r) + \int_r^1 u'(s)'v''(s)\mathrm{d}s - \int_r^1 v''(s)u'(s)\mathrm{d}s + \lambda F(v(r)) \\ &\quad + \lambda G(u(r)) \\ &= -u'(1)v'(1) + u'(r)v'(r) + \lambda F(v(r)) + \lambda G(u(r)) \\ &= -u'(1)v'(1) + J(r). \end{aligned}$$

Thus $J(r)$ is the translation of $E(r)$ by $u'(1)v'(1)$.

Remark 3.2. For the scalar case, the functional used in [3] was

$$J(r) = \frac{p-1}{p}|u'(r)|^p \;+\; 2\lambda F(u(r)).$$

In this case, $E(r)$ is given by (3.1) can be expressed explicitly as,

$$E(r) = -\int_r^1 (p-1)|u'(s)|^{p-2}u''(s)u'(s)\mathrm{d}s \;+\; \lambda F(u(r)).$$

Using the fact that $-u'(s) = |u'(s)|$, and integrating, we obtain

$$\begin{aligned} E(r) &= \int_r^1 (p-1)|u'(s)|^{p-1}u''(s)\,\mathrm{d}s + \lambda F(u(r)) \\ &= \frac{p-1}{p}|u'(s)|^p \operatorname{sgn}(u'(s))\Big|_r^1 + \lambda F(u(r)) \\ &= \frac{p-1}{p}|u'(r)|^p - \frac{p-1}{p}|u'(1)|^p + \lambda F(u(r)) \\ &= J(r) - \frac{p-1}{p}|u'(1)|^p. \end{aligned}$$

Therefore, $J(r)$ is the translation of $E(r)$ by $\frac{p-1}{p}|u'(1)|^p$.

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