

## PARTICLE SYSTEMS WITH QUASI-HOMOGENEOUS INITIAL STATES AND THEIR OCCUPATION TIME FLUCTUATIONS

TOMASZ BOJDECKI<sup>1</sup>

*Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland*  
email: tobojd@mimuw.edu.pl

LUIS G. GOROSTIZA<sup>2</sup>

*Centro de Investigación y de Estudios Avanzados, 07000 México, D.F, Mexico*  
email: lgorosti@math.cinvestav.mx

ANNA TALARCZYK<sup>1</sup>

*Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland*  
email: annatal@mimuw.edu.pl

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### *Abstract*

We consider particle systems in  $\mathbb{R}$  with initial configurations belonging to a class of measures that obey a quasi-homogeneity property, which includes as special cases homogeneous Poisson measures and many deterministic measures (simple example: one atom at each point of  $\mathbb{Z}$ ). The particles move independently according to an  $\alpha$ -stable Lévy process,  $\alpha > 1$ , and we also consider the model where they undergo critical branching. Occupation time fluctuation limits of such systems have been studied in the Poisson case. For the branching system in “low” dimension the limit was characterized by a process called sub-fractional Brownian motion, and this process was attributed to the branching because it had appeared only in that case. In the present more general framework sub-fractional Brownian motion is more prevalent, namely, it also appears as a component of the limit for the system without branching in “low” dimension. A new method of proof, based on the central limit theorem, is used.

## 1 Introduction

In a series of papers [2, 3, 4, 5, 6] (and others) we studied particle systems in  $\mathbb{R}^d$  starting from a configuration determined by a random point measure  $\nu$ , and independently moving according to a standard  $\alpha$ -stable Lévy process ( $0 < \alpha \leq 2$ ). In some models the particles additionally undergo critical branching. The evolution of the system is described by the empirical process  $N = (N_t)_{t \geq 0}$ ,

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where  $N_t(A)$  is the number of particles in the set  $A \subset \mathbb{R}^d$  at time  $t$ . The object of interest is the limit of the time-rescaled and normalized occupation time fluctuation process  $X_T$  defined by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s) ds, \quad t \geq 0, \quad (1.1)$$

as  $T \rightarrow \infty$ , where  $F_T$  is a suitable deterministic norming. The process  $X_T$  is signed measure-valued, but we regard it as a process with values in the space of tempered distributions  $\mathcal{S}'(\mathbb{R}^d)$  for technical reasons, and also because in some cases the limit is genuinely  $\mathcal{S}'(\mathbb{R}^d)$ -valued. In all the cases considered in the abovementioned papers the initial measure  $\nu$  was a Poisson field, homogeneous or not. This assumption permitted to investigate convergences conveniently with the help of the Laplace transform (due to infinite divisibility). The results always exhibited the same type of phase transition: for “low” dimensions  $d$  the limit process was the Lebesgue measure multiplied by a real long-range dependent process, whereas for “high” dimensions the limit was an  $\mathcal{S}'(\mathbb{R}^d)$ -valued process with independent increments. A natural question is what happens for non-Poisson initial measures  $\nu$ . Miłoś [10, 11] considered (critical) branching systems where  $\nu$  was an equilibrium measure. In that model the limits have a similar dimension phase transition.

In the present paper we investigate what happens with initial measures of other types, for example, some measures that are deterministic or almost deterministic. For simplicity we consider  $d = 1$  and assume that the motions are either without branching or with the simplest critical binary branching. We define a class  $\mathcal{M}$  of measures  $\nu$  which contains in particular homogeneous Poisson measures (which are “completely random”), and quasi-homogeneous deterministic measures (e.g., one atom at each point of  $\mathbb{Z}$ , which is “completely deterministic”), and we develop a unified approach that permits to obtain limits of  $X_T$  for any initial measure  $\nu \in \mathcal{M}$ . By a quasi-homogeneous deterministic point measure on  $\mathbb{R}$  we mean any measure defined by the following procedure: Given a positive integer  $k$ , in each interval  $[j, j + 1)$ ,  $j \in \mathbb{Z}$ , we fix  $k$  points. For a general  $\nu \in \mathcal{M}$ , each interval  $[j, j + 1)$  contains  $\theta_j$  points chosen at random, and  $\theta_j$ ,  $j \in \mathbb{Z}$ , are i.i.d. random variables. The main feature of those measures is this form of quasi-homogeneity and independence on the family of intervals  $[j, j + 1)$ . The idea of the proofs in this general framework is simpler than that in our previous papers, and is based on the central limit theorem.

Some of the results we obtain are unexpected. It turns out that the only case where new limits appear is the non-branching case with  $(d =) 1 < \alpha$ . They have the form  $K\lambda\xi$ , where  $K$  is a constant,  $\lambda$  is the Lebesgue measure, and  $\xi$  is the sum of two independent processes, one of them is a sub-fractional Brownian motion (see (2.3)), and the other one is a new (centered continuous long-range dependent) Gaussian process (see (2.4) and Remark 2.3(c)). The process  $\xi$  depends on the initial measure  $\nu$  only through  $E\theta_0$  and  $\text{Var}\theta_0$ . In particular, for a deterministic  $\nu$  this process reduces to a sub-fractional Brownian motion, and in the homogeneous Poisson case, as well as for any  $\nu$  with  $E\theta_0 = \text{Var}\theta_0$ , it yields a fractional Brownian motion (see Remark 2.3(a)(b) for more comments). This result seems surprising since in all earlier papers sub-fractional Brownian motion was related only to branching systems, and was consequently attributed to the branching, but in the present framework this process is more “natural” than fractional Brownian motion. On the other hand, fractional Brownian motion, which is typically related to systems in equilibrium (in particular the non-branching system with initial homogeneous Poisson measure), now appears also for a wider class of quasi-homogeneous initial measures. In all the remaining cases the limits are (up to constants) the same, and with the same normings  $F_T$ , as those for homogeneous Poisson models [3, 4].

The results show that within the class  $\mathcal{M}$  the fluctuations caused by the branching are so large that  $X_T$  “forgets” the randomness of the initial state of the system (it “remembers”  $E\theta_0$  only). On

the other hand, for low dimensions it does distinguish between  $\nu \in \mathcal{M}$  and the equilibrium initial state [10] (which is not in  $\mathcal{M}$  because the branching introduces spatial dependence). Another conclusion is that for high dimensions (which for  $d = 1$  amounts to small  $\alpha$ ), the fluctuation process “forgets” the initial measure, as long as it is in some sense homogeneous (i.e.,  $\nu \in \mathcal{M}$ ), and this property holds for both the branching and non-branching systems, and also for branching systems in equilibrium.

In this paper we are interested mainly in identifying the limit processes, therefore we do not prove convergence in functional form, but only for finite-dimensional distributions. Presumably, convergence also holds in  $C([0, \tau], \mathcal{S}'(\mathbb{R}))$  for any  $\tau > 0$ .

We have not found results in the literature concerning occupation times for particle systems starting from a deterministic or quasi-deterministic point measure. Some kinds of quasi-homogeneity of initial configurations for systems of independent particles, different from those in this paper, appear in other contexts in [14] and [8] (see Remark 2.5 (h)).

The following notation is used in the paper.

$\lambda$ : Lebesgue measure.

$\mathcal{S}(\mathbb{R})$ : space of  $C^\infty$  rapidly decreasing function on  $\mathbb{R}$ .

$\mathcal{S}'(\mathbb{R})$ : space of tempered distributions (topological dual of  $\mathcal{S}(\mathbb{R})$ ).

$\langle \cdot, \cdot \rangle$ : duality on  $\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ .

$\Rightarrow_f$ : weak convergence of finite-dimensional distributions of  $\mathcal{S}'(\mathbb{R})$ -valued processes.

$p_t(x)$ : transition probability density of the standard symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}$ .

$\mathcal{T}_t$ : semigroup determined by  $p_t$ , i.e.,  $\mathcal{T}_t \varphi = p_t * \varphi$ .

$G$ : potential operator determined by  $p_t$  for  $\alpha < 1$ , i.e.,

$$G\varphi(x) = \int_0^\infty \mathcal{T}_t \varphi(x) dt = C_\alpha \int_{\mathbb{R}} \frac{\varphi(y)}{|x-y|^{1-\alpha}} dy, \quad C_\alpha = \frac{\Gamma(\frac{1-\alpha}{2})}{2^\alpha \pi^{1/2} \Gamma(\frac{\alpha}{2})}.$$

Generic constants are written  $C, C_i$ , with possible dependencies in parenthesis.

In Section 2 we describe the particle system, formulate the results and discuss them. Section 3 contains the proof of Theorem 2.2(a), which is the only case where a new type of limit appears. The other proofs are in general more elaborate (although the new method is simpler than the one used before), specially for the branching system due to the non-linearity caused by the branching, and they can be found in [7]. That paper, which is a full-length version of the present one, contains more references and an example where convergence in  $C([0, \tau], \mathcal{S}'(\mathbb{R}))$  is proved (Proposition 2.6).

## 2 Results

We start with detailed description of the particle system.

Let  $\theta$  be a non-negative integer-valued random variable with distribution

$$P(\theta = k) = p_k, \quad k = 0, 1, 2, \dots, \quad (2.1)$$

such that  $E\theta^3 < \infty$ . This moment condition is a technical assumption satisfied by all cases of interest in this paper, but we suppose that finiteness of the second moment could be sufficient.

Let  $\theta_j, j \in \mathbb{Z}$ , be independent copies of  $\theta$ , and for each  $j \in \mathbb{Z}$  and  $k = 1, 2, \dots$ , let  $\rho_k^j = (\rho_{k,1}^j, \dots, \rho_{k,k}^j)$  be a random vector with values in  $[j, j+1)^k$ . We assume that  $(\theta_j, (\rho_k^j)_{k=1,2,\dots})$ ,  $j \in \mathbb{Z}$ , are independent. These objects determine a random point measure  $\nu$  on  $\mathbb{R}$  in the following

way: For each  $j$ ,  $\theta_j$  is the number of points in the interval  $[j, j + 1)$ , and for each  $k$ , if  $\theta_j = k$ , the positions of those points are determined by  $\rho_k^j$ . In other words,

$$\nu = \sum_{j \in \mathbb{Z}} \sum_{n=1}^{\theta_j} \delta_{\kappa_{j,n}}, \quad (2.2)$$

where  $\kappa_{j,n} = \rho_{\theta_j,n}^j$  and  $\delta_a$  is the Dirac measure at  $a \in \mathbb{R}$ . We denote by  $\mathcal{M}$  the class of all such measures  $\nu$ .

**Remark 2.1.** (a) If  $\theta \equiv k$  and, for each  $j$ , the  $\rho_k^j$  are not random, then  $\nu$  is a quasi-homogeneous deterministic measure mentioned in Introduction. The simplest example is  $\nu = \sum_{j \in \mathbb{Z}} \delta_j$ .

(b) If  $\theta$  is a standard Poisson random variable and, for each  $j$ ,  $\rho_{k,1}^j, \dots, \rho_{k,k}^j$  are independent, uniformly distributed on  $[j, j + 1)$ , then  $\nu$  given by (2.2) is the homogeneous Poisson measure (with intensity measure  $\lambda$ ).

Fix  $\alpha \in (0, 2]$  and  $\nu \in \mathcal{M}$ . Assume that at the initial time  $t = 0$  there is a collection of particles in  $\mathbb{R}$  with positions determined by  $\nu$ . As time evolves, these particles move independently according to the symmetric  $\alpha$ -stable Lévy process. We consider systems either without branching, or with critical binary branching (i.e., 0 or 2 particles with probability 1/2 each case) at rate  $V$ . For the corresponding empirical process, we define an  $\mathcal{S}'(\mathbb{R})$ -valued process  $X_T$  by (1.1).

Before stating the first theorem we recall the definition of sub-fractional Brownian motion. A *sub-fractional Brownian motion* with parameter  $H$  ( $0 < H < 1$ ) is a centered continuous Gaussian process  $\zeta^H$  with covariance

$$C^H(s, t) = E \zeta_s^H \zeta_t^H = s^{2H} + t^{2H} - \frac{1}{2}((s+t)^{2H} + |s-t|^{2H}), \quad s, t \geq 0. \quad (2.3)$$

We will need another centered Gaussian process  $\vartheta^H$  with covariance

$$Q^H(s, t) = \frac{1}{2} \operatorname{sgn}(2H - 1)((s+t)^{2H} - s^{2H} - t^{2H}), \quad s, t \geq 0, \quad (2.4)$$

( $0 < H < 1$ ). Existence of this process (for  $H \neq 1/2$ ) follows from the formula

$$Q^H(s, t) = C \int_0^s \int_0^t (r+r')^{2H-2} dr' dr = C_1 \int_0^s \int_0^t \int_{\mathbb{R}} e^{-r|x|^{1/(2-2H)}} e^{-r'|x|^{1/(2-2H)}} dx dr' dr,$$

which implies positive-definiteness of  $Q^H$ .

**Theorem 2.2.** *For the system without branching,*

(a) *if  $1 < \alpha$  and*

$$F_T = T^{1-1/2\alpha}, \quad (2.5)$$

*then*

$$X_T \Rightarrow_f K_1 \lambda (\sqrt{E\theta} \zeta^H + \sqrt{\operatorname{Var} \theta} \vartheta^H), \quad (2.6)$$

*where  $\zeta^H, \vartheta^H$  are independent,  $H = 1 - 1/2\alpha \in (1/2, 3/4]$ , and*

$$K_1 = \left( \frac{\Gamma(2-2H)}{2\pi\alpha H(2H-1)} \right)^{1/2};$$

(b) if  $\alpha = 1$  and  $F_T = \sqrt{T \log T}$ , then  $X_T \Rightarrow_f K_2 \lambda \beta$ , where  $\beta$  is a standard Brownian motion in  $\mathbb{R}$ , and

$$K_2 = \sqrt{\frac{2}{\pi} E\theta};$$

(c) if  $1 > \alpha$  and  $F_T = \sqrt{T}$ , then  $X_T \Rightarrow_f X$ , where  $X$  is an  $\mathcal{S}'(\mathbb{R})$ -valued homogeneous Wiener process with covariance

$$E\langle X(t), \varphi \rangle \langle X(s), \psi \rangle = 2E\theta(s \wedge t) \int_{\mathbb{R}} \varphi(x) G \psi(x) dx, \quad \varphi, \psi \in \mathcal{S}'(\mathbb{R}).$$

**Remark 2.3.** (a) Let  $\xi^H = \sqrt{E\theta} \zeta^H + \sqrt{\text{Var}\theta} \vartheta^H$  be the process in the limit (2.6) of Theorem 2.2(a). It is interesting that the component processes  $\zeta^H$  and  $\vartheta^H$ , both of which do not have stationary increments, are “carried” by  $E\theta$  and  $\text{Var}\theta$ , respectively. We do not have a “particle picture” interpretation of this fact. If  $E\theta = \text{Var}\theta$ , in particular if  $\nu$  is homogeneous Poisson (see Remark 2.1(b)), then  $\xi^H$  is, up to a constant factor, a fractional Brownian motion with Hurst parameter  $H$ , whose covariance is  $\frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H})$  and it has stationary increments. Thus we recover the result known in that case [3]. The system with initial homogeneous Poisson is in equilibrium, and this causes stationary increments, but now we see that this can happen more generally. On the other hand, if  $\theta$  is deterministic, then  $\xi^H$  is a sub-fractional Brownian motion. Moreover, in general the randomness of the  $\rho$ 's in the definition of  $\nu \in \mathcal{M}$  (see (2.2)) does not play any role in the limit. The presence of  $\lambda$  in (2.6) represents perfect spatial correlation, which is due to the strong recurrence of the particle motion (for  $d < \alpha$ ).

(b) The long time dependent behavior of Gaussian processes is usually characterized by the covariance of increments of the process on intervals separated by distance  $\tau$ , as  $\tau \rightarrow \infty$ . For the process  $\vartheta^H$  that behavior is asymptotic decay like  $\tau^{2H-2}$  (the same as for fractional Brownian motion), and for sub-fractional Brownian motion it is  $\tau^{2H-3}$  [2]. Thus, the long time dependent behavior of the process  $\xi^H$  is determined by  $\vartheta^H$  if  $\theta$  is random, and by  $\zeta^H$  if  $\theta$  is deterministic. Only in the case of Theorem 2.2(a) the fluctuation process remembers if  $\theta$  is random or not, and randomness of  $\theta$  causes longer range dependence.

(c) Sub-fractional Brownian motion was introduced in connection with occupation time fluctuations of branching systems and studied in [2]. It has also been discussed by other authors [1, 13, 15, 16, 17, 18, 19] as a subject of independent interest. The Gaussian process with covariance (2.4) was introduced independently in [9] regarding a decomposition of bi-fractional Brownian motion, and it has also been used in [1, 13] for decompositions relating fractional Brownian motion and sub-fractional Brownian motion. The special case  $E\theta = \text{Var}\theta$  of our result coincides with the decomposition of fractional Brownian motion for  $H > 1/2$ .

**Theorem 2.4.** For the branching system,

(a) if  $1/2 < \alpha < 1$  and  $F_T = T^{(3-1/\alpha)/2}$ , then  $X_T \Rightarrow_f K_3 \lambda \zeta^H$ , where  $\zeta^H$  is a sub-fractional Brownian motion with parameter  $H = (3 - 1/\alpha)/2 \in (1/2, 1)$ , and

$$K_3 = \left( \frac{E\theta V \Gamma(2 - 2H)}{2\pi\alpha H(2H - 1)} \right)^{1/2};$$

(b) if  $\alpha = 1/2$  and  $F_T = \sqrt{T \log T}$ , then  $X_T \Rightarrow_f K_4 \lambda \beta$ , where  $\beta$  is a standard Brownian motion, and

$$K_4 = \sqrt{\frac{2V}{\pi} E\theta};$$

(c) if  $\alpha < 1/2$  and  $F_T = \sqrt{T}$ , then  $X_T \Rightarrow_f X$ , where  $X$  is an  $\mathcal{S}'(\mathbb{R})$ -valued homogeneous Wiener process with covariance

$$E\langle X(t), \varphi \rangle \langle X(s), \psi \rangle = E\theta(s \wedge t) \int_{\mathbb{R}} \left( 2\varphi(x)G\psi(x) + V(G\varphi(x))(G\psi(x)) \right) dx, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}).$$

**Remark 2.5.** (a) In the branching case the results are, up to the constant  $E\theta$  in the limits, the same as in the homogeneous Poisson case (Theorems 2.2 in [3] and [4]).

(b) The condition  $\alpha < 1$  in Theorem 2.4(a) corresponds to  $\alpha < d$  in [4]. In the homogeneous Poisson case for  $d \leq \alpha$  we obtained limits of a similar form as for  $\alpha < d < 2\alpha$ , by introducing high density, i.e., considering systems with initial intensity  $H_T\lambda$ ,  $H_T \rightarrow \infty$  sufficiently fast [5]; the high density counteracts the tendency to local extinction caused by the critical branching. The same procedure can be applied in the present case, yielding the limits for  $1 \leq \alpha \leq 2$ , if the intervals  $[j, j+1)$  are replaced by  $[j/H_T, (j+1)/H_T)$ .

(c) As in [3] and [4], Theorems 2.2 and 2.4 can be extended to systems in  $\mathbb{R}^d$ , where the intervals  $[j, j+1)$  are replaced by cubes  $[j_1, j_1+1) \times \cdots \times [j_d, j_d+1)$ ,  $j_1, \dots, j_d \in \mathbb{Z}$ .

(d) Comparing parts (a) of Theorems 2.2 and 2.4, we see that the branching weakens the influence of the initial configuration.

(e) The presence of  $\text{Var}\theta$  in the non-branching model (Theorem 2.2(a)) is due to the fact that the fluctuations of  $\theta$  are transported forever by the motion of the particles. In the branching model (Theorem 2.4(a)), since the trees tend to extinction by criticality, the effect of the fluctuations of  $\theta$  become more and more negligible, so  $\text{Var}\theta$  does not appear in the limit (but on the average there is always something left in every region of space, represented by  $\lambda$ ).

(f) A possible reason why sub-fractional Brownian motion is the limit in the non-branching model with deterministic  $\theta$  (Remark 2.3(a)) and in the branching model (Theorem 2.4(a)) is that both systems have local spatial inhomogeneities, at the initial time for the non-branching model, which are transported by the motions of the particles, and at each time when a particle reproduces or dies in the branching model. But this does not explain why those possible causes give rise to precisely sub-fractional Brownian motion in both models. The different values of  $\alpha$  in Theorems 2.2(a) and 2.4(a), which also play a role, are related to recurrence properties of the systems.

(g) A process with a decomposition of the same form as that of  $\xi^H$  (with the same component processes  $\zeta^H$  and  $\vartheta^H$ ) appears in [12] in the occupation time fluctuation limit of a branching particle system with some infinitely divisible initial states. Since the initial conditions of the models are quite different, this suggests that there could be a deeper connection between branching and non-branching systems.

(h) Theorems 2.2 and 2.4 can also be extended to other models. For example, in [8] a model is studied in a different context with independent  $\alpha$ -stable motions without branching and initial positions of particles  $(j + \rho)_{j \in \mathbb{Z}}$ , where  $\rho$  is a random variable uniformly distributed on  $[0, 1]$ , independent of the motions. It is easy to see, by a standard conditioning argument (considering the characteristic function and conditioning on  $\rho$ ), that for models with or without branching and with this initial configuration, the limits are the same as for deterministic  $\rho$ , i.e., they are given by Theorems 2.2 and 2.4.

### 3 Proofs

#### 3.1 Auxiliary facts related to the stable density

We recall the self-similarity property of  $p_t$ ,

$$p_{at}(x) = a^{-1/\alpha} p_t(a^{-1/\alpha} x), \quad x \in \mathbb{R}, \quad a > 0. \tag{3.1}$$

and the bound

$$p_1(x) \leq \frac{C}{1 + |x|^{1+\alpha}}. \tag{3.2}$$

Since  $p_t(\cdot)$  is decreasing on  $\mathbb{R}_+$  and symmetric, then by (3.1) we have

$$p_t(x + y) \leq g_t(x) := \begin{cases} t^{-1/\alpha} p_1(0), & \text{if } |x| \leq 2 \\ p_t(\frac{x}{2}), & \text{if } |x| > 2 \end{cases}, \quad x \in \mathbb{R}, \quad |y| \leq 1. \tag{3.3}$$

Denote

$$\phi_m(x) = \frac{1}{1 + |x|^m}, \quad m > 0. \tag{3.4}$$

For  $\varphi \in \mathcal{S}(\mathbb{R})$ ,  $|\varphi(x)| \leq C(\varphi, m)\phi_m(x)$ . This, and an obvious inequality,

$$\frac{1}{1 + |x + y|^m} \leq C(m) \frac{1 + |y|^m}{1 + |x|^m}, \quad m > 0, \tag{3.5}$$

imply that for any  $n \in \mathbb{N}$ ,  $t_1, \dots, t_n > 0$ ,  $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R})$  and non-negative, we have

$$\mathcal{T}_{t_1}(\varphi_1 \mathcal{T}_{t_2}(\varphi_2 \dots \mathcal{T}_{t_n} \varphi_n) \dots)(x + y) \leq C \mathcal{T}_{t_1}(\phi_m \mathcal{T}_{t_2}(\phi_m \dots \mathcal{T}_{t_n} \phi_m) \dots)(x), \quad |y| \leq a, \quad m > 0, \tag{3.6}$$

where the constant  $C$  depends on  $m, a, \varphi_1, \dots, \varphi_n$ .

#### 3.2 Scheme of proofs

As announced in the Introduction, we will give a detailed proof of Theorem 2.2(a) only, nevertheless, it seems worthwhile to present a general scheme, based on the central limit theorem, which can be applied to the proofs of all parts of Theorems 2.2 and 2.4 (see [7] for details).

Let  $N^x$  denote the empirical process of the system (with or without branching) started from a single particle at  $x$ , and let  $N^{(j)}, j \in \mathbb{Z}$ , be the empirical process for the particles which at time  $t = 0$  belong to  $[j, j + 1)$ , i.e.,

$$N^{(j)} = \sum_{n=1}^{\theta_j} N^{\kappa_{j,n}}, \tag{3.7}$$

according to the description at the beginning of Section 2 (see (2.2)). Note that  $N^{(j)}, j \in \mathbb{Z}$ , are independent. The process  $X_T$  defined in (1.1) can be written as

$$X_T(t) = \sum_{j \in \mathbb{Z}} \frac{1}{F_T} \int_0^{Tt} (N_s^{(j)} - EN_s^{(j)}) ds. \tag{3.8}$$

The first step in the argument is to prove that for any  $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ , and  $s, t \geq 0$ ,

$$\lim_{T \rightarrow \infty} E \langle X_T(t), \varphi \rangle \langle X_T(s), \psi \rangle = E \langle X(t), \varphi \rangle \langle X(s), \psi \rangle, \tag{3.9}$$

where  $X$  is the corresponding limit process. Without loss of generality we may assume that  $\varphi, \psi \geq 0$ .

Using (3.8) we have

$$\begin{aligned} E\langle X_T(t), \varphi \rangle \langle X_T(s), \psi \rangle &= \sum_{j \in \mathbb{Z}} \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} E\langle N_r^{(j)}, \varphi \rangle \langle N_{r'}^{(j)}, \psi \rangle dr' dr \\ &- \sum_{j \in \mathbb{Z}} \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} E\langle N_r^{(j)}, \varphi \rangle E\langle N_{r'}^{(j)}, \psi \rangle dr' dr. \end{aligned} \quad (3.10)$$

Using (3.7), (2.1) and the fact that  $E\langle N_t^x, \varphi \rangle = \mathcal{T}_t \varphi(x)$  in both non-branching and (critical) branching cases, and defining, for  $x \in \mathbb{R}$ ,  $n \leq k$ , random variables

$$h_{k,n}(x) = \rho_{k,n}^{[x]} - x, \quad (3.11)$$

where  $[x]$  is the largest integer  $\leq x$ , we rewrite (3.10) as

$$\begin{aligned} E\langle X_T(t), \varphi \rangle \langle X_T(s), \psi \rangle &= \sum_{k=0}^{\infty} p_k \sum_{n=1}^k I(T; k, n) + \sum_{k=0}^{\infty} p_k \sum_{\substack{n,m=1 \\ n \neq m}}^k II(T; k, n, m) \\ &- \sum_{k=0}^{\infty} p_k \sum_{n=1}^k \sum_{\ell=0}^{\infty} p_{\ell} \sum_{m=1}^{\ell} III(T; k, n; \ell, m), \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} I(T; k, n) &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \sum_{j \in \mathbb{Z}} E\langle N_r^{\rho_{k,n}^j}, \varphi \rangle \langle N_{r'}^{\rho_{k,n}^j}, \psi \rangle dr' dr \\ &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \int_{\mathbb{R}} E\langle N_r^{x+h_{k,n}(x)}, \varphi \rangle \langle N_{r'}^{x+h_{k,n}(x)}, \psi \rangle dx dr' dr, \quad (3.13) \\ II(T; k, n, m) &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \sum_{j \in \mathbb{Z}} E\left(\mathcal{T}_r \varphi(\rho_{k,n}^j) \mathcal{T}_{r'} \psi(\rho_{k,m}^j)\right) dr' dr \\ &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \int_{\mathbb{R}} E\left(\mathcal{T}_r \varphi(x+h_{k,n}(x)) \mathcal{T}_{r'} \psi(x+h_{k,m}(x))\right) dx dr' dr \quad (3.14) \end{aligned}$$

(in the first equality for  $II$  we used independence of systems starting from different points),

$$\begin{aligned} III(T; k, n; \ell, m) &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \sum_{j \in \mathbb{Z}} E\mathcal{T}_r \varphi(\rho_{k,n}^j) E\mathcal{T}_{r'} \psi(\rho_{\ell,m}^j) dr' dr \\ &= \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \int_{\mathbb{R}} E\mathcal{T}_r \varphi(x+h_{k,n}(x)) E\mathcal{T}_{r'} \psi(x+h_{\ell,m}(x)) dx dr' dr. \end{aligned} \quad (3.15)$$

Note that

$$|h_{k,n}(x)| \leq 1, \quad x \in \mathbb{R}. \quad (3.16)$$

In each case we show convergence of  $I, II$  and  $III$ , thus proving (3.9). (It is shown that  $I, II, III$  are bounded, so the passage to the limit in each sum in (3.12) is justified).



Next, we show that

$$\langle X(t), \varphi \rangle \Rightarrow \langle X(t), \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}), t \geq 0.$$

To this end, by (3.8) and (3.9) it suffices to prove that the Lyapunov condition

$$\lim_{T \rightarrow \infty} \sum_{j \in \mathbb{Z}} \frac{1}{F_T^3} E \left| \int_0^{Tt} (\langle N_r^{(j)}, \varphi \rangle - E \langle N_r^{(j)}, \varphi \rangle) dr \right|^3 = 0$$

is satisfied, and this property follows if we show that

$$\lim_{T \rightarrow \infty} \sum_{j \in \mathbb{Z}} \frac{1}{F_T^3} E \left( \int_0^{Tt} \langle N_r^{(j)}, \varphi \rangle dr \right)^3 = 0, \quad t \geq 0, \varphi \in \mathcal{S}(\mathbb{R}), \varphi \geq 0. \quad (3.17)$$

It is clear that convergence in law of linear combinations  $\sum_{k=1}^m a_k \langle X_T(t_k), \varphi_k \rangle$  can be obtained analogously from (3.9) and (3.17), thus establishing the claimed convergence  $X_T \Rightarrow_f X$ .

In order to give (3.17) a more tractable form we use (2.1), (3.7), and the trivial inequality  $(a_1 + \dots + a_k)^3 \leq 3k^2(a_1^3 + \dots + a_k^3)$ ,  $a_1, \dots, a_k \geq 0$ , obtaining

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \frac{1}{F_T^3} E \left( \int_0^{Tt} \langle N_r^{(j)}, \varphi \rangle dr \right)^3 &= \sum_{j \in \mathbb{Z}} \frac{1}{F_T^3} \sum_{k=0}^{\infty} p_k E \left( \sum_{n=1}^k \int_0^{Tt} \langle N_r^{\rho_{k,n}^j}, \varphi \rangle dr \right)^3 \\ &\leq 3 \frac{1}{F_T^3} \sum_{k=0}^{\infty} p_k k^2 \sum_{n=1}^k \sum_{j \in \mathbb{Z}} E \left( \int_0^{Tt} \langle N_r^{\rho_{k,n}^j}, \varphi \rangle dr \right)^3 \\ &\leq 3E\theta^3 \sup_{\substack{n, k \in \mathbb{Z}_+ \\ n \leq k}} \frac{1}{F_T^3} \int_{\mathbb{R}} E \left( \int_0^{Tt} \langle N_r^{x+h_{k,n}(x)}, \varphi \rangle dr \right)^3 dx \end{aligned}$$

(see (3.11)). So, to prove (3.17) it suffices to show that

$$\lim_{T \rightarrow \infty} \sup_{\substack{n, k \in \mathbb{Z}_+ \\ n \leq k}} \frac{1}{F_T^3} \int_{\mathbb{R}} E \left( \int_0^{Tt} \langle N_r^{x+h_{k,n}(x)}, \varphi \rangle dr \right)^3 dx = 0, \quad t \geq 0, \varphi \in \mathcal{S}(\mathbb{R}), \varphi \geq 0. \quad (3.18)$$

### 3.3 Proof of Theorem 2.2(a)

Following the scheme we show that

$$\lim_{T \rightarrow \infty} I(T; k, n) = p_1(0) \int_0^t \int_0^s |r - r'|^{-1/\alpha} dr' dr \int_{\mathbb{R}} \varphi(x) dx \int_{\mathbb{R}} \psi(x) dx, \quad (3.19)$$

$$\begin{aligned} \lim_{T \rightarrow \infty} II(T; k, n, m) &= \lim_{T \rightarrow \infty} III(T; k, n; \ell, m) \\ &= p_1(0) \int_0^t \int_0^s (r + r')^{-1/\alpha} dr' dr \int_{\mathbb{R}} \varphi(x) dx \int_{\mathbb{R}} \psi(x) dx. \end{aligned} \quad (3.20)$$

(see (3.13)-(3.15)). It is easy to see that by (3.12), (2.3) and (2.4), this yields (3.9).

Let  $\eta$  denote the standard  $\alpha$ -stable Lévy process in  $\mathbb{R}$ . As we consider the model without branching, we have, for  $r > r'$ ,

$$\begin{aligned} E(N_r^{x+h_{k,n}(x)}, \varphi) \langle N_{r'}^{x+h_{k,n}(x)}, \psi \rangle &= E\varphi(x+h_{k,n}(x)+\eta_r)\psi(x+h_{k,n}(x)+\eta_{r'}) \\ &= \mathcal{T}_{r'}(\psi \mathcal{T}_{r-r'}\varphi)(x+h_{k,n}(x)). \end{aligned} \quad (3.21)$$

Putting this into (3.13) and omitting the subscripts  $k, n$ , we obtain

$$I(T) = I_1(T) + I_2(T), \quad (3.22)$$

where

$$I_1(T) = \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \mathbb{1}_{\{r>r'\}} \int_{\mathbb{R}^3} p_{r'}(x+h(x)-y)\psi(y)p_{r-r'}(y-z)\varphi(z)dzdydxdr'dr, \quad (3.23)$$

$$I_2(T) = \frac{1}{F_T^2} \int_0^{Tt} \int_0^{Ts} \mathbb{1}_{\{r\leq r'\}} \int_{\mathbb{R}^3} p_r(x+h(x)-y)\varphi(y)p_{r-r'}(y-z)\psi(z)dzdydxdr'dr. \quad (3.24)$$

In  $I_1$  we substitute  $\tilde{r} = r/T, \tilde{r}' = r'/T$ , use (2.5) and (3.1), and then we substitute  $\tilde{x} = T^{-1/\alpha}(x-y)$ , arriving at

$$\begin{aligned} I_1(T) &= \int_0^t \int_0^s \mathbb{1}_{\{r>r'\}} \int_{\mathbb{R}^3} p_{r'}(x+T^{-1/\alpha}h(T^{1/\alpha}x+y))\psi(y) \\ &\quad \cdot p_{r-r'}(T^{-1/\alpha}(y-z))\varphi(z)dzdydxdr'dr. \end{aligned}$$

By (3.16) and (3.1), the expression under the integrals converges pointwise, as  $T \rightarrow \infty$ , to

$$p_{r'}(x)p_{r-r'}(0)\psi(y)\varphi(z) = p_{r'}(x)|r-r'|^{-1/\alpha}p_1(0)\psi(y)\varphi(z),$$

and by (3.3), for  $T > 1$ , it is bounded by  $g_{r'}(x)|r-r'|^{-1/\alpha}p_1(0)\psi(y)\varphi(z)$ , which is integrable over  $[0, t] \times [0, s] \times \mathbb{R}^3$ , since  $\alpha > 1$ .  $I_2(T)$  can be treated analogously, hence by (3.22) we obtain (3.19).

Next we take  $II$ . In (3.14) we substitute  $\tilde{r} = r/T, \tilde{r}' = r'/T$ , and by (2.5) we have

$$\begin{aligned} II(T; k, n, m) &= T^{1/\alpha}E \int_0^t \int_0^s \int_{\mathbb{R}^3} p_{Tr}(x+h_{k,n}(x)-y)p_{Tr'}(x+h_{k,m}(x)-z)\varphi(y)\psi(z)dx dy dz dr' dr. \end{aligned}$$

We use (3.1) and substitute  $\tilde{x} = T^{-1/\alpha}(x-z)$ , obtaining

$$\begin{aligned} II(T; k, n, m) &= E \int_0^t \int_0^s \int_{\mathbb{R}^3} p_r(x+T^{-1/\alpha}(z-y+h_{k,n}(T^{1/\alpha}x+z))) \\ &\quad \cdot p_{r'}(x+T^{-1/\alpha}h_{k,m}(T^{1/\alpha}x+z))\varphi(y)\psi(z)dx dy dz dr'. \end{aligned}$$

The integrand converges pointwise to  $p_r(x)p_{r'}(x)\varphi(y)\psi(z)$ , and (3.16) implies that for  $T > 1$ , it is bounded by  $r^{-1/\alpha}p_1(0)g_{r'}(x)\varphi(y)\psi(z)$  (see (3.3)), which is integrable. As  $\int_{\mathbb{R}} p_r(x)p_{r'}(x)dx = (r+r')^{-1/\alpha}p_1(0)$ , we obtain the limit (3.20) for  $II$ .

Note that in this argument the only property of  $h_{k,n}$  we have used is (3.16), therefore it is immediately seen that the limit of  $III$  can be obtained in the same way (see (3.15)). This completes the proof of (3.9).

It remains to show (3.18). The expression under  $\lim_{T \rightarrow \infty} \sup_{n,k}$  in (3.18), similarly as in (3.21), can be written as

$$\begin{aligned} & \frac{3!}{F_T^3} \int_{\mathbb{R}} \int_0^{Tt} \int_r^{Tt} \int_{r'}^{Tt} \mathcal{T}_r(\varphi \mathcal{T}_{r'-r}(\varphi \mathcal{T}_{r''-r'}\varphi))(x + h_{k,n}(x)) dr'' dr' dr dx \\ & \leq \frac{6}{F_T^3} \int_{\mathbb{R}} \int_0^{Tt} \int_0^{Tt} \int_0^{Tt} \mathcal{T}_r(\varphi \mathcal{T}_{r'}(\varphi \mathcal{T}_{r''}\varphi))(x + h_{k,n}(x)) dr'' dr' dr dx \leq CJ(T), \end{aligned} \quad (3.25)$$

where

$$J(T) = \frac{1}{F_T^3} \int_{\mathbb{R}} \int_0^{Tt} \int_0^{Tt} \int_0^{Tt} \mathcal{T}_r(\phi_2 \mathcal{T}_{r'}(\phi_2 \mathcal{T}_{r''}\phi_2))(x) dr'' dr' dr dx, \quad (3.26)$$

and  $\phi_2$  is given by (3.4). In the last inequality in (3.25) we have used (3.6) and (3.16). Hence, for (3.18) it is enough to show that

$$\lim_{T \rightarrow \infty} J(T) = 0. \quad (3.27)$$

Note that in this argument we have not used the assumption on  $\alpha$ .

After obvious substitutions, using (2.5) and the invariance of Lebesgue measure for  $\mathcal{T}_t$ , we have from (3.26)

$$J(T) = T^{3/2\alpha} \int_0^t \int_0^t \int_0^t \int_{\mathbb{R}^3} \phi_2(x) p_{Tr'}(x - y) \phi_2(y) p_{Tr''}(y - z) \phi_2(z) dz dy dx dr'' dr' dr.$$

By (3.1),

$$J(T) \leq T^{3/2\alpha - 2/\alpha} p_1^3(0) t \left( \int_0^t s^{-1/\alpha} ds \right)^2 \left( \int_{\mathbb{R}} \phi_2(x) dx \right)^3 \rightarrow 0 \text{ as } T \rightarrow \infty,$$

since  $1 < \alpha$ , so (3.27) is proved. □

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