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LIPSCHITZ PERCOLATION

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Abstract

We prove the existence of a (random) Lipschitz function $F : \mathbb{Z}^{d-1} \to \mathbb{Z}^+$ such that, for every $x \in \mathbb{Z}^{d-1}$, the site $(x, F(x))$ is open in a site percolation process on $\mathbb{Z}^d$. The Lipschitz constant may be taken to be 1 when the parameter p of the percolation model is sufficiently close to 1.

1 Introduction

Let $d \geq 1$ and $p \in (0,1)$. The site percolation model on the hypercubic lattice $\mathbb{Z}^d$ is obtained by designating each site $x \in \mathbb{Z}^d$ **open** with probability p, and otherwise **closed**, with different sites receiving independent states. The corresponding probability measure on the sample space $\Omega = \{0,1\}^{\mathbb{Z}^d}$ is denoted by $\mathbb{P}_p$, and the expectation by $\mathbb{E}_p$. We write $\mathbb{Z}^+ = \{1, 2, \dots\}$, and $\|\cdot\|$ for the 1-norm on $\mathbb{Z}^d$.

Theorem 1. *For any $d \geq 2$, if $p > 1-(2d)^{-2}$ then there exists a.s. a (random) function $F:\mathbb{Z}^{d-1}\to\mathbb{Z}^+$ with the following properties.*

(i) *For each $x\in\mathbb{Z}^{d-1}$, the site $(x,F(x))\in\mathbb{Z}^d$ is open.*

(ii) *For any $x,y\in\mathbb{Z}^{d-1}$ with $\|x-y\|=1$ we have $|F(x)-F(y)|\leq 1$.*

(iii) *For any isometry θ of $\mathbb{Z}^{d-1}$ the functions F and $F\circ\theta$ have the same laws, and the random field $(F(x):x\in\mathbb{Z}^{d-1})$ is ergodic under each translation of $\mathbb{Z}^{d-1}$.*

(iv) *There exists $A=A(p,d)<\infty$ such that*

$$\mathbb{P}_p(F(0)>k)\leq A\nu^k, \qquad k\geq 0.$$

where $\nu=2d(1-p)<1$.

We may think of $((x,F(x)):x\in\mathbb{Z}^{d-1})$ as a random surface, or a Lipschitz embedding of $\mathbb{Z}^{d-1}$ in $\mathbb{Z}^d$. When $d=2$, the existence of such an embedding for large p is a consequence of the fact that two-dimensional directed percolation has a non-trivial critical point. The result is less straightforward when $d\geq 3$.

The event that there exists an F satisfying (i) and (ii) is clearly increasing, and invariant under translations of $\mathbb{Z}^{d-1}$, therefore there exists p_{L} such that the event occurs with probability 1 if $p>p_{\mathrm{L}}$ and 0 if $p<p_{\mathrm{L}}$. Theorem 1 implies that $p_{\mathrm{L}}\leq 1-(2d)^{-2}$. This upper bound may be improved to $1-(2d-1)^{-2}$ as indicated at the end of Section 4, but we do not attempt to optimize it here. (A similar remark applies to the forthcoming Theorem 2.) The inequality $p_{\mathrm{L}}>0$ also holds, because site percolation on $\mathbb{Z}^d$ with next-nearest neighbour edges has a non-trivial critical point.

Theorem 1 is concerned with the critical value p_{L} for the existence of a Lipschitz embedding with constant 1. Suppose, instead, we seek a Lipschitz embedding with some Lipschitz constant k, and let $p_{\mathrm{L}}(k)$ be the associated critical value. We partition $\mathbb{Z}^d$ into translates of the set $T=\{re_d:1\leq r\leq s\}$ where e_d is a unit vector in the direction of increasing dth coordinate, and we declare any set of the partition **occupied** if it contains one or more open sites. Each such set is occupied with probability $1-(1-p)^s$, and it follows that

$$p_{\mathrm{L}}(2s-1)\leq 1-(1-p_{\mathrm{L}})^{1/s}, \qquad s\geq 1.$$

In particular, $p_{\mathrm{L}}(k)\downarrow 0$ as $k\to\infty$.

Some history of the current paper, and some implications of the work, are summarized in Section 2. In Section 3 we present a variant of Theorem 1 involving finite surfaces. The principal combinatorial estimate appears in Section 4, and the proofs of the theorems may be found in Section 5. Further properties of Lipschitz embeddings will be presented in [8].

2 Background and applications

The percolation model is one of the most studied models for a disordered medium, and the reader is referred to [5] for a recent account of the theory. The basic question is to determine for which values of p there exists an infinite self-avoiding walk of open sites. There exists a critical value p_{c}, depending on the choice of underlying lattice, such that such a walk exists a.s. when $p>p_{\mathrm{c}}$, and not when $p<p_{\mathrm{c}}$. It is clear that $p_{\mathrm{c}}(\mathbb{Z})=1$, and it is fundamental that $p_{\mathrm{c}}(\mathbb{Z}^d)<1$ when $d\geq 2$. Similarly, there exists a critical probability $\vec{p}_{\mathrm{c}}$ for the existence of an infinite open self-avoiding walk that is non-decreasing in each coordinate, and $\vec{p}_{\mathrm{c}}(\mathbb{Z}^d)<1$ for $d\geq 2$. The existence of certain types of open surface has also been studied, see for example [1, 4, 7, 10].

The purpose of this note is to prove the existence of a non-trivial critical point for the existence of a type of open Lipschitz surface within site percolation on $\mathbb{Z}^d$ with $d \geq 2$. The existence of such surfaces is interesting in its own right, and in addition there are several applications to be developed elsewhere. We make a remark about the history of the current note. Theorem 1 was first proved by a subset of the current authors, using an argument based on a subcritical branching random walk, summarized in Section 6. This proof involves an exploration process which is of independent interest. The simpler proof presented in Sections 4 and 5 was found subsequently by the remaining authors.

Theorem 1 has several applications and extensions described in detail in [3, 8, 9]. Paper [3] is devoted to the movement of an interface through a field of obstacles, and it is proved that there exists a supersolution to the so-called Edwards–Wilkinson equation. Paper [8] is concerned with the existence (or non-existence) of embeddings of $\mathbb{Z}^m$ within the set of open sites of $\mathbb{Z}^n$, subject to certain geometrical constraints. In particular, it is proved that infinite words indexed by $\mathbb{Z}^{d-1}$ may be embedded in $\mathbb{Z}^d$, thus answering a question posed by Ron Peled and described in [6]. In [9], an extension of the method of proof of Theorem 1 is used to obtain an improved lower bound on the critical probability for entanglement percolation in $\mathbb{Z}^3$ (see also [2, 7]).

3 Local covers

We next state a variant of Theorem 1 that is in a sense stronger. Let $d \geq 2$ and consider site percolation with parameter p on $\mathbb{Z}^d$. Write $\mathbb{Z}_0^+ = \{0, 1, \dots\}$. Let $x \in \mathbb{Z}^{d-1}$. A **local cover** of x is a function $L : \mathbb{Z}^{d-1} \to \mathbb{Z}_0^+$ such that:

(i) for all $y \in \mathbb{Z}^{d-1}$, if $L(y) > 0$ then $(y, L(y))$ is open;

(ii) for any $y, z \in \mathbb{Z}^{d-1}$ with $\|y - z\| = 1$ we have $|L(y) - L(z)| \leq 1$;

(iii) $L(x) > 0$.

If x has a local cover, then the minimum of all local covers of x is itself a local cover of x; we call this the **minimal** local cover of x and denote it L_x. Define its **radius**

$$\rho_x := \sup\Big\{ \big\|(x, 0) - (y, L_x(y))\big\| : y \in \mathbb{Z}^{d-1} \text{ such that } L_x(y) > 0 \Big\},$$

and note that $\rho_x \leq \infty$. If x has no local cover, we set $\rho_x = \infty$.

Theorem 2. *For any $d \geq 2$ and $p \in (0, 1)$ such that $q := 1 - p < (2d)^{-2}$, there exists $A = A(p, d) < \infty$ such that*

$$\mathbb{P}_p(\rho_0 \geq n) \leq A[(2d)^2 q]^n, \qquad n \geq 0.$$

4 Principal estimate

The key step is to identify an appropriate set of dual paths that are blocked by a Lipschitz surface of the type sought in Theorem 1. Such paths will be allowed to move downwards (that is, in the direction of decreasing d-coordinate), with or without a simultaneous horizontal move, but whenever they move upwards, they must do so to a closed site.

Let $e_1, \dots, e_d \in \mathbb{Z}^d$ be the standard basis vectors of $\mathbb{Z}^d$. We define a **Λ-path** from u to v to be any finite sequence of distinct sites $u = x_0, x_1, \dots, x_k = v$ of $\mathbb{Z}^d$ such that for each $i = 1, 2, \dots, k$:

$$x_i - x_{i-1} \in \{\pm e_d\} \cup \{-e_d \pm e_j : j = 1, \dots, d-1\}. \tag{1}$$

A Λ-path is called **admissible** if in addition for each $i = 1, 2, \dots, k$:

$$\text{if } x_i - x_{i-1} = e_d \text{ then } x_i \text{ is closed.}$$

Denote by $u \rightarrowtail v$ the event that there exists an admissible Λ-path from u to v, and write

$$\tau_p(u) = \mathbb{P}_p(0 \rightarrowtail u).$$

The next lemma is the basic estimate used in the proofs. For $u = (u_1, u_2, \dots, u_d) \in \mathbb{Z}^d$, we write $h(u) = u_d$ for its **height**, and

$$r(u) = \|(u_1, u_2, \dots, u_{d-1})\| = \sum_{i=1}^{d-1} |u_i|$$

for its **displacement** in $\mathbb{Z}^{d-1}$. For $x \in \mathbb{R}$, $x^+ = \max\{0, x\}$ (respectively, $x^- = -\min\{0, x\}$) denotes the positive (respectively, negative) part of x.

Lemma 3. *Let $d \ge 2$ and $a = 2d$, and take $p \in (0,1)$ such that $q := 1 - p \in (0, a^{-2})$. For $h \in \mathbb{Z}$ and $r \in \mathbb{Z}_0^+$ satisfying $r \ge h^-$,*

$$\sum_{\substack{u \in \mathbb{Z}^d: \\ h(u) \ge h,\, r(u) \ge r}} \tau_p(u) \le \frac{1}{(1-aq)(1-a^2q)} (aq)^h (a^2 q)^r.$$

Proof. Fix $r \ge 0$, and let $h \in \mathbb{Z}$ satisfy $r \ge h^-$. Let

$$T = T_{r,h} = \{u \in \mathbb{Z}^d : h(u) \ge h,\ r(u) \ge r\}.$$

Let $N(u)$ be the number of admissible Λ-paths (of all finite lengths) from 0 to u, and note that

$$\sum_{u \in T} \tau_p(u) = \sum_{u \in T} \mathbb{P}_p(N(u) > 0) \le \sum_{u \in T} \mathbb{E}_p N(u).$$

Let π be a Λ-path beginning at 0. Let U and D be the respective numbers of steps in π that lie in each of the sets

$$\{e_d\}; \quad \{-e_d\} \cup \{-e_d \pm e_j : j = 1, \dots, d-1\}.$$

(The letters U, D stand for 'upwards' and 'downwards'.) Thus, the length of π is $U + D$, the final endpoint u of π satisfies $h(u) = U - D$ and $r(u) \le D$, and π is admissible with probability q^U, where $q := 1 - p$. Also, the number of Λ-paths π beginning at 0 with given values of U and D is at most a^{U+D}, where $a := 2d$.
Therefore,

$$\sum_{u \in T} \mathbb{E}_p N(u) \le \sum_{\substack{U, D \ge 0: \\ U - D \ge h,\ D \ge r}} a^{U+D} q^U.$$

Assume that $a^2 q < 1$ (i.e., $p > 1 - (2d)^{-2}$). Summing over U, the last expression equals

$$\frac{1}{1 - aq} \sum_{D \ge r} a^D (aq)^{(h+D)^+}.$$

Since $D \ge r \ge h^-$, we have $(h+D)^+ = h + D$, and the last sum equals

$$\sum_{D \ge r} (aq)^h (a^2 q)^D = \frac{(aq)^h (a^2 q)^r}{1 - a^2 q}. \qquad \square$$

Remark. The number of Λ-paths of k steps is no greater than $(2d)^k$. Only minor changes are required to the proofs if one restricts the class of Λ-paths to those satisfying (1) for which $x_i - x_{i-1} \neq -e_d$ for all i. The number of such paths is no greater than $(2d-1)^k$, and this leads to improved versions of Theorems 1 and 2 with $2d$ replaced by $2d-1$. The details are omitted.

5 Proofs of Theorems 1 and 2

We give two proofs of Theorem 1: one directly from Lemma 3, and the other via Theorem 2. The second proof gives a worse exponent in the inequality of Theorem 1(iv). We sketch a third approach in the next section.

1st proof of Theorem 1. Take p and $a = 2d$ as in Lemma 3. Let

$$T_- := \mathbb{Z}^{d-1} \times \{\dots, -1, 0\}$$

and define the random set of sites

$$G := \{v \in \mathbb{Z}^d : u \rightarrowtail v \text{ for some } u \in T_-\}.$$

Since an admissible path may always be extended by a downwards step (provided the new site is not already in the path), if $v \in G$ then $v - e_d \in G$. Using Lemma 3 with $r = 0$, we have for $h > 0$ and suitable $A < \infty$,

$$\mathbb{P}_p(he_d \in G) \le \sum_{u \in T_-} \mathbb{P}_p(u \rightarrowtail he_d) = \sum_{u \in T_{0,h}} \tau_p(u) \le A(aq)^h. \tag{2}$$

Hence, by the Borel–Cantelli lemma, a.s. for every $x \in \mathbb{Z}^{d-1}$, only finitely many of the sites $(x, h) = x + he_d$ for $h > 0$ lie in G.
For $x \in \mathbb{Z}^{d-1}$, let

$$F(x) := \min\{t > 0 : (x, t) \notin G\}.$$

Since $(x, F(x)) \notin G$ and $(x, F(x) - 1) \in G$, the site $(x, F(x))$ is necessarily open. The required property (iii) of the theorem follows by fact that $\mathbb{P}_p$ is a product measure, and (iv) is an immediate consequence of (2). To check (ii), consider any $x, y \in \mathbb{Z}^{d-1}$ with $\|x - y\| = 1$. Since $(x, F(x)-1) \in G$, and an admissible path may be extended in the diagonal direction $(y - x) - e_d$, we have $(y, F(x) - 2) \in G$, whence $F(y) > F(x) - 2$. $\square$

Proof of Theorem 2. We begin with an explicit construction of the minimal local cover L_x of $x \in \mathbb{Z}^{d-1}$, whenever x possesses a local cover. Let H_x be the set of endpoints of admissible paths from $(x, 0)$ that use no site of $\mathbb{Z}^{d-1} \times \{-1, -2, \dots\}$. By the definition of admissibility, H_x does not depend on the states of sites with height less than or equal to 0.
Let $a^2 q < 1$. If $\mathrm{rad}(H_0) = \sup\{\|u\| : u \in H_0\}$ satisfies $\mathrm{rad}(H_0) \ge k$, by the definition of admissible paths, there exists $u \in H_0$ with $h(u) = 0$ and $r(u) \ge k$. Therefore, by Lemma 3,

$$\begin{aligned} \mathbb{P}_p(\mathrm{rad}(H_0) \ge k) &\le \sum_{\substack{u \in \mathbb{Z}^d: \\ h(u) \ge 0,\ r(u) \ge k}} \tau_p(u) \\ &\le A(a^2 q)^k, \qquad k \ge 0, \end{aligned}$$

for some $A = A(p, d) < \infty$.

On the event that $|H_0| < \infty$, the minimal local cover of 0 is given by

$$L_0(y) = \min\{h \in \mathbb{Z}^+ : (y,h) \notin H_0\};$$

that is, the corresponding surface consists of the sites immediately above H_0. The claim follows. $\square$

2nd proof of Theorem 1 with different exponent in part (iv). Let $p > 1 - (2d)^{-2}$, as in Theorem 2, and let H_x be as in the proof. Let

$$F(x) := 1 + \sup\{h : (x,h) \in H_y \text{ for some } y \in \mathbb{Z}^{d-1}\}.$$

Given the general observations above, it suffices to prove that F satisfies part (iv) of Theorem 1. Now, for $k \geq 1$,

$$\begin{aligned}\mathbb{P}_p(F(0) > k) &= \mathbb{P}_p((0,k) \in H_y \text{ for some } y) \\ &\leq \mathbb{P}_p(\rho_y \geq k + \|y\| \text{ for some } y) \\ &\leq \sum_{y \in \mathbb{Z}^{d-1}} \mathbb{P}_p(\rho_0 \geq k + \|y\|),\end{aligned}$$

and this decays to 0 exponentially in k, by Theorem 2. $\square$

6 Sketch proof using branching random walk

This section contains a summary of an alternative approach to the problem, using a branching random walk to bound the size of a minimal local cover. Write $\Delta = \mathbb{Z}^{d-1} \times \mathbb{Z}^+$, and recall the **height** $h(u)$ of site $u \in \mathbb{Z}^d$. The minimal cover L at the origin 0 is in one–one correspondence with the set

$$S := \{(x, L(x)) : x \in \mathbb{Z}^{d-1},\ L(x) > 0\}$$

of open sites. The set S may be constructed iteratively as follows. Let C be the height of the lowest open site above 0, that is, $C := \inf\{n \geq 1 : ne_d \text{ is open}\}$. Clearly, S contains no site of the form $(0,k)$, $1 \leq k < C$, and in addition no site in the pyramid

$$P := \{u \in \mathbb{Z}^d : \|u\| < C\}.$$

Let $u \in \Delta$ be such that $\|u\| = C$. If all such u are open, then $S = \{u \in \Delta : \|u\| = C)\}$. Any such u that is closed is regarded as a **child** of the origin. Each such child u is labelled with the height of the lowest open site above it, that is, with the **label** $h(u) + \inf\{n \geq 1 : u + ne_d \text{ is open}\}$. The process is iterated for each such child, and so on to later generations. If the ensuing procedure terminates after a finite number of steps, then we have constructed the set S corresponding to the minimal local cover of 0.

A full analysis of the above procedure would require specifying the order in which children are considered, as well as understanding the interactions between different pyramids. Rather than do this, we will treat the families of different children as independent, thereby over-counting the total size and extent of the process. That is, we construct a dominating branching random walk, as follows.

Let $\xi = (\xi(z) : z \in \mathbb{Z})$ be a random measure on $\mathbb{Z}$ with $\xi(z) \in \{0, 1, 2, \dots\}$ a.s. The corresponding branching random walk begins with a single particle located at 0, that is, $\xi_0 := \delta_0$, the point mass. This particle produces offspring $\xi_1 := \xi$. For $n \geq 2$, ξ_n is obtained from ξ_{n-1} as follows: each particle of ξ_{n-1} has (independent) offspring with the same law as ξ, shifted according to the position of the parent. Assume there exists $\mu > 0$ such that

$$\alpha := \mathbb{E}\left(\sum_{z\in\mathbb{Z}} e^{\mu z}\xi(z)\right) < 1, \tag{3}$$

and define

$$S_n := \sum_{z\in\mathbb{Z}} e^{\mu z}\xi_n(z).$$

It is standard that S_n/α^n is a (non-negative) martingale. In particular, S_n/α^n converges a.s., whence $S_n \to 0$ a.s. as $n \to \infty$.

We next describe the law of ξ arising in the current setting. Let Q be the set of all closed $u \in \mathbb{Z}^d$ satisfying $u \neq 0$ and

$$\sum_{i=1}^{d-1} |u_i| = -u_d,$$

and think of Q as the set of children of the initial particle at 0. Each child is allocated a **location** in $\mathbb{Z}$ equal to the height of the lowest open site above it. More precisely, the location of the child corresponding to $u \in Q$ is defined as $h(u) + \inf\{n \geq 1 : u + ne_d \text{ is open}\}$, and $\xi_n(z)$ is simply the number of children with location z. The corresponding BRW is written BRW(ξ).

The number of children with height $-n$ is binomially distributed with parameters $(\tau_n, 1-p)$, where $\tau_n \leq 2(2n+1)^{d-1}$, and the height of each tower has a geometric distribution. Following an elementary calculation, there exist $\mu > 0$ and $p_1 = p_1(d) \in (0,1)$ such that: for $p \in (p_1, 1)$, we have $\alpha < 1$ in (3).

We now compare BRW(ξ) and the local cover of 0. With C as above, consider BRW(ξ) with all locations shifted by height C, written $C +$ BRW(ξ). Each child of the origin in the percolation model is a child in $C +$ BRW(ξ), and its label in the former equals its location in the latter. The first generation of $C +$ BRW(ξ) may also contain children with negative heights. In subsequent generations, the models are different, but it may be seen that $C +$ BRW(ξ) dominates the percolation model in the sense that the set of locations in $C +$ BRW(ξ) with positive heights dominates (stochastically) the set of labels in the percolation model.

With ξ'_n the nth generation in $C +$ BRW(ξ), let

$$N := \sup\{n : \xi'_n(z) > 0 \text{ for some } z > 0\}.$$

By the above domination, if $\mathbb{P}(N < \infty) = 1$, then the local cover of 0 is (a.s.) finite. By Markov's inequality,

$$\begin{aligned}\mathbb{P}(\xi'_n(z) > 0 \text{ for some } z > 0) &= \mathbb{P}(\xi_n(z) > 0 \text{ for some } z > -C)\\ &\leq \mathbb{E}(S_n)\mathbb{E}(e^{\mu C})\\ &= \alpha^n \mathbb{E}(e^{\mu C}).\end{aligned}$$

By the Borel–Cantelli lemma, $\mathbb{P}(N < \infty) = 1$, and the finiteness of the local cover at 0 follows.

Substantially more may be obtained by a more careful analysis of the maximum displacement of particles in the kth generation of $C +$ BRW(ξ). In particular, for $p > p_1$, one may deduce that $\mathbb{P}_p(\rho_0 \geq n)$ decays to 0 faster than a quantity that is exponential in some power of n, and this implies the existence of the Lipschitz function of Theorem 1, as in the second proof of Section 5. The details of these arguments are omitted.

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