

ASYMPTOTIC CONSTANTS FOR MINIMAL DISTANCE IN THE CENTRAL LIMIT THEOREM

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Abstract

In this paper, we generalize the asymptotic result of Esseen (1958) concerning the Wasserstein distance of order one in the mean central limit theorem to the Wasserstein distances of order r for r in $]1, 2]$.

1 Introduction and main results.

In this paper we continue the research started in Rio (2009), concerning the Wasserstein distances in the central limit theorem. Let Ω be a probability space, rich enough to generate the set of probability laws on $\mathbb{R} \times \mathbb{R}$. Let d be a pseudodistance on the set of real-valued random variables, such that $d(X, Y)$ depends only on the law of (X, Y) . Then, according to Paul Lévy (see Note B in Fréchet (1950) for this fact) the minimal distance \hat{d} associated to d is defined by $\hat{d}(\mu, \nu) = \inf\{d(X, Y) : X \sim \mu, Y \sim \nu\}$, where the infimum is taken over all random vectors (X, Y) with respective marginal laws μ and ν . When $E = \mathbb{R}$, $r \geq 1$ and $d(X, Y) = \|X - Y\|_r$, we denote by W_r the so defined minimal distance on the space \mathcal{M}_r of probability laws with a finite absolute moment of order r . This distance is usually called Wasserstein distance of order r . The distances W_r are homogeneous of degree 1.

Throughout the paper, X_1, X_2, \dots is a sequence of independent and identically distributed real-valued random variables with mean zero and finite positive variance. We set $S_n = X_1 + X_2 + \dots + X_n$ and $v_n = \text{Var } S_n$. We denote by μ_n the law of $v_n^{-1/2} S_n$ and by γ_v the normal law with mean 0 and variance v . In a recent paper, Rio (2009) proved that, for r in $[1, 2]$,

$$(1.1) \quad \limsup_{n \rightarrow \infty} \sqrt{n} W_r(\mu_n, \gamma_1) < \infty$$

as soon as $\mathbb{E}(|X_1|^{r+2}) < \infty$. An interesting problem is then to find the limit in (1.1). Let F_n denote the distribution function of μ_n and Φ denote the distribution function of the standard normal.

Esseen (1958) proved that

$$(1.2) \quad \lim_{n \rightarrow \infty} \sqrt{n} \|F_n - \Phi\|_p = A_p(\mu_1),$$

where $A_p(\mu_1)$ is some nonnegative explicit constant. In the specific case $p = 1$,

$$\|F_n - \Phi\|_1 = W_1(\mu_n, \gamma_1).$$

Consequently Esseen's result gives the asymptotic constant in (1.1) for $r = 1$. Zolotarev (1964) provided the following representation. Let Y and U be independent random variables, with respective distributions the standard normal law and the uniform law over $[-1/2, 1/2]$. Then

$$A_1(\mu_1) = \|(\alpha_3/6)(1 - Y^2) + (h/\sigma)U\|_1,$$

where σ is the standard deviation of X_1 , $\alpha_3 = \sigma^{-3}\mathbb{E}(X_1^3)$ and h is the span of the distribution of X_1 for lattice distributions and 0 otherwise. For $r > 1$, it is known since a long time (cf. Dall'Aglio (1956) and Fréchet (1957) for more about this) that

$$(1.3) \quad W_r(\mu_n, \gamma_1) = \|F_n^{-1} - \Phi^{-1}\|_r,$$

and consequently $W_r(\mu_n, \gamma_1) \neq \|F_n - \Phi\|_r$ in general. In the next two theorems we describe the asymptotic behaviour of $W_r(\mu_n, \gamma_1)$ as n tends to ∞ , for r in $]1, 2]$. As for $r = 1$, this behaviour depends on whether or not X_1 has a lattice distribution.

Theorem 1.1. *Let Y be a random variable with standard normal law. Set $\alpha_3 = (\mathbb{E}(X_1^2))^{-3/2}\mathbb{E}(X_1^3)$. Let r be any real in $]1, 2]$. If the distribution of X_1 is not a lattice distribution and if $\mathbb{E}(|X_1|^{r+2}) < \infty$, then*

$$\lim_{n \rightarrow \infty} n^{1/2} W_r(\mu_n, \gamma_1) = (|\alpha_3|/6) \|1 - Y^2\|_r.$$

We now state the results for lattice distributions. We will consider either smoothed sums or unsmoothed sums. For lattice distributions taking values in the arithmetic progression $\{a + kh : k \in \mathbb{Z}\}$ (h being maximal), the smoothed sums \bar{S}_n are defined by

$$\bar{S}_n = S_n + hU,$$

where U is a random variable with uniform distribution over $[-1/2, 1/2]$, independent of S_n . We denote by $\bar{\mu}_n$ the law of $v_n^{-1/2}\bar{S}_n$. Theorem 1.2 gives the asymptotic constants for smoothed or unsmoothed sums.

Theorem 1.2. *Let X_1, X_2, \dots be centered and independent identically distributed lattice random variables with variance σ^2 , taking values in the arithmetic progression $\{a + kh : k \in \mathbb{Z}\}$ (h being maximal). Let Y and U be two independent random variables, with respective laws the standard normal law and the uniform law over $[-1/2, 1/2]$. Let r be any real in $]1, 2]$. Assume that $\mathbb{E}(|X_1|^{r+2}) < \infty$. Then*

$$(a) \quad \lim_{n \rightarrow \infty} n^{1/2} W_r(\bar{\mu}_n, \gamma_1) = (|\alpha_3|/6) \|1 - Y^2\|_r$$

and

$$(b) \quad \lim_{n \rightarrow \infty} n^{1/2} W_r(\mu_n, \gamma_1) = \|(\alpha_3/6)(1 - Y^2) + (h/\sigma)U\|_r.$$

Remark 1.1. The constants appearing here are represented as in Zolotarev (1964), and consequently (b) still holds for $r = 1$. Proceeding as in Esseen (1958) one can prove that (a) is true in the case $r = 1$.

In the specific case $r = 2$, the asymptotic constant can be computed more explicitly. Let us state the corresponding result.

Corollary 1.3. *Let X_1, X_2, \dots be centered and independent identically distributed lattice random variables with variance σ^2 and finite moment of order 4, taking values in the arithmetic progression $\{a + kh : k \in \mathbb{Z}\}$ (h being maximal). Then, with the same notations as in Theorems 1.1 and 1.2,*

$$\lim_{n \rightarrow \infty} n^{1/2} W_2(\mu_n, \gamma_1) = \frac{1}{6} \sqrt{3(h/\sigma)^2 + 2\alpha_3^2}.$$

Remark 1.1. In the non lattice case, as shown by Theorem 1.1, the above result holds with $h = 0$.

Example 1: symmetric sign. Assume that the random variables are symmetric signs, that is $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. In that case $h = 2$, $\sigma = 1$ and $\alpha_3 = 0$. Then the asymptotic constant in Corollary 1.3 is $1/\sqrt{3}$.

Example 2: Poisson distribution. Let $\lambda > 0$ and assume that $X_1 + \lambda$ has the Poisson distribution $\mathcal{P}(\lambda)$. Then $h = 1$, $\sigma^2 = \lambda$ and $\alpha_3 = \lambda^{-1/2}$. In that case Corollary 1.3 gives the asymptotic constant $\frac{1}{6}(5/\lambda)^{1/2}$.

2 Non lattice distributions or lattice distributions and smoothed sums.

In this section, we prove Theorem 1.1. for non lattice distributions and Theorem 1.2(a) for lattice distributions and smoothed sums. Dividing the random variables by the standard deviation of X_1 , we may assume that the random variables X_1, X_2, \dots satisfy $\mathbb{E}(X_1^2) = 1$. Let \bar{F}_n denote the distribution function of $\bar{\mu}_n$. The first step is the result below of pointwise convergence, which is known as the Cornish-Fisher expansion. As pointed in Hall (1992, Theorem 2.4 and final comments to Chapter 2), the Cornish-Fisher expansion of first order does not need the Cramer condition. To be exhaustive, we give a proof of this result

Lemma 2.1. *If μ_1 is not a lattice distribution, then, for any u in $]0, 1[$,*

$$(a) \quad \sqrt{n}(F_n^{-1}(u) - \Phi^{-1}(u)) = \frac{\alpha_3}{6}((\Phi^{-1}(u))^2 - 1) + o(1),$$

as n tends to ∞ . If μ_1 is a lattice distribution, then

$$(b) \quad \sqrt{n}(\bar{F}_n^{-1}(u) - \Phi^{-1}(u)) = \frac{\alpha_3}{6}((\Phi^{-1}(u))^2 - 1) + o(1).$$

Furthermore the convergence in (a) and (b) are uniform on $[\delta, 1 - \delta]$, for any positive δ .

Proof of Lemma 2.1. Let $\phi = \Phi'$ denote the density of the standard normal. We start from Esseen's (1945) estimates

$$(2.1) \quad F_n(x) = \Phi(x) + \frac{\alpha_3}{6}(1 - x^2)\phi(x)n^{-1/2} + o(n^{-1/2}),$$

which holds true for non lattice distributions, and

$$(2.2) \quad \bar{F}_n(x) = \Phi(x) + \frac{\alpha_3}{6}(1-x^2)\phi(x)n^{-1/2} + o(n^{-1/2}),$$

which holds true for lattice distributions. Moreover these estimates hold uniformly in x as n tends to ∞ . Let

$$\Psi(x) = \Phi(x) + \frac{\alpha_3}{6}(1-x^2)\phi(x)n^{-1/2} \text{ and } Q(u) = \Phi^{-1}(u) + \frac{\alpha_3}{6}((\Phi^{-1}(u))^2 - 1)n^{-1/2}.$$

We start by proving that, for any positive A , uniformly in x over $[-A, A]$,

$$(2.3) \quad Q(\Psi(x)) = x + O(1/n).$$

First

$$\sup_{x \in \mathbb{R}} |1 - x^2| \phi(x) = (2\pi)^{-1/2}.$$

For n large enough, $\Psi(x)$ lies in $[\Phi(-2A), \Phi(2A)]$ for any x in $[-A, A]$. Then, by the Taylor formula at order 2 applied at $\Phi(x)$ with the increment $\frac{\alpha_3}{6}(1-x^2)\phi(x)n^{-1/2}$,

$$|\Phi^{-1}(\Psi(x)) - x - \frac{\alpha_3}{6}(1-x^2)n^{-1/2}| \leq \frac{\alpha_3^2}{72\pi n} \sup_{u \in [\Phi(-2A), \Phi(2A)]} |(\Phi^{-1})''(u)|.$$

Now

$$(\Phi^{-1})''(u) = \Phi^{-1}(u)(\phi(\Phi^{-1}(u)))^{-2}.$$

Hence

$$\sup_{u \in [\Phi(-2A), \Phi(2A)]} |(\Phi^{-1})''(u)| = \sup_{x \in [0, 2A]} (x/\phi^2(x)) = 2A/\phi^2(2A) < \infty,$$

which ensures that, uniformly in x over $[-A, A]$,

$$(2.4) \quad \Phi^{-1}(\Psi(x)) = x + \frac{\alpha_3}{6}(1-x^2)n^{-1/2} + O(1/n).$$

Now, uniformly in x over $[-A, A]$,

$$\begin{aligned} Q(\Psi(x)) &= \Phi^{-1}(\Psi(x)) + \frac{\alpha_3}{6}((\Phi^{-1}(\Psi(x)))^2 - 1)n^{-1/2} \\ &= x + \frac{\alpha_3}{6}(1-x^2)n^{-1/2} + O(1/n) + \frac{\alpha_3}{6}(x^2 - 1)n^{-1/2} + O(1/n) \end{aligned}$$

by (2.4) applied twice. The estimate (2.3) follows.

Starting from (2.1), we now prove Lemma 2.1(a). The proof of Lemma 2.1(b) (omitted) can be done exactly in the same way, starting from (2.2). From (2.1), for n large enough, $F_n(x)$ lies in $[\Phi(-2A), \Phi(2A)]$ for any x in $[-A, A]$. Then, by (2.1) again and the Rolle theorem

$$Q(F_n(x)) - Q(\Psi(x)) = o(n^{-1/2})$$

uniformly in x over $[-A, A]$. Consequently, by (2.3), there exists some sequence $(\varepsilon_n)_n$ of positive reals converging to 0 such that

$$|x - Q(F_n(x))| \leq n^{-1/2}\varepsilon_n,$$

for any x in $[-A, A]$. It follows that

$$(2.5) \quad |F_n^{-1}(u) - Q(F_n(F_n^{-1}(u)))| \leq n^{-1/2} \varepsilon_n,$$

provided that $F_n^{-1}(u)$ lies in $[-A, A]$. By (2.1) again this condition holds true for n large enough if u belongs to $[\Phi(-A/2), \Phi(A/2)]$. The estimate (2.1) also implies that the jumps of F_n are uniformly bounded by $o(n^{-1/2})$. Hence

$$\sup_{u \in]0, 1[} |F_n(F_n^{-1}(u)) - u| = o(n^{-1/2}),$$

which ensures that

$$(2.6) \quad \sup_{u \in [\Phi(-A/2), \Phi(A/2)]} |Q(F_n(F_n^{-1}(u))) - Q(u)| = o(n^{-1/2}).$$

Finally, by (2.5) and (2.6), Lemma 2.1(a) holds true with $\delta = \Phi(-A/2)$. ■

Proof of Theorem 1.1. Recall that, for laws μ and ν with respective distribution functions F and G , $W_r(\mu, \nu) = \|F^{-1} - G^{-1}\|_r$. Now, by Lemma 2.1, for any positive δ ,

$$\lim_{n \rightarrow \infty} n^{r/2} \int_{\delta}^{1-\delta} |F_n^{-1}(u) - \Phi^{-1}(u)|^r du = (|\alpha_3|/6)^r \int_{\delta}^{1-\delta} |\Phi^{-1}(u)|^2 - 1|^r du$$

and

$$\lim_{n \rightarrow \infty} n^{r/2} \int_{\delta}^{1-\delta} |\bar{F}_n^{-1}(u) - \Phi^{-1}(u)|^r du = (|\alpha_3|/6)^r \int_{\delta}^{1-\delta} |\Phi^{-1}(u)|^2 - 1|^r du.$$

Since $\Phi^{-1}(u)$ has the standard normal distribution under the Lebesgue measure over $[0, 1]$, Theorem 1.1 will follow from the above inequality if we prove that, for N large enough,

$$(2.7a) \quad \lim_{\delta \searrow 0} \sup_{n \geq N} \left(n^{r/2} \int_0^1 \mathbb{1}_{\inf(u, 1-u) < \delta} |F_n^{-1}(u) - \Phi^{-1}(u)|^r du \right) = 0$$

and

$$(2.7b) \quad \lim_{\delta \searrow 0} \sup_{n \geq N} \left(n^{r/2} \int_0^1 \mathbb{1}_{\inf(u, 1-u) < \delta} |\bar{F}_n^{-1}(u) - \Phi^{-1}(u)|^r du \right) = 0.$$

Now $|F_n^{-1}(u) - \bar{F}_n^{-1}(u)| \leq n^{-1/2} h$, which ensures that

$$\lim_{\delta \searrow 0} \sup_{n > 0} \left(n^{r/2} \int_0^1 \mathbb{1}_{\inf(u, 1-u) < \delta} |\bar{F}_n^{-1}(u) - F_n^{-1}(u)|^r du \right) = 0.$$

Hence (2.7b) follows from (2.7a) via the triangle inequality. The proof of (2.7a) will be done via Theorem 6.1 in Rio (2009), which is based on estimates of Borisov, Panchenko and Skilyagina (1998) for smooth functions of S_n . For F and G distribution functions on the real line, let

$$\kappa_{r,2}(F, G) = \int_0^1 \left(1 + (|F^{-1}(u)| + |G^{-1}(u)|)^2 / 4 \right) |F^{-1}(u) - G^{-1}(u)|^r du.$$

Theorem 6.1 in Rio (2009) states that, if $n^{r/2} \geq \mathbb{E}(|X_1|^{r+2})$ (recall X_1 has unit variance), then

$$\kappa_{r,2}(F_n, \Phi) \leq C n^{-r/2} \mathbb{E}(|X_1|^{r+2})$$

for some positive constant C depending only on r . Now

$$\int_0^1 \mathbb{1}_{\inf(u, 1-u) < \delta} |F_n^{-1}(u) - \Phi^{-1}(u)|^r du \leq \frac{4}{1 + (\Phi^{-1}(\delta))^2} \kappa_{r,2}(F_n, \Phi).$$

Hence, for $N = (\mathbb{E}(|X_1|^{r+2}))^{2/r}$,

$$\sup_{n \geq N} \left(n^{r/2} \int_0^1 \mathbb{1}_{\inf(u, 1-u) < \delta} |F_n^{-1}(u) - \Phi^{-1}(u)|^r du \right) \leq \frac{4C \mathbb{E}(|X_1|^{r+2})}{1 + (\Phi^{-1}(\delta))^2},$$

which implies (2.7a). Hence Theorem 1.1 holds true.

3 Lattice distributions.

In this section we prove Theorem 1.2(b). Again we may assume that $\sigma^2 = 1$. Let

$$\Delta_n(t) = n^{1/2}(F_n^{-1}(t) - \Phi^{-1}(t)).$$

From (2.7a), it is enough to prove that, for any δ in $]0, 1/2[$,

$$(3.1) \quad \lim_{n \rightarrow \infty} \int_{\delta}^{1-\delta} |\Delta_n(t)|^r du = \int_0^1 \int_{\delta}^{1-\delta} |(\alpha_3/6)(1 - |\Phi^{-1}(t)|^2) + hu - h/2|^r dudt.$$

In order to prove (3.1) we will use Lemma 2.1(b). For any distribution function F and any real x , let $F(x-0) = \lim_{t \uparrow x} F(t)$. Let x_0 be the smallest number in the lattice $\{n^{-1/2}(a + kh) : k \in \mathbb{Z}\}$ such that $F_n(x_0 - 0) \geq \delta$. For any relative integer l , set $x_l = x_0 + lhn^{-1/2}$ and $a_l = F_n(x_l)$. Then, for any l in \mathbb{Z} such that $0 < a_{l-1} < a_l < 1$, we have

$$(3.2) \quad F_n^{-1}(t) = x_l \text{ for any } t \in]a_{l-1}, a_l],$$

since $a_{l-1} = F_n(x_l - 0)$. Furthermore, it can easily be proven that

$$(3.3) \quad \bar{F}_n^{-1}((1-u)a_{l-1} + ua_l) = x_l + n^{-1/2}h(u - 1/2) \text{ for any } u \in [0, 1].$$

Throughout the sequel, let $t_l(u) = (1-u)a_{l-1} + ua_l$. Let m be the largest integer such that $F_n(x_m) \leq 1 - \delta$. It comes from Esseen's estimates that $a_{l-1} < a_l$ for any l in $[1, m]$, for n large enough. Then, for l in $[1, m]$ and u in $[0, 1]$, by Lemma 2.1(b) together with (3.3),

$$\Delta_n(t_l(u)) = \frac{\alpha_3}{6} (|\Phi^{-1}(t_l(u))|^2 - 1) - h(u - 1/2) + o(1)$$

uniformly in $l \in [1, m]$ and $u \in [0, 1]$. It follows that

$$\int_{a_0}^{a_m} |\Delta_n(t)|^r du = \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t_l(u))|^2 - 1) - h(u - \frac{1}{2}) \right|^r du + o(1).$$

Now $|a_0 - \delta| + |1 - \delta - a_m| = O(n^{-1/2})$ and $|\Delta_n(t)|$ is uniformly bounded on $[\delta, 1 - \delta]$. It follows that the integrals from δ to a_0 and from a_m to $1 - \delta$ tend to 0 as n tends to ∞ . Hence

$$(3.4) \quad \int_{\delta}^{1-\delta} |\Delta_n(t)|^r du = \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t_l(u))|^2 - 1) - h(u - \frac{1}{2}) \right|^r du + o(1).$$

Define now

$$M_l(u) = \sup_{t \in [a_{l-1}, a_l]} \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t)|^2 - 1) - h(u - \frac{1}{2}) \right|^r$$

and

$$m_l(u) = \inf_{t \in [a_{l-1}, a_l]} \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t)|^2 - 1) - h(u - \frac{1}{2}) \right|^r.$$

Let

$$I_n = \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t_l(u))|^2 - 1) - h(u - \frac{1}{2}) \right|^r du$$

and

$$J_n = \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 \int_0^1 \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t_l(v))|^2 - 1) - h(u - \frac{1}{2}) \right|^r dudv.$$

Then

$$\sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 m_l(u) du \leq I_n \wedge J_n \leq I_n \vee J_n \leq \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 M_l(u) du,$$

and consequently

$$|I_n - J_n| \leq \sum_{l=1}^m (a_l - a_{l-1}) \int_0^1 (M_l(u) - m_l(u)) du.$$

Let

$$C = \sup_{u \in [0,1]} \sup_{t \in [\delta, 1-\delta]} \left| \frac{\partial}{\partial t} \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t)|^2 - 1) - h(u - \frac{1}{2}) \right|^r \right|.$$

Clearly C is finite and $M_l(u) - m_l(u) \leq C(a_l - a_{l-1})$, whence

$$(3.5) \quad |I_n - J_n| \leq C \sum_{l=1}^m (a_l - a_{l-1})^2.$$

Now

$$\lim_{n \rightarrow \infty} \max_{l \in [1, m]} (a_l - a_{l-1}) = 0,$$

which ensures that the upper bound in (3.5) converges to 0 as n tends to ∞ . It follows that $(I_n - J_n)$ converges to 0 as n tends to ∞ . Finally

$$J_n = \int_0^1 \int_{a_0}^{a_m} \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t)|^2 - 1) - h(u - \frac{1}{2}) \right|^r dudt,$$

so that, repeating the arguments of the proof of (3.4), we have:

$$(3.6) \quad \lim_{n \rightarrow \infty} J_n = \int_0^1 \int_{\delta}^{1-\delta} \left| \frac{\alpha_3}{6} (|\Phi^{-1}(t)|^2 - 1) - h(u - \frac{1}{2}) \right|^r dudt.$$

Both (3.4), (3.6) and the convergence of $(I_n - J_n)$ to 0 then imply (3.1). Theorem 1.2(b) is proved.

■

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