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HITTING PROPERTIES OF A RANDOM STRING

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Abstract We consider Funaki's model of a random string taking values in $\mathbf{R}^d$. It is specified by the following stochastic PDE,

$$\frac{\partial u(x)}{\partial t} = \frac{\partial^2 u(x)}{\partial x^2} + \dot{W}.$$

where $\dot{W} = \dot{W}(x,t)$ is two-parameter white noise, also taking values in $\mathbf{R}^d$. We find the dimensions in which the string hits points, and in which it has double points of various types. We also study the question of recurrence and transience.

Keywords and phrases Gaussian field, space-time white noise, stochastic PDE.

AMS subject classifications (1991) Primary, 60H15; Secondary, 35R60, 35L05.

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1 Introduction

In this paper we study hitting problems, double points and recurrence questions for the following model of a random string, first introduced by Funaki [Fun83]:

$$\frac{\partial u(x)}{\partial t} = \frac{\partial^2 u(x)}{\partial x^2} + \dot{W}. \tag{1.1}$$

Here $\dot{W}(x,t)$ is a $\mathbf{R}^d$ valued space-time white noise and $(u_t(x) : t \geq 0,\ x \in \mathbf{R})$ is a continuous $\mathbf{R}^d$ valued process. We give details of the meaning of this equation below. We also consider the analogous random loop, that is a solution indexed over $x \in \mathbf{T} = \mathbf{R}$ (mod 1), the circle. We give a brief motivation for this model. Under Newton's law of motion, the equation for the damped motion of a particle of mass m in a force field F is

$$m\frac{\partial^2 x(t)}{\partial t^2} = F(x(t)) - \lambda\frac{\partial x(t)}{\partial t}.$$

However if the particle has low mass, or the force field and damping are strong, then the motion is well approximated by Aristotle's law

$$\frac{\partial x(t)}{\partial t} = \lambda^{-1}F(x(t)),$$

which says that the velocity is proportional to the force. In the same way, the usual equation for an elastic string can be approximated by a heat equation. Allowing the string to move in $\mathbf{R}^d$ leads us to look for $\mathbf{R}^d$-valued solutions. If the string is influenced by white noise we arrive at the equation (1.1). Simple linear scaling allows us to set all parameters to one.

Before proceeding further, we now give an outline of our main results, which are stated in greater detail in theorems 1,2 and 3, later in the paper. We say that the random string hits a point $z \in \mathbf{R}^d$ if $u_t(x) = z$ for some $t > 0$, $x \in \mathbf{R}$. We shall show the following properties hold, each as an almost sure event:

- The random string hits points if and only if $d < 6$;
- For fixed $t_0 > 0$, there exist points x, y such that $u_{t_0}(x) = u_{t_0}(y)$ if and only if $d < 4$.
- There exist points (t,x) and (t,y) such that $u_t(x) = u_t(y)$ if and only if $d < 8$;
- There exist points (t,x) and (s,y) such that $u_t(x) = u_s(y)$ if and only if $d < 12$.

The string is Hölder continuous of any order less than 1/2 in space and 1/4 in time. This suggests that the range of the process $(t,x) \to u_t(x)$ might be 6 dimensional and leads to the guess that dimension $d = 6$ is the critical for hitting points and $d = 12$ is critical for double points. The second assertion in the above list is obvious, once we show that a for certain version of the string, $x \to u_{t_0}(x)$ is a Brownian motion parameterized by x. (See Remark 1 after Theorem 2, at the beginning of Section 5). It is well known that Brownian motion has double points if and only if $d < 4$ (see [Kni81], [DEK50], and [DEKT57]).

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Thanks to the Markov property and potential theory, we have detailed information about the hitting behavior of Brownian motion and other stochastic processes. Much less is known about the hitting behavior of random fields. There are 2 prominent exceptions to this statement. The first involves random fields of the form

$$Z(t_1, \ldots, t_n) = (X^{(1)}_{t_1}, \ldots, X^{(n)}_{t_n})$$

where $X^{(1)}_{t_1}, \ldots, X^{(n)}_{t_n}$ are independent processes. See Fitzsimmons and Salisbury [FS89] for this kind of work. The second case involves the Brownian sheet and other random fields with the multi-parameter Markov property (see Orey and Pruitt [OP73], Hirsch and Song [HS95], and Khoshnevesen and Shi [KS99]). In both of these cases, there is good information about what kind of sets the process will hit with positive probability. The above examples rely heavily on the Markov properties for the random fields in question. Peres [Per96] has also done some beautiful work with applications to the hitting properties of random fields. In addition to hitting questions, Orey and Pruitt [OP73] also studied the recurrence and transience of the Brownian sheet.

We employ two main methods in the proof. The finiteness of hitting times and the existence of double points of various types is deduced, below the critical dimensions, by a straightforward inclusion-exclusion argument. The more interesting direction is the proof of the non-existence of such events in critical dimensions. For this we make use of a stationary pinned version of the random string, constructed in section 2. Starting the string off as a two sided Brownian motion leads to a solution for which the distribution of $(u_{t+t_0}(x) - u_{t_0}(0) : t \geq 0,\, x \in \mathbf{R})$ does not depend on t_0. The word 'pinned' refers to looking at the image of the string under the map $f \to f - f(0)$. The image under this map is still a Markov process and the law of a two sided Brownian motion is its unique stationary distribution. The stationary pinned string has a simple scaling property that allows us to use scaling arguments, for example to show the Lebesgue measure of the range is zero in suitable critical dimensions. An absolute continuity argument, given in section 3, will show that our results hold for the string, not just in its stationary version, but also for the random loop. Sections 4 and 5 contain the arguments for hitting points and for double points respectively. It makes sense to ask for recurrence properties of the stationary string, in the same spirit as Orey and Pruitt's results for the Brownian sheet. In section 6 we show that the random string is recurrent if and only if $d \leq 6$. We finish this introduction by briefly discussing existence of solutions to (1.1), giving a simple inclusion-exclusion type lemma, and introducing common notations used in the text.

The components $\dot{W}_1(x,t), \ldots, \dot{W}_d(x,t)$ of the vector noise $\dot{W}(x,t)$ are independent space-time white noises, which are generalized Gaussian processes with covariance given by $E\left[\dot{W}_i(x,t)\dot{W}_i(y,s)\right] = \delta(t-s)\delta(x-y)$. That is, $W_i(f)$ is a random field indexed by functions $f \in \mathbf{L}^2([0,\infty) \times \mathbf{R})$, and for two such test functions $f, g \in \mathbf{L}^2([0,\infty) \times \mathbf{R})$ we have

$$E\left[W_i(f)W_i(g)\right] = \int_0^\infty \int f(t,x)g(t,x)\,dx dt.$$

Heuristically,

$$W_i(f) = \int_0^\infty \int f(t,x) W(dx\, dt)$$

We suppose that the noise is adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $\mathcal{F}$ is complete and $(\mathcal{F}_t)$ is right continuous, in that $W(f)$ is $\mathcal{F}_t$-measurable whenever f is supported in $[0,t] \times \mathbf{R}$.

The initial conditions play an unimportant role in the properties we study for the solutions to (1.1). We may take any initial conditions that are suitable for the deterministic heat equation. To be concrete, and so that we may apply results from the literature, we shall take initial conditions in $\mathcal{E}_{exp}$, the space of continuous functions of at most exponential growth, defined by $\mathcal{E}_{exp} = \cup_{\lambda>0} \mathcal{E}_\lambda$ where

$$\mathcal{E}_\lambda = \{ f \in C(\mathbf{R}, \mathbf{R^d}) : \ |f(x)| \exp(-\lambda |x|) \to 0 \text{ as } x \to \pm\infty \}.$$

We define a solution to (1.1) to be a $(\mathcal{F}_t)$ adapted, continuous random field $(u_t(x) : t \geq 0, \ x \in \mathbf{R})$ satisfying

(i) $u_0 \in \mathcal{E}_{exp}$ almost surely and is adapted to $\mathcal{F}_0$,

(ii) for each $t > 0$ there exists $\lambda > 0$ so that $u_s \in \mathcal{E}_\lambda$, for all $s \leq t$ almost surely,

(iii) for each $t > 0$ and $x \in \mathbf{R}$, the following Green's function representation holds

$$u_t(x) = \int G_t(x-y) u_0(y) dy + \int_0^t \int G_{t-r}(x-y) W(dy\, dr). \tag{1.2}$$

Here $G_t(x)$ is the fundamental solution $G_t(x) = (4\pi t)^{-1/2} \exp(-x^2/4t)$. We note that for initial conditions u_0 that are deterministic, or that are Gaussian fields independent of $\mathcal{F}_0$, the solutions are Gaussian fields. For any deterministic initial condition in $\mathcal{E}_{exp}$ there is a version of the solution (1.2) satisfying the regularity condition (ii), and the laws of these solutions form a Markov family in time.

For $\phi : \mathbf{R} \to \mathbf{R}^d$, we write (u_t, ϕ) for the integral $\int u_t(x)\phi(x)dx$, whenever this is well defined. The above definition of solutions is equivalent to a 'weak' formulation, in that condition (iii) may be replaced by the following: for all ϕ smooth and of compact support

$$(u_t, \phi) = (u_0, \phi) + \int_0^t (u_s, \Delta\phi) ds + \int_0^t \int \phi(y) W(dy\, ds).$$

Pardoux [Par93] or Walsh [Wal86] are references for the basic properties of SPDEs driven by space-time white noise as used above. The equivalence of the weak formulation is shown, in the case of real valued stochastic PDE solutions, in Shiga ([Shi94]).

We shall make frequent use of the following inclusion-exclusion type lemma.

Lemma 1. *Suppose that $(A_i : i = 1, \ldots, n)$ are events and $A = \bigcup_{i=1}^n A_i$ Then*

$$P(A) \geq \frac{\left[\sum_{i=1}^n P(A_i) \right]^2}{\sum_{i=1}^n P(A_i) + 2 \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)}. \tag{1.3}$$

This lemma is similar in spirit to the standard inclusion-exclusion bound

$$P(A) \geq \sum_{i=1}^n P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j).$$

For both lower bounds, one must find a lower bound for $\sum_{i=1}^n P(A_i)$. But to obtain a useful lower bound using Lemma 1, one often only need show that $\sum_{1\le i<j\le n} P(A_i \cap A_j)$ is comparable to $(\sum_{i=1}^n P(A_i))^2$, while for the standard inclusion-exclusion bound one must show that it is strictly less. The well-known proof is easy, but so short that we include it. For a real variable $Z \ge 0$ we have

$$E[Z] = E[Z\mathbf{1}(Z>0)] \le \left[E(Z^2)\, P(Z>0)\right]^{1/2}$$

by the Cauchy–Schwarz inequality. The lemma follows from this inequality, after rearranging, by taking $Z = \sum_i \mathbf{1}(A_i)$ and noting $P(A) = P(Z>0)$.

After submitting the paper, we noticed that below the critical dimension, our results can be derived from [GH80], Theorem 22.1, which gives conditions under which the occupation measure of a Gaussian process is absolutely continuous with respect to Lebesgue measure. It would then follow from Fubini's theorem that the range of the process has positive measure. Then, using our Girsanov-type result from Lemma 2, and adding a random drift to the process, it would follow that the probability of hitting 0 is positive. We leave these details to the reader, if he is interested. We give our own proof of these results, because it is simple and follows from first principles. Our argument may also be better adapted to nonlinear SPDE, whose solutions are not Gaussian.

Finally, here is some notation. For $x \in \mathbf{R}^d$ and $r \ge 0$ we write $B_r(x)$ for the box $\{y \in \mathbf{R}^d : |y_i - x_i| < r\}$. For use in our inclusion-exclusion arguments we define space and time grids of points by

$$t_{i,n} = i2^{-4n} \text{ and } \quad x_{j,n} = j2^{-2n} \text{ for } n,\, i,\, j \in \mathbf{Z}. \tag{1.4}$$

Throughout the paper $C(d,T,\ldots)$ will denote a constant, whose dependency will be listed, but whose value is unimportant and may change from line to line. Constants $c_1, c_2, \ldots$ with subscripts denote specific constants which do not change and may be referred to later in the text.

2 The stationary pinned string

The motivation for the pinned string comes from the following calculation. Starting from zero initial conditions, the solution to (1.1) is given by $u_t(x) = \int_0^t \int G_{t-r}(x-z) W(dz\,dr)$. The variance of the first component is given by

$$\begin{aligned} E\left[\left(u_t^1(x)\right)^2\right] &= \int_0^t \int G_r(x-z)^2 dz dr \\ &= \frac{1}{\sqrt{8\pi}} \int_0^t r^{-1/2} dr \\ &= \frac{1}{\sqrt{2\pi}} t^{1/2} \end{aligned}$$

and diverges to infinity as $t \to \infty$. However the variance of a spatial increment has the following limit as $t \to \infty$.

$$\begin{aligned} E\left[\left(u_t^1(x) - u_t^1(y)\right)^2\right] &= \int_0^t \int (G_r(x-z) - G_r(y-z))^2 dz dr \\ &\to \int_0^\infty \int (G_r(x-z) - G_r(y-z))^2 dz dr \\ &= |x-y|. \end{aligned}$$

One way to calculate the double integral above and justify the final equality is to apply Plancherel's theorem, using the Fourier transform $\hat{G}_r(\theta) = (2\pi)^{-1/2} \exp(-r\theta^2)$ of G_r, to rewrite this double integral as

$$\begin{aligned} &\frac{1}{2\pi} \int_0^\infty \int \exp(-2r\theta^2) |\exp(i(x-y)\theta) - 1|^2 d\theta dr \\ &= \frac{|x-y|}{2\pi} \int_0^\infty \int \exp(-2s\eta^2) |\exp(i\eta) - 1|^2 d\eta ds \\ &= \frac{|x-y|}{4\pi} \int \frac{|\exp(i\eta) - 1|^2}{\eta^2} d\eta, \end{aligned}$$

using the substitutions $\eta = \theta(x-y)$ and $s = r|x-y|^{-2}$. This shows the answer is of the form $c_0|x-y|$ and the remaining integral can be evaluated, for example by contour integration, showing that $c_0 = 1$. The limiting variance $|x-y|$ is exactly that of a two sided Brownian motion. The idea is to start a solution with this covariance structure and check that the spatial increments are stationary in time.

Motivated by the calculation above, we take an initial function $(U_0(x) : x \in \mathbf{R})$ which is a two-sided $\mathbf{R}^d$-valued Brownian motion satisfying $U_0(0) = 0$ and $E[(U_0(x) - U_0(y))^2] = |x-y|$, and which is independent of the white noise $\dot{W}$. This can be created in the following way: take an independent space-time white noise $\tilde{W}$ and let

$$U_0(x) = \int_0^\infty \int \left(G_r(x-z) - G_r(z)\right) \tilde{W}(dz\, dr).$$

Indeed, the calculation above shows that this integral has the correct covariance. We may assume, by extending the probability space if needed, that U_0 is $\mathcal{F}_0$-measurable. The solution to (1.1) driven by the noise $W(x,s)$ is then given by

$$\begin{aligned} U_t(x) &= \int G_t(x-z) U_0(z) dz + \int_0^t \int G_r(x-z) W(dz\, dr) \qquad (2.1) \\ &= \int_0^\infty \int \left(G_{t+r}(x-z) - G_{t+r}(z)\right) \tilde{W}(dz\, dr) \\ &\quad + \int_0^t \int G_r(x-z) W(dz\, dr). \end{aligned}$$

We call a continuous version of this process the stationary pinned string. Note that the components $(U_t^i(x) : t \geq 0,\, x \in \mathbf{R})$ for $i = 1, \ldots, d$ are independent and identically distributed.

Proposition 1. *The components* $(U_t^{(i)}(x) : x \in \mathbf{R}, t \geq 0)$ *of the stationary pinned string are mean zero Gaussian fields with the following covariance structure: for* $x, y \in \mathbf{R}$, $t \geq 0$

$$E\left[\left(U_t^{(i)}(x) - U_t^{(i)}(y)\right)^2\right] = |x-y|, \tag{2.2}$$

and for $x, y \in \mathbf{R}$, $0 \leq s < t$

$$E\left[\left(U_t^{(i)}(x) - U_s^{(i)}(y)\right)^2\right] = (t-s)^{1/2} F\left(|x-y|(t-s)^{-1/2}\right) \tag{2.3}$$

where

$$F(a) = (2\pi)^{-1/2} + \frac{1}{2} \int \int G_1(a-z) G_1(a-z') \left(|z| + |z'| - |z-z'|\right) dz dz'.$$

$F(x)$ *is smooth function, bounded below by* $(2\pi)^{-1/2}$, *and* $F(x)/|x| \to 1$ *as* $|x| \to \infty$. *Furthermore there exists* $c_1 > 0$ *so that for all* $x, y \in \mathbf{R}$, $0 \leq s \leq t$

$$c_1 \left(|x-y| + |t-s|^{1/2}\right) \leq E\left[\left(U_t^{(i)}(x) - U_s^{(i)}(y)\right)^2\right] \leq 2\left(|x-y| + |t-s|^{1/2}\right). \tag{2.4}$$

Proof. Aiming for (2.2), a simple calculation using the isometry for stochastic integrals, and the independence of the two integrals in (2.1), gives

$$\begin{aligned}
&E\left[\left(U_t^{(i)}(x)-U_t^{(i)}(y)\right)^2\right] \\
&\quad = E\left[\left(\int_0^\infty\int (G_{t+r}(x-z)-G_{t+r}(y-z))\,\tilde{W}(dz\,dr)\right)^2\right] \\
&\qquad\quad +E\left[\left(\int_0^t\int (G_r(x-z)-G_r(y-z))\,W(dz\,dr)\right)^2\right] \\
&\quad = \int_0^\infty\int (G_r(x-z)-G_r(y-z))^2\,dzdr \\
&\quad = |x-y|.
\end{aligned}$$

To calculate (2.3) we use the fact that

$$U_t(x)=\int G_{t-s}(x-z)U_s(z)dz+\int_s^t\int G_{t-r}(x-z)W(dz\,dr)$$

so that

$$\begin{aligned}
&E\left[\left(U_t^{(i)}(x)-U_s^{(i)}(y)\right)^2\middle|\mathcal{F}_s\right] \\
&\quad = E\Bigg[\Bigg(\int G_{t-s}(x-z)(U_s^{(i)}(z)-U_s^{(i)}(y))dz \\
&\qquad\quad +\int_s^t\int G_{t-r}(x-z)W(dz\,dr)\Bigg)^2\Bigg|\mathcal{F}_s\Bigg] \\
&\quad = \left(\int G_{t-s}(x-z)(U_s^{(i)}(z)-U_s^{(i)}(y))dz\right)^2+\int_s^t\int G_{t-r}^2(x-z)dzdr \\
&\quad = \left(\int G_{t-s}(x-z)(U_s^{(i)}(z)-U_s^{(i)}(y))dz\right)^2+\left(\frac{|t-s|}{2\pi}\right)^{1/2}.
\end{aligned}$$

Using (2.2) we have

$$\begin{aligned}
&E\left[\left(\int G_{t-s}(x-z)\Big(U_s^{(i)}(z)-U_s^{(i)}(y)\Big)dz\right)^2\right] \\
&\quad = \int G_{t-s}(x-z)\int G_{t-s}(x-z') \\
&\qquad\quad \cdot E\Big[\Big(U_s^{(i)}(z)-U_s^{(i)}(y)\Big)\Big(U_s^{(i)}(z')-U_s^{(i)}(y)\Big)\Big]dzdz' \\
&\quad = \frac{1}{2}\int G_{t-s}(x-z)\int G_{t-s}(x-z') \\
&\qquad\quad \cdot E\Big[\Big(U_s^{(i)}(z)-U_s^{(i)}(y)\Big)^2+\Big(U_s^{(i)}(z')-U_s^{(i)}(y)\Big)^2 \\
&\qquad\qquad -\Big(U_s^{(i)}(z)-U_s^{(i)}(z')\Big)^2\Big]dzdz' \\
&\quad = \frac{1}{2}\int G_{t-s}(x-z)\int G_{t-s}(x-z')\left(|z-y|+|z'-y|-|z-z'|\right)dzdz'.
\end{aligned}$$

The scaling $G_r(x) = r^{-1/2} G_1(x/r^{1/2})$ now leads to the covariance formula (2.3). The function $F(a)$ can be expressed in terms of exponentials and Gaussian error functions. However, the form given makes it clear that $F(a) \geq (2\pi)^{-1/2}$. The calculations needed to establish the other properties of $F(a)$ are straightforward and omitted.

The upper bound in (2.4) follows directly from (2.2) and (2.3). This upper bound implies that there is a continuous version of $(t,x) \to U_t(x)$ and that this version satisfies the growth estimates (ii) in the definition of a solution (1.2). The lower bound in (2.4) is immediate from (2.2) in the case $s = t$. If $|x-y| < |t-s|^{1/2}$ then we argue that

$$\begin{aligned} E\left[\left(U_t^{(i)}(x) - U_s^{(i)}(y)\right)^2\right] &= |t-s|^{1/2} F(|x-y|\,|t-s|^{-1/2}) \\ &\geq \frac{1}{\sqrt{2\pi}} |t-s|^{1/2} \\ &\geq \frac{1}{2\sqrt{2\pi}} \left(|x-y| + |t-s|^{1/2}\right). \end{aligned}$$

If $0 < |t-s|^{1/2} \leq |x-y|$ then we argue that

$$\begin{aligned} E\left[\left(U_t^{(i)}(x) - U_s^{(i)}(y)\right)^2\right] &= |t-s|^{1/2} F(|x-y|\,|t-s|^{-1/2}) \\ &\geq |x-y| \inf\{F(z)/z : z \geq 1\} \\ &\geq \frac{1}{2} \inf\{F(z)/z : z \geq 1\} \left(|x-y| + |t-s|^{1/2}\right). \end{aligned}$$

Combining the three cases, we may take

$$c_1 = \min\left((8\pi)^{-1/2}, \inf\{F(z)/2z : z \geq 1\}\right).$$

This finishes the proof. □

Corollary 1. *The stationary pinned string has the following properties:*

- **Translation invariance** *For any* $t_0 \geq 0$ *and* $x_0 \in \mathbf{R}$ *the field*

$$(U_{t_0+t}(x_0 \pm x) - U_{t_0}(x_0) : x \in \mathbf{R}, t \geq 0)$$

has the same law as the stationary pinned string.

- **Scaling** *For* $L > 0$ *the field*

$$(L^{-1} U_{L^4 t}(L^2 x) : x \in \mathbf{R}, t \geq 0)$$

has the same law as the stationary pinned string.

- **Time reversal** *For any* $T > 0$ *the field*

$$(U_{T-t}(x) - U_T(0) : x \in \mathbf{R},\ 0 \leq t \leq T)$$

has the same law as the stationary pinned string over the interval $[0, T]$.

Proof. The covariance formulae (2.2) and (2.3), together with $U_0(0) = 0$, characterize the law of the stationary pinned string. The translation invariance, scaling and time reversal follow immediately by checking that this covariance structure is preserved. □

3 Absolute Continuity Results

There are general criteria for the absolute continuity of Gaussian random fields (see for example Ibragimov and Rozanov [IR78]). For our differential equation setting we found it easier to exploit Girsanov's theorem (see Dawson [Daw78] and Nualart and Pardoux [NP94] for applications to stochastic PDEs). The following lemma deals with solutions to the perturbed equation

$$\frac{\partial v_t(x)}{\partial x} = \frac{\partial^2 v_t(x)}{\partial x^2} + h_t(x) + \dot{W}(x,t). \tag{3.1}$$

where $h_t(x) : [0,\infty) \times \mathbf{R} \to \mathbf{R}^d$ is an adapted, continuous function. Solutions to (3.1) are defined as in the introduction, with an extra drift term; for example the weak formulation has the extra integral $\int_0^t (h_s, \phi) ds$.

Lemma 2. *Suppose* $(u_t(x))$ *is a solution to (1.1) and* $(v_t(x))$ *is a solution to (3.1). Suppose also that they have the same deterministic initial condition* $u_0(x) = v_0(x) = f \in \mathcal{E}_{exp}$*. Then either of the following two conditions on* $h_t(x)$ *is sufficient to imply that the laws* $P_u^{(T)}$ *and* $P_v^{(T)}$ *of the solutions* $(u_t(x))$ *and* $(v_t(x))$*, on the region* $(t,x) \in [0,T] \times \mathbf{R}$*, are mutually absolutely continuous.*

(a) The drift $h_t(x)$ *is deterministic and satisfies* $\int_0^T \int |h_t(x)|^2 dx dt < \infty$.

(b) The drift $h_t(x)$ *has compact support* A *and is independent of* $(W(dx\, dt) : (t,x) \in A)$.

The lemma is an easy consequence of Girsanov's change of measure theorem. Indeed suppose $(v_t(x))$ is a solution to (3.1), started at f, on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Define

$$\frac{dQ}{dP} = \exp\left(\int_0^T \int h_t(x) \cdot W(dx\, dt) - \frac{1}{2} \int_0^T \int |h_t(x)|^2 dx\, dt \right). \tag{3.2}$$

In the case $h_t(x)$ is deterministic the stochastic integral is Gaussian and the exponential defines a martingale. Then under Q the process $v_t(x)$ is a solution to (1.1) started at f with respect to a new noise defined by $\tilde{W}(f) = W(f) - \int_0^t (h_s, f) ds$. This is easiest to check using the weak formulation of the equation and Levy's characterization of a space-time white noise W (see Walsh [Wal86] Chapter 3). In a similar way one can obtain a solution of (3.1) starting from a solution to (1.1).

In case (b), the same proof works once one knows that the exponential in (3.2) defines a true martingale. Since $h_t(x)$ is continuous and adapted the stochastic integral in (3.2) is well defined and the formula for dQ/dP defines a positive supermartingale. It is sufficient to check that it has expectation 1 to ensure it is a martingale. Let $\mathcal{G}$ be the σ-field generated by $(W(dx\, dt) : (t, x) \in A)$. Conditioned on $\mathcal{G}$, the stochastic integral is Gaussian, and so

$$E\left[\exp\left(\int_0^T \int h_t(x) \cdot W(dx\, dt) - \frac{1}{2} \int_0^T \int |h_t(x)|^2 dx dt \right) \middle| \mathcal{G} \right] = 1.$$

Taking a further expectation shows the exponential has expectation 1.

One consequence of the absolute continuity is that, under the conditions of the lemma, solutions to (3.1) are unique in law and satisfy the Markov property.

Corollary 2. *Suppose $(u_t(x))$ and $(\tilde{u}_t(x))$ are both solutions to (1.1). For compact sets $A \subseteq (0, \infty) \times \mathbf{R}$ the laws of the fields $(u_t(x) : (t, x) \in A)$ and $(\tilde{u}_t(x) : (t, x) \in A)$ are mutually absolutely continuous.*

Proof We may suppose that the initial functions $u_0 = f$ and $\tilde{u}_0 = g$ are fixed elements of $\mathcal{E}_{\exp}$, and that the two solutions are defined on the same probability space and with respect to the same noise W. The case where u_0 and $\tilde{u}_0$ are random then follows by using the Markov property at time zero. We may also suppose that A is a rectangle and choose a $\mathbf{C}^\infty$ function $\psi_t(x)$ that equals 1 on A and has compact support inside $(0, \infty) \times \mathbf{R}$. Define

$$v_t(x) = u_t(x) + \psi_t(x) \int G_t(x - y)(g(y) - f(y)) dy.$$

Then using the representation (1.2) we see that $v_t(x) = \tilde{u}_t(x)$ for $(t, x) \in A$. Also $v_0 = f$ and it is straightforward to check that $(v_t(x))$ is a solution to (3.1) with

$$h_t(x) = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x} \right) \left(\psi_t(x) \int G_t(x - y)(g(y) - f(y)) dy \right).$$

Note that $h_t(x)$ is smooth, deterministic and of compact support and so certainly satisfies the hypothesis of Lemma (2). The result then follows from Lemma (2) by taking T large enough that $A \subseteq [0, T] \times \mathbf{R}$. □

Corollary 3. *Suppose* $(u_t(x))$ *is a solution to (1.1) and* $z \in \mathbf{R}^d$*. For any compact set* $A \subseteq (0,\infty) \times \mathbf{R}$ *the laws of the fields* $(u_t(x) : (t,x) \in A)$ *and* $(z + u_t(x) : (t,x) \in A)$ *are mutually absolutely continuous.*

Proof The proof is similar to the proof of the previous corollary, but one defines $v_t(x) = u_t(x) + z\,\psi_t(x)$ and changes $h_t(x)$ accordingly. □

Our next aim is to show sufficient absolute continuity to allow us to transfer our results from the random string to the random loop. A continuous adapted process $(\tilde{u}_t(x) : t \geq 0,\ x \in \mathbf{T})$ is a solution to the random loop form of (1.1) if it satisfies (1.2) where $G_t(x)$ is replaced by the Green's function for the heat equation on the circle and the stochastic integral is only over the circle $\mathbf{T}$. This requires only a white noise $W(dx\,dt)$ defined on $t \geq 0, x \in \mathbf{T}$.

The corollary below implies that the properties we prove about the random string in theorems 1,2 and 3 hold also for the random loop.

Corollary 4. *Suppose* $(u_t(x) : t \geq 0,\ x \in \mathbf{R})$ *is a solutions to (1.1) and* $(\tilde{u}_t(x) : t \geq 0, x \in \mathbf{T})$ *is a solution to (1.1) on the circle. For any compact set* $A \subseteq (0,\infty) \times (0,1)$ *the laws of the fields* $(u_t(x) : (t,x) \in A)$ *and* $(\tilde{u}_t(x) : (t,x) \in A)$ *are mutually absolutely continuous.*

Proof We may suppose that the initial functions $u_0 = f \in \mathcal{E}_{exp}$ and $\tilde{u}_0 = g \in C(\mathbf{T})$ are deterministic. The case where u_0 and $\tilde{u}_0$ are random then follows by using the Markov property at time zero. We also suppose that they are defined on the same probability space and the noise driving $(\tilde{u}_t(x))$ is the restriction to the circle of the noise W driving $(u_t(x))$.

We use a standard symmetry trick to extend the solution $(\tilde{u}_t(x))$ over the real line. We may extend the solution to $(\tilde{u}^{(per)}_t(x) : t \geq 0,\ x \in \mathbf{R})$ by making it periodic with period one. We also extend the noise to a noise $W^{(per)}(dx\,dt)$ over the whole line by making it periodic. Note that $\tilde{u}^{(per)}_t(x) = \tilde{u}_t(x)$ and $W^{(per)}(dx\,dt) = W(dx\,dt)$ for $t \geq 0,\ x \in \mathbf{T}$. Then $(\tilde{u}^{(per)}_t(x))$ satisfies (1.2) over the whole line, with the Green's function for the whole line but with the periodic noise $W^{(per)}(dx\,dt)$.

We again take a $\mathcal{C}^\infty$ function $\psi_t(x)$ that equals 1 on A and still has compact support inside $(0,\infty) \times (0,1)$. Define

$$\begin{aligned} v_t(x) &= u_t(x) + \psi_t(x) \int G_t(x-y) \left(g^{(per)}(y) - f(y) dy \right) \\ &\quad + \psi_t(x) \int_0^t \int G_{t-s}(x-y) \left(W^{(per)}(dy\,ds) - W(dy\,ds) \right). \end{aligned}$$

Then using the representation (1.2) we see that $v_t(x) = \tilde{u}_t(x)$ for $(t,x) \in A$. Also $v_0 = f$ and it is straightforward to check that $(v_t(x))$ is a solution to (3.1) with

$$\begin{aligned} h_t(x) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x} \right) \left(\psi_t(x) \int G_t(x-y) \left(g^{(per)}(y) - f(y) \right) dy \right) \\ &\quad + \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x} \right) \Bigg(\psi_t(x) \int_0^t \int G_{t-s}(x-y) \\ &\qquad\qquad \cdot \left(W^{(per)}(dy\,ds) - W(dy\,ds) \right) \Bigg). \end{aligned}$$

Note that $h_t(x)$ has compact support. We claim that $h_t(x)$ is also smooth. The only term in $h_t(x)$ for which this is not clear is the stochastic integral

$$I(t,x) = \int_0^t \int G_{t-s}(x-y) \left(W^{(per)}(dy\,ds) - W(dy\,ds) \right).$$

However since $W^{(per)}(dy\,ds) - W(dy\,ds) = 0$ for $y \in (0,1)$ the function $I(t,x)$ solves the deterministic heat equation in the region $[0,\infty) \times (0,1)$, with zero initial conditions and continuous random boundary values $I(t,1)$ and $I(t,0)$. Hence it is smooth in this region and since $\psi_t(x)$ is also supported in this region the claim follows.

Since $W^{(per)}(dy\,ds) - W(dy\,ds) = 0$ for $y \in (0,1)$, the perturbation $(h_t(x))$ is adapted to the σ-field $\mathcal{G}$ generated by the noise $W(f)$ for f supported outside $(0,1) \times [0,\infty)$. Hence the integrand $h_t(x)$ is independent of the noise $(W(dx\,dt) : t \geq 0,\, x \in \mathbf{T})$, and we can apply part (b) of Lemma 2. □

Corollary 5. *Suppose $(u_t(x))$ is a solution to (1.1). Suppose also that A^+ is a compact set in the half space $\mathbf{H}^+ = (0,\infty)\times(0,\infty)$ and A^- is a compact set in the half space $\mathbf{H}^- = (0,\infty)\times(-\infty,0)$. Then the law of the pair of fields*

$$\big((u_t(x) : (t,x) \in A^+), (u_t(x) : (t,x) \in A^-) \big)$$

is mutually absolutely continuous with respect to the law of

$$\Big((U_t(x) : (t,x) \in A^+), (\tilde{U}_t(x) : (t,x) \in A^-) \Big)$$

where $(U_t(x))$ and $(\tilde{U}_t(x))$ are independent copies of the stationary pinned string.

Proof We may suppose that the initial function $u_0 = f$ is deterministic. Suppose also $(u_t(x))$ is driven by a noise $W(dx\,dt)$. On the same probability space, construct solutions $(u_t^+(x))$ (respectively $(u_t^-(x))$) to (1.1) on the half space $\mathbf{H}^+$ (respectively $\mathbf{H}^-$) with zero initial conditions and with Dirichlet boundary conditions along the axis $\{x = 0\}$. The noise driving $(u_t^+(x))$ (respectively $(u_t^-(x))$) is $(W(dx\,dt) : (t,x) \in \mathbf{H}^+)$ (respectively $(W(dx\,dt) : (t,x) \in \mathbf{H}^-)$). We can represent the solution $u_t^+(x)$ by

$$u_t^+(x) = \int_0^t \int G_{t-s}(x-y) W^+(dy\,ds), \quad \text{for } t \geq 0,\ x \geq 0$$

where $W^+(dx\,dt)$ is the odd extension of the noise $(W(dx\,dt) : (t,x) \in \mathbf{H}^+)$ defined by

$$W^+([-b,-a] \times [s,t]) = -W([a,b] \times [s,t]), \quad \text{for all } 0 \leq s \leq t,\ 0 \leq a \leq b.$$

A similar representation holds for $u_t^-(x)$ using an odd extension of $(W(dx\,dt) : (t,x) \in \mathbf{H}^-)$. Note that $(u_t^+(x))$ and $(u_t^-(x))$ are independent.

Now choose $\psi_t^+(x)$ (respectively $\psi_t^-(x)$) smooth, equal to 1 on A^+ (respectively on A^-), and supported in $\mathbf{H}^+$ (respectively in $\mathbf{H}^-$). Define

$$\begin{aligned} v_t(x) &= u_t(x) + \psi_t^+(x)\Big(- \int G_t(x-y) f(y) dy \\ &\qquad + \int^t \int G_{t-s}(x-y) \left(W^+(dy\,ds) - W(dy\,ds)\right) \Big) \\ &\quad + \psi_t^-(x)\Big(- \int G_t(x-y) f(y) dy \\ &\qquad + \int^t \int G_{t-s}(x-y) \left(W^-(dy\,ds) - W(dy\,ds)\right) \Big). \end{aligned}$$

We now argue as in Corollary 4. $v_t(x)$ agrees with $u_t^+(x)$ on A^+ and with $u_t^-(x)$ on A^-. Also it solves (3.1) with a suitable drift $h_t(x)$ that satisfies the assumptions of Lemma 2 (b). So the law of the pair

$$\big((u_t(x) : (t,x) \in A^+), (u_t(x) : (t,x) \in A^-)\big)$$

is mutually absolutely continuous with respect the law of the pair

$$\big((u_t^+(x) : (t,x) \in A^+), (u_t^-(x) : (t,x) \in A^-)\big)\,.$$

But this second pair is independent and a similar argument to the above shows that $(u_t^+(x) : (t,x) \in A^+)$ (respectively $(u_t^-(x) : (t,x) \in A^-)$) is absolutely continuous with respect to $(U_t(x) : (t,x) \in A^+)$ (respectively $(\tilde{U}_t(x) : (t,x) \in A^+)$). □

4 Hitting Points

For a R^d-valued function $u_t(x)$ indexed over $(t,x) \in A \subseteq [0,\infty) \times \mathbf{R}$, we say that $(u_t(x) : (t,x) \in A)$ hits the point $z \in \mathbf{R}^d$ if $u_t(x) = z$ for some $(t,x) \in A$. The aim of this section is to prove the following result.

Theorem 1. *Suppose $(u_t(x) : t \geq 0, x \in \mathbf{R})$ is a solution to (1.1).*

(a) If $d \leq 5$ then $P((u_t(x) : (t,x) \in A)$ hits $z) > 0$ for all $z \in \mathbf{R}^d$ and all $A \subseteq [0,\infty) \times \mathbf{R}$ with non-empty interior.

(b) If $d \geq 6$ then $P((u_t(x) : t > 0, x \in \mathbf{R})$ hits $z) = 0$ for all $z \in \mathbf{R}^d$.

To prove the theorem we shall need the following lemma which gives covariance estimates on the events of the stationary pinned string hitting a small ball.

Lemma 3. *There exist constants $0 < c_2, c_3 < \infty$, depending only on the dimension d, so that the following bounds hold: for all $s,t \in [1,2]$, $x,y \in [-2,2]$ and $\delta \in (0,1]$*

$$P\Big(U_t(x) \in B_\delta(0)\Big) \geq c_2 \delta^d, \tag{4.1}$$

$$P\Big(U_t(x) \in B_\delta(0), U_s(y) \in B_\delta(0)\Big) \leq c_3 \delta^{2d} \left(|t-s|^{1/2} + |x-y|\right)^{-d/2}. \tag{4.2}$$

Proof of Lemma 3. The Gaussian variable $U_t^{(i)}(x)$ has mean zero and variance $t^{1/2}F(|x|t^{-1/2}) \geq (2\pi)^{-1/2}$ for $x \in [-2,2]$, $t \in [1,2]$. So it has a density which is bounded above by $(2\pi)^{-1/4}$ and (4.1) follows by the independence of the coordinates $U_t^{(i)}$. An analogous upper bound also holds.

To prove (4.2) we consider the mean zero Gaussian vector $(X,Y) = (U_t^{(i)}(x), U_s^{(i)}(y))$. The covariance (2.3) gives an expression for $\sigma_X^2 = E[X^2]$, $\sigma_Y^2 = E[Y^2]$ and $\rho_{X,Y}^2 = E[(X-Y)^2]$. The law of $X - E[X|Y]$ is Gaussian and a routine calculation shows its variance is given by

$$\mathrm{Var}\,(X - E[X|Y]) = \frac{\Big(\rho_{X,Y}^2 - (\sigma_X - \sigma_Y)^2\Big)\Big((\sigma_X + \sigma_Y)^2 - \rho_{X,Y}^2\Big)}{4\sigma_X^2}. \tag{4.3}$$

For mean zero Gaussian Z the probability $P(\mu + Z \in B_\delta(0))$ is maximized at $\mu = 0$. So we can bound the probability

$$P\Big(X \in B_\delta(0)|Y\Big) \leq C\delta \cdot \Big(\mathrm{Var}(X - E[X|Y])\Big)^{-1/2}$$

and hence obtain

$$P\Big(X \in B_\delta(0), Y \in B_\delta(0)\Big) \leq C\delta^2\Big(\mathrm{Var}(X - E[X|Y])\Big)^{-1/2}. \tag{4.4}$$

The covariance (2.3) implies that σ_X is bounded and bounded away from zero, for $t \in [1,2]$ and $|x| \leq 2$. The inequality (2.4) implies that $\rho_{X,Y}^2 \geq c_1\left(|t-s|^{1/2} + |x-y|\right)$. The differentiability of $F(z)$ and the mean value theorem combine to show that $|\sigma_X - \sigma_Y| \leq C\left(|t-s| + |x-y|\right)$. Using these bounds in (4.3) shows there exists $C > 0$ and $\epsilon > 0$ so that

$$\mathrm{Var}\Big(X - E[X|Y]\Big) \geq C\left(|t-s|^{1/2} + |x-y|\right) \tag{4.5}$$

whenever $t, s \in [1,2]$, $x, y \in [-2,2]$ and $|t-s| + |x-y| \leq \epsilon$. The variance $\mathrm{Var}(X - E(X|Y))$ is a continuous function of $s, t \in [1,2]$ and $x, y \in [-2,2]$. It vanishes in this region only on $s = t, x = y$ and hence is bounded below when $|t-s| + |x-y| \geq \epsilon$. So, changing the constant C if necessary, the lower bound (4.5) holds without the restriction $|t-s| + |x-y| \leq \epsilon$. Substituting (4.5) into (4.4), and using the independence of coordinates $U^{(i)}$ gives the desired bound. □

Proof of Theorem 1 in $d \leq 5$. We start with a series of five easy reductions. First, the projection of a solution into a lower dimension is still a solution. Thus we need only argue in dimension $d = 5$. Second, it is enough to prove the result when A is a compact rectangle in $(0,\infty) \times \mathbf{R}$. Third, by the absolute continuity from Corollary 2 it is enough to prove the result for the stationary pinned string. Fourth, the absolute continuity from Corollary 3 shows that $P\left((u_t(x) : (t,x) \in A) \text{ hits } z\right)$ is either zero for all z or strictly positive for all z. So it is enough to prove the result when $z = 0$. Fifth and finally, the scaling of the stationary pinned string implies that it is enough to consider the rectangle $A = [1,2] \times [0,1]$. To see this, note that if $P\left((U_t(x) : (t,x) \in A) \text{ hits } 0\right) > 0$ then, as above, the absolute continuity results imply for a solution $(u_t(x))$ to (1.1) started at any $f \in \mathcal{E}_{exp}$, and for any $z \in \mathbf{R}^d$, that $P\left((u_t(x) : (t,x) \in A) \text{ hits } z\right) > 0$. Then by applying the Markov property at a time t_0 one sees that

$$P\Big((U_{t+t_0}(x + x_0) : (t,x) \in A) \text{ hits } 0\Big) > 0$$

for any $t_0 \geq 1$ and $x_0 \in \mathbf{R}$. The scaling of the stationary pinned string gives, for any $0 < r < s$ and $a < b$,

$$
\begin{aligned}
&P\Big((U_t(x) : (t,x) \in [r,s] \times [a,b]) \text{ hits } 0\Big) \\
&\quad = \; P\left((U_t(x) : (t,x) \in [L^4 r, L^4 s] \times [L^2 a, L^2 b]) \text{ hits } 0\right) \\
&\quad \geq \; P\left((U_{t+(L^4 r-1)}(x + L^2 a) : (t,x) \in A) \text{ hits } 0\right) \\
&\quad > \; 0
\end{aligned}
$$

provided we pick, as we may, L large enough that $L^4 r \geq 1$, $L^4(s-r) \geq 1$, and $L^2(b-1) \geq 1$.

Now fix $A = [1,2] \times [0,1]$ for the rest of this proof. Recall the grid of points $t_{i,n}$ and $x_{i,n}$ defined in (1.4). Let

$$
\delta_n = 2^{-6n/5}
$$

and define events

$$
\mathcal{B}_{i,j,n} = \left\{ U_{1+t_{i,n}}(x_{j,n}) \in B_{\delta_n}(0) \right\}, \quad \mathcal{B}_n = \bigcup_{i=1}^{2^{4n}} \bigcup_{j=1}^{2^{2n}} \mathcal{B}_{i,j,n}.
$$

We shall show that $P(\mathcal{B}_n) \geq p_0 > 0$ for all n. Then, using continuity of U and the compactness of A,

$$
P\Big((U_t(x) : (t,x) \in A) \text{ hits the point } 0\Big) \geq P(\mathcal{B}_n \text{ infinitely often}) \geq p_0.
$$

We shall apply Lemma 1 to the events $\mathcal{B}_{i,j,n}$. First, (4.1) applied in dimension $d = 5$ implies that

$$
\sum_{i=1}^{2^{4n}} \sum_{j=1}^{2^{2n}} P\left(\mathcal{B}_{i,j,n}\right) \geq c_2 2^{6n} \delta_n^5 = c_2. \tag{4.6}
$$

Second, using (4.2),

$$
\begin{aligned}
&\sum_{i=1}^{2^{4n}}\sum_{j=1}^{2^{2n}}\sum_{\tilde{i}=1}^{2^{4n}}\sum_{\tilde{j}=1}^{2^{2n}} P\left(\mathcal{B}_{i,j,n}\cap\mathcal{B}_{\tilde{i},\tilde{j},n}\right)\mathbf{1}\left((i,j)\neq(\tilde{i},\tilde{j})\right) \qquad (4.7)\\
\leq\;& 2\sum_{i=1}^{2^{4n}}\sum_{j=1}^{2^{2n}}\sum_{k=0}^{2^{4n}}\sum_{\ell=-2^{2n}}^{2^{2n}} P\left(\mathcal{B}_{i,j,n}\cap\mathcal{B}_{i+k,j+\ell,n}\right)\mathbf{1}\left((k,\ell)\neq(0,0)\right)\\
\leq\;& 2c_3 2^{6n}\delta_n^{10}\sum_{k=0}^{2^{4n}}\sum_{\ell=-2^{2n}}^{2^{2n}}\left(\left|k2^{-4n}\right|^{1/2}+\left|\ell 2^{-2n}\right|\right)^{-5/2}\mathbf{1}\left((k,\ell)\neq(0,0)\right)\\
\leq\;& 2^2c_3 2^{11n}\delta_n^{10}\sum_{k=0}^{2^{4n}}\sum_{\ell=0}^{2^{2n}}\left(k^{1/2}+|\ell|\right)^{-5/2}\mathbf{1}\left((k,\ell)\neq(0,0)\right)\\
\leq\;& 2^2 3^{5/2}c_3 2^{11n}\delta_n^{10}\sum_{k=1}^{2^{4n}+1}\sum_{\ell=1}^{2^{2n}+1}\left(k^{1/2}+|\ell|\right)^{-5/2}\\
\leq\;& 2^2 3^{5/2}c_3 2^{11n}\delta_n^{10}\int_0^{2^{4n}+1}\int_0^{2^{2n}+1}\left(x^{1/2}+y\right)^{-5/2}dydx\\
\leq\;& 2^3 3^{3/2}c_3 2^{11n}\delta_n^{10}\int_0^{2^{4n}+1}x^{-3/4}dx\\
=\;& 2^{21/4}3^{3/2}c_3 2^{12n}\delta_n^{10}\\
\leq\;& 2^6 3^2 c_3
\end{aligned}
$$

Using Lemma 1, together with (4.6) and (4.7), we obtain

$$P(\mathcal{B}_n)\geq\frac{c_2^2}{1+2^6 3^2 c_3}>0$$

for all $n\geq 1$. This completes the proof that points can be hit in dimensions $d\leq 5$. The reader can check that the above proof would fail if we replace $d=5$ by $d=6$.

Proof of Theorem 1 in $d\geq 6$. We again make some reductions. By considering projections of the string into lower dimensions, it is enough to consider dimension $d=6$. It is enough to show that $P(u_t(x)=z$ for some $(t,x)\in A)=0$ for a bounded rectangle A. It is then enough to consider the stationary pinned string and again, using scaling, it is enough to consider $A=[0,1)\times[0,1)$. Finally, since the probability $P(U_t(x)=z$ for some $(t,x)\in A)$ is either zero for all z or strictly positive for all z, the problem can be tackled by studying the range of the process, defined by

$$U(A)=\{U_t(x):(t,x)\in A\}\subseteq\mathbf{R}^6.$$

Indeed, if we denote the Lebesgue measure of $U(A)$ by $m(U(A))$ then

$$E[m(U(A))]=\int_{\mathbf{R}^6}P(U_t(x)=z\text{ for some }(t,x)\in A)\,dz,$$

which is zero if and only if the integrand is identically zero.

Subdivide A into eight disjoint rectangles $A_1, \ldots, A_8$, each a translate of $[0, 1/4) \times [0, 1/2)$. The scaling and translation invariance of the stationary pinned string and of the Lebesgue measure in dimension $d = 6$ implies that $E[m(U(A_i))] = (1/8)E[m(U(A))]$ for all $i = 1, \ldots, 8$. However by an 'inclusion-exclusion' type argument

$$m(U(A)) \leq \sum_{i=1}^{8} m(U(A_i)) - m\Big(U(A_1) \cap U(A_2)\Big).$$

Taking expectation of both sides shows that $E[m(U(A_1) \cap U(A_2))] = 0$. We may suppose that $A_1 = [0, 1/4) \times [0, 1/2)$ and $A_2 = [1/4, 1/2) \times [0, 1/2)$. Let $\mathcal{H}$ be the σ-field generated by $(U_{1/4}(x) : x \in \mathbf{R})$. Next, we use the Markov property of solutions and time reversal for the stationary pinned string. Conditioned on $\mathcal{H}$, the laws of $(U_t(x) : 1/4 \leq t \leq 1/2,\, x \in \mathbf{R})$ and $(U_{1/4-t}(x) : 1/4 \leq t \leq 1/2,\, x \in \mathbf{R})$ are identical and independent. So

$$\begin{aligned}
0 &= E\Big[m(U(A_1) \cap U(A_2))\Big] \\
&= \int_{\mathbf{R^6}} E\Big[\mathbf{1}(x \in U(A_1))\mathbf{1}(x \in U(A_2)\Big] dx \\
&= E\left(\int_{\mathbf{R^6}} E\left[\mathbf{1}(x \in U(A_1))\mathbf{1}(x \in U(A_2))\Big|\mathcal{H}\right] dx\right) \\
&= E\left(\int_{\mathbf{R^6}} E\left[\mathbf{1}(x \in U(A_1))\Big|\mathcal{H}\right] E\left[\mathbf{1}(x \in U(A_2))\Big|\mathcal{H}\right] dx\right) \\
&= E\left(\int_{\mathbf{R^6}} E\left[\mathbf{1}(x \in U(A_1))\Big|\mathcal{H}\right]^2 dx\right)
\end{aligned} \tag{4.8}$$

This implies that $E\left[\mathbf{1}(x \in U(A_1))\Big|\mathcal{H}\right] = 0$ for almost every x, almost surely. But then we have

$$E\left[m(U(A))\right] = 8E\left[m(U(A_1))\right] = E\left(\int_{\mathbf{R^6}} E\left[\mathbf{1}(x \in U(A_1))\Big|\mathcal{H}\right] dx\right) = 0$$

and therefore $m(U(A)) = 0$ almost surely, which concludes the proof . □

5 Double points

We consider two kinds of double points. For a $\mathbf{R}^d$ valued function $u_t(x)$, we say that $(u_t(x) : (t, x) \in A)$ has a double point at $z \in \mathbf{R}^d$ if there exist (t, x), $(t, y) \in A$, with $x \neq y$, so that $u_t(x) = u_t(y) = z$. We say that the range of the function $(u_t(x) : (t, x) \in A)$ has a double point z if there exist (t, x), $(s, y) \in A$, with $(t, x) \neq (s, y)$, such that $u_t(x) = u_s(y) = z$. The aim of this section is to prove the following result.

Theorem 2. *Suppose $(u_t(x) : t \geq 0, x \in \mathbf{R})$ is a solution to (1.1), and let $A \subseteq (0, \infty) \times \mathbf{R}$ have non-empty interior. The following statements hold almost surely.*

(a) If $d \leq 7$, then $(u_t(x) : (t, x) \in A)$ has a double point.

(b) If $d \geq 8$, then $(u_t(x) : t > 0, x \in \mathbf{R})$ has no double points.

(c) If $d \leq 11$, then the range of $(u_t(x) : (t,x) \in A)$ has a double point.

(d) If $d \geq 12$, then the range of $(u_t(x) : t > 0, x \in \mathbf{R})$ has no double points.

Remarks

1. One could also consider double points at a fixed time, that is, fix $t > 0$ and ask if there exist $x \neq y$ so that $u_t(x) = u_t(y)$. However, the covariance structure (2.2) implies that the process $x \to U_t(x) - U_t(0)$ is a two sided Brownian motion. As mentioned in the introduction, it is well known that there are double points, with non-zero probability, if and only if $d < 4$. Absolute continuity then shows the same holds true for general solutions to (1.1).

2. Parts (a) and (c) on the existence of double points follow by an inclusion-exclusion argument similar to that in theorem 1. We illustrate this by giving the argument for part (c), which is the more complicated, and leave the details of part (a) to the reader. In proof of non-existence, we need some small tricks to reduce the argument to the scaling property of the stationary string.

Proof of Theorem 2 (c): existence of double points of the range in dimensions $d \leq 11$. We can again make various reductions, arguing as in the proof of theorem 1. By projection it is enough to argue in dimensions $d = 11$. It is enough to consider bounded A, and hence by absolute continuity, enough to consider the stationary pinned string. Scaling and translation invariance for the stationary string again imply it is enough to consider one fixed rectangle, say $A = [0,4] \times [0,1]$.

For the rest of this proof we set

$$A_1 = [0,1] \times [0,1] \quad \text{and} \quad A_2 = [3,4] \times [0,1]$$

and $\delta_n = 2^{-12n/11}$. Define the events

$$\mathcal{B}_{i,j,k,\ell,n} = \left\{ U_{t_{i,n}}(x_{j,n}) - U_{3+t_{k,n}}(x_{\ell,n}) \in B_{\delta_n}(0) \right\}, \quad \mathcal{B}_n = \bigcup_{i,k=1}^{2^{4n}} \bigcup_{j,\ell=1}^{2^{2n}} \mathcal{B}_{i,j,k,\ell,n}.$$

We will show $P(\mathcal{B}_n) \geq p_0 > 0$ for all n. Then, by continuity and compactness, we have

$$\begin{aligned} & P\,(\text{the range of } (U_t(x) : (t,x) \in [0,4] \times [0,1]) \text{ has a double point}) \\ & \quad \geq \quad P(\{U_t(x) : (t,x) \in A_1\} \cap \{U_t(x) : (t,x) \in A_2\} \neq \emptyset) \\ & \quad \geq \quad P(\mathcal{B}_n \text{ infinitely often}) \geq p_0. \end{aligned}$$

We need the following lemma on the covariance structure of the events $\mathcal{B}_{i,j,k,\ell,n}$.

Lemma 4. *Suppose that $s_i, t_i, x_i, y_i \in [0,1]$ for $i = 1,2$. There exist constants $0 < c_4, c_5 < \infty$, depending only on the dimension d, so that for all $0 < \delta \leq 1$*

$$P\left(U_{t_1}(x_1) - U_{3+s_1}(y_1) \in B_\delta(0)\right) \geq c_4 \delta^d \tag{5.1}$$

$$\begin{aligned} & P\left(U_{t_1}(x_1) - U_{3+s_1}(y_1) \in B_\delta(0), U_{t_2}(x_2) - U_{3+s_2}(y_2) \in B_\delta(0)\right) \\ & \quad \leq \quad c_5 \delta^{2d} \left(|t_1 - t_2|^{1/2} + |s_1 - s_2|^{1/2} + |x_1 - x_2| + |y_1 - y_2| \right)^{-d/2}. \end{aligned} \tag{5.2}$$

We delay the proof of this lemma until after we complete the main argument. Using estimate (5.1), we conclude that

$$\sum_{i,k=1}^{2^{4n}} \sum_{j,\ell=1}^{2^{2n}} P\left(\mathcal{B}_{i,j,k,\ell,n}\right) \geq c_4 2^{12n} \delta_n^{11} = c_4. \tag{5.3}$$

Using estimate (5.2), we find that

$$\begin{aligned}
&\sum_{i_1,i_2,k_1,k_2=1}^{2^{4n}} \sum_{j_1,j_2,\ell_1,\ell_2=1}^{2^{2n}} P\left(\mathcal{B}_{i_1,j_1,k_1,\ell_1,n} \cap \mathcal{B}_{i_2,j_2,k_2,\ell_2,n}\right) \mathbf{1}_{((i_1,j_1,k_1,\ell_1)\neq(i_2,j_2,k_2,\ell_2))} \\
&\leq 2 \sum_{i_1,k_1=1}^{2^{4n}} \sum_{j_1,\ell_1=1}^{2^{2n}} \sum_{i_2,k_2=-2^{4n}}^{2^{4n}} \sum_{j_2,\ell_2=-2^{2n}}^{2^{2n}} P\left(\mathcal{B}_{i_1,j_1,k_1,\ell_1,n} \cap \mathcal{B}_{i_1+i_2,j_1+j_2,k_1+k_2,\ell_1+\ell_2,n}\right) \\
&\qquad \cdot \mathbf{1}_{((i_2,j_2,k_2,\ell_2)\neq(0,0,0,0))} \\
&\leq c_5 2^{n+1} \delta_n^{22} \sum_{i_2,k_2=-2^{4n}}^{2^{4n}} \sum_{j_2,\ell_2=-2^{2n}}^{2^{2n}} \Big(|i_2|^{1/2} + |j_2| + |k_2|^{1/2} + |\ell_2| \Big)^{-11/2} \\
&\qquad \cdot \mathbf{1}_{((i_2,j_2,k_2,\ell_2)\neq(0,0,0,0))}.
\end{aligned}$$

It is straightforward, as in the proof of theorem 1, to bound this quadruple sum by a constant, independent of n. Using this and (5.3) in Lemma 1 completes the proof that $P(\mathcal{B}_n) \geq p_0 > 0$.

One way to show that the probability of double points in the range is actually one is to use scaling and a zero one law. Alternatively one can use the following argument. Suppose that the probability of double points is strictly less than one. Then,

$$\begin{aligned}
&P\Big((U_t(x) : (t,x) \in [0,4] \times [0,1]) \text{ has no double points}\Big) \\
&= P\Big((U_t(x) : (t,x) \in [0,16] \times [0,2]) \text{ has no double points}\Big) \quad \text{(by scaling)} \\
&\leq P\Big((U_t(x) : (t,x) \in [0,4] \times [0,1]) \\
&\qquad \text{and } (U_t(x) : (t,x) \in [12,16] \times [0,1]) \text{ have no double points}\Big) \\
&< P\Big((U_t(x) : (t,x) \in [0,4] \times [0,1]) \text{ has no double points}\Big)
\end{aligned}$$

The strict inequality in the last line follows by applying the Markov property at time $t = 4$, the absolute continuity results and the translation invariance of the stationary string. Thus we have $p < p$, where p is the probability of double points. This is a contradiction.

Proof of Lemma 4. The proof follows the argument used for Lemma 3, with the change that we now consider the Gaussian pair

$$(X,Y) = \left(U_{t_1}(x_1) - U_{3+s_1}(y_1), U_{t_2}(x_2) - U_{3+s_2}(y_2)\right).$$

The covariance (2.3) implies that σ_X and σ_Y are bounded above and away from zero as s_i, t_i, x_i, y_i range over $[0,1]$. Using the identity

$$(a-b+c-d)^2 = (a-b)^2 + (c-d)^2 + (a-d)^2 + (b-c)^2 - (a-c)^2 - (b-d)^2$$

we may use (2.3) to find, for $t_1 \neq t_2$ and $s_1 \neq s_2$,

$$\begin{aligned} \rho^2_{X,Y} &= E\left((U_{t_1}(x_1) - U_{t_2}(x_2) + U_{3+s_2}(y_2) - U_{3+s_1}(y_1))^2\right) \\ &= |t_2 - t_1|^{1/2} F\left(|x_2 - x_1|\,|t_2 - t_1|^{-1/2}\right) \\ &\quad + |s_2 - s_1|^{1/2} F\left(|y_2 - y_1|\,|s_2 - s_1|^{-1/2}\right) \\ &\quad + H_{t_1 - s_1}(x_1 - y_1) + H_{t_2 - s_2}(x_2 - y_2) \\ &\quad - H_{t_1 - s_2}(x_1 - y_2) - H_{t_2 - s_1}(x_2 - y_1) \end{aligned} \tag{5.4}$$

where $H_r(z) = |3 + r|^{1/2} F\left(|z| \cdot |3 + r|^{-1/2}\right)$. Small changes are needed for the cases where $t_1 = t_2$ or $s_1 = s_2$, but these are easy and left to the reader. The function $H_r(z)$ is smooth for $r, z \in [-1, 1]$. The last four terms on the right hand side of (5.4) are differences of H at the four vertices of a parallelogram. Using the mean value theorem twice, these can be expressed as a double integral of second derivatives of H over the parallelogram. Hence the contribution of these last four terms is bounded by the size of the second derivatives and the area of the parallelogram and is thus at most $C(|t_2 - t_1|^2 + |s_2 - s_1|^2 + |x_2 - x_1|^2 + |y_2 - y_1|^2)$. Using (2.4) to bound the first two terms on the right hand side of (5.4) from below we find there exists $\epsilon > 0$ so that

$$\rho^2_{X,Y} \geq \frac{c_1}{2}\left(|t_2 - t_1|^{1/2} + |s_2 - s_1|^{1/2} + |x_2 - x_1| + |y_2 - y_1|\right),$$

whenever $t_i, s_i, x_i, y_i \in [0, 1]$ and $|t_2 - t_1| + |s_2 - s_1| + |x_2 - x_1| + |y_2 - y_1| \leq \epsilon$. The rest of the argument exactly parallels that of Lemma 3 and is omitted. □

Proof of Theorem 2 (b): non-existence of double points in dimensions $d \geq 8$. By a projection argument we need work only in dimension $d = 8$. It is enough to show there are no double points for $(u_t(x) : (t, x) \in A)$ for compact $A \subseteq (0, \infty) \times \mathbf{R}$, and hence by absolute continuity we can work with the stationary pinned string. We shall show that

$$P\left(0 \in \{U_t(x) - U_t(-y) : t,\, x,\, y \in [1, 2)\}\right) = 0. \tag{5.5}$$

By scaling and translation invariance this implies, for all $t_0, L \geq 0$ and $x_0 \in \mathbf{R}$, that

$$P\left(0 \in \left\{U_t(x) - U_t(-y) : t \in [t_0, t_0 + L^4),\ x, y \in [x_0 + L^2, x_0 + 2L^2)\right\}\right) = 0.$$

Taking a countable union of such events shows that there are no double points. Define

$$V(t, x, y) = U_{1+t}(1 + x) - U_{1+t}(-1 - y), \quad \text{for } t, x, y \in [0, 1).$$

We must show that $P(V(t, x, y) = 0$ for some $(t, x, y) \in [0, 1)^3) = 0$. Define, using an independent copy $\tilde{U}_t(x)$ of the stationary string,

$$\tilde{V}(t, x, y) = U_{1+t}(1 + x) - \tilde{U}_{1+t}(-1 - y), \quad \text{for } t, x, y \in [0, 1).$$

Corollary 5 implies that the laws of $(V(t, x, y) : (t, x, y) \in [0, 1)^3)$ and $(\tilde{V}(t, x, y) : (t, x, y) \in [0, 1)^3)$ are mutually absolutely continuous. Hence we may work with $\tilde{V}$ in place of V. The absolute continuity from Corollary 3 implies that $P(\tilde{V}(t, x, y) = z$ for some $(t, x, y) \in [0, 1)^3)$ is either zero for all z or strictly positive for all z. Hence, as in theorem 1 part (b), it is enough to show that $E(m(\tilde{V}([0,1)^3))) = 0$. We can now apply scaling. Subdivide $A = [0,1)^3$ into a disjoint union of 16 rectangles $(A_i : 1 = 1,\ldots,16)$ each of the form $A_i = (t_i, x_i, y_i) + A_0$, with $(t_i, x_i, y_i) \in A$ and $A_0 = [0,1/4) \times [0,1/2)^2$. Using the independence of U and $\tilde{U}$ and the scaling for the stationary pinned string with $L = 2^{-1/2}$, we obtain the following equality in law:

$$
\begin{aligned}
&m(\tilde{V}(A_i)) \\
&= m\left(\left\{U_{1+t}(1+x) - \tilde{U}_{1+t}(-1-y) : (t,x,y) \in (t_i,x_i,y_i) + A_0\right\}\right) \\
&\stackrel{\mathcal{L}}{=} m\Big(\big\{2^{-1/2}\left(U_t(x) - \tilde{U}_s(-y)\right) : \\
&\qquad\qquad (t,x,y) \in (4+4t_i, 2+2x_i, 2+2y_i) + A)\big\}\Big) \\
&= \frac{1}{16} m\left(\left\{U_t(x) - \tilde{U}_s(-y) : (t,x,y) \in (4+4t_i, 2+2x_i, 2+2y_i) + A)\right\}\right) \\
&= \frac{1}{16} m\Big(\Big\{\left[U_{4+4t_i+t}(2x_i + x) - U_{3+4t_i}(2x_i)\right] \\
&\qquad - \left[\tilde{U}_{4+4t_i+t}(2y_i + y) - \tilde{U}_{3+4t_i}(2y_i)\right] : (t,x,y) \in A)\Big\}\Big) \\
&\stackrel{\mathcal{L}}{=} \frac{1}{16} m(\tilde{V}(A)).
\end{aligned}
$$

The third equality uses the scale factor $(\sqrt{2})^d = (\sqrt{2})^8 = 16$; the fourth equality uses the fact that Lebesgue measure is unchanged by translation; the final equality in law uses the translation invariance of the stationary pinned string.

Using the inclusion-exclusion argument from the proof of theorem 1, we obtain

$$E\left[m\left(\tilde{V}(A_i) \cap \tilde{V}(A_j)\right)\right] = 0 \text{ for } i \neq j.$$

The rest of the argument is similar to the theorem 1 part (b). We may assume that

$$A_1 = [0,1/4) \times [0,1/2)^2, \quad A_2 = [1/4,1/2) \times [0,1/2)^2.$$

Define, for $(t,x,y) \in A_0$,

$$
\begin{aligned}
V^{(1)}(t,x,y) &= \left(U_{(5/4)+t}(1+x) - U_{5/4}(0)\right) - \left(\tilde{U}_{(5/4)+t}(-1-y) - \tilde{U}_{5/4}(0)\right), \\
V^{(2)}(t,x,y) &= \left(U_{(5/4)-t}(1+x) - U_{5/4}(0)\right) - \left(\tilde{U}_{(5/4)-t}(-1-y) - \tilde{U}_{5/4}(0)\right).
\end{aligned}
$$

Note that

$$m\left(\tilde{V}(A_1) \cap \tilde{V}(A_2)\right) = m\left(V^{(1)}(A_0) \cap V^{(2)}(A_0)\right).$$

Let $\mathcal{H}$ denote the σ-field generated by $(U_{5/4}(x), \tilde{U}_{5/4}(x) : x \in \mathbf{R})$. Using the Markov property, the time reversal and translation invariance of the stationary pinned string, the processes $V^{(1)}$ and $V^{(2)}$ are, conditioned on $\mathcal{H}$, independent and identically distributed. Now we can argue exactly as in (4.8) in the proof of theorem 1 part (b) to conclude that $E(m(\tilde{V}(A))) = 16E(m(\tilde{V}(A_1)) = 0$ which finishes the proof of (5.5). □

Proof of Theorem 2 (d): non-existence of double points of the range in dimensions $d \geq 12$. Only small changes are needed from the proof of part (b). Again by a projection argument we need work only in dimension $d = 12$. It is enough to show there are no double points in the range $(u_t(x) : (t,x) \in A)$ for compact sets $A \subseteq (0,\infty) \times \mathbf{R}$, and hence by absolute continuity we can work with the stationary pinned string. We shall show, for any $a \in \mathbf{R}$, that

$$P\left(0 \in \{U_t(x) - U_s(y) : (t,x,s,y) \in [3,4) \times [0,1) \times [0,1) \times [a,a+1)\}\right) = 1. \tag{5.6}$$

By scaling and translation invariance this implies, for all $t_0, L \geq 0$ and $x_0, y_0 \in \mathbf{R}$, that

$$\begin{aligned} P\Big(0 \in \big\{U_t(x) - U_s(y) : (t,s,x,y) \in [t_0, t_0 + L^4) \\ \times [t_0 + 3L^4, t_0 + 4L^4) \times [x_0, x_0 + L^2) \times [y_0, y_0 + L^2)\big\}\Big) \\ = \; 1. \end{aligned}$$

Taking a countable union of such events shows that there are no double points $U_t(x) = U_s(y)$ where $t \neq s$. Combining this with the result of part (b) of the theorem concludes the proof.

Define

$$V(t,s,x,y) = U_{3+t}(x) - U_{1-s}(a+y), \quad \text{for } t,s,x,y \in [0,1).$$

We must show that $P(V(t,s,x,y) = 0 \text{ for some } (t,s,x,y) \in [0,1)^4) = 0$. Define, using an independent copy $\tilde{U}_t(x)$ of the stationary string,

$$\tilde{V}(t,s,x,y) = U_{1+t}(x) - \tilde{U}_{1+s}(y), \quad \text{for } t,s,x,y \in [0,1).$$

We claim that the laws of $(V(t,s,x,y) : (t,s,x,y) \in [0,1)^4)$ and $(\tilde{V}(t,s,x,y) : (t,s,x,y) \in [0,1)^4)$ are mutually absolutely continuous. Indeed, let $\mathcal{H}$ be the σ-field generated by $(U_2(x) : x \in \mathbf{R})$. We can use the Markov property and the time reversal property of the stationary pinned string to conclude the following. The processes $(U_{3+t}(x) : (t,x) \in [0,1)^2)$ and $(U_{1-s}(a+y) : (s,y) \in [0,1)^2)$ are conditionally independent, with respect to $\mathcal{H}$. Also, each is a solution to (1.1). Now the claim follows by applying the absolute continuity from Corollary 2.

By the claim we may work with $\tilde{V}$ in place of V. The absolute continuity from Corollary 3 implies that $P(\tilde{V}(t,s,x,y) = z \text{ for some } (t,s,x,y) \in [0,1)^4)$ is either zero for all z or strictly positive for all z. Hence, as in theorem 1 part (b), it is enough to show that $E(m(\tilde{V}([0,1)^4))) = 0$. We can now apply scaling. Subdivide $A = [0,1)^4$ into a disjoint union of 64 rectangles $(A_i : 1 = 1, \ldots, 64)$ each of the form $A_i = (t_i, s_i, x_i, y_i) + A_0$, with $(t_i, s_i, x_i, y_i) \in A$ and $A_0 = [0,1/4)^2 \times [0,1/2)^2$. Using the independence of U and $\tilde{U}$ and the scaling for the stationary pinned string with $L = 2^{-1/2}$, we obtain the following equality in law:

$$\begin{aligned} & m(\tilde{V}(A_i)) \\ & = \; m\left(\left\{U_{1+t}(x) - \tilde{U}_{1+s}(y) : (t,s,x,y) \in (t_i, s_i, x_i, y_i) + A_0\right\}\right) \\ & \stackrel{\mathcal{L}}{=} \; m\Big(\Big\{2^{-1/2}\left(U_t(x) - \tilde{U}_s(y)\right) \\ & \qquad : (t,s,x,y) \in (4 + 4t_i, 4 + 4s_i, 2x_i, 2y_i) + A\Big\}\Big) \\ & = \; \frac{1}{64} m\left(\left\{U_t(x) - \tilde{U}_s(y) : (t,s,x,y) \in (4 + 4t_i, 4 + 4s_i, 2x_i, 2y_i) + A\right\}\right) \\ & = \; \frac{1}{64} m\Big(\Big\{\Big[U_{4+4t_i+t}(2x_i + x) - U_{3+4t_i}(2x_i)\Big] \\ & \qquad - \Big[\tilde{U}_{4+4s_i+s}(2y_i + y) - \tilde{U}_{3+4s_i}(2y_i)\Big] : (t,s,x,y) \in A)\Big\}\Big) \\ & \stackrel{\mathcal{L}}{=} \; \frac{1}{64} m(\tilde{V}(A)). \end{aligned}$$

We again obtain, using the inclusion-exclusion argument from the proof of theorem 1,

$$E\left[m\left(\tilde{V}(A_i)\cap\tilde{V}(A_j)\right)\right]=0 \text{ for } i\neq j.$$

We may assume that

$$A_1=[0,1/4)^2\times[0,1/2)^2,\quad A_2=[1/4,1/2)^2\times[0,1/2)^2.$$

Defining, for $(t,s,x,y)\in A_0$,

$$\begin{aligned} V^{(1)}(t,s,x,y) &= \left(U_{(5/4)+t}(x)-U_{5/4}(0)\right)-\left(\tilde{U}_{(5/4)+s}(x)-\tilde{U}_{5/4}(0)\right), \\ V^{(2)}(t,s,x,y) &= \left(U_{(5/4)-t}(x)-U_{5/4}(0)\right)-\left(\tilde{U}_{(5/4)-s}(x)-\tilde{U}_{5/4}(0)\right), \end{aligned}$$

we note that

$$m\left(\tilde{V}(A_1)\cap\tilde{V}(A_2)\right)=m\left(V^{(1)}(A_0)\cap V^{(2)}(A_0)\right).$$

Arguing exactly as in the proof of part (b) we may conclude that

$$E[m(\tilde{V}(A))]=64E[m(\tilde{V}(A_1))]=0,$$

which finishes the proof of (5.6). □

6 Transience and recurrence

For the N-parameter Brownian sheet in d dimensions, Orey and Pruitt [OP73] gave necessary and sufficient conditions on d and N for recurrence. In this section, we will study the same question for the stationary pinned string $(U_t(x))$ in $\mathbf{R}^d$. We say that a continuous function $(f_t(x): t\geq 0, x\in\mathbf{R})$ is recurrent if for any $\delta>0$ there exist sequences (x_n), (t_n), with $\lim_{n\to\infty}t_n=\infty$, so that $f_{t_n}(x_n)\in B_\delta(0)$. The aim of this section is to prove the following result.

Theorem 3. *The stationary pinned string $(U_t(x))$ in $\mathbf{R}^d$ is almost surely recurrent if $d\leq 6$ and almost surely not recurrent if $d\geq 7$.*

To help in the proof of Theorem 3, we first establish the following 0-1 law. Define

$$\begin{aligned} \mathcal{G}_N=\sigma\left\{U_0(x):|x|>N\right\} \\ \vee\sigma\left\{W(\varphi):\varphi(t,x)=0 \text{ if } 0\leq t\leq N \text{ and } |x|\leq N\right\}. \end{aligned}$$

We then set $\mathcal{G}=\bigcap_{N=1}^{\infty}\mathcal{G}_N$. We can show that $\mathcal{G}$ is trivial, using the independence of U_0 and W, and the arguments used to prove Kolmogorov's 0-1 law on the triviality of the Brownian tail σ-field.

Lemma 5. *Let $\mathcal{R}(\delta)$ be the event that there exist sequences (x_n), (t_n), with $t_n\to\infty$, so that $U_{t_n}(x_n)\in B_\delta(0)$, and let $\mathcal{R}=\cap_{\delta>0}\mathcal{R}_\delta$.*
Then $(\mathcal{R}(\delta):\delta>0)$ and $\mathcal{R}$ are all tail events in $\mathcal{G}$.

Proof of Lemma 5. For $N \geq 1$ and $t \geq N$, define

$$f_t^{(N)}(x) = \int_{-N}^{N} G_t(x-y)U_0(y)dy + \int_0^N \int_{-N}^{N} G_{t-s}(x-y)W(dy\,ds)$$

and set $U_t^{(N)}(x) = U_t(x) - f_t^{(N)}(x)$. Then subtracting $f_t^{(N)}(x)$ from the representation for $U_t(x)$ given in (2.1) shows that $(U_t^{(N)}(x) : t \geq N,\, x \in \mathbf{R})$ is $\mathcal{G}_N$-measurable. We claim that

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} \left| f_t^{(N)}(x) \right| = 0 \tag{6.1}$$

almost surely, for each $N \geq 1$. Assuming this claim then, since $B_\delta(0)$ is an open box, we see that the event $\mathcal{R}(\delta)$ is unchanged, up to a null set, if we replace $U_t(x)$ by $U_t^{(N)}(x)$ in its definition, implying that $\mathcal{R}(\delta)$ is a tail event.

To prove the claim (6.1), note that $f_t^{(N)}(x) = \int G_{t-N}(x-z)g^{(N)}(z)dz$, where

$$g^{(N)}(z) = \int_{-N}^{N} G_N(x-y)U_0(y)dy + \int_0^N \int_{-N}^{N} G_{N-s}(x-y)W(dy\,ds).$$

It is straightforward to show that $g^{(N)}(x)$ is almost surely in L^1. Then the inequality $\|f_t^{(N)}\|_\infty \leq (4\pi t)^{-1/2}\|g^{(N)}\|_1$ implies the claim (6.1). □

Proof of Theorem 3 in dimensions $d \leq 6$. By projection, it suffices to deal with the case $d = 6$. We will use an inclusion-exclusion argument again, working with values of the string $U_t(x)$ when t and x are integers. Fix $\delta \in (0,1]$, and, for integers i, j define

$$\mathcal{R}_{i,j} = \{U_i(j) \in B_\delta(0)\}, \quad \mathcal{R}(N,\delta) = \bigcup_{i=N}^{N^2} \bigcup_{0\leq j\leq i^{1/2}} \mathcal{R}_{i,j}.$$

Our aim is to use an inclusion-exclusion argument to show that $P(\mathcal{R}(N,\delta)) \geq p_0 > 0$ for all N sufficiently large. Then, using the definition in Lemma 5, we have

$$P(\mathcal{R}(\delta)) \geq P(\mathcal{R}(N,\delta) \text{ infinitely often}) \geq p_0 > 0.$$

By the zero-one law $P(\mathcal{R}(\delta)) = 1$ for any $\delta > 0$, which will complete the proof of recurrence.

The variance estimates (2.4) on $U_t(x)$ imply that there exist constants c_6, $c_7 > 0$, depending only on δ, so that for $i = 0, 1, \ldots$ and $j \in \mathbf{Z}$ with $(i,j) \neq (0,0)$

$$c_6(i^{1/2} + |j|)^{-3} \leq P(\mathcal{R}_{i,j}) \leq c_7(i^{1/2} + |j|)^{-3}. \tag{6.2}$$

So, for sufficiently large N,

$$\begin{aligned} \sum_{i=N}^{N^2} \sum_{0\leq j\leq i^{1/2}} P(\mathcal{R}_{i,j}) &\geq c_6 \sum_{i=N}^{N^2} \sum_{0\leq j\leq i^{1/2}} (i^{1/2} + |j|)^{-3} \\ &\geq \frac{c_6}{2} \int_N^{N^2} \int_0^{x^{1/2}} (x^{1/2} + y)^{-3} dy\, dx \\ &= \frac{3c_6}{16} \log(N). \end{aligned} \tag{6.3}$$

A similar calculation, using the upper bound in (6.2), shows that for sufficiently large N

$$\sum_{i=N}^{N^2} \sum_{0 \le j \le i^{1/2}} P(\mathcal{R}_{i,j}) \le 4c_7 \log(N). \tag{6.4}$$

From Lemma 3, we have

$$\begin{aligned} &P\left(U_1(x) \in B_\delta(0), U_{1+s}(x+y) \in B_\delta(0)\right) \\ &\qquad \le c_3 \delta^{12} (s^{1/2} + |y|)^{-3} \quad \text{whenever } x,\ y \in [-2,2],\ s \in [0,1]. \end{aligned}$$

Using the scaling for the stationary pinned string, with the choice $L = t^{1/4}$, we obtain $c_8 > 0$, depending only on δ so that

$$P\left(U_t(x) \in B_\delta(0), U_{t+s}(x+y) \in B_\delta(0)\right) \le c_8 (t^{1/2} + |x|)^{-3} (s^{1/2} + |y|)^{-3} \tag{6.5}$$

whenever $t \ge 1$, $|x|$, $|y| \le 2t^{1/2}$ and $s \in [0,t]$. We need the bound (6.5) for a larger set of parameters. Since the stationary string is a solution to (1.1) we have

$$U_{t+s}(x+y) = \int G_s(x+y-z) U_t(z) dz + \int_0^s \int G_{s-r}(x+y-z) W(dz\, dr)$$

so that

$$\begin{aligned} &\mathrm{Var}\Big(U_{t+s}(x+y) - E[U_{t+s}(x+y)|\mathcal{F}_t]\Big) \\ &= \mathrm{Var}\left(\int_0^s \int G_{s-r}(x+y-z) W(dz\, dr)\right) \\ &= C s^{1/2}. \end{aligned}$$

Hence

$$P(U_{t+s}(x+y) \in B_\delta(0)|\mathcal{F}_t) \le C s^{-3/2} \le 3^3\, C \cdot (s^{1/2} + |y|)^{-3},$$

provided $|y| \le 2s^{1/2}$. Using this we see that the bound (6.5) also holds, after possibly modifying the value of c_8, whenever $|y| \le 2s^{1/2}$.

Now we can estimate the covariance term for the event $\mathcal{R}(N,\delta)$.

$$\begin{aligned} &\sum_{i=N}^{N^2} \sum_{0 \le j \le i^{1/2}} \sum_{\tilde{i}=N}^{N^2} \sum_{0 \le \tilde{j} \le \tilde{i}'^{1/2}} P(\mathcal{R}_{i,j} \cap \mathcal{R}_{\tilde{i},\tilde{j}}) \mathbf{1}\left((i,j) \ne (\tilde{i},\tilde{j})\right) \\ &\le\ 2 \sum_{i=0}^{N^2} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N^2} \sum_{\ell=-\infty}^{\infty} P(\mathcal{R}_{i,j} \cap \mathcal{R}_{i+k,j+\ell}) \\ &\qquad \mathbf{1}\Big((k,\ell) \ne (0,0),\ |j| \le i^{1/2},\ |l| \le (i+k)^{1/2}\Big) \\ &\le\ 2c_8 \sum_{i=0}^{N^2} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N^2} \sum_{\ell=-\infty}^{\infty} (i^{1/2} + |j|)^{-3} (k^{1/2} + |\ell|)^{-3} \mathbf{1}\left((k,\ell) \ne (0,0)\right). \end{aligned} \tag{6.6}$$

To justify the second inequality, we note that for values of $k \geq i/3$ we have $|l| \leq 2(i+k)^{1/2} \leq 2k^{1/2}$, and we may apply (6.5). For values of $k \leq i/3$ we have $|l| \leq (i+k)^{1/2} \leq 2i^{1/2}$ and $j \leq i^{1/2}$, and again we may apply (6.5). Now we bound the quadruple sum in (6.6), for sufficiently large N, by

$$C\left(\int_1^{N^2}\int_0^\infty (x^{1/2}+y)^{-3} dy\, dx\right)^2 \leq c_9\left(\log(N)\right)^2$$

where c_9 depends only on δ. Using (6.3), (6.4) and (6.6) with Lemma 1, we see that $P(\mathcal{R}(N,\delta)) \geq p_0 > 0$ for sufficiently large N, completing the proof. □

Proof of Theorem 3 in dimensions $d \geq 7$. It again suffices, by a projection argument, to work in dimension $d = 7$. The strategy is to study the string along a grid of points, show that 'recurrence on this grid' is impossible, and then to control the pieces between the grid points. We define squares in the (t,x) plane as follows. Let $S_{i,j} = [i, i+1] \times [j, j+1]$ for $i = 1, 2, \ldots$ and $j \in \mathbf{Z}$. We will divide the squares $S_{i,j}$ into rectangles. To this end, let $m(i,j)$ be the unique integer such that

$$m(i,j)^3 \leq \left(i^{1/2} + |j|\right)^{1/4} < (m(i,j)+1)^3. \tag{6.7}$$

We divide each square $S_{i,j}$ into $m(i,j)^3$ rectangles, each a translate of $[0, m^{-2}] \times [0, m^{-1}]$, where $m = m(i,j)$. We say these rectangles are of type m. Let $M(m)$ be the number of rectangles of type m, let $(R_k^{(m)} : k = 1, \ldots, M(m))$ be an enumeration of the rectangles of type m, and let $(t_k^{(m)}, x_k^{(m)})$ be the point in $R_k^{(m)}$ with smallest (x,t) coordinates. Fix $\delta > 0$. Then, using the lower bound on the variance of $U_t(x)$ in (2.4), and the subdivision of $S_{i,j}$ into $m(i,j)^3$ rectangles, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{k=1}^{M(m)} P\left(U_{t_k^{(m)}}\left(x_k^{(m)}\right) \in B_{2\delta}(0)\right) &\leq C\sum_{i=1}^{\infty}\sum_{j\in\mathbf{Z}} m(i,j)^3\left(i^{1/2}+|j|\right)^{-7/2} \\
&\leq C\sum_{i=1}^{\infty}\sum_{j\in\mathbf{Z}}\left(i^{1/2}+|j|\right)^{-13/4} \\
&< \infty.
\end{aligned} \tag{6.8}$$

The finiteness of the double sum follows by bounding it by a suitable integral in the usual way. By the Borel-Cantelli lemma, the string, evaluated at the grid points $(t_k^{(m)}, x_k^{(m)})$, will eventually leave the box $B_{2\delta}(0)$. We now interpolate between the grid points. Using the boundedness of the variance of $U_t(x)$ over $(x,t) \in [0,1]^2$, we first apply Borel's inequality for Gaussian fields (see [Adl90] chapter II) to find constants $0 < c_{10}, c_{11} < \infty$ so that

$$P\left(\sup_{(t,x)\in[0,1]^2} |U_t(x)| \geq \delta\right) \leq c_1 \exp(-c_2\delta^2)$$

for all $\lambda > 0$. Now by translation invariance and then scaling we have, for any $m \geq 1$ and

$1 \leq k \leq M(m)$,

$$P\left(\sup_{(t,x)\in R_k^{(m)}} |U_t(x) - U_{t_k^{(m)}}(x_k^{(m)})| \geq \delta\right) \tag{6.9}$$
$$= P\left(\sup_{(t,x)\in[0,1/m^2]\times[0,1/m]} |U_t(x)| \geq \delta\right)$$
$$= P\left(\sup_{(t,x)\in[0,1]^2} |U_t(x)| \geq m^{1/2}\delta\right)$$
$$\leq c_1 \exp\left(-c_2 m\delta^2\right).$$

We can bound for the number $M(m)$ of rectangles of type m as follows. $M(m)$ equals m^3 times the number of squares $S_{i,j}$ with $m(i,j) = m$. Now, (6.7) implies that $i \leq (m+1)^{24}$ and $|j| \leq (m+1)^{12}$. So a crude bound on $M(m)$ is given by $M(m) \leq Cm^3m^{24}m^{12} = Cm^{39}$. Combining this with (6.9) we have

$$\sum_{m=1}^{\infty}\sum_{k=1}^{M(m)} P\left(\sup_{(t,x)\in R_k^{(m)}} \left|U_t(x) - U_{t_k^{(m)}}(x_k^{(m)})\right| \geq \delta\right) < \infty.$$

Combining this with (6.8) we may apply the Borel-Cantelli lemma to conclude that the probability of recurrence is zero, completing the proof. □

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