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OPTIONAL PROCESSES WITH NON-EXPLODING REALIZED POWER VARIATION ALONG STOPPING TIMES ARE LÀGLÀD

CHRISTOPH KÜHN
Frankfurt MathFinance Institute, Goethe-Universität, D-60054 Frankfurt a.M., Germany
email: ckuehn@math.uni-frankfurt.de

MARC TEUSCH
Deloitte Germany
email: mteusch@deloitte.de

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Abstract

We prove that an optional process of non-exploding realized power variation along stopping times possesses almost surely làglàd paths. This result is useful for the analysis of some imperfect market models in mathematical finance. In the finance applications variation naturally appears along stopping times and not pathwise. On the other hand, if the power variation were only taken along deterministic points in time, the assertion would obviously be wrong.

1 Introduction

In financial market models with proportional transaction costs and effective friction trading strategies have to be almost surely of finite variation in order to avoid infinite losses (see Campi and Schachermayer [2]). In models with a "large" trader having a smooth impact on the price process of an illiquid stock, as introduced by Bank and Baum [1] and Çetin, Jarrow, and Protter [3], a trading strategy should be of non-exploding *quadratic* variation. Besides the interest in its own, the result of this note is of use for the analysis of these models. It guarantees that trading strategies possess limits from the left and from the right, i.e. left and right jumps of the process can be defined and used for the analysis, even if one does not start with càglàd strategies (resp. càdlàg, depending on the precise interpretation of a strategy) from the very beginning. Let the real-valued process X model the number of shares of the illiquid stock the trader plans to hold, $T_0 = 0$, and $T_1 \leq T_2 \leq \ldots$ the stopping times at which he rebalances his portfolio. Roughly speaking in these models appears some transaction costs term of the order $\sum_{k=1,\ldots,n}(X_{T_k} - X_{T_{k-1}})^2$. Thus the realized quadratic variation, naturally arising along stopping times and not pathwise, should be non-exploding when passing to a time-continuous limit.

In contrast to the total variation, for the quadratic variation the restriction to stopping times is

crucial. Namely, it is well-known that for any $r \in \mathbb{R}_+ \cup \{+\infty\}$ and for almost all paths $B_\cdot(\omega)$ of a Brownian motion on $[0,t]$ with $t>0$ there exist sequences $(\tau_n)_{n\in\mathbb{N}}$ of grids $\tau_n = (t_0^n, t_1^n, \ldots, t_{k_n}^n)$ (depending on ω) with $0 = t_0^n \le t_1^n \le \ldots \le t_{k_n}^n = t$ and $\max_k |t_k^n - t_{k-1}^n| \to 0$ such that $\lim_{n\to\infty}\sum_{k=1}^{k_n}(B_{t_k^n}(\omega) - B_{t_{k-1}^n}(\omega))^2 = r$ (see the footnote on page 192 of Lévy [7] and Freedman [5], Proposition 70 and the arguments given on pages 48 and 49). This means that in a pathwise sense the quadratic variation does not exist and is exploding, but of course if grid points are restricted to stopping times the realized quadratic variation converges to t in probability.
On the other hand, if the power variation is only taken along deterministic points in time, a non-exploding variation does obviously not imply that paths possess left and right limits, see Example 3.1 for an easy counterexample. The reason for this is that for processes having neither left- nor right-continuous paths arbitrary sequences of grids with vanishing mesh do not always capture the entire variation.

2 Main part

Throughout the note we fix a terminal time $T \in \mathbb{R}_+$ and a complete probability space $(\Omega, \mathscr{F}, P)$ equipped with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. The optional σ-field is the σ-field $\mathscr{O}$ on $\Omega\times[0,T]$ that is generated by all adapted processes with càdlàg paths (considered as mappings on $\Omega\times[0,T]$). A stochastic process that is $\mathscr{O}$-measurable is called optional.

Definition 2.1 (Non-exploding realized power variation)**.** *Let $p>0$. We say that a real-valued optional process X has non-exploding realized power variation of order $p>0$ if for any sequence $(\tau_n)_{n\in\mathbb{N}}$ of grids $\tau_n = (T_0^n, T_1^n, \ldots, T_{k_n}^n)$ with $0 = T_0^n \le T_1^n \le \ldots \le T_{k_n}^n = T$ stopping times and $\max_{k=1,\ldots,k_n}|T_k^n - T_{k-1}^n| \to 0$ in probability we have that*

$$P\left(\limsup_{n\to\infty}\sum_{k=1,\ldots,k_n}|X_{T_k^n} - X_{T_{k-1}^n}|^p < \infty\right) = 1.$$

Definition 2.2. *A function $f:[0,T]\to\mathbb{R}$ is called làglàd ("avec des limites à gauche et des limites à droite") if for any $t\in(0,T]$ the limit $\lim_{s<t,s\to t} f(s)$ exists as an element of $\mathbb{R}$ and for any $t\in[0,T)$ the limit $\lim_{s>t,s\to t} f(s)$ exists as an element of $\mathbb{R}$.*

Theorem 2.3. *Let X be a real-valued optional process. We have that $\{\omega\in\Omega \mid X_\cdot(\omega) \text{ is làglàd}\} \in \mathscr{F}$. If X has non-exploding realized power variation of some order $p>0$ (in the sense of Definition 2.1), then $P(\{\omega\in\Omega \mid X_\cdot(\omega) \text{ is làglàd}\}) = 1$.*

The proof of Theorem 2.3 uses a section theorem for optional sets. Example 3.2 shows that the assertion of Theorem 2.3 would not hold under the slightly weaker assumption that X is only progressively measurable instead of optional.

Lemma 2.4. *A function $x:[0,T]\to\mathbb{R}$ is not làglàd if and only if there exists an $M>0$ such that for all $l\in\mathbb{N}$, $l\ge 2$, and $\delta>0$ there exist an $\eta>0$ and points $0<t_1<t_2<\ldots<t_l<T$ with $\eta \le |t_k - t_{k-1}| \le \delta$ and $|x(t_k) - x(t_{k-1})|^p \ge M$ for $k=2,\ldots,l$.*

The proof of implication "⇐" shows that the equivalence also holds without introducing a maximal distance δ between two neighboring points, but the lemma is needed in the current form.

Proof of Lemma 2.4. "⇒": Assume that x has no limit from the left at $t \in (0,T]$ or this limit lies in $\{-\infty,\infty\}$. In both cases there exists a sequence $(t_n)_{n\in\mathbb{N}}$ strictly increasing to t s.t. $(x(t_n))_{n\in\mathbb{N}}$ is no Cauchy sequence. Thus there exists an $M>0$ and a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ s.t. $|x(t_{n_k})-x(t_{n_{k-1}})|^p \geq M$ for all $k\in\mathbb{N}$. As $|t-t_{n_k}|\to 0$ we can find for any $l\geq 2$ and $\delta>0$ a k_0 s.t. $|t_{n_k}-t_{n_{k-1}}|\leq\delta$ for $k=k_0+1,\dots,k_0+l$ and the finitely many non-vanishing distances are bounded away from zero. In the case of a missing (finite) limit from the right the argument is the same.
"⇐": Assume that there is an $M>0$ s.t. for all $l\in\mathbb{N}$, $l\geq 2$, there are $0<t_1<t_2<\ldots<t_l<T$ with $|x(t_k)-x(t_{k-1})|^p\geq M$ for $k=2,\dots,l$. For given $n\in\mathbb{N}$ we choose l large enough s.t. $[(l-1)/3]/n>T$ where $[r]:=\max\{m\in\mathbb{N}\cup\{0\}\mid m\leq r\}$. Then, there is a $l_0\in\{1,2,\dots,l-3\}$ s.t. $t_{l_0}<t_{l_0+1}<t_{l_0+2}<t_{l_0+3}\leq t_{l_0}+1/n$. Thus there is a sequence of quadruples $((t_{1,n},t_{2,n},t_{3,n},t_{4,n}))_{n\in\mathbb{N}}$ with

$$\begin{aligned}&t_{1,n}<t_{2,n}<t_{3,n}<t_{4,n},\quad |t_{2,n}-t_{1,n}|\leq\frac{1}{n},\ |t_{3,n}-t_{2,n}|\leq\frac{1}{n},\ |t_{4,n}-t_{3,n}|\leq\frac{1}{n}\quad\text{and}\\ &|x(t_{2,n})-x(t_{1,n})|^p\geq M,\ |x(t_{3,n})-x(t_{2,n})|^p\geq M,\ |x(t_{4,n})-x(t_{3,n})|^p\geq M\quad\forall n\in\mathbb{N}.\end{aligned}$$

By compactness of $[0,T]$, the sequence possesses a subsequence such that all components converge to some $t^*\in[0,T]$. Either $t_{2,n}<t^*$ for infinitely many n from the subsequence or $t_{3,n}>t^*$ for infinitely many n from the subsequence. By $t_{2,n}-1/n\leq t_{1,n}<t_{2,n}$ and $|x(t_{2,n})-x(t_{1,n})|^p\geq M$, the former would contradict to the existence of the left limit of x at t^*. The latter would contradict to the existence of the right limit of x at t^*. □

Definition 2.5. *Let $x:[0,T]\to\mathbb{R}$, $M\geq 0$, $\eta>0$, and $\delta>\eta$. A collection of points $s_0,s_1,\dots,s_i$, $i\in\mathbb{N}$, with $s_0=0$ and $s_i=T$ is called admissible if it satisfies the following: if $B_j\neq\emptyset$ then $s_j\in B_j$ and if $B_j=\emptyset$ then $s_j=(s_{j-1}+3\delta+\eta/2)\wedge T$, where*

$$B_j=\left\{s\in[s_{j-1}+\eta/2,(u_j+\eta/2)\wedge T]\mid |x(s)-x(s_{j-1})|^p\geq\frac{M}{2^p}\right\}\tag{2.1}$$

$$\text{and } u_j=\inf\left\{s\geq s_{j-1}+\eta/2\mid |x(s)-x(s_{j-1})|^p\geq\frac{M}{2^p}\right\}\wedge\left(s_{j-1}+3\delta\right)\wedge T\tag{2.2}$$

for $j=1,\dots,i$.

Lemma 2.6. *Let $x:[0,T]\to\mathbb{R}$, $M\geq 0$, $\eta>0$, and $\delta>\eta$. There exists an $i\in\mathbb{N}$ and an admissible collection $s_0,s_1,\dots,s_i$ in the sense of Definition 2.5. Any admissible collection $s_0,s_1,\dots,s_i$, $i\in\mathbb{N}$, satisfies*

$$|s_j-s_{j-1}|\leq 4\delta,\quad j=1,\dots,i,\tag{2.3}$$

and

$$\text{if } s_j<T \text{ then } s_j\geq s_{j-1}+\frac{\eta}{2},\quad j=1,\dots,i.\tag{2.4}$$

Assume that there are points $0<t_1<t_2<\ldots<t_l<T$ for some $l\in\mathbb{N}$, $l\geq 2$, with

$$\eta\leq|t_j-t_{j-1}|\leq\delta\qquad\text{and}\qquad|x(t_j)-x(t_{j-1})|^p\geq M\qquad\text{for } j=2,\dots,l.\tag{2.5}$$

Then, any admissible collection $s_0,s_1,\dots,s_i$ also satisfies

$$\sum_{j=1}^{i}|x(s_j)-x(s_{j-1})|^p\geq\frac{M}{2^p}\left[\frac{l}{4}\right],\tag{2.6}$$

where $[r]:=\max\{n\in\mathbb{N}\cup\{0\}\mid n\leq r\}$.

Proof of Lemma 2.6. $s_0, s_1, \dots$ can be constructed recursively. We have that $s_j \ge s_{j-1} + \eta/2$ as long as $s_j < T$. Thus T is attained after finitely many steps. By $\eta \le \delta$, (2.3) and (2.4) are obviously statisfied. Now assume the existence of $0 < t_1 < t_2 < \ldots < t_l < T$ satisfying (2.5).
Step 1: Let us firstly prove (2.6) only for the case that $l = 4$. Let j_0 be such that $s_{j_0} < t_1 \le s_{j_0+1}$.
Case 1: $t_1 < s_{j_0} + \eta/2$. Then,

$$s_{j_0} + \eta/2 \le t_2 < t_3 \le (s_{j_0} + 3\delta) \wedge T \tag{2.7}$$

holds true (note that the first inequality holds due to $s_{j_0} + \eta/2 \le t_1 + \eta/2 \le t_2 - \eta + \eta/2 \le t_2$). In addition, by $|x(t_3) - x(t_2)|^p \ge M$, we have that either $|x(t_2) - x(s_{j_0})|^p \ge M2^{-p}$ or $|x(t_3) - x(s_{j_0})|^p \ge M2^{-p}$. This implies that $u_{j_0+1} \le t_3$ and by (2.7) $B_{j_0+1} \neq \emptyset$. Thus $|x(s_{j_0+1}) - x(s_{j_0})|^p \ge M2^{-p}$.
Case 2: $t_1 \ge s_{j_0} + \eta/2$. If $t_2 \le s_{j_0} + 3\delta$ we can argue as in Case 1, but with t_1, t_2 satisfying (2.7) instead of t_2, t_3. Otherwise either $B_{j_0+1} \neq \emptyset$ and we are done anyway or $s_{j_0+1} + \eta/2 \le t_3 < t_4 \le s_{j_0+1} + 3\delta$ and we can argue as in Case 1, but with t_3, t_4, s_{j_0+1} instead of t_2, t_3, s_{j_0}.
Step 2: Let us now prove (2.6) for arbitrary $l \in \mathbb{N}$. If $l \ge 5$ we have by Step 1 that the variation up to some s_j with $s_j \le t_4 + \eta/2$ is at most $M2^{-p}$. It remains to show that the variation on the interval $[s_j, T]$ is at most $M2^{-p}\left(\left[\frac{l}{4}\right] - 1\right)$. As $t_5 > t_4 + \eta/2 \ge s_j$ the points $t_5, \dots, t_l \in (s_j, T)$ are available and the assertion follows by induction over $[l/4]$. □

Proof of Theorem 2.3. *Step 1:* By Lemma 2.4 we have

$$\{\omega \in \Omega \mid X_\cdot(\omega) \text{ is not làglàd}\} = \bigcup_{m\in\mathbb{N}} \bigcap_{l\in\mathbb{N}\setminus\{1\}} \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} A_{m,l,n,k} \tag{2.8}$$

where

$$\begin{aligned} A_{m,l,n,k} := \Big\{\omega \in \Omega \mid \exists 0 < t_1 < t_2 < \ldots < t_l < T \text{ with } \tfrac{1}{k} \le |t_i - t_{i-1}| \le \tfrac{1}{n} \\ \text{and } |X_{t_i}(\omega) - X_{t_{i-1}}(\omega)|^p \ge \tfrac{1}{m} \text{ for } i = 2, \dots, l.\Big\}. \end{aligned}$$

As X is optional it is $(\mathscr{F} \otimes \mathscr{B}) - \mathscr{B}$-measurable. Thus the mapping $(\omega, t_1, t_2) \mapsto (X_{t_1}(\omega), X_{t_2}(\omega))$ is $(\mathscr{F} \otimes \mathscr{B}^2) - \mathscr{B}^2$-measurable. Furthermore the mapping $(x_1, x_2) \mapsto |x_2 - x_1|^p$ is $\mathscr{B}^2 - \mathscr{B}$-measurable by continuity. Thus the composition $(\omega, t_1, t_2) \mapsto |X_{t_2}(\omega) - X_{t_1}(\omega)|^p$ is $(\mathscr{F} \otimes \mathscr{B}^2) - \mathscr{B}$-measurable. This implies that the mapping

$$(\omega, t_1, \dots, t_l) \mapsto (|X_{t_2}(\omega) - X_{t_1}(\omega)|^p, \dots, |X_{t_l}(\omega) - X_{t_{l-1}}(\omega)|^p)$$

is $(\mathscr{F} \otimes \mathscr{B}^l) - \mathscr{B}^l$-measurable and we obtain that

$$\left\{(\omega, t_1, \dots, t_l) \mid \frac{1}{k} \le |t_i - t_{i-1}| \le \frac{1}{n},\ |X_{t_i}(\omega) - X_{t_{i-1}}(\omega)|^p \ge \frac{1}{m},\ i = 1, \dots, l\right\} \in \mathscr{F} \otimes \mathscr{B}^l.$$

By completeness of $\mathscr{F}$, the projection of a set in $\mathscr{F} \otimes \mathscr{B}^l$ onto Ω is in $\mathscr{F}$ (see e.g. Theorem 1.32 combined with Theorem 1.36 of He, Wang, and Yan [6]). This means that $A_{m,l,n,k} \in \mathscr{F}$ and thus by (2.8) $\{\omega \in \Omega \mid X_\cdot(\omega) \text{ is làglàd}\} \in \mathscr{F}$.
Step 2: Assume that

$$P(\{\omega \in \Omega \mid X_\cdot(\omega) \text{ is not làglàd}\}) > 0 \tag{2.9}$$

and let $p > 0$. We have to show that X is *not* of non-exploding realized power variation of order p in the sense of Definition 2.1.

The idea of the proof is as follows. We want to construct a sequence of grids $(\widetilde{\tau}_i)_{i\in\mathbb{N}}$ with $|\widetilde{T}^i_j - \widetilde{T}^i_{j-1}| \leq \frac{1}{i}$ such that on a set of positive probability the power variation of X along $\widetilde{\tau}_i$ exceeds i for all $i \in \mathbb{N}$. For this we use a section theorem to construct recursively stopping times that constitute an admissible collection in the sense of Definition 2.5 for approximately all paths. By Lemma 2.4 the paths which are not làglàd possess „enough" power variation. It follows from Lemma 2.6 that the variation also appears along the timepoints of an admissible collection which we have constructed with stopping times.
Let us now start with the formal proof. By (2.9), there exists an $m \in \mathbb{N}$ such that $P\left(\bigcap_{l\in\mathbb{N}\setminus\{1\}}\bigcap_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} A_{m,l,n,k}\right) =: r > 0$. Given $i \in \mathbb{N}$, choose $n = n(i) = 4i$ and $l = l(i) \geq 2$ large enough such that

$$\frac{1}{2^p m}\left[\frac{l(i)}{4}\right] \geq i.$$

Choose $k = k(i) \in \mathbb{N}$ such that $k(i) > 4i$ and

$$P\left(\bigcap_{l\in\mathbb{N}\setminus\{1\}}\bigcap_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} A_{m,l,n,k} \setminus A_{m,l(i),n(i),k(i)}\right) \leq r2^{-(i+2)}$$

(by choosing $k(i)$ large enough the latter can be achieved as $A_{m,l(i),n(i),k} \uparrow \cup_{\kappa\in\mathbb{N}} A_{m,l(i),n(i),\kappa}$ for $k \uparrow \infty$ and $\cup_{\kappa\in\mathbb{N}} A_{m,l(i),n(i),\kappa} \supset \cap_{l\in\mathbb{N}\setminus\{1\}} \cap_{n\in\mathbb{N}} \cup_{\kappa\in\mathbb{N}} A_{m,l,n,\kappa}$).
For every $i \in \mathbb{N}$ we want to construct stopping times $T^i_0 \leq T^i_1 \leq \ldots$. Assuming that the stopping time T^i_{j-1} is already specified we define

$$S^i_j := \inf\left\{t \geq T^i_{j-1} + \frac{1}{2k} \mid |X_t - X_{T^i_{j-1}}|^p \geq \frac{1}{2^p m}\right\} \wedge \left(T^i_{j-1} + \frac{3}{4i}\right) \wedge T$$

and

$$\Gamma^i_j := \left\{(\omega, t) \mid T^i_{j-1}(\omega) + \frac{1}{2k} \leq t \leq \left(S^i_j(\omega) + \frac{1}{2k}\right) \wedge T, \quad |X_t(\omega) - X_{T^i_{j-1}(\omega)}(\omega)|^p \geq \frac{1}{2^p m}\right\}.$$

S^i_j and Γ^i_j are the "random versions" of u_j and B_j in (2.2) resp. (2.1). Note that $X_{T^i_{j-1}}$ is $\mathscr{F}_{T^i_{j-1}}$-measurable as X is optional (see e.g. Theorem 3.12 of He, Wang, and Yan [6]). Thus S^i_j is the debut of an optional set and therefore a wide-sense stopping time (see e.g. Theorem 4.30 of [6]). Consequently, $S^i_j + 1/(2k)$ is a stopping time and Γ^i_j an optional set. For technical reasons define

$$\widetilde{\Gamma}^i_j := \left(\Gamma^i_j \bigcap (\Omega \times [0,T))\right) \bigcup \left(((\Omega \setminus \pi_\Omega(\Gamma^i_j)) \times [0,T)) \bigcap [[(T^i_{j-1} + 3/(4i) + 1/(2k)) \wedge T]]\right),$$

where $\pi_\Omega(\Gamma^i_j)$ denotes the projection of $\Gamma^i_j \subset \Omega \times [0,T]$ onto Ω. $\widetilde{\Gamma}^i_j$ is also optional. Now we recursively define $[0,T]$-valued stopping times $T^i_0, T^i_1, \ldots$. Let $T^i_0 := 0$. By a section theorem for optional sets (see e.g. Theorem 4.7 of [6]), there exists a $[0,T]$-valued stopping time T^i_j with $P(T^i_j < T) \geq P(\pi_\Omega(\widetilde{\Gamma}^i_j)) - r2^{-(i+j+2)}$ such that

$$(\omega, T^i_j(\omega)) \in \widetilde{\Gamma}^i_j \qquad \text{for all } \omega \text{ with } T^i_j(\omega) < T. \tag{2.10}$$

Let us comment the construction of the set $\widetilde{\Gamma}^i_j$ and the stopping time T^i_j. Fixing an ω the ω-section of $\widetilde{\Gamma}^i_j$ mimics the admissibility condition from Definition 2.5. Namely, if $\omega \in \pi_\Omega(\Gamma^i_j)$, i.e.

there is a $s \in [0,T]$ with $(\omega,s) \in \Gamma^i_j$, the next point t of the collection has to satisfy $(\omega,t) \in \Gamma^i_j$. If $\omega \notin \pi_\Omega(\Gamma^i_j)$, then we require that $t = (T^i_{j-1}(\omega)+3/(4i)+1/(2k)) \wedge T$. In addition, $\Omega \times \{T\}$ is taken out of the set $\widetilde{\Gamma}^i_j$. This is one way to distinguish the case that $T^i_j(\omega) = T$ occurs as the selection of the stopping time from the optional set fails from the case that $T^i_j(\omega) = T$ and T is the only admissible successor of $T^i_{j-1}(\omega)$ in the sense of Definition 2.5. (2.10) guarantees that $T^i_j(\omega) = T$ if T is the only admissible successor of $T^i_{j-1}(\omega)$ or if we have that already $T^i_{j-1}(\omega) = T$. Putting together, on all paths without any failing selection we obtain admissible collections of timepoints in the sense of Definition 2.5. In the following, this idea is written down mathematically.
Denote $B_{i,j} := \{T^i_j < T\} \bigcup (\Omega \setminus \pi_\Omega(\widetilde{\Gamma}^i_j))$. It follows that

$$P(B_{i,j}) \geq 1 - r2^{-(i+j+2)}.$$

Let $\omega \in \bigcap_{j \in \mathbb{N}} B_{i,j}$, i.e. for any j we have either $T^i_j(\omega) < T$ or $\omega \notin \pi_\Omega(\widetilde{\Gamma}^i_j)$. In the case that $T^i_j(\omega) < T$ (2.10) yields that $(\omega, T^i_j(\omega)) \in \Gamma^i_j$ if $\omega \in \pi_\Omega(\Gamma^i_j)$ and $T^i_j(\omega) = T^i_{j-1}(\omega) + 3/(4i) + 1/(2k)$ if $\omega \notin \pi_\Omega(\Gamma^i_j)$. In the case that $\omega \notin \pi_\Omega(\widetilde{\Gamma}^i_j)$ we have by (2.10) that $T^i_j(\omega) = T$ and either $(\omega, T) \in \Gamma^i_j$ or $((\omega,T) \notin \Gamma^i_j$ and $(T^i_{j-1}(\omega) + 3/(4i) + 1/(2k)) \wedge T = T)$. This shows that $T^i_0(\omega), T^i_1(\omega), \ldots$ is an admissible collection of points in the sense of Definition 2.5 for the parameters $M = 1/m$, $\eta = 1/k(i)$, and $\delta = 1/(4i)$. Namely, the ω-section of Γ^i_j corresponds to B_j and $(T^i_{j-1}(\omega)+3/(4i)+1/(2k)) \wedge T$ corresponds to $(s_{j-1}+3\delta+\eta/2) \wedge T$ in the lemma. (2.4) tells us that at the latest after $[2kT]+1$ steps (which does not depend on ω) T is attained. If $\omega \notin \bigcap_{j\in\mathbb{N}} B_{i,j}$, T is also attained at the latest after $[2kT]+1$ steps by (2.10). To guarantee that neighboring points have a distance not greater than $1/i$ *everywhere* we recursively define new stopping times $\widetilde{T}^i_0, \widetilde{T}^i_1, \ldots, \widetilde{T}^i_{[2kT]+1}$. Let $\widetilde{T}^i_0 := 0$. Given $\widetilde{T}^i_{j-1}$ define $\widetilde{T}^i_j := T^i_j \wedge (\widetilde{T}^i_{j-1} + 1/i)$. However, by (2.3), $\widetilde{T}^i_j = T^i_j$ on the set $\bigcap_{j' \in \mathbb{N}} B_{i,j'}$. Thus we have that $|\widetilde{T}^i_j - \widetilde{T}^i_{j-1}| \leq 1/i$ and for any $\omega \in \bigcap_{j\in\mathbb{N}} B_{i,j}$ $\widetilde{T}^i_0(\omega), \widetilde{T}^i_1(\omega), \ldots, \widetilde{T}^i_{[2Tk]+1}(\omega)$ is an admissible collection of points in the sense of Definition 2.5 as well.
Let $\omega \in A_{m,l(i),n(i),k(i)} \bigcap \left(\bigcap_{j\in\mathbb{N}} B_{i,j}\right)$. For the mapping $t \mapsto X_t(\omega)$ (2.5) is satisfied, with the parameters $M = 1/m$, $l = l(i)$, $\eta = 1/k(i)$, and $\delta = 1/(4i)$. It follows from Lemma 2.6 that

$$\sum_{j=1}^{[2kT]+1} |X_{\widetilde{T}^i_j(\omega)}(\omega) - X_{\widetilde{T}^i_{j-1}(\omega)}(\omega)|^p \geq \frac{1}{2^p m}\left[\frac{l}{4}\right] \geq i, \; i \in \mathbb{N}. \tag{2.11}$$

We arrive at

$$\begin{aligned} &P\left(\limsup_{i\in\mathbb{N}} \sum_{j=1}^{[2k(i)T]+1} |X_{\widetilde{T}^i_j} - X_{\widetilde{T}^i_{j-1}}|^p = \infty\right) \\ &\geq P\left(\bigcap_{i\in\mathbb{N}}\left(A_{m,l(i),n(i),k(i)} \bigcap \left(\bigcap_{j\in\mathbb{N}} B_{i,j}\right)\right)\right) \\ &\geq r - r\sum_{i=1}^{\infty} 2^{-(i+2)} - r\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 2^{-(i+j+2)} = \frac{r}{2} > 0. \end{aligned} \tag{2.12}$$

Together with $|\widetilde{T}^i_j - \widetilde{T}^i_{j-1}| \leq 1/i$, (2.12) shows that the variation is *not* non-exploding in the sense of Definition 2.1. ☐

3 Counterexamples

The following easy example shows that the assertion of Theorem 2.3 would be wrong, if the variation were only considered along deterministic points in time.

Example 3.1. *Let $(\xi_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables which are uniformly distributed on $[0,T]$. Define*

$$X_t(\omega) := \begin{cases} 1 : \text{if } t = \xi_n(\omega) \text{ for some } n \in \mathbb{N}. \\ 0 : \text{otherwise.} \end{cases}$$

For simplicity define $\mathscr{F}_t$ for all $t \in [0,T]$ as the completion of the sigma-algebra generated by $(\xi_n)_{n\in\mathbb{N}}$. Then, X is optional as it can be written as $X = \sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} X^{n,m}$ with the càdlàg adapted processes $X^{n,m}$ defined by

$$X^{n,m} := \sum_{k=1}^{2^m} 1_{\left\{\omega\in\Omega \ | \ \xi_n(\omega)\in\left[\frac{k-1}{2^m}T, \frac{k}{2^m}T\right)\right\}\times\left[\frac{k-1}{2^m}T, \frac{k}{2^m}T\right)}.$$

Adaptedness of $X^{n,m}$ holds as already $\mathscr{F}_0$ contains all information about ξ_n. The variation of X along deterministic times vanishes with probability one as for any fixed t X_t vanishes with probability one. On the other hand, for almost all ω $(\xi_n(\omega))_{n\in\mathbb{N}}$ is dense in $[0,T]$. Thus, with probability one the paths of X are not làglàd.

The following example shows that the assertion of Theorem 2.3 would not hold under the slightly weaker assumption that X is only *progressively measurable* instead of optional. A process X is called progressively measurable if for any $t \in [0,T]$ the restriction of X on $\Omega \times [0,t]$ is $\mathscr{F}_t \otimes \mathscr{B}([0,t])$-measurable where $\mathscr{B}([0,t])$ denotes the Borel σ-field on $[0,t]$.

Example 3.2. *Let $(B_t)_{t\in[0,T]}$ be a standard one-dimensional Brownian motion and $(\mathscr{F}^B_t)_{t\in[0,T]}$ be the usual augmentation of the natural filtration of B, i.e. $(\mathscr{F}^B_t)_{t\in[0,T]}$ is right-continuous and complete. For each ω the set $\{t \mid B_t(\omega) \neq 0\}$ is the disjoint union of open intervals, the excursion intervals of the path $B_\cdot(\omega)$ from 0. Now define*

$$A := \{(\omega,t) \mid t \text{ is the left-hand endpoint of an excursion interval of the path } B_\cdot(\omega)\},$$

i.e. for each $(\omega,t) \in A$ we have that $B_t(\omega) = 0$ and there exists an $\varepsilon > 0$ s.t. $B_s(\omega) \neq 0$ for all $s \in (t, t+\varepsilon)$.
The set A is an example due to Dellacherie and Meyer for a progressive set which is not optional (see [4], page 128). The progressively measurable process $X := 1_A$ is not làglàd as for almost all ω there are both infinitely many timepoints s with $(\omega,s) \in A$ and infinitely many timepoints s with $(\omega,s) \notin A$ in any right neighbourhood of $t = 0$. On the other hand, the power variation of X does not explode along stopping times in the sense of Definition 2.1 as we have $P(X_\tau = 0) = 1$ for any stopping time τ. The latter follows by the strong Markov property of B w.r.t. $(\mathscr{F}^B_t)_{t\in[0,T]}$. The process $(B_{t+\tau} - B_\tau)_{t\geq 0}$ is a standard Brownian motion and stochastically independent of τ which implies that τ cannot be the starting point of an excursion with positive probability.

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