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ON THE CONVERGENCE OF STOCHASTIC INTEGRALS DRIVEN BY PROCESSES CONVERGING ON ACCOUNT OF A HOMOGENIZATION PROPERTY

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Abstract: We study the limit of functionals of stochastic processes for which an homogenization result holds. All these functionals involve stochastic integrals. Among them, we consider more particularly the Lévy area and those giving the solutions of some SDEs. The main question is to know whether or not the limit of the stochastic integrals is equal to the stochastic integral of the limit of each of its terms. In fact, the answer may be negative, especially in presence of a highly oscillating first-order differential term. This provides us some counterexamples to the theory of good sequence of semimartingales.

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Introduction

Among all the results on homogenization, the probabilistic approach is related to the intuitive idea of a particle in the highly heterogeneous media, but whose "statistical behavior" is close to that of a Brownian motion. The variance of this Brownian motions gives the effective coefficient of the media. Though there could exist some systems which are sensitive to some functional of trajectories. For example, the trajectories of the particles control a differential equation, or a differential one-form is integrated along them. Thus, one may ask if it is legitimate to substitute the trajectories of a Brownian motion to the trajectories of the particles. In other words, does the effective coefficient provide sufficient information to compute some approximations of such functionals? We show in this article that the answer may be negative.

In this article, we deal with operators of type

$$L^{\varepsilon} = \frac{1}{2} a_{i,j}(\cdot/\varepsilon) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\varepsilon} b_i(\cdot/\varepsilon) \frac{\partial}{\partial x_i} + c_i(\cdot/\varepsilon) \frac{\partial}{\partial x_i},$$

where a and b are periodic. Let us denote by X^{ε} the process generated by L^{ε}, and $\overline{b}$ the average of b with respect to the invariant measure of L^1 acting on the space of periodic functions. It is well known (see for example [1]) that the process $\widetilde{\mathsf{X}}^{\varepsilon} = (\mathsf{X}^{\varepsilon}_t - \overline{b}t/\varepsilon)_{t \geqslant 0}$ converges in distribution to a stochastic process $\overline{\mathsf{X}}$ given by $\overline{\mathsf{X}}_t = x + \sigma^{\mathrm{eff}} \mathsf{B}_t + c^{\mathrm{eff}} t$, where B is a Brownian motion, c^{eff} is a constant vector and $a^{\mathrm{eff}} = \sigma^{\mathrm{eff}} (\sigma^{\mathrm{eff}})^{\mathrm{T}}$ is a constant matrix called the effective coefficient of the media.

We are interested in three types of problems which are strongly linked:

(i) Let H^{ε} be a family of processes adapted to the filtration generated by X^{ε} and such that $(\mathsf{H}^{\varepsilon}, \widetilde{\mathsf{X}}^{\varepsilon})$ converges to $(\mathsf{H}, \overline{\mathsf{X}})$. Is the limit of $\int_0^t \mathsf{H}^{\varepsilon}_s \, \mathrm{d}\widetilde{\mathsf{X}}^{\varepsilon}_s$ as ε goes to 0 equal to $\int_0^t \mathsf{H}_s \, \mathrm{d}\overline{\mathsf{X}}_s$? We show with some examples that it could be true, but also that $\int_0^{\cdot} \mathsf{H}^{\varepsilon}_s \, \mathrm{d}\widetilde{\mathsf{X}}^{\varepsilon}_s$ may not converge at all, or that a corrective term appears.

(ii) We consider the convergence of the Lévy area of $(\widetilde{\mathsf{X}}^{i,\varepsilon}, \widetilde{\mathsf{X}}^{j,\varepsilon})$ for $i, j = 1, \dots, N$,

$$A^{i,j}_{s,t}(\widetilde{\mathsf{X}}^{\varepsilon}) = \frac{1}{2} \int_s^t (\widetilde{\mathsf{X}}^{i,\varepsilon}_r - \widetilde{\mathsf{X}}^{i,\varepsilon}_s) \, \mathrm{d}\widetilde{\mathsf{X}}^{j,\varepsilon}_r - \frac{1}{2} \int_s^t (\widetilde{\mathsf{X}}^{j,\varepsilon}_r - \widetilde{\mathsf{X}}^{j,\varepsilon}_s) \, \mathrm{d}\widetilde{\mathsf{X}}^{i,\varepsilon}_r.$$

It is shown that $A^{i,j}_{s,t}(\widetilde{\mathsf{X}}^{\varepsilon})$ converges to $A^{i,j}_{s,t}(\overline{\mathsf{X}}) + \overline{\psi}_{i,j}(t-s)$, where $\overline{\psi}_{i,j}$ is a constant. Some heuristic arguments of this fact could be found in [10].

(iii) Finally, we consider then the problem of the convergence of the solution Y^{ε} of some SDE driven by $\widetilde{\mathsf{X}}^{\varepsilon}$. Here again, interchanging the functional giving the solution of SDE from $\widetilde{\mathsf{X}}^{\varepsilon}$ and the passage to the limit does not always provide the limit of Y^{ε}.

Yet it has to be noted that if $b = 0$, then it is possible to interchange the passage to the limit and the functionals such that the one giving the Lévy area or the solution of an SDE.

The forthcoming article [10] also explains how the problems (ii) and (iii) are related. We summarize this link here: In [11] (see also [9, 12]), T. Lyons gives a pathwise definition of $\mathsf{Z}_t = z + \int_0^t f(\mathsf{X}_s)\,\mathrm{d}\mathsf{X}_s$ and $\mathsf{Y}_t = y + \int_0^t f(\mathsf{Y}_s)\,\mathrm{d}\mathsf{X}_s$ when X is a general process of finite p-variation with $p \in [2,3)$, provided one knows, for a piecewise smooth approximation X^δ of X, the limit of

$$A^{i,j}_{s,t}(\mathsf{X}^\delta) = \frac{1}{2}\int_s^t (\mathsf{X}^{i,\delta}_r - \mathsf{X}^{i,\delta}_s)\,\mathrm{d}\mathsf{X}^{j,\delta}_r - \frac{1}{2}\int_s^t (\mathsf{X}^{j,\delta}_r - \mathsf{X}^{j,\delta}_s)\,\mathrm{d}\mathsf{X}^{i,\delta}_r$$

for any $(i,j) \in \{1,\dots,N\}^2$. Moreover, the maps $\mathfrak{K} : \mathsf{X} \mapsto \mathsf{Z}$ and $\mathfrak{I} : \mathsf{X} \mapsto \mathsf{Y}$ are continuous in the topology of p-variation. The Lévy area $A_{0,t}(\mathsf{X}) = (A^{i,j}_{0,t}(\mathsf{X}))_{i,j=1,\dots,N}$ is a possible limit of $(A_{0,t}(\mathsf{X}^\delta))_{\delta>0}$. But there also exists some approximations X^δ of the trajectories of X such that $A_{0,t}(\mathsf{X}^\delta)$ converges to $A_{0,t}(\mathsf{X}) + \psi t$ for an antisymmetric matrix ψ. As explained in [10], with $A(\mathsf{X})$ as a limit of $A(\mathsf{X}^\delta)$, Y and Z are equal in distribution to the Stratonovich integrals $\mathsf{Z}_t = z + \int_0^t f(\mathsf{X}_s) \circ \mathrm{d}\mathsf{X}_s$ and $\mathsf{Y}_t = y + \int_0^t f(\mathsf{Y}_s) \circ \mathrm{d}\mathsf{X}_s$, while with $A_{s,t}(\mathsf{X}) + \psi(t-s)$, a drift is added to the previous integrals. Thus, using the continuity of $\mathfrak{K}$ and $\mathfrak{I}$, the asymptotic behavior of $A(\widetilde{\mathsf{X}})$ provides the limit of stochastic integrals or solutions of SDEs.

Although conditions (conditions UCV and UT) to ensure that one may interchange limits and stochastic integrals driven by semimartingales are now well known, the problem of interchanging stochastic integrals and the limit of stochastic process obtained by the homogenization theory seems, at the best of our knowledge, to have never been treated. Yet the part of this work concerning the limit of SDEs uses some tools and results developed to deal with averaging of SDEs or Backward Stochastic Differential Equations [14, 15, 16, 17]. Besides, the notion of good sequence of semimartingales and conditions UCV and UT (see section 1.2) are widely used throughout this article, even to construct counterexamples. Moreover, the results in this article give some natural counterexamples to the theory of good sequence of semimartingales.

1 Notation, assumptions and review of some results

We denote by $\mathsf{X}^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \mathsf{X}$ the convergence in distribution of a family $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ of random variables to X.

Moreover, we use the Einstein summation convention, which means that all the repeated indices shall be summed over.

1.1 Homogenization

Let $a = (a_{i,j})_{i,j=1}^N$ be a family of measurable, bounded functions with value in the space of symmetric matrices and uniformly elliptic: There exist some positive constants λ and Λ such that

$$\forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N,\ \lambda|\xi|^2 \leqslant a_{i,j}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2. \tag{1}$$

We assume that a is continuous and that $\frac{\partial a_{i,j}}{\partial x_j}$ exists and is bounded for any $i,j = 1,\dots,N$. Let also $b = (b_i)_{i=1}^N$ and $c = (c_i)_{i=1}^N$ be two families of measurable functions with values in $\mathbb{R}^N$. We assume that b and c are bounded by Λ. Let $\sigma = (\sigma_{i,j})_{i,j=1}^N$ be a bounded, measurable function such that $\sigma(x)\sigma^{\mathrm{T}}(x) = a(x)$.

These assumptions are sufficient to ensure the existence of a unique (in law) solution to the stochastic differential equations (2) and (3) below.

We use the expression "*periodic media*" when the coefficients a, b and c are 1-periodic. We are interested in the homogenization property of the family of semimartingales X^ε given by one of the following assumptions.

Assumption 1. *Homogenization in periodic media without a fast oscillating first-order differential term:*

$$\mathsf{X}_t^\varepsilon = x + \int_0^t \sigma(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}\mathsf{B}_s^\varepsilon + \int_0^t c(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s. \tag{2}$$

Assumption 2. *Homogenization in periodic media with a fast oscillating first-order differential term:*

$$\mathsf{X}_t^\varepsilon = x + \int_0^t \sigma(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}\mathsf{B}_s^\varepsilon + \frac{1}{\varepsilon}\int_0^t b(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s + \int_0^t c(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s. \tag{3}$$

Assumption 1 is contained in Assumption 2, but the presence of an highly-oscillating differential first-order term b leads to different results.

We denote by ${}^\varepsilon\mathsf{X}$ the solution of the SDE

$${}^\varepsilon\mathsf{X}_t = x/\varepsilon + \int_0^t \sigma({}^\varepsilon\mathsf{X}_s)\,\mathrm{d}\mathsf{B}_s^\varepsilon + \int_0^t b({}^\varepsilon\mathsf{X}_s)\,\mathrm{d}s + \varepsilon\int_0^t c({}^\varepsilon\mathsf{X}_s)\,\mathrm{d}s. \tag{4}$$

We remark that X^ε and ${}^\varepsilon\mathsf{X}$ are linked by the following relation: $(\mathsf{X}_t^\varepsilon)_{t\geqslant 0}$ is equal in distribution to the process $(\varepsilon \cdot {}^\varepsilon\mathsf{X}_{t/\varepsilon^2})_{t\geqslant 0}$.

Let us denote by $\mathbb{T}^N$ the N-dimensional torus $\mathbb{R}^N/\mathbb{Z}^N$. The space of measurable, square-integrable functions on $\mathbb{T}^N$ is denoted by $\mathrm{L}^2(\mathbb{T}^N)$, and is equipped with the norm $\|u\|_{\mathrm{L}^2(\mathbb{T}^N)} = \left(\int_{\mathbb{T}^N} |u(x)|^2\,\mathrm{d}x\right)^{1/2}$. The completion of smooth, periodic functions on $\mathbb{T}^N$ with respect to the norm $\|u\|_{\mathrm{H}^1(\mathbb{T}^N)} = \left(\int_{\mathbb{T}^N} |u(x)|^2\,\mathrm{d}x + \int_{\mathbb{T}^N} \|\nabla u(x)\|^2\,\mathrm{d}x\right)^{1/2}$ is denoted by $\mathrm{H}^1(\mathbb{T}^N)$. Moreover, the subspace of functions in $\mathrm{H}^1(\mathbb{T}^N)$ with a null mean-value (*i.e.*, $\int_{\mathbb{T}^N} u(x)\,\mathrm{d}x = 0$) is denoted by $\mathrm{H}_0^1(\mathbb{T}^N)$. With an abuse of notation, if $f = (f_1, \dots, f_N)$ is a measurable vector valued function, we still denote by $\|f\|_{\mathrm{L}^2(\mathbb{T}^N)}$ the norm $\left(\sum_{i=1}^N \|f_i\|_{\mathrm{L}^2(\mathbb{T}^N)}^2\right)^{1/2}$.

One remarkable feature of the space $\mathrm{H}_0^1(\mathbb{T}^N)$ is that it satisfied the *Poincaré inequality*: there exists a constant C such that for any function u in $\mathrm{H}^1(\mathbb{T}^N)$,

$$\left\|u - \int_{\mathbb{T}^N} u(x)\,\mathrm{d}x\right\|_{\mathrm{L}^2(\mathbb{T}^N)} \leqslant C\,\|u\|_{\mathrm{H}^1(\mathbb{T}^N)}. \tag{5}$$

Let L be the operator $L = \frac{1}{2}a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + b_i\frac{\partial}{\partial x_i}$. It could be shown that there exists a unique solution m to

$$L^* m = 0,\ m \in \mathrm{H}^1(\mathbb{T}^N) \text{ and } \int_{\mathbb{T}^N} m(x)\,\mathrm{d}x = 1, \tag{6}$$

where L^* is the adjoint of the operator L seen as an operator acting on the space of periodic functions.

Generally, under Assumption 2, X^ε_t does not converge, but $\mathsf{X}^\varepsilon_t - \overline{b}t/\varepsilon$ converge with

$$\overline{b} = \int_{\mathbb{T}^N} b_i(x) m(x) \,\mathrm{d}x. \tag{7}$$

Proposition 1. *With the previous notations, there exists a constant, symmetric and non-degenerate $N \times N$-matrix σ^{eff}, together with a constant vector c^{eff} and a N-dimensional Brownian motion B such that*

$$\widetilde{\mathsf{X}}^\varepsilon \overset{\text{dist.}}{=} (\mathsf{X}^\varepsilon_t - t\overline{b}/\varepsilon)_{t\geqslant 0} \underset{\varepsilon\to 0}{\Longrightarrow} \overline{\mathsf{X}} \text{ with } \overline{\mathsf{X}}_t = x + \sigma^{\mathrm{eff}}\mathsf{B}_t + tc^{\mathrm{eff}}.$$

The coefficients σ^{eff} and c^{eff} may be constructed explicitly from the coefficients of L^1 (see (12) and (13) below). Furthermore, σ^{eff} does not depend on the value of c. If $c = 0$, then $c^{\mathrm{eff}} = 0$.

A special case appears when $\overline{b} = 0$. This happens when L is a divergence form operator, that is $b_i = \frac{1}{2}\frac{\partial a_{i,j}}{\partial x_j} + \frac{\partial V}{\partial x_i}$ for some periodic function V. But this could also happen if the generalized principal eigenvalue of the operator is equal to 0, which means that (see Section 8.2 in [18, 19] for example)

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}_x\left[\inf\{\, s \geqslant 0 \,|\, |\mathsf{X}_s| \geqslant n\,\} > t\right] = 0.$$

We give the sketch of the proof of Proposition 1 under Assumptions 1 and 2. For details, the reader is referred to [1, 13, 14] for example. Some of the notations used in this proof will be used below.

Sketch of the proof. The idea is to find some functions $u_1, \cdots, u_N$ that are periodic and such that

$$\widetilde{\mathsf{X}}^{i,\varepsilon}_t - \widetilde{\mathsf{X}}^{i,\varepsilon}_0 + \varepsilon u_i(\mathsf{X}^\varepsilon_t/\varepsilon) - \varepsilon u_i(\mathsf{X}^\varepsilon_0/\varepsilon) = \mathsf{M}^{i,\varepsilon}_t + \int_0^t c_j\left(\delta_{i,j} + \frac{\partial u_i}{\partial x_j}\right)(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s, \tag{8}$$

where M^ε is a local martingale with cross-variations

$$\langle \mathsf{M}^{i,\varepsilon}, \mathsf{M}^{j,\varepsilon}\rangle_t \overset{\text{dist.}}{=} \varepsilon^2 \int_0^{t/\varepsilon^2} a_{p,q}\left(\delta_{i,p} + \frac{\partial u_i}{\partial x_p}\right)\left(\delta_{j,q} + \frac{\partial u_j}{\partial x_q}\right)(\mathsf{X}^1_s)\,\mathrm{d}s.$$

The functions $u_1, \dots, u_N$ belong to $\mathrm{H}^1_0(\mathbb{T}^N)$ and are solutions to

$$u_i(x) = 0 \text{ under Assumption 1}, \tag{9}$$

$$Lu_i = -b_i + \overline{b}_i \text{ under Assumption 2}. \tag{10}$$

In (10), the existence of u_i is given by the Fredholm alternative, hence the importance of $\overline{b}$.

The projection of the process generated by L on the torus $\mathbb{T}$ is ergodic with respect to the measure $m(x)\,\mathrm{d}x$ whose density m is solution to (6).

For any periodic, integrable function f, we know as a consequence of the Poincaré inequality (5) that L (which acts on periodic functions) has a spectral gap and that for any $t > 0$,

$$\sup_{x\in\mathbb{R}^N} \mathbb{E}_x \left[\left| \frac{\varepsilon^2}{t} \int_0^{t/\varepsilon^2} f({}^{\varepsilon}\mathsf{X}_s)\,\mathrm{d}s - \int_{\mathbb{T}^N} f(x)m(x)\,\mathrm{d}x \right| \right] \leqslant g(\varepsilon)\,\|f\|_{\mathrm{L}^1(\mathbb{T}^N)}\,, \tag{11}$$

for some constants function g such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ with a rate that does not depend on f.

As a and $u_1, \cdots, u_N$ are periodic, the inequality (11) implies that the cross-variations of M^ε converge to

$$\langle \mathsf{M}^{i,\varepsilon}, \mathsf{M}^{j,\varepsilon}\rangle_t \xrightarrow[\varepsilon\to 0]{proba} ta^{\mathrm{eff}}_{i,j} \overset{\mathrm{def}}{=} t\int_{\mathbb{T}^N} a_{p,q}\left(\delta_{i,p} + \frac{\partial u_i}{\partial x_p}\right)\left(\delta_{j,q} + \frac{\partial u_j}{\partial x_q}\right)(x)m(x)\,\mathrm{d}x. \tag{12}$$

The Central Limit Theorem for martingales [2, Theorem 1.4, p. 339] implies that M^ε converge in distribution to a martingale M with cross-variations $\langle \mathsf{M}^i, \mathsf{M}^j\rangle_t = ta^{\mathrm{eff}}_{i,j}$. We define σ^{eff} to be the square-root of the matrix a^{eff}. Then, there exists a N-dimensional Brownian motion B such that $\mathsf{M} = \sigma^{\mathrm{eff}}\mathsf{B}$.

Again with (4) and (11),

$$\int_0^t c_j\left(\delta_{i,j} + \frac{\partial u_i}{\partial x_j}\right)(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{proba} tc^{\mathrm{eff}}_i \overset{\mathrm{def}}{=} t\int_{\mathbb{T}^N} c_j\left(\delta_{i,j} + \frac{\partial u_i}{\partial x_j}\right)(x)m(x)\,\mathrm{d}x. \tag{13}$$

In fact, this convergence holds in the space of continuous functions (see for example Corollary 1.3 in [8, p. 58]).

The boundedness of u_i for $i = 1, \ldots, N$ implies that $\widetilde{\mathsf{X}}^\varepsilon$ converges in distribution to $\overline{\mathsf{X}}$, where $\overline{\mathsf{X}}_t = x + \mathsf{M}_t + tc^{\mathrm{eff}}$. ∎

Remark 1. The first-order differential term c may be treated by using the Girsanov theorem, as in [6, 7]. In view of (4), This allows to understand why c does not "interact" with the diffusive behavior of the limit $\overline{\mathsf{X}}$, in difference to b.

1.2 A criteria of convergence of stochastic integral driven by a semimartingale

We give in this section a criterion under which the limit of stochastic integrals driven by convergent semimartingales is the stochastic integral of the limits. We took the following definitions and results from the review article [5].

For a semimartingale X and a stochastic process H, we denote by $\mathsf{H}\cdot\mathsf{X}$, when it exists, the continuous stochastic process $\int_0^\cdot \mathsf{H}_s\,\mathrm{d}\mathsf{X}_s$.

Definition 1 (Good sequence, Definition 7.3 in [5, p. 22]**).** A sequence of càdlàg (right-continuous with left limit) semimartingales $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ is said to be a *good sequence* if $\mathsf{X}^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \mathsf{X}$, and for any sequence $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ of càdlàg processes such that H^ε is adapted to the filtration generated by X^ε and $(\mathsf{H}^\varepsilon, \mathsf{X}^\varepsilon) \underset{\varepsilon\to 0}{\Longrightarrow} (\mathsf{H}, \mathsf{X})$, then X is a semimartingale with respect to the smallest filtration $\mathcal{H} = (\mathcal{H}_t)_{t\geqslant 0}$ generated by (H, X) satisfying the usual hypotheses, and, when all the involved stochastic integrals are defined,

$$\mathsf{H}^\varepsilon \cdot \mathsf{X}^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \mathsf{H}\cdot\mathsf{X}.$$

There exist two equivalent conditions ensuring that a sequence of semimartingales is good.

Definition 2 (Condition UT, Definition 7.4 in [5, p. 22]**).** A sequence $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ of semimartingales is said to be *uniformly tight*, or to satisfy the *condition UT*, if for each $t \in (0, 1]$, the set

$$\left\{ \int_0^t \mathsf{H}^\varepsilon_{s-}\,\mathrm{d}\mathsf{X}^\varepsilon_s \,\middle|\, \begin{array}{l} \forall\varepsilon > 0,\ \mathsf{H}^\varepsilon \text{ is càdlàg and piecewise constant} \\ \text{and } \sup_{s\in[0,1]} |\mathsf{H}^\varepsilon_s| \leqslant 1 \end{array} \right\}$$

is tight.

Definition 3 (Condition UCV, Definition 7.5 in [5, p. 23]**).** A sequence of continuous semimartingales $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ is said to have *uniformly controlled variations*, or to satisfy the *condition UCV*, if for each $\alpha > 0$ and each $\varepsilon > 0$, there exists some stopping time $T^{\varepsilon,\alpha}$ such that $\mathbb{P}\left[T^{\varepsilon,\alpha} \leqslant \alpha \right] \leqslant \frac{1}{\alpha}$ and

$$\sup_{\varepsilon>0} \sup_{i=1,\ldots,N} \mathbb{E}\left[\langle \mathsf{M}^{i,\varepsilon}, \mathsf{M}^{i,\varepsilon}\rangle_{1\wedge T^{\varepsilon,\alpha}} + \int_0^{1\wedge T^{\varepsilon,\alpha}} |\,\mathrm{d}\mathsf{N}^{i,\varepsilon}_s| \right] < +\infty,$$

where $\mathsf{X}^\varepsilon = \mathsf{X}^\varepsilon_0 + \mathsf{M}^\varepsilon + \mathsf{N}^\varepsilon$ is the decomposition of X^ε as the sum of a local martingale and a process locally of finite variation.

Remark 2. The conditions UT and UCV have been developed for càdlàg processes. Yet the definition of the condition UCV is more complicated for discontinuous processes, since the jumps have to be taken into account.

The following Theorem summarizes the main results about good sequences.

Theorem 1 (Theorems 7.6, 7.7 and 7.10 in [5])**.** *Let $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ be a sequence of semimartingales converging in distribution to some process X. Then the sequence $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ is good if and only if it satisfies the condition UT and if and only if it satisfies the condition UCV.*

We end this section by a lemma, that provides some interpretation of a condition close to be the condition UCV. Of course, the homogenization result gives some examples in which the assumptions on the following lemma are not satisfied.

Lemma 1. *Let $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ be a family of semimartingales with the decomposition $\mathsf{X}^\varepsilon = \mathsf{X}^\varepsilon_0 + \mathsf{M}^\varepsilon + \mathsf{N}^\varepsilon$ and such that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\langle \mathsf{M}^{i,\varepsilon}, \mathsf{M}^{i,\varepsilon}\rangle_1 + \int_0^1 |\,\mathrm{d}\mathsf{N}^{i,\varepsilon}_s| \right] < +\infty \tag{14}$$

and $(\mathsf{X}^\varepsilon, \mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)$ converges in distribution to the process $(\mathsf{X}, \mathsf{M}, \mathsf{N})$ on the space of continuous functions on $[0, 1]$. Then X is a semimartingale with decomposition $\mathsf{X} = \mathsf{X}_0 + \mathsf{M} + \mathsf{N}$.

Proof. With Corollary VI.6.6 in [3, p. 342], it is well known that M is a martingale with respect to the filtration generated by $(\mathsf{X},\mathsf{M},\mathsf{N})$.

On the other hand, if $(z^\varepsilon)_{\varepsilon>0}$ is a family of smooth functions of finite variation on $[0,1]$ converging uniformly to z, then

$$\sum_{i=0}^{k-1}|z_{t_{i+1}}-z_{t_i}|\leqslant\liminf_{\varepsilon\to 0}\sum_{i=0}^{k-1}|z^\varepsilon_{t_{i+1}}-z^\varepsilon_{t_i}|\leqslant\liminf_{\varepsilon\to 0}\int_0^1|\mathrm{d}z^\varepsilon_s|,$$

where $0\leqslant t_0\leqslant\cdots\leqslant t_k\leqslant 1$ is any partition of $[0,1]$. Hence, z is also of finite variation and $\int_0^1|\mathrm{d}z_s|\leqslant\liminf_{\varepsilon\to 0}\int_0^1|\mathrm{d}z^\varepsilon_s|$. Thus, it is clear from (14) that N is of integrable variation. ∎

Of course, in view of the homogenization result with a highly oscillating first order differential term, the condition (14) will not be satisfied, since the limit of the term of finite variation in the decomposition of X^ε is a martingale.

2 Good sequence and homogenization

In view of the results of Section 1.2, the first natural question to solve our problem is to know if $(\widetilde{\mathsf{X}}^\varepsilon)_{\varepsilon>0}$ is a good sequence of semimartingales. If yes, the problem of interchanging stochastic integrals and limits is already solved. Although this is not always true, let us start by a positive answer.

Proposition 2. (i) *Under Assumption 1, the sequence of semimartingales* $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ *is a good sequence.*

(ii) *Under Assumption 2, and with the notations of the proof of Proposition 1,* $(\widetilde{\mathsf{X}}^\varepsilon+\varepsilon u(\mathsf{X}^\varepsilon/\varepsilon))_{\varepsilon>0}$ *and* $(\mathsf{M}^\varepsilon)_{\varepsilon>0}$*, where* u *is defined by* (10) *and* M^ε *is defined by* (8)*, are good sequences of semimartingales.*

Proof. Proof of (i). Under Assumption 1, the process X^ε is $\mathsf{X}^\varepsilon_t=\mathsf{M}^\varepsilon_t+\int_0^t c(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s$, with

$$\langle\mathsf{M}^{i,\varepsilon},\mathsf{M}^{i,\varepsilon}\rangle_t=\int_0^t a_{i,i}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s\leqslant\sup_{x\in\mathbb{T}^N}|a_{i,i}(x)|t.$$

In addition, $|c(x)|$ is bounded by Λ and then for any $\varepsilon>0$, $\int_0^t|c(\mathsf{X}^\varepsilon_s/\varepsilon)|\,\mathrm{d}s\leqslant\Lambda t$. Thus, $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ satisfies the condition UCV and is a good sequence of semimartingales.

Proof of (ii). Under Assumption 2, the proof is the same for $(\mathsf{M}^\varepsilon)_{\varepsilon>0}$ and $(\widetilde{\mathsf{X}}^\varepsilon+\varepsilon u(\mathsf{X}^\varepsilon/\varepsilon))_{\varepsilon>0}$, since ∇u is bounded. ∎

However, under Assumption 2, *i.e.*, when there is a highly oscillating first-order differential term, $(\widetilde{\mathsf{X}}^\varepsilon)_{\varepsilon\geq 0}$ is not a good sequence in general. Otherwise, according to Theorem 7.12 in [5, p. 30], $\langle\widetilde{\mathsf{X}}^\varepsilon,\widetilde{\mathsf{X}}^\varepsilon\rangle\underset{\varepsilon\to 0}{\Longrightarrow}\langle\overline{\mathsf{X}},\overline{\mathsf{X}}\rangle$. While under Assumption 2, (11) yields

$$(\langle\widetilde{\mathsf{X}}^{i,\varepsilon},\widetilde{\mathsf{X}}^{j,\varepsilon}\rangle_t)_{t\in[0,1]}\xrightarrow[\varepsilon\to 0]{proba}(t\overline{a}_{i,j})_{t\in[0,1]}\text{ with }\overline{a}_{i,j}=\int_{\mathbb{T}^N}a_{i,j}(x)m(x)\,\mathrm{d}x.$$

But $\langle\overline{\mathsf{X}}^i,\overline{\mathsf{X}}^j\rangle_t=ta^{\mathrm{eff}}_{i,j}$, and generally, $a^{\mathrm{eff}}\neq\overline{a}$. For example, in dimension one, if $b(x)=\frac{1}{2}a'(x)$ then $a^{\mathrm{eff}}=\left(\int_0^1 a(x)^{-1}\,\mathrm{d}x\right)^{-1}$ while $\overline{a}=\int_0^t a(x)\,\mathrm{d}x$.

2.1 Some counterexamples

As we have seen that nothing special happens under Assumption 1, we work from now under Assumption 2.

In presence of a highly oscillating first-order differential term, we easily find some new counterexamples to the fact that the limit of the stochastic integral is the stochastic integral of the limit. However, there are cases for which interchanging limits and stochastic integrals is possible.

Example 1. Let f is a function of class $\mathcal{C}^2$ on $\mathbb{R}^N$ with compact support. Then, by the Itô formula,

$$f(\widetilde{\mathsf{X}}_1^\varepsilon) = f(x) + \int_0^1 \nabla f(\widetilde{\mathsf{X}}_s^\varepsilon)\,\mathrm{d}\widetilde{\mathsf{X}}_s^\varepsilon + \frac{1}{2}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(\widetilde{\mathsf{X}}_s^\varepsilon)\,\mathrm{d}\langle \widetilde{\mathsf{X}}^{i,\varepsilon}, \widetilde{\mathsf{X}}^{j,\varepsilon}\rangle_s.$$

It is now clear that jointly with the convergence of $\widetilde{\mathsf{X}}^\varepsilon$ to $\overline{\mathsf{X}}$ (see Lemma 2 below),

$$\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(\widetilde{\mathsf{X}}_s^\varepsilon)\,\mathrm{d}\langle \widetilde{\mathsf{X}}^{i,\varepsilon}, \widetilde{\mathsf{X}}^{j,\varepsilon}\rangle_s \underset{\varepsilon\to 0}{\Longrightarrow} \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(\overline{\mathsf{X}}_s)\overline{a}_{i,j}\,\mathrm{d}s$$

and that $f(\widetilde{\mathsf{X}}_1^\varepsilon) \Rightarrow f(\overline{\mathsf{X}}_1)$. With the Itô formula applied to $\overline{\mathsf{X}}$, we deduce that

$$\int_0^1 \nabla f(\widetilde{\mathsf{X}}_s^\varepsilon)\,\mathrm{d}\widetilde{\mathsf{X}}_s^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \int_0^1 \nabla f(\overline{\mathsf{X}}_s)\,\mathrm{d}\overline{\mathsf{X}}_s + \frac{1}{2}\int_0^1 (a_{i,j}^{\mathrm{eff}} - \overline{a}_{i,j})\frac{\partial^2 f}{\partial x_i \partial x_j}(\overline{\mathsf{X}}_s)\,\mathrm{d}s. \tag{15}$$

Hence, when $g = (g_1, \cdots, g_N)$ is a $\mathcal{C}^1$ vortex-free vector field with compact support on $\mathbb{R}^N$, then $\int_0^1 g(\widetilde{\mathsf{X}}_s^\varepsilon)\,\mathrm{d}\widetilde{\mathsf{X}}_s^\varepsilon$ does not converge in general to $\int_0^1 g(\overline{\mathsf{X}}_s)\,\mathrm{d}\overline{\mathsf{X}}_s$.

Remark 3. In dimension 1, the convergence of (15) may be seen as a special case of a more general result presented in [20]: If $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is a family of semimartingales such that $(\mathsf{Y}^\varepsilon, \langle \mathsf{Y}^\varepsilon\rangle)_{\varepsilon>0} \underset{\varepsilon\to 0}{\Longrightarrow} (\mathsf{Y}, \mathsf{V})$ and f is analytic, then $\int_0^1 f(\mathsf{Y}_s^\varepsilon)\,\mathrm{d}\mathsf{Y}_s^\varepsilon$ converges to $\int_0^1 f(\mathsf{Y}_s)\,\mathrm{d}\mathsf{Y}_s + \frac{1}{2}\int_0^1 f'(\mathsf{Y}_s)\,\mathrm{d}\langle \mathsf{Y}\rangle_s - \frac{1}{2}\int_0^1 f'(\mathsf{Y}_s)\,\mathrm{d}\mathsf{V}_s$.

Example 2. Let h be a bounded, 1-periodic function on $\mathbb{R}^N$ with value in $\mathbb{R}^N$. The process Y^ε defined by $\mathsf{Y}_t^\varepsilon = \sqrt{\varepsilon}h(\mathsf{X}_t^\varepsilon/\varepsilon)$ converges to 0 in probability, so that $\mathsf{Y}^\varepsilon \cdot \mathsf{M}^{\mathsf{X}^\varepsilon}$ converges to 0, where $\mathsf{M}^{\mathsf{X}^\varepsilon}$ is the martingale part of X^ε. But, if we set $\widetilde{b} = b - \overline{b}$,

$$\frac{\sqrt{\varepsilon}}{\varepsilon}\int_0^1 h(\mathsf{X}_s^\varepsilon/\varepsilon)\widetilde{b}(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s \overset{\text{dist.}}{=} \varepsilon\sqrt{\varepsilon}\int_0^{1/\varepsilon^2} (h\widetilde{b} - d)(\mathsf{X}_s^1)\,\mathrm{d}s + \frac{1}{\sqrt{\varepsilon}}d,$$

where $d = \int_{\mathbb{T}^N} h(x)b(x)m(x)\,\mathrm{d}x - \overline{b}\int_{\mathbb{T}^N} h(x)m(x)\,\mathrm{d}x$. Using the homogenization procedure, it is clear that $\varepsilon\sqrt{\varepsilon}\int_0^{t/\varepsilon^2}(hb - d)(\mathsf{X}_s^1)\,\mathrm{d}s$ converges to 0. But generally, $d \neq 0$. Hence, there exists some processes Y^ε such that Y^ε converges in distribution to 0, but $\mathsf{Y}^\varepsilon \cdot \mathsf{X}^\varepsilon$ does not converge.

2.2 An example in which the interchange is possible

In this section and the next one, we use the following hypothesis on a family $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ of stochastic processes.

Hypothesis 1. *For any $\varepsilon > 0$, let H^ε be a predictable process with respect to the minimal admissible filtration $\mathcal{F}^\varepsilon$ generated by $\widetilde{\mathsf{X}}^\varepsilon$. Let also H a the predictable process adapted to the minimal admissible filtration $\mathcal{F}$ generated by $\overline{\mathsf{X}}$. Moreover, $(\mathsf{H}^\varepsilon, \widetilde{\mathsf{X}}^\varepsilon) \xrightarrow[\varepsilon\to 0]{} (\mathsf{H}, \overline{\mathsf{X}})$ in the space of càdlàg functions with the Skorohod topology. There is no need for H^ε and H to be continuous.*

Let us give an example of family of processes $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ for which the limit of stochastic integral with respect to $\widetilde{\mathsf{X}}^\varepsilon$ is the stochastic integral of the limits of $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ and $(\widetilde{\mathsf{X}}^\varepsilon)_{\varepsilon>0}$. Of course, we work under Assumption 2, *i.e.*, in presence of a highly oscillating first-order term. And for that, the variations of H^ε shall be "slow" enough.

Proposition 3. *In addition to Hypothesis 1, we assume that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\,\|\mathsf{H}^\varepsilon\|_\infty\,\right] < +\infty.$$

We assume that for each $\varepsilon > 0$, there exists a (random) partition $0 = t_1 < \cdots < t_{n_\varepsilon} = 1$ of $[0,1]$ with n_ε terms such that

$$n_\varepsilon \varepsilon \xrightarrow[\varepsilon\to 0]{} 0 \tag{16}$$

$$\mathbb{E}\left[\varepsilon^{-1}\|\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon\|_\infty\right] \xrightarrow[\varepsilon\to 0]{} 0, \tag{17}$$

where

$$\overline{\mathsf{H}}^\varepsilon(t) = \sum_{i=1}^{n_\varepsilon} \mathsf{H}^\varepsilon_{t_i} \mathbf{1}_{[t_i, t_{i+1})}(t).$$

Then, $\mathsf{H}^\varepsilon \cdot \widetilde{\mathsf{X}}^\varepsilon$ converges in distribution to $\mathsf{H} \cdot \overline{\mathsf{X}}$.

Proof. We set $\mathsf{Y}^\varepsilon_t = \widetilde{\mathsf{X}}^\varepsilon_t + \varepsilon u(\mathsf{X}^\varepsilon_t/\varepsilon)$. It is clear from (17) that $\|\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon\|_\infty \xrightarrow[\varepsilon\to 0]{proba} 0$. Now,

$$\mathsf{H}^\varepsilon \cdot \widetilde{\mathsf{X}}^\varepsilon = (\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon) \cdot \widetilde{\mathsf{X}}^\varepsilon + (\overline{\mathsf{H}}^\varepsilon - \mathsf{H}^\varepsilon) \cdot \mathsf{Y}^\varepsilon + \mathsf{H}^\varepsilon \cdot \mathsf{Y}^\varepsilon + \mathsf{R}^\varepsilon$$

with

$$\mathsf{R}^\varepsilon_t = -\varepsilon \sum_{i=1}^{k \text{ s.t. } t_k < t} \mathsf{H}^\varepsilon_{t_i}(u(\mathsf{X}^\varepsilon_{t_{i+1}}/\varepsilon) - u(\mathsf{X}^\varepsilon_{t_i}/\varepsilon)).$$

With (16) and the fact that u is bounded,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant 1} |\mathsf{R}^\varepsilon_t|\right] \leqslant \varepsilon n_\varepsilon \mathbb{E}\left[\,\|\mathsf{H}^\varepsilon\|_\infty\,\right] 2\|u\|_\infty \xrightarrow[\varepsilon\to 0]{} 0.$$

With Proposition 2, $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is a good sequence of semimartingales and converges jointly with $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ to $(\overline{\mathsf{X}}, \mathsf{H})$. So, $\mathsf{H}^\varepsilon \cdot \mathsf{Y}^\varepsilon$ converges to $\mathsf{H} \cdot \overline{\mathsf{X}}$ and $(\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon) \cdot \mathsf{Y}^\varepsilon$ converges to 0.

It remains to study $(\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon) \cdot \widetilde{\mathsf{X}}^\varepsilon$, which is equal at time t to

$$\int_0^t (\mathsf{H}^\varepsilon_{s-} - \overline{\mathsf{H}}^\varepsilon_{s-})\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s + \frac{1}{\varepsilon}\int_0^t (\mathsf{H}^\varepsilon_{s-} - \overline{\mathsf{H}}^\varepsilon_{s-})\widetilde{b}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s,$$

where $\widetilde{b} = b - \overline{b}$ and $\mathsf{M}^{\mathsf{X}^\varepsilon}$ is the martingale part of $\widetilde{\mathsf{X}}^\varepsilon$.

As $\mathbb{E}\left[\sup_{0\leqslant t\leqslant 1} |\mathsf{M}^{\mathsf{X}^\varepsilon}_t|^2\right] \leqslant \Lambda T$, it is clear that $(\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon) \cdot \mathsf{M}^{\mathsf{X}^\varepsilon}$ converges to 0 as ε goes to 0. Furthermore, (17) implies that

$$\mathbb{E}\left[\frac{1}{\varepsilon}\left|\int_0^1 (\mathsf{H}^\varepsilon_s - \overline{\mathsf{H}}^\varepsilon_s)\widetilde{b}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s\right|\right] \leqslant \mathbb{E}\left[\frac{1}{\varepsilon}\|\widetilde{b}\|_\infty \|\mathsf{H}^\varepsilon - \overline{\mathsf{H}}^\varepsilon\|_\infty\right] \xrightarrow[\varepsilon\to 0]{} 0.$$

We have then proved that $\mathsf{H}^\varepsilon \cdot \widetilde{\mathsf{X}}^\varepsilon$ converges in distribution to $\mathsf{H} \cdot \overline{\mathsf{X}}$. ∎

2.3 Integration of good semimartingales

For two (càdlàg) semimartingales X and Y, the *quadratic covariation process* is defined to be

$$[\mathsf{X}, \mathsf{Y}]_t = \mathsf{X}_t \mathsf{Y}_t - \mathsf{X}_0 \mathsf{Y}_0 - \int_0^t \mathsf{X}_{s-}\,\mathrm{d}\mathsf{Y}_s - \int_0^t \mathsf{Y}_{s-}\,\mathrm{d}\mathsf{X}_s.$$

Proposition 4. *We are still under Assumption 2. In addition to Hypothesis 1, we assume furthermore that* $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ *is a good sequence of* $\mathcal{F}^\varepsilon$*-semimartingales, and that* H *is also a semimartingale. Then, there exists a martingale* N *such that, if* M *is the martingale part of* $\overline{\mathsf{X}}$*,*

$$\langle \mathsf{N}^i, \mathsf{N}^j \rangle_t = t\int_{\mathbb{T}^N} a(x)\nabla u_i(x)\nabla u_j(x) m(x)\,\mathrm{d}x,$$
$$\langle \mathsf{M}^i, \mathsf{N}^j \rangle_t = t\int_{\mathbb{T}^N} a_{p,q}(x)\left(\delta_{p,i} + \frac{\partial u_i(x)}{\partial x_p}\right)\frac{\partial u_j(x)}{\partial x_j}(x) m(x)\,\mathrm{d}x$$

and

$$\mathsf{H}^\varepsilon \cdot \widetilde{\mathsf{X}}^\varepsilon \xRightarrow[\varepsilon\to 0]{} \mathsf{H} \cdot \overline{\mathsf{X}} - [\mathsf{N}, \mathsf{H}] \tag{18}$$

in the Skorohod topology.

Proof. We still use the notations of the proof of Proposition 1 and we denote by $\mathsf{M}^{\mathsf{X}^\varepsilon}$ the martingale part of $\widetilde{\mathsf{X}}^\varepsilon$, *i.e.*, $\mathsf{M}^{\mathsf{X}^\varepsilon}_t = \int_0^t \sigma(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}\mathsf{B}_s$. Moreover, we set $\mathsf{Y}^\varepsilon_t = \widetilde{\mathsf{X}}^\varepsilon_t - \widetilde{\mathsf{X}}^\varepsilon_0 + \varepsilon u(\mathsf{X}^\varepsilon_t/\varepsilon) - \varepsilon u(\mathsf{X}^\varepsilon_0/\varepsilon)$. Let M^ε and V^ε be the martingale part and the process of finite variation whose sum gives Y^ε (see (8)). Let us remember that $\mathsf{M}^\varepsilon_0 = 0$ and that $\mathsf{X}^\varepsilon_0 = \widetilde{\mathsf{X}}^\varepsilon_0 = x$ for any $\varepsilon > 0$.

An integration by parts leads to

$$\mathsf{H}^\varepsilon \cdot \widetilde{\mathsf{X}}^\varepsilon = (\widetilde{\mathsf{X}}^\varepsilon - x - \mathsf{Y}^\varepsilon) \times \mathsf{H}^\varepsilon + \mathsf{Y}_0^\varepsilon \times \mathsf{H}_0^\varepsilon \\ + \mathsf{H}^\varepsilon \cdot \mathsf{Y}^\varepsilon - (\widetilde{\mathsf{X}}^\varepsilon - x - \mathsf{Y}^\varepsilon) \cdot \mathsf{H}^\varepsilon - [\widetilde{\mathsf{X}}^\varepsilon - \mathsf{Y}^\varepsilon, \mathsf{H}^\varepsilon].$$

From Hypothesis 1, $(\mathsf{H}^\varepsilon, \widetilde{\mathsf{X}}^\varepsilon)$ converges in distribution to $(\mathsf{H}, \overline{\mathsf{X}})$. The quantity $\sup_{t\in[0,1]} |\widetilde{\mathsf{X}}_t^\varepsilon - x - \mathsf{Y}_t^\varepsilon|$ converges almost surely to 0. Then $(\mathsf{H}^\varepsilon, \mathsf{Y}^\varepsilon)$ also converges in distribution to $(\mathsf{H}, \overline{\mathsf{X}})$. Owing to Proposition 2(ii), $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is a good sequence of semimartingales. So, $(\mathsf{H}^\varepsilon, \mathsf{Y}^\varepsilon, \mathsf{H}^\varepsilon \cdot \mathsf{Y}^\varepsilon)$ converges in distribution to to $(\mathsf{H}, \overline{\mathsf{X}}, \mathsf{H} \cdot \overline{\mathsf{X}})$. Using the fact that $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ is a good sequence of semimartingales, $(\widetilde{\mathsf{X}}^\varepsilon - x - \mathsf{Y}^\varepsilon) \cdot \mathsf{H}^\varepsilon$ converges to 0. Besides, $\mathsf{H}^\varepsilon \times (\widetilde{\mathsf{X}}^\varepsilon - x - \mathsf{Y}^\varepsilon)$ converge to 0 since H^ε converges.

As $\widetilde{\mathsf{X}}^\varepsilon - \mathsf{Y}^\varepsilon$ is continuous, according to Proposition I.4.9 in [3, p. 52],

$$[\widetilde{\mathsf{X}}^\varepsilon - \mathsf{Y}^\varepsilon, \mathsf{H}^\varepsilon] = [\mathsf{M}^{\mathsf{X}^\varepsilon} - \mathsf{M}^\varepsilon, \mathsf{H}^\varepsilon].$$

For any $\varepsilon > 0$, we set $\mathsf{N}^\varepsilon = \mathsf{M}^\varepsilon - \mathsf{M}^{\mathsf{X}^\varepsilon}$. The martingale $(\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)$, which takes its values in $\mathbb{R}^{2N}$, has cross-variations for $i, j = 1, \dots, 2N$,

$$\langle (\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)^i, (\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)^j \rangle_t = \sum_{p,q=1,\dots,N} \int_0^t a_{p,q} \left(\delta_{p,i} + \frac{\partial u_i}{\partial x_p}\right)\left(\delta_{q,j} + \frac{\partial u_j}{\partial x_q}\right)(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s.$$

with the convention that $u_{i+N} = u_i$ for $i = 1, \dots, N$. It is clear that $(\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)_{\varepsilon>0}$ satisfies the condition UCV. Besides, $(\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)$ converges on account of (11) and the Central Limit for martingales to (M, N), where M is the limit of M^ε and N is a martingale whose cross-variations are, for $i, j = 1, \dots, N$,

$$\langle \mathsf{N}^i, \mathsf{N}^j \rangle_t = t \int_{\mathbb{T}^N} a_{p,q} \frac{\partial u_i}{\partial x_p} \frac{\partial u_j}{\partial x_q}(x) m(x)\,\mathrm{d}x.$$

As N^ε and M^ε are continuous, the quadratic co-variations of $(\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)$ are equal to their cross-variations.

Using the fact that $(\mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon)$ is continuous, we deduce that

$$(\mathsf{H}^\varepsilon, \mathsf{Y}^\varepsilon, \mathsf{H}^\varepsilon \cdot \mathsf{Y}^\varepsilon, \mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon, \langle \mathsf{M}^\varepsilon, \mathsf{M}^\varepsilon \rangle, \langle \mathsf{N}^\varepsilon, \mathsf{N}^\varepsilon \rangle, \langle \mathsf{M}^\varepsilon, \mathsf{N}^\varepsilon \rangle)_{\varepsilon>0}$$

is tight in the Skorohod topology. Moreover, this sequence converges in the Skorohod topology to $(\mathsf{H}, \mathsf{Y}, \mathsf{H} \cdot \overline{\mathsf{X}}, \mathsf{M}, \mathsf{N}, \langle \mathsf{M}, \mathsf{M} \rangle, \langle \mathsf{N}, \mathsf{N} \rangle, \langle \mathsf{M}, \mathsf{N} \rangle)$.

Let $\mathcal{H}$ is the filtration generated by $(\mathsf{H}, \mathsf{M}, \mathsf{N})$. As $(\mathsf{H}^\varepsilon)_{\varepsilon>0}$ and $(\mathsf{Y}^\varepsilon, \mathsf{N}^\varepsilon)_{\varepsilon>0}$ are good sequences, the semimartingales H and (M, N) are $\mathcal{H}$-semimartingales. Since

$$\mathsf{H}^\varepsilon \mathsf{N}^\varepsilon = \mathsf{H}_0^\varepsilon \mathsf{N}_0^\varepsilon + \mathsf{H}^\varepsilon \cdot \mathsf{N}^\varepsilon + \mathsf{N}^\varepsilon \cdot \mathsf{H}^\varepsilon + [\mathsf{N}^\varepsilon, \mathsf{H}^\varepsilon],$$

the process $(\mathsf{H}^\varepsilon, \mathsf{Y}^\varepsilon, \mathsf{N}^\varepsilon, [\mathsf{N}^\varepsilon, \mathsf{H}^\varepsilon])$ converges in the Skorohod topology to the process $(\mathsf{H}, \overline{\mathsf{X}}, \mathsf{N}, [\mathsf{N}, \mathsf{H}])$.

Combining these results, we obtain (18). ∎

3 Convergence of the Lévy Area

Let us define the *Lévy area* $A^{i,j}(\mathsf{Z})$ of two coordinates i and j of a semimartingale Z with values in $\mathbb{R}^N$ by

$$A^{i,j}_{s,t}(\mathsf{Z}) = \frac{1}{2}\int_s^t (\mathsf{Z}^i_r - \mathsf{Z}^i_s)\,\mathrm{d}\mathsf{Z}^j_r - \frac{1}{2}\int_s^t (\mathsf{Z}^j_r - \mathsf{Z}^j_s)\,\mathrm{d}\mathsf{Z}^i_r.$$

The quantity $A^{i,j}_{s,t}(\mathsf{Z})$ is only defined as a limit in probability but corresponds intuitively to the area of the surface contained between the curve $r \in [s,t] \mapsto (\mathsf{Z}^i_r, \mathsf{Z}^j_r)$ and the chord $[(\mathsf{Z}^i_s, \mathsf{Z}^j_s), (\mathsf{Z}^i_t, \mathsf{Z}^j_t)]$.

The study $A^{i,j}_{s,t}(\mathsf{Z})$ can be reduced to the study of $A^{i,j}_{0,t}(\mathsf{Z})$, since $A^{i,j}_{0,t}(\mathsf{Z}) = A^{i,j}_{0,s}(\mathsf{Z}) + A^{i,j}_{s,t}(\mathsf{Z}) + (\mathsf{Z}^j_t - \mathsf{Z}^j_s)(\mathsf{Z}^i_s - \mathsf{Z}^i_0)$ for any $0 \leqslant s \leqslant t$.

The area between a path and its chord is a functional which may not be continuous with respect to the uniform norm: It is easily proved that $t \mapsto (n^{-1}\cos(nt), n^{-1}\sin(nt))$ converges uniformly to 0 as $n \to \infty$, while its area between 0 and 2π remains constant.

Under Assumption 1, it is immediate that $A(\mathsf{X}^\varepsilon)$ converges to $A(\overline{\mathsf{X}})$, since the sequence $(\mathsf{X}^\varepsilon)_{\varepsilon>0}$ is a good sequence. We now work under Assumption 2. We recall that $\widetilde{\mathsf{X}}^\varepsilon_t = \mathsf{X}^\varepsilon_t - t\overline{b}/\varepsilon$. From the results of Section 1.1, $\widetilde{\mathsf{X}}^\varepsilon$ converges in distribution to the semimartingale $\overline{\mathsf{X}}$ given in Proposition 1.

Proposition 5. *Under Assumption 2, let us define for* $i, j = 1, \dots, N$*,*

$$\overline{\psi}_{i,j} = \frac{1}{2}\int_{\mathbb{T}^2}\Bigg(a_{j,i}\left(\frac{\partial u_j}{\partial x_j} - \frac{\partial u_i}{\partial x_i}\right) - a_{j,j}\frac{\partial u_i}{\partial x_j} + a_{i,i}\frac{\partial u_j}{\partial x_i} + (b_i - \overline{b}_i)u_j - (b_j - \overline{b}_j)u_i\Bigg)(x)m(x)\,\mathrm{d}x.$$

If $\mathsf{X}^\varepsilon_0 = \overline{\mathsf{X}}_0 = x$*, then*

$$A^{i,j}_{0,\cdot}(\widetilde{\mathsf{X}}^\varepsilon) \xrightarrow[\varepsilon\to 0]{} (A^{i,j}_{0,t}(\overline{\mathsf{X}}) + \overline{\psi}_{i,j}t)_{t\geqslant 0}.$$

in the space of continuous functions.

We remark that $\overline{\psi}_{i,j} = -\overline{\psi}_{j,i}$. In [10], we give some heuristic interpretation of the result of this proposition.

Proof. We may assume without loss of generality that the dimension of the space N is equal to 2 and that $(i,j) = (1,2)$. For $i = 1, 2$, we set $\mathsf{Y}^{i,\varepsilon}_t = \mathsf{M}^{i,\varepsilon}_t + \int_0^t c_j\left(\delta_{i,j} + \frac{\partial u_i}{\partial x_j}\right)(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s$, where M^ε is defined in (8) in the proof of Proposition 1. Hence, we know from Section 1.1 that Y^ε converges to $\overline{\mathsf{X}} - x$, since $\mathsf{X}^\varepsilon_0 = x$. Moreover, $\mathsf{Y}^\varepsilon_t - (\widetilde{\mathsf{X}}^\varepsilon_t - \widetilde{\mathsf{X}}^\varepsilon_0) = \varepsilon u(\mathsf{X}^\varepsilon_t/\varepsilon) - \varepsilon u(\mathsf{X}^\varepsilon_0/\varepsilon)$.

We use the following decomposition, since $\widetilde{\mathsf{X}}^\varepsilon_0 = x$:

$$(\widetilde{\mathsf{X}}^{1,\varepsilon} - \widetilde{\mathsf{X}}^{1,\varepsilon}_0)\cdot\widetilde{\mathsf{X}}^{2,\varepsilon} = (\widetilde{\mathsf{X}}^{1,\varepsilon} - x_1)\cdot(\widetilde{\mathsf{X}}^{2,\varepsilon} - x_2 - \mathsf{Y}^{2,\varepsilon}) + (\widetilde{\mathsf{X}}^{1,\varepsilon} - x_1)\cdot\mathsf{Y}^{2,\varepsilon}.$$

One knows that $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is a good sequence of semimartingales (see Proposition 2) and that $\widetilde{\mathsf{X}}^{1,\varepsilon}-\widetilde{\mathsf{X}}^{1,\varepsilon}_0$ converges in distribution to $\overline{\mathsf{X}}^1-x_1$. So $(\widetilde{\mathsf{X}}^{1,\varepsilon}-\widetilde{\mathsf{X}}^{1,\varepsilon}_0)\cdot\mathsf{Y}^{2,\varepsilon}$ converges in distribution to $(\overline{\mathsf{X}}^1-x_1)\cdot\overline{\mathsf{X}}^2$.

An integration by parts on $\Psi=(\widetilde{\mathsf{X}}^{1,\varepsilon}-\widetilde{\mathsf{X}}^{1,\varepsilon}_0)\cdot(\widetilde{\mathsf{X}}^{2,\varepsilon}-\widetilde{\mathsf{X}}^{2,\varepsilon}_0-\mathsf{Y}^{2,\varepsilon})$ yields

$$\begin{aligned}\Psi(t)=\int_0^t u_2(\mathsf{X}^\varepsilon_s/\varepsilon)(b_1(\mathsf{X}^\varepsilon_s/\varepsilon)-\overline{b}_1)\,\mathrm{d}s+\varepsilon\int_0^t u_2(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^{2,\varepsilon}}_s\\ +\varepsilon u_2(\mathsf{X}^\varepsilon_0/\varepsilon)(\widetilde{\mathsf{X}}^{1,\varepsilon}_t-\widetilde{\mathsf{X}}^{1,\varepsilon}_0)+\langle\mathsf{M}^{2,\varepsilon}-\mathsf{M}^{\mathsf{X}^{2,\varepsilon}},\mathsf{M}^{\mathsf{X}^{1,\varepsilon}}\rangle_t\\ +(\widetilde{\mathsf{X}}^{1,\varepsilon}_t-\widetilde{\mathsf{X}}^{1,\varepsilon}_0)(\widetilde{\mathsf{X}}^{2,\varepsilon}_t-\widetilde{\mathsf{X}}^{2,\varepsilon}_0-\mathsf{Y}^{2,\varepsilon}_t).\end{aligned}$$

Since u is bounded, $\varepsilon\int_0^t u_2(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^{1,\varepsilon}}_s$ converges in probability to 0 uniformly in t. Clearly, the product $(\widetilde{\mathsf{X}}^{1,\varepsilon}_t-\widetilde{\mathsf{X}}^{1,\varepsilon}_0)(\widetilde{\mathsf{X}}^{2,\varepsilon}_t-\widetilde{\mathsf{X}}^{2,\varepsilon}_0-\mathsf{Y}^{2,\varepsilon}_t)=\varepsilon(\widetilde{\mathsf{X}}^{1,\varepsilon}_t-x_1)(u_2(\mathsf{X}^\varepsilon_0/\varepsilon)-u_2(\mathsf{X}^\varepsilon_t/\varepsilon))$ converges uniformly in t to 0, since $\widetilde{\mathsf{X}}^{1,\varepsilon}-x_1$ converges in distribution. From (11), one has

$$\int_0^t u_2(\mathsf{X}^\varepsilon_s/\varepsilon)(b_1(\mathsf{X}^\varepsilon_s/\varepsilon)-\overline{b}_1)\,\mathrm{d}s\xrightarrow[\varepsilon\to0]{proba}t\int_{\mathbb{T}^2}(b_1(x)-\overline{b}_1)u_2(x)m(x)\,\mathrm{d}x\overset{\text{def}}{=}tD_{2,1}.$$

Similarly,

$$\begin{aligned}\langle\mathsf{M}^{2,\varepsilon}-\mathsf{M}^{\mathsf{X}^{2,\varepsilon}},\mathsf{M}^{\mathsf{X}^{1,\varepsilon}}\rangle_t=\int_0^t\left(a_{2,1}\frac{\partial u_2}{\partial x_2}+a_{2,2}\frac{\partial u_2}{\partial x_1}\right)(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s\\ \xrightarrow[\varepsilon\to0]{proba}t\int_{\mathbb{T}^2}\left(a_{2,1}\frac{\partial u_2}{\partial x_2}+a_{2,2}\frac{\partial u_2}{\partial x_1}\right)(x)m(x)\,\mathrm{d}x\overset{\text{def}}{=}tC_{2,1}.\end{aligned}$$

The previous convergences hold in fact in the space of continuous functions.

Similar computations for $(\widetilde{\mathsf{X}}^{2,\varepsilon}-x_2)\cdot\widetilde{\mathsf{X}}^{1,\varepsilon}$ leads to the result with $\overline{\psi}_{1,2}=\frac{1}{2}(C_{2,1}-C_{1,2}+D_{2,1}-D_{1,2})$. ∎

4 Convergence of solutions of SDEs

In [10], we consider the convergence of $\int_0^t f(\widetilde{\mathsf{X}}^\varepsilon_s)\,\mathrm{d}\widetilde{\mathsf{X}}^\varepsilon_s$ and the convergence of the solution Y^ε of the SDE $\mathsf{Y}^\varepsilon_t=y+\int_0^t f(\mathsf{Y}^\varepsilon_s)\,\mathrm{d}\widetilde{\mathsf{X}}^\varepsilon_s$. Here, we consider the more general problem, where one of the component of f is "fast", that is

$$\mathsf{Y}^\varepsilon_t=\mathsf{Y}^\varepsilon_0+\int_0^t f(\mathsf{X}^\varepsilon_s/\varepsilon,\mathsf{X}^\varepsilon_s-\overline{b}s/\varepsilon,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}\widetilde{\mathsf{X}}^\varepsilon_s\text{ with }\mathsf{Y}^\varepsilon_0=y.\tag{19}$$

The tools to deal with SDEs are taken from those used in the theory of averaging Backward Stochastic Differential Equations [14, 15].

Let f be a function defined from $\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}$ to $\mathbb{R}^N$. We assume that f is such that, for any $\varepsilon>0$, there exists a unique strong solution Y^ε to the SDE (19). We assume that the function $x\mapsto f(x,\cdot,\cdot)$ is periodic. The process Y^ε takes its value in $\mathbb{R}$, but nothing prevent us to consider processes Y^ε in $\mathbb{R}^m$ for any integer m. We are interested in the convergence of Y^ε.

4.1 Without highly oscillating first-order differential term

Let f be a function on $\mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}$. We say that $(z, y) \mapsto f(\cdot, z, y)$ is *equi-continuous* if the modulus of continuity of this function does not depend on the first variable x.

We work under the following assumption on the function f.

Hypothesis 2. *We assume that there exists a constant K such that $|f(x, z, y)|$ is bounded by K for any $(x, z, y) \in \mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}$, and that $(z, y) \mapsto f(\cdot, z, y)$ is equi-continuous.*

Proposition 6. *We work under Assumption 1 (So $\overline{b} = 0$ and $\widetilde{\mathsf{X}}^\varepsilon = \mathsf{X}^\varepsilon$) and Hypothesis 2. Let $\alpha(z, y) = (\alpha_{i,j}(y, z))_{i,j=1}^N$ be a function on $\mathbb{R}^N \times \mathbb{R}$ such that*

$$\alpha\alpha^{\mathrm{T}}(z, y) = \int_{\mathbb{T}^N} a_{i,j}(x) f_i(x, z, y) f_j(x, z, y) m(x) \,\mathrm{d}x. \tag{20}$$

Then there exists some Brownian Motion B on an extension of the probability space on which $\overline{\mathsf{X}}$ is defined such that Y^ε converges in distribution to the unique solution of the SDE

$$\mathsf{Y}_t = y + \int_0^t \alpha(\overline{\mathsf{X}}_s, \mathsf{Y}_s) \,\mathrm{d}\mathsf{B}_s.$$

We denote by $\overline{f}$ the function on $\mathbb{R}^N \times \mathbb{R}$ defined by

$$\overline{f}(z, y) = \int_{\mathbb{T}^N} f(x, z, y) m(x) \,\mathrm{d}x.$$

The following lemma is particularly useful, and its proof may be found in [14, 15].

Lemma 2. *Let f be an equi-continuous function. If $(\mathsf{X}^\varepsilon, \mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is tight, then for any $\kappa > 0$,*

$$\sup_{x \in \mathbb{R}^N} \mathbb{P}_x \left[\left| \int_0^t f(\mathsf{X}_s^\varepsilon/\varepsilon, \mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon) \,\mathrm{d}s - \int_0^t \overline{f}(\mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon) \,\mathrm{d}s \right| \geqslant \kappa \right] \xrightarrow[\varepsilon \to 0]{} 0.$$

Proof of Proposition 6. According to Remark 1, we assume that $c = 0$. Under Assumption 1, the process Y^ε is a continuous martingale. Moreover, the sequence $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is tight in the space of continuous functions. For that, we remark that

$$\langle \mathsf{Y}^\varepsilon \rangle_t = \int_0^t a_{i,j}(\mathsf{X}_s^\varepsilon/\varepsilon) f_i(\mathsf{X}_s^\varepsilon/\varepsilon, \mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon) \,\mathrm{d}s,$$

the coefficients $a_{i,j}$ and the functions f_i are bounded. So, it is clear that $(\langle \mathsf{Y}^\varepsilon \rangle)_{\varepsilon>0}$ is tight, and it follows from Theorem 4.13 in [3, p. 322], that the sequence $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is also tight.

Let Y be a limit point for this sequence. We know from Corollary VI.6.6 in [3, p. 342] that Y is itself a martingale and that $\langle \mathsf{Y}^\varepsilon \rangle$ converges to $\langle \mathsf{Y} \rangle$. With Lemma 2, $\langle \mathsf{Y}^\varepsilon \rangle$ converges in distribution to the process $\langle \mathsf{Y} \rangle = \int_0^\cdot \overline{a_{i,j} f_i f_j}(\overline{\mathsf{X}}_s, \mathsf{Y}_s) \,\mathrm{d}s$. So, for any limit point Y of the sequence Y_t^ε, and any function α from $\mathbb{R}^N \times \mathbb{R}$ satisfying (20), there exists a Brownian Motion B on an extension of the probability space (see for example Theorem 3.4.2 in [4, p. 170]) such that

$$\mathsf{Y}_t = y + \int_0^t \alpha(\overline{\mathsf{X}}_s, \mathsf{Y}_s) \,\mathrm{d}\mathsf{B}_s$$

and the quadratic variation of Y is $\langle \mathsf{Y} \rangle = \int_0^\cdot \overline{a_{i,j} f_i f_j}(\overline{\mathsf{X}}_s, \mathsf{Y}_s) \,\mathrm{d}s$. Due to the uniqueness of the solution of the martingale problem for $(\overline{\mathsf{X}}, \mathsf{Y})$, the limit is unique in distribution. ∎

We have to remark that nothing proves that Y and B are adapted to the filtration generated by $\overline{\mathsf{X}}$, since Y is just a martingale with respect to the filtration generated by $\overline{\mathsf{X}}$ and itself. Yet, if $(x, z, y) = f(z, y)$, then the martingale Y is a martingale with respect to the filtration generated by $\overline{\mathsf{X}}$. For that, one has just to remark that

$$\mathsf{Y}_t^\varepsilon - y = \int_0^t f(\mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon)\,\mathrm{d}\mathsf{X}_s^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \int_0^t f(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}\overline{\mathsf{X}}_s = \mathsf{Y}_t - y.$$

According to the Yamada and Watanbe's result (see for example Section 5.3.D in [4, p. 309]), Y is the unique strong solution to the SDE $\mathsf{Y}_t = y + \int_0^t f(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}\overline{\mathsf{X}}_s$ and is then adapted to the filtration generated by $\overline{\mathsf{X}}$.

4.2 With a highly oscillating first-order differential term

Under Assumption 2, the situation is more complicated and Y^ε does not always converge. However, we may prove that there exists a function u on $\mathbb{T}^N \times \mathbb{R}^N \times \mathbb{R}$ such that

$$Lu(\cdot, z, y) = -f(\cdot, z, y)b(\cdot) + \overline{fb}(z, y) \tag{21}$$

where $\overline{fb}(z, y) = \int_{\mathbb{T}^N} b(x)f(x, z, y)m(x)\,\mathrm{d}x$. If $f(x, z, y) = f(z, y)$, then $\overline{fb}(z, y) = f(z, y)\overline{b}$.

Proposition 7. *We assume that $(z, y) \mapsto f(\cdot, z, y)$ is of class $\mathcal{C}^2$, and that this function together with all its first and second derivatives are equi-continuous and bounded on $\mathbb{R}^N \times \mathbb{R}$ with respect to the first variable. The sequence $\left(\mathsf{Y}^\varepsilon - \frac{1}{\varepsilon}\int_0^\cdot \overline{fb}(\widetilde{\mathsf{X}}_s^\varepsilon, \mathsf{Y}_s^\varepsilon)\,\mathrm{d}s\right)_{\varepsilon>0}$ converges in distribution to the unique solution Y of*

$$\begin{aligned}\mathsf{Y}_t = \mathsf{Y}_0 + \int_0^t \alpha(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}\mathsf{B}_s + \frac{1}{2}\int_0^t \overline{\frac{\partial^2 u}{\partial x_i \partial y} a_{i,j} f_j}(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s - \overline{b}\int_0^t \overline{\nabla_z u}(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s \\ + \frac{1}{2}\int_0^t \overline{a_{i,j}\frac{\partial^2 u}{\partial x_i \partial z_j}}(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s + \int_0^t \overline{\nabla_z u \cdot fb}(\overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s,\end{aligned} \tag{22}$$

where $\alpha(z, y)$ is such that

$$\alpha(z, y)\alpha^{\mathrm{T}}(z, y) = \int_{\mathbb{T}^N} a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}(x, z, y)m(x)\,\mathrm{d}x$$

and B is a $(\sigma(\overline{\mathsf{X}}_s, \mathsf{Y}_s; 0 \leqslant s \leqslant t))_{t\geqslant 0}$-standard Brownian Motion.

The following proposition is the central point of the proof. It means that the continuity of f is transfered to u.

Proposition 8 ([16, 17]**).** *Under the hypotheses on f of Proposition 7, the function $(z, y) \mapsto u(\cdot, z, y)$ given by (21) is twice differentiable with continuous derivatives up to order 2. Furthermore, this function and its derivatives up to order 2 are uniformly bounded on $\mathbb{R}^N \times \mathbb{R}$. Moreover, for $i, j = 1, \dots, N$, $(y, z) \mapsto \nabla_x u(\cdot, z, y)$, $(y, z) \mapsto \nabla_z u(\cdot, z, y)$, $(y, z) \mapsto \frac{\partial^2 u}{\partial x_i \partial y}(\cdot, z, y)$ and $(y, z) \mapsto \frac{\partial^2 u}{\partial x_i \partial z_j}(\cdot, z, y)$ are equi-continuous and bounded on each compact uniformly with respect to the first variable.*

Proof. The proof relies on the formula $u(x,z,y) = \int_0^{+\infty} P_t(f(x,z,y)b(x) - \overline{fb}(z,y))\,\mathrm{d}t$, where $(P_t)_{t>0}$ is the semi-group generated by L seen as an operator acting on periodic functions. The continuity and differentiability of u follows from the fact that this semi-group admits a probability transition function which is differentiable. See [16, 17] for details. ∎

Proof of Proposition 7. The Itô formula implies that

$$\begin{aligned}
&\mathsf{Y}^\varepsilon_t - \frac{1}{\varepsilon}\int_0^t \overline{fb}(\widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}s + \varepsilon u(\mathsf{X}^\varepsilon_t/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_t, \mathsf{Y}^\varepsilon_t) \\
&\quad = \mathsf{Y}^\varepsilon_0 + \varepsilon u(\mathsf{X}^\varepsilon_0/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_0, \mathsf{Y}^\varepsilon_0) + \int_0^t (f + \nabla_x u)(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s \\
&\qquad + \frac{1}{2}\int_0^t f_j \frac{\partial^2 u}{\partial x_i \partial y}(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s) a_{i,j}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x_j \partial z_i}(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s) a_{i,j}(\mathsf{X}^\varepsilon_s/\varepsilon)\,\mathrm{d}s \\
&\qquad + \int_0^t \nabla_z u(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s) \cdot fb(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}s \\
&\qquad - \int_0^t \bar{b}\nabla_z u(\mathsf{X}^\varepsilon_s/\varepsilon, \widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}s + \mathsf{V}^\varepsilon_t,
\end{aligned}$$

where V^ε contains all the terms of order ε and consequently decreases to 0 as $\varepsilon \to 0$.

Then, one may use Proposition 8 and Lemma 2 to prove that the sequence $\left(\mathsf{Y}^\varepsilon - \frac{1}{\varepsilon}\int_0^\cdot \overline{fb}(\widetilde{\mathsf{X}}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}s\right)_{\varepsilon>0}$ is tight and that any limit Y of this sequence is a solution to (22). With the martingale problem, the limit is unique in distribution. ∎

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