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EXACT CONVERGENCE RATE FOR THE MAXIMUM OF STANDARDIZED GAUSSIAN INCREMENTS

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Abstract

We prove an almost sure limit theorem on the exact convergence rate of the maximum of standardized gaussian random walk increments. This gives a more precise version of Shao's theorem (Shao, Q.-M., 1995. On a conjecture of Révész. Proc. Amer. Math. Soc. 123, 575-582) in the gaussian case.

1 Introduction

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. standard gaussian variables. Let $S_n = \sum_{i=1}^n \xi_i$, $S_0 = 0$ be the gaussian random walk and define the maximum of standardized gaussian random walk increments by

$$L_n = \max_{0 \le i < j \le n} \frac{S_j - S_i}{\sqrt{j - i}}.$$
 (1)

Recently, the precise behaviour of L_n and related quantities has become important in statistics [15], [16], [5], [8], [2]. For example, in the context of nonparametric function estimation, the statistical multiscale paradigm suggests to select a function from a set of candidate functions such that the resulting residuals immitate L_n , i.e. they behave as white noise simultaneously on all scales, see [5], [8], [2].

It follows from a more general result of Shao [14], see Theorem 2.1 below, that

$$\lim_{n \to \infty} L_n / \sqrt{2 \log n} = 1 \text{ a.s.}$$

Our goal is to determine the exact convergence rate in Shao's theorem.

Theorem 1.1. We have

$$\limsup_{n \to \infty} \sqrt{2 \log n} \left(L_n - \sqrt{2 \log n} \right) / \log \log n = 3/2 \ a.s.$$

and

$$\liminf_{n \to \infty} \sqrt{2 \log n} \left(L_n - \sqrt{2 \log n} \right) / \log \log n = 1/2 \ a.s.$$

For comparison, we cite a result of [12], see also [19], [11], [18], [1] for extensions and improvements. Let $\{\xi_i\}_{i=1}^{\infty}$ be a stationary centered gaussian sequence with covariance function $r_n = \mathbb{E}(\xi_1 \xi_n), r_1 = 1$. Suppose that $r_n = O(n^{-\varepsilon}), n \to \infty$, for some $\varepsilon > 0$. Then, with $M_n = \max_{i=1,\dots,n} \xi_i$,

$$\limsup_{n \to \infty} \sqrt{2 \log n} \left(M_n - \sqrt{2 \log n} \right) / \log \log n = 1/2 \text{ a.s.}$$

and

$$\liminf_{n\to\infty} \sqrt{2\log n} \left(M_n - \sqrt{2\log n} \right) / \log \log n = -1/2 \text{ a.s.}$$

There is a continuous-time version of Theorem 1.1. To state it, let $\{B(x), x \geq 0\}$ be the standard Brownian motion. For n > 1 define

$$L_n^{\mathrm{Br},1} = \sup_{\substack{x_1,x_2 \in [0,1] \\ x_2 - x_1 \geq 1/n}} \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} \quad \text{and} \quad L_n^{\mathrm{Br},2} = \sup_{\substack{x_1,x_2 \in [0,n] \\ x_2 - x_1 \geq 1}} \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}}.$$

Although for each fixed n the distributions of $L_n^{\mathrm{Br},1}$ and $L_n^{\mathrm{Br},2}$ coincide, the distributions of stochastic processes $\{L_n^{\mathrm{Br},1}, n>1\}$ and $\{L_n^{\mathrm{Br},2}, n>1\}$ are clearly different.

Theorem 1.2. For i = 1, 2 we have

$$\limsup_{n \to \infty} \sqrt{2 \log n} \left(L_n^{Br,i} - \sqrt{2 \log n} \right) / \log \log n = 5/2 \ a.s.$$

and

$$\liminf_{n \to \infty} \sqrt{2 \log n} \left(L_n^{Br,i} - \sqrt{2 \log n} \right) / \log \log n = 3/2 \ a.s.$$

The lim sup part of Theorem 1.1 and Theorem 1.2 may be generalized to the following integral test.

Theorem 1.3. Let $\{f_n\}_{n=1}^{\infty}$ be a positive non-decreasing sequence. Then

$$\mathbb{P}[L_n > f_n \text{ for infinitely many values of } n] = 1 \text{ iff } \sum_{n=1}^{\infty} f_n e^{-f_n^2/2} = \infty$$

and, for i=1,2,

$$\mathbb{P}[L_n^{Br,i} > f_n \text{ for infinitely many values of } n] = 1 \text{ iff } \sum_{n=1}^{\infty} f_n^3 e^{-f_n^2/2} = \infty.$$

In the case i = 1, the second statement is a consequence of the Chung-Erdös-Sirao integral test [4]. Since $L_n^{\text{Br},2}$ may be treated by essentially the same method as L_n , we omit the proof of the second half of the above theorem.

The corresponding result for M_n instead of L_n and $L_n^{\text{Br},i}$ reads as follows, see [11, Th.B]:

$$\mathbb{P}[M_n > f_n \text{ for infinitely many values of } n] = 1 \text{ iff } \sum_{n=1}^{\infty} f_n^{-1} e^{-f_n^2/2} = \infty.$$

The limiting behavior as $n \to \infty$ of the distribution of L_n was studied in [15] (alternatively, see Th. 1.3 of [9]) and that of $L_n^{\text{Br},i}$, i = 1, 2, in Th 1.6 of [9]. It was shown there that the appropriately normalized distributions of L_n and $L_n^{\text{Br},i}$ converge to the Gumbel (double-exponential) distribution. The next theorem is a simple consequence of these results.

Theorem 1.4. We have

$$\lim_{n \to \infty} \sqrt{2 \log n} \left(L_n - \sqrt{2 \log n} \right) / \log \log n = 1/2$$

and, for i = 1, 2,

$$\lim_{n \to \infty} \sqrt{2\log n} \left(L_n^{Br,i} - \sqrt{2\log n} \right) / \log \log n = 3/2,$$

where the convergence is in probability.

Above, we have considered increments of *all* possible lengths of the gaussian random walk and Brownian motion. The increments of *fixed* length were extensively studied in the past, the most well-known result being the Erdös-Rényi-Shepp law of large numbers. Results similar to ours in the case of increments of fixed length were obtained in [7], [6], [13].

2 The non-gaussian case

It seems difficult to obtain the exact convergence rate for the maximum of standardized increments in the case of non-gaussian summands. Let us recall the following result of [14], see also [17].

Theorem 2.1. Let $\{\xi_i\}_{i=1}^{\infty}$ be i.i.d. random variables. Suppose that $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 = 1$ and that $\varphi(t) = \log \mathbb{E}e^{t\xi_1}$ exists finitely in some interval containing 0. Let $S_n = \sum_{i=1}^n \xi_i$, $S_0 = 0$ and define L_n as in (1). Then

$$\lim_{n \to \infty} L_n / \sqrt{2 \log n} = \alpha^* \qquad a.s.,$$

where $\alpha^* \in [1, \infty]$ is a constant defined as follows. Let $I(t) = \sup_{x \in \mathbb{R}} (xt - \varphi(x)), \ \alpha(c) = \sup\{t \geq 0 : I(t) \leq 1/c\}$. Then

$$\alpha^* = \sup_{c>0} (\alpha(c)\sqrt{c/2}). \tag{2}$$

In particular, if ξ_i are standard normal, then $\alpha^* = 1$. If the supremum in (2) is attained at some number $c^* \in (0, \infty)$ then Shao's proof shows that the almost sure limiting behavior of L_n coincides with the behavior of the Erdös-Rényi statistic

$$L_n^{c^*} = \max_{\substack{0 \le i < j \le n \\ j-i = [c^* \log n]}} \frac{S_j - S_i}{\sqrt{j-i}}.$$

If the supremum is attained at, say, $c^* = \infty$, then dominating in L_n are terms of the form $(S_j - S_i)/\sqrt{j-i}$ with $j-i \gg \log n$. Now, in the case of standard normal variables we have $\alpha(c) = \sqrt{2/c}$ for every $c \in (0, \infty)$. Thus, in the gaussian case, dominating in L_n are terms of the form $(S_j - S_i)/\sqrt{j-i}$ with $j-i \approx c \log n$, where c varies in $(0,\infty)$. Our proofs use this fact extensively. As the above discussion suggests, the exact convergence rate in the case of non-gaussian summands may depend strongly on the way in which the supremum in (2) is attained. Note also that for statistical purposes the square-root normalization in (1) seems to be natural only in the gaussian case. See [16] for a different normalization having a natural statistical interpretation.

In the rest of the paper we prove Theorem 1.3 and Theorem 1.1.

3 Standardized Brownian motion increments

We denote by $\mathbb{H} = \{t = (x, y) \in \mathbb{R}^2 | y > 0\}$ the upper half-plane. Let $\{B(x), x \in \mathbb{R}\}$ be the standard Brownian motion. Then the gaussian field $\{X(t), t = (x,y) \in \mathbb{H}\}\$ of standardized Brownian motion increments is defined by

$$X(t) = \frac{B(x+y) - B(x)}{\sqrt{y}}. (3)$$

We shall need the following theorem, see Theorem 2.1 and Example 2.10 in [3], which describes the precise asymptotics of the high excursion probability of the field X.

Theorem 3.1. Let $K \subset \mathbb{H}$ be a compact set with positive Jordan measure. Then, for some constant $C_K > 0$,

$$\mathbb{P}\left[\sup_{t\in K}X(t)>u\right]\sim C_Ku^3e^{-u^2/2}\ as\ u\to +\infty.$$

The next theorem, although not stated explicitly in [3], may be proved by the methods of [3], cf. also [10, Lemma 12.2.4].

Theorem 3.2. Let $K \subset \mathbb{H}$ be a compact set with positive Jordan measure. Let $u \to +\infty$ and $q \to +0$ in such a way that $qu^2 \to a$ for some constant a > 0. Then, for some constant $C_{K,a} > 0$,

$$\mathbb{P}\left[\sup_{t\in K\cap q\mathbb{Z}^2}X(t)>u\right]\sim C_{K,a}u^3e^{-u^2/2}\ as\ u\to +\infty.$$

Proof of Theorem 1.3. 4

We prove only the first statement. In the sequel, C, C', C'', etc. are constants whose values

are irrelevant and may change from line to line. Suppose that $\sum_{n=1}^{\infty} f_n e^{-f_n^2/2} < \infty$. We are going to prove that a.s. only finitely many events $L_n > f_n$ occur. A simple argument, see e.g. [4], shows that we may suppose that $\frac{1}{2}\sqrt{2\log n} < f_n < 2\sqrt{2\log n}.$

Let $\{t_n\}_{n=0}^{\infty}$ be an increasing integer sequence such that $t_0 = 0$ and $t_n = [n \log n]$ for sufficiently large n. Note that

$$C'(\log n)^{1/2} < f_{t_n} < C''(\log n)^{1/2}.$$
 (4)

It is easy to see that

$$\sum_{n=1}^{\infty} f_{t_n}^3 \exp(-f_{t_n}^2/2) < \infty. \tag{5}$$

For $m, k \in \mathbb{Z}_{\geq 0}$ define $l_{m,k} = t_{2^k(m+1)} - t_{2^k m}$. We need the following technical lemma.

Lemma 4.1. There is a constant c_1 such that for all $m, k \in \mathbb{Z}_{\geq 0}$ we have $l_{m+1,k} < c_1 l_{m,k}$.

Proof. Let N be a sufficiently large integer. Suppose first that m=0 and k>N. Then

$$l_{m+1,k} = [2^{k+1}\log 2^{k+1}] - [2^k\log 2^k] < 5[2^k\log 2^k] = 5(t_{2^k} - t_0) = 5l_{m,k}.$$

Suppose now that $2^k m > N$. Then

$$\begin{split} l_{m+1,k} &\leq 2^k (m+2) \log(2^k (m+2)) - 2^k (m+1) \log(2^k (m+1)) + 1 \\ &= \int_{2^k (m+1)}^{2^k (m+2)} (1 + \log u) du + 1 = \int_{2^k m}^{2^k (m+1)} (1 + \log(u + 2^k)) du + 1 \\ &\leq \int_{2^k m}^{2^k (m+1)} 2(1 + \log u) du - 2 < 2l_{m,k}. \end{split}$$

Note that for all but finitely many pairs $(m,k) \in \mathbb{Z}^2_{\geq 0}$ we have either m=0 and k>N or $2^k m>N$. This finishes the proof of the lemma.

Put $c_2 = c_1 + 1$. For $m, k \in \mathbb{Z}_{\geq 0}$ let the set $\mathcal{A}_{m,k}$ be defined by

$$\mathcal{A}_{m,k} = \{(i,j) \in \mathbb{Z}^2 : t_{2^k m} \le j \le t_{2^k (m+1)}, l_{m,k} \le j - i \le c_2 l_{m,k} \}.$$

For $m \in \mathbb{Z}_{>0}$ define

$$\mathcal{B}_m = \{(i, j) \in \mathbb{Z}^2 : t_m < j \le t_{m+1}, 1 \le j - i < l_{m,0} \}.$$

We are going to show that each pair $(i,j) \in \mathbb{Z}^2$ such that $0 \leq i < j$ is contained in some $\mathcal{A}_{m,k}$, $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ or in some \mathcal{B}_m , $m \in \mathbb{Z}_{\geq 0}$. Take some $0 \leq i < j < \infty$. We may suppose that $j \geq t_1$, since otherwise $(i,j) \in \mathcal{B}_0$. Let $\tilde{k} = \max\{k : t_{2^k} \leq j\}$. For each $k = 0, \ldots, \tilde{k}$ there is $m(k) \in \mathbb{Z}_{>0}$ such that the interval $[t_{2^k m(k)}, t_{2^k (m(k)+1)}]$ contains j. Note that $m(\tilde{k}) = 1$. For $k = 0, \ldots, \tilde{k}$ set $d_k = l_{m(k),k}$ and let $d_{\tilde{k}+1} = t_{2^{\tilde{k}+1}}$. Then it follows from Lemma 4.1 that $d_{k+1} < c_2 d_k$ for each $k = 0, \ldots, \tilde{k}$. If $j - i < d_0$, then $(i,j) \in \mathcal{B}_{m(0)}$. Otherwise, we can find a number $k = k(i,j) \in \{0,\ldots,\tilde{k}\}$ such that $j-i \in [d_k,d_{k+1}]$ (note that $j-i \leq j \leq t_{2^{\tilde{k}+1}} = d_{\tilde{k}+1}$). Thus, we have $j \in [t_{2^k m(k)},t_{2^k (m(k)+1)}]$ and $j-i \in [d_k,d_{k+1}] \subset [d_k,c_2d_k]$. It follows that $(i,j) \in \mathcal{A}_{m(k),k}$.

We are going to show that a.s. only finitely many events

$$A_{m,k} = "\max_{(i,j) \in \mathcal{A}_{m,k}} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_{2^k}m}", k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{> 0}$$

and

$$B_m = \max_{(i,j) \in \mathcal{B}_m} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_m}, m \in \mathbb{Z}_{\geq 0}$$

occur. Recall that $\{X(t), t = (x, y) \in \mathbb{H}\}$ is the field of standardized Brownian motion increments defined in (3). Let K be the set $\{(x, y) \in \mathbb{H} : y \in [1, c_2], x + y \in [0, 1]\}$. Note that the

sets $\mathcal{A}_{m,k}$ and $K \cap l_{m,k}^{-1}\mathbb{Z}^2$ may be identified by taking $i = t_{2^k m} + l_{m,k} x$, $j = i + l_{m,k} y$. Using this and the scaling property of Brownian motion it is easy to see that the gaussian vector

$$\left\{\frac{S_j - S_i}{\sqrt{j - i}}; (i, j) \in \mathcal{A}_{m, k}\right\}$$

has the same distribution as

$${X(t); t \in K \cap l_{m,k}^{-1} \mathbb{Z}^2}.$$

It follows that

$$\mathbb{P}[A_{m,k}] \le \mathbb{P}[\max_{t \in K} X(t) > f_{t_{2^k m}}].$$

Thus, by Theorem 3.1, we obtain

$$\mathbb{P}[A_{m,k}] \le C f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2).$$

In order to be able to apply the Borel-Cantelli lemma we have to show that

$$\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2) < \infty.$$

It is easy to verify that $\log t_{2^k m} < C \log t_i$ for all $i \in [2^k (m-1) + 1, 2^k m], k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{>0}$. It follows that

$$(\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2) < C2^{-k} \sum_{i=2^k (m-1)+1}^{2^k m} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2).$$

and thus, by summing over m,

$$\sum_{m=1}^{\infty} (\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2) < C 2^{-k} \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2).$$

It follows, using (4), the previous inequality and (5),

$$\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2) < C_1 \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} (\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2)$$

$$< C_2 \sum_{k=0}^{\infty} 2^{-k} \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2)$$

$$= 2C_2 \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2)$$

$$< C_3 \sum_{i=1}^{\infty} f_{t_i}^3 \exp(-f_{t_i}^2/2) < \infty.$$

By the Borel-Cantelli lemma, a.s. only finitely many events $A_{m,k}, k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{>0}$ occur. Now we are going to show that $\sum_{m=0}^{\infty} \mathbb{P}[B_m] \leq \infty$. Let \mathcal{N} be standard gaussian random variable. Since $\#\mathcal{B}_m \leq (t_{m+1} - t_m)^2$, we have

$$\sum_{m=1}^{\infty} \mathbb{P}[B_m] \le \sum_{m=1}^{\infty} (t_{m+1} - t_m)^2 \mathbb{P}[\mathcal{N} > f_{t_m}]$$

$$\le C_1 \sum_{m=1}^{\infty} (\log m)^2 f_{t_m}^{-1} \exp(-f_{t_m}^2/2)$$

$$\le C_2 \sum_{m=1}^{\infty} f_{t_m}^3 \exp(-f_{t_m}^2/2) < \infty.$$

Thus, by the Borel-Cantelli lemma, a.s. only finitely many events B_m , $m \in \mathbb{Z}_{\geq 0}$ occur. Since any pair (i,j) such that $0 \leq i < j < \infty$ is contained in some $\mathcal{A}_{m,k}$, $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ (resp. some \mathcal{B}_m , $m \in \mathbb{Z}_{\geq 0}$), with probability 1 for all but finitely many (i,j) we have $(S_j - S_i)/\sqrt{j-i} \leq f_{t_{2k_m}} \leq f_j$ (resp. $(S_j - S_i)/\sqrt{j-i} \leq f_{t_m} \leq f_j$). It follows that a.s. only finitely many events $L_n > f_n$ occur.

 $(S_j - S_i)/\sqrt{j-i} \le f_{t_{2^k m}} \le f_j$ (resp. $(S_j - S_i)/\sqrt{j-i} \le f_{t_m} \le f_j$). It follows that a.s. only finitely many events $L_n > f_n$ occur. Now suppose that $\sum_{n=1}^{\infty} f_n e^{-f_n^2/2} = \infty$. Recall that $t_n = [n \log n]$ for sufficiently large n and let $l_n = [\log n]$. Again, it is not difficult to see that we may suppose that $\frac{1}{2}\sqrt{2\log n} < f_n < 2\sqrt{2\log n}$ and even $f_n/\sqrt{2\log n} \to 1$ as $n \to \infty$. It follows easily that $\sum_{n=1}^{\infty} f_{t_n}^3 e^{-f_{t_n}^2/2} = \infty$. We have to show that a.s. $L_n > f_n$ for infinitely many values of n. For n sufficiently large let

$$C_n = \{(i, j) \in \mathbb{Z}^2 : i \in [t_{n-1}, t_{n-1} + 1/4l_n], j - i \in [1/4l_n, 1/2l_n]\}.$$

Then the events

$$C_n = \max_{(i,j)\in\mathcal{C}_n} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_n}$$

are independent. Let $K = [0, 1/4] \times [1/4, 1/2]$. The sets C_n and $K \cap l_n^{-1}\mathbb{Z}^2$ may be identified by taking $i = t_{n-1} + l_n x$, $j = i + l_n y$. Thus, the distribution of the gaussian vector

$$\left\{ \frac{S_j - S_i}{\sqrt{j - i}}; (i, j) \in \mathcal{C}_n \right\}$$

coincides with the distribution of

$$\{X(t); t\in K\cap l_n^{-1}\mathbb{Z}^2\},$$

and we have, by Theorem 3.2,

$$\sum_{n=1}^{\infty} \mathbb{P}[C_n] \ge C \sum_{n=1}^{\infty} f_{t_n}^3 \exp(-f_{t_n}^2/2) = \infty.$$

By the Borel-Cantelli lemma infinitely many events C_n occur a.s. and thus, for infinitely many pairs (i,j) we have $(S_j - S_i)/\sqrt{j-i} > f_{t_n} > f_j$. This finishes the proof.

5 Proof of Theorem 1.1.

The lim sup part of the theorem follows from Theorem 1.3 by setting $f_n(\varepsilon) = \sqrt{2\log n} + (3/2 + \varepsilon) \log \log n / \sqrt{2\log n}$ and noting that $\sum_{n=1}^{\infty} f_n(\varepsilon) e^{-f_n(\varepsilon)^2/2} < \infty$ iff $\varepsilon > 0$.

We prove the liminf part. Let $f_n = \sqrt{2 \log n} + (1/2 + \varepsilon) \log \log n / \sqrt{2 \log n}$ for some $\varepsilon > 0$. Then $\lim_{n \to \infty} \mathbb{P}[L_n < f_n] = 1$ by [9, Th. 1.3]. Thus, $L_n < f_n$ for infinitely many values of n a.s.

Let $f_n(\varepsilon) = \sqrt{2\log n} + (1/2 - \varepsilon) \log \log n / \sqrt{2\log n}$ for some $\varepsilon > 0$. It remains to show that $L_n < f_n(\varepsilon)$ for at most finitely many values of n a.s. Let $l_n = [\log n]$. For $k = 1, \ldots, [n/(2l_n)]$ define

$$\mathcal{A}_k^{(n)} = \{(i,j) \in \mathbb{Z}^2 : (2k-1)l_n \le i \le 2kl_n, l_n/4 \le j-i \le l_n/2\}$$

and let $A_k^{(n)}$ be the event

$$A_k^{(n)} = \max_{(i,j) \in \mathcal{A}_b^{(n)}} \frac{S_j - S_i}{\sqrt{j - i}} < f_n(\varepsilon).$$

Then

$$\mathbb{P}[L_n < f_n(\varepsilon)] < \mathbb{P}[\bigcap_{k=1}^{[n/(2l_n)]} A_k^{(n)}] = \mathbb{P}[A_1^{(n)}]^{[n/(2l_n)]}. \tag{6}$$

Let $K = [0,1] \times [1/4,1/2]$. Again, we may identify $\mathcal{A}_k^{(n)}$ and $K \cap l_n^{-1}\mathbb{Z}^2$ by taking $i = (2k-1)l_n + l_n x$, $j = i + l_n y$. Thus, the gaussian vector

$$\left\{ \frac{S_j - S_i}{\sqrt{j - i}}; (i, j) \in \mathcal{A}_k^{(n)} \right\}$$

has the same distribution as

$$\{X(t); t \in K \cap l_n^{-1} \mathbb{Z}^2\},\$$

and we have, by Theorem 3.2,

$$\mathbb{P}[A_1^{(n)}] = 1 - \mathbb{P}[A_1^{(n)c}] < 1 - cf_n(\varepsilon)^3 \exp(-f_n(\varepsilon)^2/2). \tag{7}$$

A simple calculation using (6) and (7) shows that

$$\mathbb{P}[L_n < f_n(\varepsilon)] < \exp(-c(\log n)^{\varepsilon}).$$

It follows from the Borel-Cantelli lemma and the above inequality with ε replaced by $\varepsilon/2$ that a.s. only finitely many events $L_{n_i} < f_{n_i}(\varepsilon/2)$ take place, where $n_i = 2^i$. To finish the proof note that $f_{n_{i+1}}(\varepsilon) < f_{n_i}(\varepsilon/2)$, n > N. Since each n is contained in an interval of the form $[n_i, n_{i+1}]$, we have $L_n \ge L_{n_i} \ge f_{n_i}(\varepsilon/2) > f_{n_{i+1}}(\varepsilon) \ge f_n(\varepsilon)$ for all but finitely many values of n

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