Elect. Comm. in Probab. **12** (2007), 434–441

ELECTRONIC
COMMUNICATIONS
in PROBABILITY

ON THE STRONG LAW OF LARGE NUMBERS FOR D-DIMENSIONAL ARRAYS OF RANDOM VARIABLES

LE VAN THANH
Department of Mathematics, Vinh University, Nghe An 42118, Vietnam
email: lvthanhvinh@yahoo.com

Submitted February 21? 2007, *accepted in final form* August 13, 2007

AMS 2000 Subject classification: 60F15
Keywords: Strong law of large number, almost sure convergence, d-dimensional arrays of random variables

Abstract
In this paper, we provide a necessary and sufficient condition for general d-dimensional arrays of random variables to satisfy strong law of large numbers. Then, we apply the result to obtain some strong laws of large numbers for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables.

1 Introduction

Let $\mathbb{Z}_+^d$, where d is a positive integer, denote the positive integer d-dimensional lattice points. The notation $\mathbf{m} \prec \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, ..., m_d)$ and $\mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d$, means that $m_i \leqslant n_i$, $1 \leqslant i \leqslant d$. Let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants, and let $\mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d$, we denote $|\mathbf{n}| = \prod_{i=1}^d n_i$, $|\mathbf{n}(\alpha)| = \prod_{i=1}^d n_i^{\alpha_i}$, $I(\mathbf{n}) = \{(a_1, \ldots, a_d) \in \mathbb{Z}_+^d : 2^{n_i - 1} \leqslant a_i < 2^{n_i}, 1 \leqslant i \leqslant d\}$, $\overline{\mathbf{n}} = (2^{n_1-1}, \ldots, 2^{n_d-1})$.
Consider a d-dimensional array $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $S_{\mathbf{n}} = \sum_{\mathbf{i} \prec \mathbf{n}} X_{\mathbf{i}}$, and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. In Section 2, we provide a necessary and sufficient condition for

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ almost surely (a.s.)}$$

to hold. This condition springs from a recent result of Chobanyan, Levental and Mandrekar [1] which provided a condition for strong law of large numbers (SLLN) in the case $d = 1$ (see Chobanyan, Levental and Mandrekar [1, Theorem 3.3]). Some applications to SLLN for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables are made in Section 3.

2 Result

We can now state our main result.

THEOREM 2.1. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. For $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d$, set

$$T_{\mathbf{m}} = \frac{1}{|\overline{\mathbf{m}}(\alpha)|} \max_{\mathbf{k} \in I(\mathbf{m})} \Big| \sum_{\overline{\mathbf{m}} \prec \mathbf{i} \prec \mathbf{k}} X_{\mathbf{i}} \Big|.$$

Then

$$\lim_{|\mathbf{m}| \to \infty} T_{\mathbf{m}} = 0 \text{ a.s.} \tag{2.1}$$

if and only if

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.2}$$

Proof. To prove Theorem 2.1, we will need the following lemmma. The proof of the following lemma is just an application of Kronecker's lemma with d-dimensional indices as was so kindly pointed out to the author by the referee.

LEMMA 2.1. Let $\{x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of constants, and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be a collection of positive constants. If

$$\lim_{|\mathbf{n}| \to \infty} x_{\mathbf{n}} = 0, \tag{2.3}$$

then

$$\lim_{|\mathbf{n}| \to \infty} \frac{1}{|\overline{\mathbf{n}}(\alpha)|} \sum_{\mathbf{k} \prec \mathbf{n}} |\overline{\mathbf{k}}(\alpha)| x_{\mathbf{k}} = 0. \tag{2.4}$$

Proof of Theorem 2.1. Let $\mathbf{m} = (m_1, \ldots, m_d),\ \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ with $\mathbf{n} \in I(\mathbf{m})$. Set

$$\mathbf{n}^{(j)} = (n_1, \ldots, n_{j-1}, 2^{m_j - 1} - 1, n_{j+1}, \ldots, n_d),\ 1 \leqslant j \leqslant d,$$

$$S_{\mathbf{n}}^{(1)} = S_{\mathbf{n}^{(1)}},$$

$$S_{\mathbf{n}}^{(d)} = \sum_{i_1 = 2^{m_1 - 1}}^{n_1} \cdots \sum_{i_{d-1} = 2^{m_{d-1} - 1}}^{n_{d-1}} \sum_{i_d = 1}^{2^{m_d - 1} - 1} X_{(i_1, \ldots, i_d)},$$

and

$$S_{\mathbf{n}}^{(j)} = \sum_{i_1 = 2^{m_1 - 1}}^{n_1} \cdots \sum_{i_{j-1} = 2^{m_{j-1} - 1}}^{n_{j-1}} \sum_{i_j = 1}^{2^{m_j - 1} - 1} \sum_{i_{j+1} = 1}^{n_{j+1}} \cdots \sum_{i_d = 1}^{n_d} X_{(i_1, \ldots, i_d)},\ 2 \leqslant j \leqslant d - 1.$$

Then

$$S_{\mathbf{n}}^{(j)} = S_{\mathbf{n}^{(j)}} - \sum_{k=1}^{j-1} S_{\mathbf{n}^{(j)}}^{(k)},\ 2 \leqslant j \leqslant d. \tag{2.5}$$

Assume that (2.1) holds. Since

$$\frac{|S_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} \leqslant \frac{1}{|\overline{\mathbf{m}}(\alpha)|} \sum_{\mathbf{k} \prec \mathbf{m}} |\overline{\mathbf{k}}(\alpha)| T_{\mathbf{k}},$$

the conclusion (2.2) holds by Lemma 2.1. Thus (2.1) implies (2.2). Now, assume that (2.2) holds. Then

$$\lim_{|\mathbf{m}|\to\infty} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{S_{\mathbf{n}}^{(1)}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.6}$$

For $1 \leqslant j \leqslant d$, by (2.5), (2.6) and the induction method, we obtain

$$\lim_{|\mathbf{m}|\to\infty} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{S_{\mathbf{n}}^{(j)}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.7}$$

Since

$$S_{\mathbf{n}} = \sum_{j=1}^{d} S_{\mathbf{n}}^{(j)} + \sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1,\ldots,i_d)},$$

we have that

$$|\sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1,\ldots,i_d)}| \leqslant |S_{\mathbf{n}}| + \sum_{j=1}^{d} |S_{\mathbf{n}}^{(j)}|.$$

This implies

$$T_{\mathbf{m}} \leqslant 2^{\alpha_1+\cdots+\alpha_d} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{|S_{\mathbf{n}}| + \sum_{j=1}^{d} |S_{\mathbf{n}}^{(j)}|}{|\mathbf{n}(\alpha)|}. \tag{2.8}$$

The conclusion (2.1) follows immediately from (2.2), (2.7) and (2.8). □

3 Applications

In this section, we present some applications of Theorem 2.1. A d-dimensional array of random variables $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be *blockwise independent* (resp., *blockwise orthogonal*) if for each $\mathbf{k} \in \mathbb{Z}_+^d$, the random variables $\{X_{\mathbf{i}}, \mathbf{i} \in I(\mathbf{k})\}$ is independent (resp., orthogonal). The concept of blockwise independence for a sequence of random variables was introduced by Móricz [9]. Extensions of classical Kolmogorov SLLN (see, e.g., Chow and Teicher [2], p. 124) to the blockwise independence case were established by Móricz [9] and Gaposhkin [4]. Móricz [9] and Gaposhkin [4] also studied SLLN problem for sequence of blockwise orthogonal random variables.

Firstly, we establish a blockwise independence and d-dimensional version of the Kolmogorov SLLN.

THEOREM 3.1. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of mean 0 blockwise independent random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. If

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|X_{\mathbf{n}}|^p}{|\mathbf{n}(\alpha)|^p} < \infty \text{ for some } 0 < p \leqslant 2, \tag{3.1}$$

then SLLN

$$\lim_{|\mathbf{n}|\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.2}$$

obtains.

In the case $0 < p \leqslant 1$, the independence hypothesis and the hypothesis that $EX_{\mathbf{n}} = 0, \mathbf{n} \in \mathbb{Z}_+^d$ are superfluous.

Proof. We need the following lemma which was proved by Thanh [11] in the case $d = 2$. If d is arbitrary positive integer, then the proof is similar and so is omitted.
LEMMA 3.1. Let $\mathbf{n} \in \mathbb{Z}_+^d$ and let $\{X_{\mathbf{i}}, \mathbf{i} \prec \mathbf{n}\}$ be a collection of $|\mathbf{n}|$ mean 0 independent random variables. Then there exists a constant C depending only on p and d such that

$$E(\max_{\mathbf{k}\prec\mathbf{n}} |S_{\mathbf{k}}|^p) \leqslant C \sum_{\mathbf{i}\prec\mathbf{n}} E|X_{\mathbf{i}}|^p \text{ for all } 0 < p \leqslant 2.$$

In the case $0 < p \leqslant 1$, the independence hypothesis and the hypothesis that $EX_{\mathbf{i}} = 0, \mathbf{i} \prec \mathbf{n}$ are superfluous, and C is given by $C = 1$. In the case $1 < p < 2$, C is given by $C = 2\big(\frac{p}{p-1}\big)^{pd}$. In the case $p = 2$, Lemma 3.1 was proved by Wichura [12] and C is given by $C = 4^d$.
Proof of Theorem 3.1. Define $T_{\mathbf{m}}, \mathbf{m} \in \mathbb{Z}_+^d$ as in Theorem 2.1. Note that for all $\mathbf{m} \in \mathbb{Z}_+^d$,

$$\begin{aligned} E|T_{\mathbf{m}}|^p &= \frac{1}{|\overline{\mathbf{m}}(\alpha)|^p} E\Big(\max_{\mathbf{k}\in I(\mathbf{m})} \big|\sum_{\overline{\mathbf{m}}\prec\mathbf{i}\prec\mathbf{k}} X_{\mathbf{i}}\big|\Big)^p \\ &\leqslant \frac{C}{|\overline{\mathbf{m}}(\alpha)|^p} \sum_{\mathbf{i}\in I(\mathbf{m})} E|X_{\mathbf{i}}|^p \text{ (by Lemma 3.1)} \\ &\leqslant 2^{\alpha_1+\cdots+\alpha_d} C \frac{\sum_{\mathbf{i}\in I(\mathbf{m})} E|X_{\mathbf{i}}|^p}{|\mathbf{i}(\alpha)|^p}. \end{aligned}$$

It thus follows from (3.1) that $\sum_{\mathbf{m}\in\mathbb{Z}_+^d} E|T_{\mathbf{m}}|^p < \infty$ whence $\lim_{|\mathbf{m}|\to\infty} T_{\mathbf{m}} = 0$ a.s. The conclusion (3.2) follows immediately from Theorem 2.1. □

The following theorem extends Theorem 3.1 and its part (ii) reduces to a result of Smythe [10] when the $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are independent and $\alpha_1 = \cdots = \alpha_d = 1$.
THEOREM 3.2. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. Assume that $\varphi(x)$ is a continuous functions on $[0,\infty)$, $\varphi(0) \geqslant 0$, $\varphi(x) > 0$ for $x > 0$, and

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E(\varphi(|X_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} < \infty. \tag{3.3}$$

If either
(i) $\varphi(x)/x \downarrow$, and $\varphi(x) \uparrow$
or
(ii) $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are blockwise independent and have mean 0, and

$$\varphi(x)/x \uparrow,\ \varphi(x)/x^2 \downarrow,$$

then SLLN (3.2) obtains.

Proof. For $\mathbf{n} \in \mathbb{Z}_+^d$, set

$$Y_{\mathbf{n}} = X_{\mathbf{n}} I(|X_{\mathbf{n}}| \leqslant |\mathbf{n}(\alpha)|),$$

$$Z_{\mathbf{n}} = X_{\mathbf{n}} I(|X_{\mathbf{n}}| > |\mathbf{n}(\alpha)|).$$

Consider the case (i) first. It follows from (3.3) that

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Y_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Y_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \text{ (by the first condition of (i))}\\ &< \infty.\end{aligned}$$

By Theorem 3.1,

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}Y_{\mathbf{i}}}{|\mathbf{n}(\alpha)|}=0 \text{ a.s.} \tag{3.4}$$

On the other hand

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{X_{\mathbf{n}}\neq Y_{\mathbf{n}}\} &= \sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{|X_{\mathbf{n}}|>|\mathbf{n}(\alpha)|\}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{\varphi(|X_{\mathbf{n}}|)\geqslant\varphi(|\mathbf{n}(\alpha)|)\}\\ &\quad\text{(by the second condition of (i))}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|X_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)}\\ &< \infty \quad \text{(by (3.3))}.\end{aligned}$$

By the Borel-Cantelli lemma,

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}(X_{\mathbf{i}}-Y_{\mathbf{i}})}{|\mathbf{n}(\alpha)|}=0 \text{ a.s.} \tag{3.5}$$

The conclusion (3.2) follows immediately from (3.4) and (3.5).
Now, consider the case (ii). It follows from (3.3) that

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(Y_{\mathbf{n}}-EY_{\mathbf{n}})^2}{|\mathbf{n}(\alpha)|^2} &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{EY_{\mathbf{n}}^2}{|\mathbf{n}(\alpha)|^2}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Y_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \quad \text{(by the last condition of (ii))}\\ &< \infty \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_{\mathbf{n}}-EZ_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} &\leqslant 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|}\\ &\leqslant 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Z_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \quad \text{(by the second condition of (ii))}\\ &< \infty.\end{aligned} \tag{3.7}$$

By Theorem 3.1, the conclusion (3.6) implies

$$\lim_{|\mathbf{n}|\to\infty} \frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Y_{\mathbf{i}} - EY_{\mathbf{i}})}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.8}$$

and the conclusion (3.7) implies

$$\lim_{|\mathbf{n}|\to\infty} \frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Z_{\mathbf{i}} - EZ_{\mathbf{i}})}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.9}$$

The conclusion (3.2) follows immediately from (3.8) and (3.9). □

REMARK 3.1. (i) According to the discussion in Smythe [10], the proof of part (ii) of Theorem 3.2 was based on the "Khintchin-Kolmogorov convergence theorem, Kronecker lemma approach". But it seems that the Kronecker lemma for d-dimensional arrays when $d \geqslant 2$ is not such a good tool as in the study of the SLLN for the case $d = 1$ (see Mikosch and Norvaisa [6]). Moreover, in the blockwise independence case, according to an example of Móricz [9], the conclusion of Theorem 3.1 (or part (ii) of Theorem 3.2) cannot in general be reached through the well-know Kronecker lemma approach for proving SLLNs even when $d = 1$.
(ii) Chung [3] proved part (i) of Theorem (3.2) (for the case $d = 1$ only) by the Kolmogorov three series theorem and the Kronecker lemma. So in his proof, the independence assumption must be required.
We now establish the Marcinkiewicz-Zygmund SLLN for d-dimensional arrays of blockwise independent identically distributed random variables. The following theorem reduces to a result of Gut [5] when the $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are independent.
THEOREM 3.3. Let $\{X, X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of blockwise independent identically distributed random variables with $EX = 0$, $E(|X|^r(\log^+|X|)^{d-1}) < \infty$ for some $1 \leqslant r < 2$. Then SLLN

$$\lim_{|\mathbf{n}|\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} = 0 \text{ a.s.} \tag{3.10}$$

obtains.

Proof. According to the proof of Lemma 2.2 of Gut [5],

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E(Y_{\mathbf{n}} - EY_{\mathbf{n}})^2}{|\mathbf{n}|^{2/r}} < \infty \tag{3.11}$$

where $Y_{\mathbf{n}} = X_{\mathbf{n}}(|X_{\mathbf{n}}| \leqslant |\mathbf{n}|^{1/r})$, $\mathbf{n} \in \mathbb{Z}_+^d$. And similarly, we also have

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|Z_{\mathbf{n}} - EZ_{\mathbf{n}}|}{|\mathbf{n}|^{1/r}} < \infty \tag{3.12}$$

where $Z_{\mathbf{n}} = X_{\mathbf{n}}(|X_{\mathbf{n}}| > |\mathbf{n}|^{1/r})$, $n \in \mathbb{Z}_+^d$. By Theorem 3.1 (with $\alpha_i = 1/r, 1 \leqslant i \leqslant d$), the conclusion (3.10) follows immediately from (3.11) and (3.12). □

Finally, we establish the SLLN for d-dimensional arrays of blockwise orthogonal random variables. The following theorem is a blockwise orthogonality version of Theorem 1 of Móricz [8]

and its proof is based on the d-dimensional version of the Rademacher-Mensov inequality (see Móricz [7]) and the method used in the proof of Theorem 3.1.
THEOREM 3.4. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of blockwise orthogonal random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. If

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|X_{\mathbf{n}}|^2}{|\mathbf{n}(\alpha)|^2} \Pi_{i=1}^d [\log(n_i+1)]^2 < \infty,$$

then SLLN (3.2) obtains.

Acknowledgements

The author are grateful to the referee for carefully reading the manuscript and for offering some very perceptive comments which helped him improve the paper.

References

[1] Chobanyan, S., Levental, S., and Mandrekar, V. Prokhorov blocks and strong law of large numbers under rearrangements. *J. Theoret. Probab.* **17** (2004), 647-672. MR2091554

[2] Chow, Y. S. and Teicher, H. *Probability Theory: Independence, Interchangeability, Martingales,* 3rd Ed. Springer-Verlag, New York, 1997. MR1476912

[3] Chung, K. L. Note on some strong laws of large numbers. *Amer. J. Math.* **69** (1947), 189-192. MR0019853

[4] Gaposhkin, V.F. On the strong law of large numbers for blockwise independent and blockwise orthogonal random variables. *Teor. Veroyatnost. i Primenen.* **39** (1994), 804-812 (in Russian). English translation in *Theory Probab. Appl.* **39** (1994), 667-684 (1995). MR1347654

[5] Gut, A. Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices. *Ann. Probability.* **6** (1978), 469-482. MR0494431

[6] Mikosch, T. and Norvaisa, R. Strong laws of large numbers for fields of Banach space valued random variables. *Probab. Th. Rel. Fields.* **74** (1987), 241-253. MR0871253

[7] Móricz, F. Moment inequalities for the maximum of partial sums of random fields. *Acta Sci. Math.* **39** (1977), 353-366. MR0458535

[8] Móricz, F. Strong limit theorems for quasi-orthogonal random fields. *J. Multivariate. Anal.* **30** (1989), 255-278. MR1015372

[9] Móricz, F. Strong limit theorems for blockwise m-dependent and blockwise quasiorthogonal sequences of random variables. *Proc. Amer. Math. Soc.* **101** (1987), 709-715. MR0911038

[10] Smythe, R.T. Strong laws of large numbers for r-dimensional arrays of random variables. *Ann. Probab.* **1** (1973), 164-170. MR0346881

[11] Thanh. L. V. Mean convergence theorems and weak laws of large numbers for double arrays of random variables. *J. Appl. Math. Stochastic Anal.* 2006, Art. ID 49561, 1 - 15. MR2220986

[12] Wichura, M. J. Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters. *Ann. Math. Statist.* **60** (1969), 681-687. MR0246359