MARKED METRIC MEASURE SPACES

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Abstract

A marked metric measure space (mmm-space) is a triple (X, r, μ), where (X, r) is a complete and separable metric space and μ is a probability measure on $X \times I$ for some Polish space I of possible marks. We study the space of all (equivalence classes of) marked metric measure spaces for some fixed I. It arises as a state space in the construction of Markov processes which take values in random graphs, e.g. tree-valued dynamics describing randomly evolving genealogical structures in population models.

We derive here the topological properties of the space of mmm-spaces needed to study convergence in distribution of random mmm-spaces. Extending the notion of the Gromov-weak topology introduced in (Greven, Pfaffelhuber and Winter, 2009), we define the marked Gromov-weak topology, which turns the set of mmm-spaces into a Polish space. We give a characterization of tightness for families of distributions of random mmm-spaces and identify a convergence determining algebra of functions, called polynomials.

1 Introduction

Metric spaces form a basic structure in mathematics. In probability theory, they build a natural set-up for the possible outcomes of random experiments. In particular, the Borel σ-algebra generated by the topology induced by a metric space is fundamental. Here, spaces such as $\mathbb{R}^d$ (equipped with the Euclidean metric), the space of càdlàg paths (equipped with the Skorohod metric) and the space of probability measures (equipped with the Prohorov metric) are frequently considered.

Recently, random metric spaces which differ from these examples, have attracted attention in probability theory. Most prominent examples are the description of *random genealogical structures* via Aldous' *Continuum Random Tree* (see [2] and [17] for many related results) or the *Kingman coalescent* [10], the *Brownian map* [18] and the connected components of the *Erdős-Renyi random graph* [1], which are all random compact metric spaces. The former two examples give rise to trees, which are special metric spaces, so-called $\mathbb{R}$-trees [7]. The latter two examples are based on random graphs and the underlying metric coincides with the graph metric.

In order to discuss convergence in distribution of random metric spaces, the space of metric spaces must be equipped with a topology such that it becomes a Polish space, i.e. a separable topological space, metrizable by a complete metric. Moreover, to be able to formulate tightness criteria for families of distributions on this space, it is necessary to identify criteria for relative compactness in this topology. Such topological properties of the space of compact metric spaces have been developed using the *Gromov-Hausdorff topology* (see [16, 3, 11]).

Many applications deal with a *random evolution of metric spaces*. In such processes, it is frequently necessary to pick a random point from the metric space according to some appropriate distribution, called the sampling measure. Therefore, a (probability) measure on the metric space must be specified and the resulting structure including this sampling measure gives rise to *metric measure spaces* (mm-spaces). First stochastic processes taking values in mm-spaces, subtree-prune and re-graft [12] and the tree-valued Fleming-Viot dynamics [14] have been constructed. In [13] it was shown that the *Gromov-weak topology* turns the space of mm-spaces into a Polish space; see also [16, Chapter $3\frac{1}{2}$]. Recently, random configurations and random *dynamics on metric spaces* in the form of random graphs have been studied as well (see [8]). Two examples are *percolation* [20] and *epidemic models on random graphs* [6].

The present paper was inspired by the study of a process of random configurations on evolving trees [5]. Such objects arise in mathematical population genetics in the context of *Moran models* or *multi-type branching processes*, where the random genealogy of a population evolves together with the (genetic) types of individuals. At any time the state of such a process is a *marked metric measure space* (mmm-space), where the measure is defined on the product of the metric space and some fixed mark/type space; see Section 2.1. Slightly more complicated structures arise in the study of spatial versions of such population models, where the mark specifies both the genetic type and the location of an individual [15].

Here we establish topological properties of the space of mmm-spaces needed to study *convergence in distribution* of *random mmm-spaces*. This requires an extension of the Gromov-weak topology to the marked case (Theorem 1), which is shown to be Polish (Theorem 2), a characterization of tightness of distributions in that space (Theorem 4) and a description of a convergence determining set of functions in the space of probability measures on mmm-spaces (Theorem 5).

2 Main results

First, we have to introduce some notation. For product spaces $X \times Y \times \cdots$, we denote the projection operators by $\pi_X, \pi_Y, \ldots$ For a Polish space E, we denote by $\mathscr{M}_1(E)$ the space of probability measures on the Borel σ-Algebra on E, equipped with the topology of weak convergence, which is denoted by $\Rightarrow$. Moreover, for $\varphi : E \to E'$ (for some other Polish space E'), the image measure of μ under φ is denoted $\varphi_*\mu$.

Let $\mathscr{C}_b(E)$ denote the set of bounded continuous functions on E and recall that a set of functions $\Pi \subseteq \mathscr{C}_b(E)$ is *separating in* $\mathscr{M}_1(E)$ iff for all E-valued random variables X, Y we have $X \overset{d}{=} Y$ if $\mathbf{E}[\Phi(X)] = \mathbf{E}[\Phi(Y)]$ for all $\Phi \in \Pi$. Moreover, Π is *convergence determining in* $\mathscr{M}_1(E)$ if for

any sequence $X, X_1, X_2, \ldots$ of E-valued random variables we have $X_n \overset{n \to \infty}{\Longrightarrow} X$ iff $\mathbf{E}[\Phi(X_n)] \xrightarrow{n \to \infty}$ $\mathbf{E}[\Phi(X)]$ for all $\Phi \in \Pi$.

Here and in the whole paper the key ingredients are complete separable metric spaces $(X, r_X), (Y, r_X), \ldots$ and probability measures $\mu_X, \mu_Y, \ldots$ on $X \times I, Y \times I, \ldots$ for a *fixed*

$$\text{complete and separable metric space } (I, r_I), \tag{1}$$

which we refer to as the *mark space*.

2.1 Marked metric measure spaces

Motivation: The present paper is motivated by genealogical structures in population models. Consider a population X of individuals, all living at the same time. Assume that any pair of individuals $x, y \in X$ has a common ancestor, and define a metric on X by setting $r_X(x, y)$ as the time to the most recent common ancestor of x and y, also referred to as their *genealogical distance*. In addition, individual $x \in X$ carries some *mark* $\kappa_X(x) \in I$ for some measurable function κ_X. In order to be able to sample individuals from the population, introduce a *sampling measure* $\nu_X \in \mathcal{M}_1(X)$ and define

$$\mu_X(dx, du) := \nu_X(dx) \otimes \delta_{\kappa_X(x)}(du). \tag{2}$$

Recall that most population models, such as branching processes, are exchangeable. On the level of genealogical trees, this leads to the following notion of equivalence of marked metric measure spaces: We call two triples (X, r_X, μ_X) and (Y, r_Y, μ_Y) equivalent if there is an isometry $\varphi : \operatorname{supp}(\nu_X) \to \operatorname{supp}(\nu_Y)$ such that $\nu_Y = \varphi_* \nu_X$ and $\kappa_Y(\varphi(x)) = \kappa_X(x)$ for all $x \in \operatorname{supp}(\nu_X)$, i.e. marks are preserved under φ.

It turns out that it requires strong restrictions on κ to turn the set of triples (X, r_X, μ_X) with μ_X as in (2) into a Polish space (see [19]). Since these restrictions are frequently not met in applications, we pass to the larger space of triples (X, r_X, μ_X) with general $\mu_X \in \mathcal{M}_1(X \times I)$ right away. This leads to the following key concept.

Definition 2.1 (mmm-spaces).
1. An *I-marked metric measure space*, or *mmm-space*, for short, is a triple (X, r, μ) such that (X, r) is a complete and separable metric space and $\mu \in \mathcal{M}_1(X \times I)$, where $X \times I$ is equipped with the product topology. To avoid set theoretic pathologies we assume that $X \in \mathscr{B}(\mathbb{R})$. In all applications we have in mind this is always the case.
2. Two mmm-spaces $(X, r_X, \mu_X), (Y, r_Y, \mu_Y)$ are equivalent if they are *measure- and mark-preserving isometric* meaning that there is a measurable $\varphi : \operatorname{supp}((\pi_X)_* \mu_X) \to \operatorname{supp}((\pi_Y)_* \mu_Y)$ such that

$$r_X(x, x') = r_Y(\varphi(x), \varphi(x')) \text{ for all } x, x' \in \operatorname{supp}((\pi_X)_* \mu_X) \tag{3}$$

and

$$\widetilde{\varphi}_* \mu_X = \mu_Y \text{ for } \widetilde{\varphi}(x, u) = (\varphi(x), u). \tag{4}$$

We denote the equivalence class of (X, r, μ) by $\overline{(X, r, \mu)}$.
3. We introduce

$$\mathbb{M}^I := \left\{ \overline{(X, r, \mu)} : (X, r, \mu) \text{ mmm-space} \right\} \tag{5}$$

and denote the generic elements of $\mathbb{M}^I$ by $\chi, y, \ldots$.

Remark 2.2 (Connection to mm-spaces). In [13], the space of metric measure spaces (mm-spaces) was considered. These are triples (X, r, μ) where $\mu \in \mathcal{M}_1(X)$. Two mm-spaces (X, r_X, μ_X) and (Y, r_Y, μ_Y) are equivalent if φ exists such that (3) holds. The set of equivalence classes of such mm-spaces is denoted by $\mathbb{M}$, which is closely connected to the structure we have introduced in Definition 2.1. Namely for $\chi = \overline{(X, r, \mu)} \in \mathbb{M}^I$, we set

$$\pi_1(\chi) := \overline{(X, r, (\pi_X)_*\mu)} \in \mathbb{M}, \qquad \pi_2(\chi) := (\pi_I)_*\mu \in \mathcal{M}_1(I). \tag{6}$$

Note that $\pi_2(\chi)$ is the distribution of marks in I and $\mathbb{M}$ can be identified with $\mathbb{M}^I$ if $\#I = 1$.

Outline: In Section 2.2, we state that the marked distance matrix distribution, arising by subsequently sampling points from an mmm-space, uniquely characterizes the mmm-space (Theorem 1). Hence, we can define the marked Gromov-weak topology based on weak convergence of marked distance matrix distributions, which turns $\mathbb{M}^I$ into a Polish space (Theorem 2). Moreover, we characterize relatively compact sets in the Gromov-weak topology (Theorem 3). In Subsection 2.3 we treat our main subject, *random* mmm-spaces. We characterize tightness (Theorem 4) and show that polynomials, specifying an algebra of real-valued functions on $\mathbb{M}^I$, are convergence determining (Theorem 5).
The proofs of Theorems 1 – 5, are given in Sections 3.1, 3.3, 4.1 and 4.3, respectively.

2.2 The Gromov-weak topology

Our task is to define a topology that turns $\mathbb{M}^I$ into a Polish space. For this purpose, we introduce the notion of the *marked distance matrix distribution*.

Definition 2.3 (Marked distance matrix distribution).
Let (X, r, μ) be an mmm-space, $\chi := \overline{(X, r, \mu)} \in \mathbb{M}^I$ and

$$R^{(X,r)} : \begin{cases} (X \times I)^{\mathbb{N}} & \to \mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}, \\ ((x_k, u_k)_{k \geq 1}) & \mapsto \left((r(x_k, x_l))_{1 \leq k < l}, (u_k)_{k \geq 1} \right). \end{cases} \tag{7}$$

The *marked distance matrix distribution of* $\chi = \overline{(X, r, \mu)}$ is defined by

$$\nu^\chi := (R^{(X,r)})_*\mu^{\mathbb{N}} \in \mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}). \tag{8}$$

For generic elements in $\mathbb{R}^{\binom{\mathbb{N}}{2}}$ and $I^{\mathbb{N}}$, we write $\underline{r} = (r_{ij})_{1 \leq i < j}$ and $\underline{u} = (u_i)_{i \geq 1}$, respectively.

In the above definition $(R^{(X,r)})_*\mu^{\mathbb{N}}$ does not depend on the particular element (X, r, μ) of the equivalence class $\chi = \overline{(X, r, \mu)}$, i.e. ν^χ is well-defined. The key property of $\mathbb{M}^I$ is that the distance matrix distribution uniquely determines mmm-spaces as the next result shows.

Theorem 1. *Let* $\chi, y \in \mathbb{M}^I$. *Then,* $\chi = y$ *iff* $\nu^\chi = \nu^y$.

This characterization of elements in $\mathbb{M}^I$ allows us to introduce a topology as follows.

Definition 2.4 (Marked Gromov-weak topology)**.**
Let $\chi, \chi_1, \chi_2, \dots \in \mathbb{M}^I$. We say that $\chi_n \xrightarrow{n \to \infty} \chi$ in the *marked Gromov-weak topology (MGW topology)* iff

$$\nu^{\chi_n} \xRightarrow{n \to \infty} \nu^{\chi} \tag{9}$$

in the weak topology on $\mathcal{M}_1\big(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}\big)$, where, as usual, $\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}$ is equipped with the product topology of $\mathbb{R}_+$ and I, respectively. $\qquad\qquad\square$

The next result implies that $\mathbb{M}^I$ is a suitable space to apply standard techniques of probability theory (most importantly, weak convergence and martingale problems).

Theorem 2. *The space $\mathbb{M}^I$, equipped with the MGW topology, is Polish.*

In order to study weak convergence in $\mathbb{M}^I$, knowledge about relatively compact sets is crucial.

Theorem 3 (Relative compactness in the MGW topology)**.**
For $\Gamma \subseteq \mathbb{M}^I$ the following assertions are equivalent:
(i) The set Γ is relatively compact with respect to the marked Gromov-weak topology.
(ii) Both, $\pi_1(\Gamma)$ is relatively compact with respect to the Gromov-weak topology on $\mathbb{M}$ and $\pi_2(\Gamma)$ is relatively compact with respect to the weak topology on $\mathcal{M}_1(I)$.

Remark 2.5 (Relative compactness in $\mathbb{M}$)**.** For the application of Theorem 3, it is necessary to characterize relatively compact sets in $\mathbb{M}$, equipped with the Gromov-weak topology. Proposition 7.1 of [13] gives such a characterization which we recall: Let $r_{12} : (\underline{r}, \underline{u}) \mapsto r_{12}$. Then the set $\pi_1(\Gamma)$ is relatively compact in $\mathbb{M}$, iff

$$\{(r_{12})_* \nu^{\chi} : \chi \in \Gamma\} \subseteq \mathcal{M}_1(\mathbb{R}_+) \text{ is tight} \tag{10}$$

and

$$\sup_{\chi = \overline{(X, r, \mu)} \in \Gamma} \mu((x, u) \in X \times I : \mu(B_\varepsilon(x) \times I) \le \delta) \xrightarrow{\delta \to 0} 0 \tag{11}$$

for all $\varepsilon > 0$, where $B_\varepsilon(x)$ is the open ε-ball around $x \in X$.

2.3 Random mmm-spaces

When showing convergence in distribution of a sequence of random mmm-spaces, it must be established that the sequence of distributions is tight and all potential limit points are the same and hence we need (i) tightness criteria (see Theorem 4) and (ii) a separating (or even convergence-determining) algebra of functions in $\mathcal{M}_1(\mathbb{M}^I)$ (see Theorem 5).

Theorem 4 (Tightness of distributions on $\mathbb{M}^I$)**.**
For an arbitrary index set J let $\{\mathscr{X}_j : j \in J\}$ be a family of $\mathbb{M}^I$-valued random variables. The set of distributions of $\{\mathscr{X}_j : j \in J\}$ is tight iff

  (i) the set of distributions of $\{\pi_1(\mathscr{X}_j) : j \in J\}$ is tight as a subset of $\mathcal{M}_1(\mathbb{M})$,

  (ii) the set of distributions of $\{\pi_2(\mathscr{X}_j) : j \in J\}$ is tight as a subset of $\mathcal{M}_1(\mathcal{M}_1(I))$.

In order to define a separating algebra of functions in $\mathscr{M}_1(\mathbb{M}^I)$, we denote by

$$\overline{\mathscr{C}}_n^{(k)} := \overline{\mathscr{C}}_n^{(k)}\big(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}\big) \tag{12}$$

the set of bounded, real-valued functions ϕ on $\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}$, which are continuous and k times continuously differentiable with respect to the coordinates in $\mathbb{R}_+^{\binom{\mathbb{N}}{2}}$ and such that $(\underline{r}, \underline{u}) \mapsto \phi(\underline{r}, \underline{u})$ depends on the first $\binom{n}{2}$ variables in $\underline{r}$ and the first n in $\underline{u}$. (The space $\overline{\mathscr{C}}_0$ consists of constant functions.) For $k = 0$, we set $\overline{\mathscr{C}}_n := \overline{\mathscr{C}}_n^{(0)}$.

Definition 2.6 (Polynomials).
1. A function $\Phi : \mathbb{M}^I \to \mathbb{R}$ is a *polynomial*, if, for some $n \in \mathbb{N}_0$, there exists $\phi \in \overline{\mathscr{C}}_n$, such that

$$\Phi(\chi) = \langle v^\chi, \phi \rangle := \int \phi(\underline{r}, \underline{u})v^\chi(d\underline{r}, d\underline{u}) \tag{13}$$

for all $\chi \in \mathbb{M}^I$. We then write $\Phi = \Phi^{n,\phi}$.
2. For a polynomial Φ the smallest number n such that there exists $\phi \in \overline{\mathscr{C}}_n$ satisfying (13) is called the *degree* of Φ.
3. We set for $k = 0, 1, \ldots, \infty$

$$\Pi^k := \bigcup_{n=0}^{\infty} \Pi_n^k, \qquad \Pi_n^k := \{\Phi^{n,\phi} : \phi \in \overline{\mathscr{C}}_n^{(k)}\}. \tag{14}$$

The following result shows that polynomials are not only separating, but even convergence determining in $\mathscr{M}_1(\mathbb{M}^I)$.

Theorem 5 (Polynomials are convergence determining in $\mathscr{M}_1(\mathbb{M}^I)$).
1. For every $k = 0, 1, \ldots, \infty$, the algebra Π^k is separating in $\mathscr{M}_1(\mathbb{M}^I)$.
2. There exists a countable algebra $\Pi_*^\infty \subseteq \Pi^\infty$ that is convergence determining in $\mathscr{M}_1(\mathbb{M}^I)$.

Remark 2.7 (Application to random mmm-spaces).
1. In order to show convergence in distribution of random mmm-spaces $\mathscr{X}_1, \mathscr{X}_2, \ldots$, there are two strategies. (i) If a limit point $\mathscr{X}$ is already specified, the property $\mathbf{E}[\Phi(\mathscr{X}_n)] \xrightarrow{n \to \infty} \mathbf{E}[\Phi(\mathscr{X})]$ for all $\Phi \in \Pi^k$ suffices for convergence $\mathscr{X}_n \xRightarrow{n \to \infty} \mathscr{X}$ by Theorem 5. (ii) If no limit point is identified yet, tightness of the sequence implies existence of limit points. Then, convergence of $\mathbf{E}[\Phi(\mathscr{X}_n)]$ as a sequence in $\mathbb{R}$ for all $\Phi \in \Pi^k$ shows uniqueness of the limiting object. Both situations arise in practice; see the proof of Theorem 1(c) in [5] for an application of the former and the proof of Theorem 4 in [5] for the latter.
2. Theorem 5 extends Corollary 3.1 of [13] in the case of unmarked metric measure spaces. As the theorem shows, convergence of polynomials is enough for convergence in the Gromov-weak topology if the limit object is known. We will show in the proof that convergence of polynomials is enough to ensure tightness of the sequence.

3 Properties of the marked Gromov-weak topology

After proving Theorem 1 in Section 3.1, we introduce the *Gromov-Prohorov metric* on $\mathbb{M}^I$ a concept of interest also by itself in Section 3.2. We will show in the proofs of Theorems 2 and 3 in Section 3.3 that this metric is complete and metrizes the MGW topology.

3.1   Proof of Theorem 1

We adapt the proof of Gromov's reconstruction theorem for metric measure spaces, given by A. Vershik – see Chapter $3\frac{1}{2}$.5 and $3\frac{1}{2}$.7 in [16] – to the marked case.

Let $x = \overline{(X, r_X, \mu_X)}, y = \overline{(Y, r_Y, \mu_Y)} \in \mathbb{M}^I$. It is clear that $\nu^x = \nu^y$ if $x = y$. Thus, it remains to show that the converse is also true, i.e. we need to show that $\nu^x = \nu^y$ implies that x and y are measure-preserving isometric (see Definition 2.1).

If $\nu^x = \nu^y$, then there exists $\nu \in \mathcal{M}_1\big((\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}) \times (\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})\big)$ putting mass 1 on the diagonal and having ν^x and ν^y as projections on the first resp. second coordinate. We define a probability measure $\mu \in \mathcal{M}_1\big((X \times I)^{\mathbb{N}} \times (Y \times I)^{\mathbb{N}}\big)$ by

$$\mu(A \times B) := \nu\big(R^{(X,r_X)}(A) \times R^{(Y,r_Y)}(B)\big), \quad A \in \mathcal{B}\big((X \times I)^{\mathbb{N}}\big), \, B \in \mathcal{B}\big((Y \times I)^{\mathbb{N}}\big).$$

Here $\mathcal{B}$ denotes the Borel-σ algebra. Then we have (recall (7)) that $(R^{(X,r_X)} \circ \pi_{(X \times I)^{\mathbb{N}}})_* \mu = \nu^x$, $(R^{(Y,r_Y)} \circ \pi_{(Y \times I)^{\mathbb{N}}})_* \mu = \nu^y$, and

$$R^{(X,r_X)} \circ \pi_{(X \times I)^{\mathbb{N}}}((\underline{x}, \underline{u}), (\underline{y}, \underline{v})) = R^{(Y,r_Y)} \circ \pi_{(Y \times I)^{\mathbb{N}}}((\underline{x}, \underline{u}), (\underline{y}, \underline{v})), \tag{15}$$

for μ-almost all $((\underline{x}, \underline{u}), (\underline{y}, \underline{v})) = (((x_1, u_1), (x_2, u_2), \dots), ((y_1, v_1), (y_2, v_2), \dots))$. Then in particular, by the Glivenko-Cantelli theorem, for μ-almost all $((\underline{x}, \underline{u}), (\underline{y}, \underline{v}))$,

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{(x_k, u_k)} \overset{n \to \infty}{\Longrightarrow} \mu_X \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} \delta_{(y_k, v_k)} \overset{n \to \infty}{\Longrightarrow} \mu_Y. \tag{16}$$

Now, take any $((\underline{x}, \underline{u}), (\underline{y}, \underline{v}))$ such that (15) and (16) hold as well as $(x_n, u_n) \in \text{supp}(\mu_X), (y_n, v_n) \in \text{supp}(\mu_Y), n \in \mathbb{N}$. By (15) we find that $\underline{u} = \underline{v}$. Define $\varphi : \text{supp}((\pi_X)_* \mu_X) \to \text{supp}((\pi_Y)_* \mu_Y)$ as the only continuous map satisfying $\varphi(x_n) = y_n, n \in \mathbb{N}$ and recall the definition of $\widetilde{\varphi}$ in (4). By (15), we obtain that $r_X(x_m, x_n) = r_Y(y_m, y_n) = r_Y(\varphi(x_m), \varphi(x_n)), m, n \in \mathbb{N}$, which extends to $\text{supp}((\pi_X)_* \mu_X)$ by continuity. In addition, by (16) and continuity, $\widetilde{\varphi}_* \mu_X = \mu_Y$ and so (X, r_X, μ_X) and (Y, r_Y, μ_Y) are measure-preserving isometric, i.e. $x = y$.

3.2   The Gromov-Prohorov metric

In this section, we define the marked Gromov-Prohorov metric on $\mathbb{M}^I$, which generates a topology which is at least as strong as the marked Gromov-weak topology, see Lemma 3.5. However, since we establish in Proposition 3.6 that both topologies have the same compact sets, we see in Proposition 3.7 that the topologies are the same, and hence, the marked Gromov-Prohorov metric metrizes the marked Gromov-weak topology. We use the same notation for φ and $\widetilde{\varphi}$ as in Definition 2.1. Recall that the topology of weak convergence of probability measures on a separable space is metrized by the Prohorov metric (see [9, Theorem 3.3.1]).

Definition 3.1 (The marked Gromov-Prohorov topology).
For $x_i = \overline{(X_i, r_i, \mu_i)} \in \mathbb{M}^I$, $i = 1, 2$, set

$$d_{\text{MGP}}(x_1, x_2) := \inf_{(Z, \varphi_1, \varphi_2)} d_{\text{Pr}}((\widetilde{\varphi}_1)_* \mu_1, (\widetilde{\varphi}_2)_* \mu_2), \tag{17}$$

where the infimum is taken over all complete and separable metric spaces (Z, r_Z), isometric embeddings $\varphi_1 : X_1 \to Z$, $\varphi_2 : X_2 \to Z$ and d_{Pr} denotes the Prohorov metric on $\mathcal{M}_1(Z \times I)$, based

on the metric $\widetilde{r}_Z = r_Z + r_I$ on $Z \times I$, metrizing the product topology. Here, d_{MGP} denotes the *marked Gromov-Prohorov metric (MGP metric)*. The topology induced by d_{MGP} is called the *marked Gromov-Prohorov topology (MGP topology)*.

Remark 3.2 (Equivalent definition of the MGP metric)**.**
For $\chi_i = \overline{(X_i, r_i, \mu_i)} \in \mathbb{M}^I$, $i = 1, 2$, denote by $X_1 \sqcup X_2$ the disjoint union of X_1 and X_2. Then,

$$d_{\mathrm{MGP}}(\chi_1, \chi_2) := \inf_{r_{X_1 \sqcup X_2}} d_{\mathrm{Pr}}((\widetilde{\varphi}_1)_* \mu_1, (\widetilde{\varphi}_2)_* \mu_2), \tag{18}$$

where the infimum is over all metrics $r_{X_1 \sqcup X_2}$ on $X_1 \sqcup X_2$ extending the metrics on X_1 and X_2 and $\varphi_i : X_i \to X_1 \sqcup X_2$, $i = 1, 2$ denote the canonical embeddings.

Remark 3.3 (d_{MGP} is a metric)**.** The fact that d_{MGP} indeed defines a metric follows from an easy extension of Lemma 5.4 in [13]. Symmetry and non-negativity are clear from the definition, and positive definiteness is a consequence of Theorem 1. Furthermore the triangle inequality holds by the following argument: For three mmm-spaces $\chi_i = \overline{(X_i, r_i, \mu_i)} \in \mathbb{M}^I$, $i = 1, 2, 3$ and any $\varepsilon > 0$, by the same construction as in Remark 3.2, we can choose a metric $r_{X_1 \sqcup X_2 \sqcup X_3}$ on $X_1 \sqcup X_2 \sqcup X_3$, extending the metrics $r_{X_1}, r_{X_2}, r_{X_3}$, such that

$$\begin{aligned} d_{\mathrm{Pr}}((\widetilde{\varphi}_1)_* \mu_1, (\widetilde{\varphi}_2)_* \mu_2) - d_{\mathrm{MGP}}(\chi_1, \chi_2) &< \varepsilon, \\ d_{\mathrm{Pr}}((\widetilde{\varphi}_2)_* \mu_2, (\widetilde{\varphi}_3)_* \mu_3) - d_{\mathrm{MGP}}(\chi_2, \chi_3) &< \varepsilon. \end{aligned} \tag{19}$$

Then, we can use the triangle inequality for the Prohorov metric on $\mathcal{M}_1((X_1 \sqcup X_2 \sqcup X_3) \times I)$ and let $\varepsilon \to 0$ to obtain the triangle inequality for d_{MGP}.

Lemma 3.4 (Equivalent description of the MGP topology)**.**
Let $\chi = \overline{(X, r_X, \mu_X)}, \chi_1 = \overline{(X_1, r_1, \mu_1)}, \chi_2 = \overline{(X_2, r_2, \mu_2)}, \ldots \in \mathbb{M}^I$. *Then,* $d_{\mathrm{MGP}}(\chi_n, \chi) \xrightarrow{n \to \infty} 0$ *if and only if there is a complete and separable metric space* (Z, r_Z) *and isometric embeddings* $\varphi_X : X \to Z, \varphi_1 : X_1 \to Z, \varphi_2 : X_2 \to Z, \ldots$ *with*

$$d_{\mathrm{Pr}}((\widetilde{\varphi}_n)_* \mu_n, (\widetilde{\varphi}_X)_* \mu_X) \xrightarrow{n \to \infty} 0. \tag{20}$$

Proof. The assertion is an extension of Lemma 5.8 in [13] to the marked case. The proof of the present lemma follows the same lines, which we sketch briefly.
First, the "if"-direction is clear. For the "only if" direction, fix a sequence $\varepsilon_1, \varepsilon_2, \cdots > 0$ with $\varepsilon_n \to 0$ as $n \to \infty$. By the same construction as in Remark 3.3, we can construct a metric r_Z on Z, defined as the completion of $X \sqcup X_1 \sqcup X_2 \sqcup \cdots$, with the property that

$$d_{\mathrm{Pr}}((\widetilde{\varphi}_n)_* \mu_n, (\widetilde{\varphi}_X)_* \mu_X) - d_{\mathrm{MGP}}(\chi_n, \chi) < \varepsilon_n, \tag{21}$$

where $\varphi_X : X \to Z$ and $\varphi_n : X_n \to Z, n \in \mathbb{N}$ are canonical embeddings. The assertion follows. $\qquad \square$

Lemma 3.5 (MGP convergence implies MGW convergence)**.**
Let $\chi, \chi_1, \chi_2, \cdots \in \mathbb{M}^I$ *be such that* $d_{\mathrm{MGP}}(\chi_n, \chi) \xrightarrow{n \to \infty} 0$. *Then,* $\chi_n \xrightarrow{n \to \infty} \chi$ *in the MGW topology.*

Proof. Let $\chi = \overline{(X, r, \mu)}, \chi_1 = \overline{(X_1, r_1, \mu_1)}, \chi_2 = \overline{(X_2, r_2, \mu_2)}, \ldots$. Take (Z, r_Z) and isometric embeddings $\varphi_X, \varphi_1, \varphi_2, \ldots$ such that (20) from Lemma 3.4 holds.
It is a consequence of Proposition 3.4.5 in [9] that $\bigcup_n \overline{\mathscr{C}_n}$ is convergence determining in $\mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$; see also the proof of Proposition 4.1. Let $\Phi \in \Pi^0$ be such that $\Phi(.) = \langle v^\cdot, \phi \rangle$ for

some $\phi \in \bigcup_{n=0}^{\infty} \overline{\mathscr{C}}_n$. Since $(\widetilde{\varphi}_n)_* \mu_n \overset{n \to \infty}{\Longrightarrow} (\widetilde{\varphi}_X)_* \mu_X$ by (20), we also have that $\left((\widetilde{\varphi}_n)_* \mu_n\right)^{\otimes \mathbb{N}} \overset{n \to \infty}{\Longrightarrow}$ $\left((\widetilde{\varphi}_X)_* \mu_X\right)^{\otimes \mathbb{N}}$ in $\mathscr{M}_1((Z \times I)^{\mathbb{N}})$. Hence we can conclude that

$$
\int \phi \left((r_Z(z_k, z_l))_{1 \le k < l}, \underline{u}\right) \left((\widetilde{\varphi}_n)_* \mu_n\right)^{\otimes \mathbb{N}} (d\underline{z}, d\underline{u})
$$
$$
\overset{n \to \infty}{\longrightarrow} \int \phi \left((r_Z(z_k, z_l))_{1 \le k < l}, \underline{u}\right) \left((\widetilde{\varphi}_X)_* \mu_X\right)^{\otimes \mathbb{N}} (d\underline{z}, d\underline{u}). \tag{22}
$$

Since $\chi = \overline{(Z, r_Z, (\widetilde{\varphi}_X)_* \mu_X)}$ and $\chi_n = \overline{(Z, r_Z, (\widetilde{\varphi}_n)_* \mu_n)}, n = 1, 2, \ldots$, this proves that $\langle \nu^{\chi_n}, \phi \rangle \overset{n \to \infty}{\longrightarrow}$ $\langle \nu^{\chi}, \phi \rangle$. Because $\Phi \in \Pi^0$ was arbitrary, we have that $\nu^{\chi_n} \overset{n \to \infty}{\Longrightarrow} \nu^{\chi}$. Then, by definition, $\chi_n \overset{n \to \infty}{\longrightarrow} \chi$ in the MGW topology. $\qquad \square$

Proposition 3.6 (Relative compactness in $\mathbb{M}^I$)**.**
Let $\Gamma \subseteq \mathbb{M}^I$. Then conditions (i) and (ii) of Theorem 3 are equivalent to
(iii) The set Γ is relatively compact with respect to the marked Gromov-Prohorov topology.

Proof. First, (iii)$\Rightarrow$(i) follows from Lemma 3.5. Thus, it remains to show (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

(i)$\Rightarrow$(ii): Note that Π^0 contains functions $\Phi(.) = \langle \nu^{\cdot}, \phi \rangle$ such that ϕ does not depend on the variables $\underline{u} \in I^{\mathbb{N}}$, as well as functions ϕ which only depend on $u_1 \in I$. Denote the former set of functions by Π_{dist} and the latter by Π_{mark}.
Assume that the sequence $\chi_1, \chi_2, \cdots \in \Gamma$ converges to $\chi \in \mathbb{M}^I$ with respect to the MGW topology. Since $\Phi(\chi_n) \overset{n \to \infty}{\longrightarrow} \Phi(\chi)$ for all $\Phi \in \Pi_{\text{dist}}$, we find that $\pi_1(\chi_n) \overset{n \to \infty}{\longrightarrow} \pi_1(\chi)$ in the Gromov-weak topology. In addition, $\Phi(\chi_n) \overset{n \to \infty}{\longrightarrow} \Phi(\chi)$ for all $\Phi \in \Pi_{\text{mark}}$ implies $\pi_2(\chi_n) \overset{n \to \infty}{\Longrightarrow} \pi_2(\chi)$. In particular, (ii) holds.

(ii)$\Rightarrow$(iii): Recall from Theorem 5 of [13] that the (unmarked) Gromov-weak and the (unmarked) Gromov-Prohorov topology coincide. For a sequence in Γ, take a subsequence $\chi_1 = \overline{(X_1, r_1, \mu_1)}, \chi_2 = \overline{(X_2, r_2, \mu_2)}, \cdots \in \Gamma$ and $\chi = \overline{(X, r_X, \mu_X)} \in \mathbb{M}^I$ such that $\pi_1(\chi_n) \overset{n \to \infty}{\longrightarrow} \pi_1(\chi) \in \mathbb{M}$ in the Gromov-Prohorov topology and

$$
d_{\text{Pr}}(\pi_2(\chi_n), \pi_2(\chi)) \overset{n \to \infty}{\longrightarrow} 0. \tag{23}
$$

Using Lemma 5.7 of [13], take a complete and separable metric space (Z, r_Z), isometric embeddings $\varphi_X : X \to Z, \varphi_1 : X_1 \to Z, \varphi_2 : X_2 \to Z, \ldots$ such that

$$
d_{\text{Pr}}((\pi_{X_n} \circ \widetilde{\varphi}_n)_* \mu_n, (\pi_X \circ \widetilde{\varphi}_X)_* \mu_X)
$$
$$
= d_{\text{Pr}}((\pi_{X_n})_*((\widetilde{\varphi}_n)_* \mu_n), (\pi_X)_*((\widetilde{\varphi}_X)_* \mu_X)) \overset{n \to \infty}{\longrightarrow} 0. \tag{24}
$$

In particular, (23) shows that $\{\pi_2(\chi_n) = (\pi_I)_*(\widetilde{\varphi}_n)_* \mu_n : n \in \mathbb{N}\}$ is relatively compact in $\mathscr{M}_1(I)$ and (24) shows that $\{(\pi_{X_n})_*((\widetilde{\varphi}_n)_* \mu_n) : n \in \mathbb{N}\}$ is relatively compact in $\mathscr{M}_1(Z)$. This implies that $\{(\widetilde{\varphi}_n)_* \mu_n : n \in \mathbb{N}\}$ is relatively compact in $\mathscr{M}_1(Z \times I)$. Hence, we can find a convergent subsequence, and (iii) follows by Lemma 3.4. $\qquad \square$

Proposition 3.7 (MGW and MGP topologies coincide)**.**
The marked Gromov-Prohorov metric generates the marked Gromov-weak topology, i.e. the marked Gromov-weak topology and the marked Gromov-Prohorov topology coincide.

Proof. Let $\chi, \chi_1, \chi_2, \dots \in \mathbb{M}^I$. We have to show that $\chi_n \xrightarrow{n\to\infty} \chi$ in the MGW topology if and only if $\chi_n \xrightarrow{n\to\infty} \chi$ in the MGP topology. The 'if'-part was shown in Lemma 3.5. For the 'only if'-direction, assume that $\chi_n \xrightarrow{n\to\infty} \chi$ in the MGW topology. It suffices to show that for all subsequences of $\chi_1, \chi_2, \dots$, there is a further subsequence $\chi_{n_1}, \chi_{n_2}, \dots$ such that

$$d_{\mathrm{MGP}}(\chi_{n_k}, \chi) \xrightarrow{k\to\infty} 0. \tag{25}$$

By Proposition 3.6 $\{\chi_n : n \in \mathbb{N}\}$ is relatively compact in the MGP topology. Therefore, for a subsequence, there exists $y \in \mathbb{M}^I$ and a further subsequence $\chi_{n_1}, \chi_{n_2}, \dots$ with $\chi_{n_k} \xrightarrow{k\to\infty} y$ in the MGP topology. By the 'if'-direction it follows that $\chi_{n_k} \xrightarrow{k\to\infty} y$ in the MGW topology, which shows that $y = \chi$ and therefore (25) holds. $\square$

3.3   Proofs of Theorems 2 and 3

Clearly, Theorem 3 was already shown in Proposition 3.6.

For Theorem 2, some of our arguments are similar to proofs in [13], where the case without marks is treated, which are also based on a similar metric. We have shown in Proposition 3.7 that the marked Gromov-Prohorov metric metrizes the marked Gromov-weak topology. Hence, we need to show that the marked Gromov-weak topology is separable, and d_{MGP} is complete.

We start with separability. Note that the marked Gromov-Prohorov topology coincides with the topology of weak convergence on $\{v^\chi : \chi \in \mathbb{M}^I\} \subseteq \mathcal{M}_1\big(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}\big)$. Hence, separability follows from separability of the topology of weak convergence on $\mathcal{M}_1\big(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}\big)$.

For completeness, consider a Cauchy sequence $\chi_1, \chi_2, \dots \in \mathbb{M}^I$. It suffices to show that there is a convergent subsequence. Note that $\pi_1(\chi_n)$ is Cauchy in $\mathbb{M}$ and $\pi_2(\chi_n)$ is Cauchy in $\mathcal{M}_1(I)$. In particular, $\{\pi_i(\chi_n) : n \in \mathbb{N}\}, i = 1, 2$ are relatively compact. By Proposition 3.6, this implies that $\{\chi_n : n \in \mathbb{N}\}$ is relatively compact in $\mathbb{M}^I$ and thus, there exists a convergent subsequence.

4   Properties of random mmm-spaces

In this section we prove the probabilistic statements which we asserted in Subsection 2.3. In particular, we prove Theorems 4 in Section 4.1 and Theorem 5 in Section 4.3. In Section 4.2 we give properties of polynomials a class of functions not only crucial for the topology of $\mathbb{M}^I$ but also to formulate martingale problems (see [5, 14]).

4.1   Proof of Theorem 4

The proof is an easy consequence of Theorem 3: By Prohorov's Theorem, the family of distributions of $\{\mathcal{X}_j : j \in J\}$ is tight iff for all $\varepsilon > 0$ there is $\Gamma_\varepsilon \subseteq \mathbb{M}^I$ relatively compact with $\inf_{j\in J} \mathbf{P}(\mathcal{X}_j \in \Gamma_\varepsilon) > 1 - \varepsilon$. By Theorem 3 the latter is the case iff for all $\varepsilon > 0$ there are relatively compact $\Gamma_\varepsilon^1 \subseteq \mathbb{M}$ and $\Gamma_\varepsilon^2 \subseteq \mathcal{M}_1(I)$ such that

$$\inf_{j\in J}\mathbf{P}(\pi_1(\mathcal{X}_j) \in \Gamma_\varepsilon^1) > 1 - \varepsilon, \qquad \inf_{j\in J}\mathbf{P}(\pi_2(\mathcal{X}_j) \in \Gamma_\varepsilon^2) > 1 - \varepsilon. \tag{26}$$

This is the same as (i) and (ii).

4.2   Polynomials

We prepare the proof of Theorem 5 with some results on polynomials. We show that polynomials separate points (Proposition 4.1) and are convergence determining in $\mathbb{M}^I$ (Proposition 4.2).

Proposition 4.1 (Polynomials form an algebra that separates points)**.**
1. For $k = 0, 1, \ldots, \infty$, the set of polynomials Π^k is an algebra. In particular, if $\Phi = \Phi^{n,\phi} \in \Pi^k_n, \Psi = \Psi^{m,\psi} \in \Pi^k_m$, then

$$(\Phi \cdot \Psi)(\chi) = \langle \nu^\chi, \phi \cdot (\psi \circ \rho^n_1) \rangle \tag{27}$$

with ρ^n_1 being the "shift"

$$\rho^n_1(\underline{r}, \underline{u}) = \big((r_{i+n,j+n})_{1 \leq i < j}, (u_{i+n})_{i \geq 1}\big). \tag{28}$$

2. For all $k = 1, 2, \ldots, \infty, \Pi^k$ separates points in $\mathbb{M}^I$, i.e. for $\chi, y \in \mathbb{M}^I$ we have $\chi = y$ iff $\Phi(\chi) = \Phi(y)$ for all $\Phi \in \Pi^k$.

Proof. 1. First, we note that the marked distance matrix distributions are exchangeable in the following sense: Let $\sigma : \mathbb{N} \to \mathbb{N}$ be injective. Set

$$R_\sigma : \begin{cases} \mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}} & \to \mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}} \\ \big((r_{ij})_{1 \leq i < j}, (u_k)_{k \geq 1}\big) & \mapsto \big((r_{\sigma(i) \wedge \sigma(j), \sigma(i) \vee \sigma(j)}), (u_{\sigma(k)})_{k \geq 1}\big). \end{cases} \tag{29}$$

Then, for $\chi \in \mathbb{M}^I$, we find that

$$(R_\sigma)_* \nu^\chi = \nu^\chi. \tag{30}$$

Next, we show that Π^k is an algebra. Clearly, Π^k is a linear space and $1 \in \Pi^k$. Next consider multiplication of polynomials. By (30), we find that $(\rho^n_1)_* \nu^\chi = \nu^\chi$. If $\Phi^{n,\phi} \in \Pi^k_n$, this implies

$$(\Phi \cdot \Psi)(\chi) = \left(\int \phi(\underline{r}, \underline{u}) \nu^\chi(d\underline{r}, d\underline{u}) \right) \cdot \left(\int \psi(\rho^n_1(\underline{r}, \underline{u})) \nu^\chi(d\underline{r}, d\underline{u}) \right)$$

$$= \int \phi(\underline{r}, \underline{u}) \psi(\rho^n_1(\underline{r}, \underline{u})) \nu^\chi(d\underline{r}, d\underline{u}) = \langle \nu^\chi, \phi \cdot (\psi \circ \rho^n_1) \rangle, \tag{31}$$

which shows that Π^k is closed under multiplication as well.
2. We turn to showing that Π^k separates points. Recall that for $\chi \in \mathbb{M}^I$, the distance matrix distribution ν^χ is an element of $\mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$. On such product spaces, the set of functions

$$\left\{ \phi(\underline{r}, \underline{u}) = \prod_{i=1}^n g_i(u_i) \prod_{l=i+1}^n f_{il}(r_{il}) : f_{il} \in \overline{\mathscr{C}}^k(\mathbb{R}_+), g_i \in \overline{\mathscr{C}}^k(I), n \in \mathbb{N} \right\} \subseteq \Pi^k \tag{32}$$

is separating in $\mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$ by Proposition 3.4.5 of [9]. If $\chi \neq y$, we have $\nu^\chi \neq \nu^y$ by Theorem 1 and hence, there exists $\phi \in \Pi^k$ with $\langle \phi, \nu^\chi \rangle \neq \langle \phi, \nu^y \rangle$ and hence Π^k separates points.
□

Proposition 4.2 (A convergence determining subset of Π^{∞})**.**
There exists a countable algebra $\Pi_*^{\infty} \subseteq \Pi^{\infty}$ that is convergence determining in $\mathbb{M}^I$, i.e. for $\chi, \chi_1, \chi_2, \dots \in \mathbb{M}^I$, we have $\chi_n \xrightarrow{n \to \infty} \chi$ iff $\Phi(\chi_n) \xrightarrow{n \to \infty} \Phi(\chi)$ for all $\Phi \in \Pi_*^{\infty}$.

Proof. The necessity is clear. For the sufficiency argue as follows. Focus on the one-dimensional marginals of marked distance matrix distributions, which are elements of $\mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$ first. On the one hand by Lemma 3.2.1 of [4], there exists a countable, linear set $V_{\mathbb{R}_+}$ of continuous, bounded functions which is convergence determining in $\mathcal{M}_1(\mathbb{R}_+)$, i.e. for $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}_1(\mathbb{R}_+)$ we have $\mu_n \xRightarrow{n \to \infty} \mu$ iff $\langle \mu_n, f \rangle \xrightarrow{n \to \infty} \langle \mu, f \rangle$ for all $f \in V_{\mathbb{R}_+}$. By an approximation argument, we can choose $V_{\mathbb{R}_+}$ even such that it only consists of infinitely often continuously differentiable functions. On the other hand there exists a countable, linear set V_I of continuous, bounded functions which is convergence determining in I. Without loss of generality, $V_{\mathbb{R}_+}$ and V_I are algebras. Since a marked distance matrix distribution ν^{χ} for $\chi \in \mathbb{M}^I$ is a probability measure on a countable product, Proposition 3.4.6 in [9] implies that the algebra

$$V := \left\{ \prod_{k=1}^{n} g_k(u_k) \prod_{l=k+1}^{n} f_{kl}(r_{kl}) : n \in \mathbb{N}, g_k \in V_I, f_{kl} \in V_{\mathbb{R}_+} \right\} \tag{33}$$

is convergence determining in $\mathcal{M}_1\big(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}\big)$. In particular,

$$\Pi_*^{\infty} := \{ \chi \mapsto \langle \nu^{\chi}, \phi \rangle : \phi \in V \} \subseteq \Pi^{\infty} \tag{34}$$

is a countable algebra that is convergence determining. Indeed, for $\chi, \chi_1, \chi_2, \dots \in \mathbb{M}^I$, we have $\chi_n \xrightarrow{n \to \infty} \chi$ in the marked Gromov-weak topology iff $\nu^{\chi_n} \xRightarrow{n \to \infty} \nu^{\chi}$ in the weak topology on $\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}$ iff $\langle \nu^{\chi_n}, \phi \rangle \xrightarrow{n \to \infty} \langle \nu^{\chi}, \phi \rangle$ for all $\phi \in V$. $\qquad \square$

4.3   Proof of Theorem 5

By Theorem 3.4.5 of [9] and Proposition 4.1, Π^k is separating in $\mathcal{M}_1(\mathbb{M}^I)$.
We will show that Π_*^{∞} from Proposition 4.2 is a countable, convergence determining algebra in $\mathcal{M}_1(\mathbb{M}^I)$. Recall V and its ingredients, V_I and $V_{\mathbb{R}_+}$ from the proof of Proposition 4.2. By Lemma 3.4.3 in [9], we have that $\mathscr{X}_n \xRightarrow{n \to \infty} \mathscr{X}$ iff (i) $\mathbf{E}[\Phi(\mathscr{X}_n)] \xrightarrow{n \to \infty} \mathbf{E}[\Phi(\mathscr{X})]$ for all $\Phi \in \Pi_*^{\infty}$ and (ii) the family of distributions of $\{\mathscr{X}_n : n \in \mathbb{N}\}$ is tight. We will show that (i) implies (ii).
By Theorem 4 we have to show that (i) implies that

$$\text{the family of distributions of } \{\pi_i(\mathscr{X}_n) : n \in \mathbb{N}\} \text{ is tight for } i = 1, 2. \tag{35}$$

Before we prove this relation we need some new objects and auxiliary facts.
For $(\underline{\underline{r}}, \underline{u}) \in \mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}$ and $\varepsilon > 0$, we set

$$
\begin{aligned}
v(\underline{\underline{r}}, \underline{u}) &:= u_1, \\
w(\underline{\underline{r}}, \underline{u}) &:= r_{12}, \\
z_{\varepsilon}(\underline{\underline{r}}, \underline{u}) &:= \limsup_{n \to \infty} \frac{1}{n} \sum_{i=2}^{n} \mathbb{1}_{\{r_{1n} < \varepsilon\}}.
\end{aligned}
\tag{36}
$$

Moreover, for a random variable $\mathscr{Y}$ with values in $\mathbb{M}^I$, we define $(\underline{R}, \underline{U})^{\mathscr{Y}}$ as the random variable with values in $\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}$, such that given $\mathscr{Y} = y$, $(\underline{R}, \underline{U})^{\mathscr{Y}}$ has distribution ν^y. We have

$$
\begin{aligned}
\mathbf{E}[\phi((\underline{R}, \underline{U})^{\mathscr{X}_n})] &= \mathbf{E}\big[\mathbf{E}[\phi((\underline{R}, \underline{U})^{\mathscr{X}_n})|\mathscr{X}_n]\big] \\
&= \mathbf{E}[\langle \nu^{\mathscr{X}_n}, \phi \rangle] \xrightarrow{n \to \infty} \mathbf{E}[\langle \nu^{\mathscr{X}}, \phi \rangle] = \mathbf{E}[\phi((\underline{R}, \underline{U})^{\mathscr{X}})],
\end{aligned}
\tag{37}
$$

for all $\phi \in V$ by Assumption (i). Since V is convergence determining in $\mathscr{M}_1(\mathbb{R}^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$, we note that

$$
(\underline{R}, \underline{U})^{\mathscr{X}_n} \xrightarrow{n \to \infty} (\underline{R}, \underline{U})^{\mathscr{X}}.
\tag{38}
$$

In order to show (35) for $i = 1$, by Theorem 3 of [13], we need to show that (38) implies

(a) $\big\{ w\big((\underline{R}, \underline{U})^{\mathscr{X}_n}\big) : n \in \mathbb{N} \big\}$ is tight,

(b) For all $\varepsilon > 0$ there exists $\delta > 0$ such that $\limsup_{n \to \infty} \mathbf{P}\big(z_\varepsilon\big((\underline{R}, \underline{U})^{\mathscr{X}_n}\big) < \delta\big) < \varepsilon$.

For (a), note that by (37)

$$
\mathbf{E}[f(w((\underline{R}, \underline{U})^{\mathscr{X}_n})] \xrightarrow{n \to \infty} \mathbf{E}[f(w((\underline{R}, \underline{U})^{\mathscr{X}})]
\tag{39}
$$

for all $f \in V_{\mathbb{R}_+}$. Hence, since $V_{\mathbb{R}_+}$ is convergence determining in $\mathbb{R}_+$, $w((\underline{R}, \underline{U})^{\mathscr{X}_n}) \xrightarrow{n \to \infty} w((\underline{R}, \underline{U})^{\mathscr{X}})$, and in particular, (a) holds.

For (b), consider the distribution of $z_\varepsilon((\underline{R}, \underline{U})^{\mathscr{X}})$. Since the single random variable $\mathscr{X}$ is tight in $\mathbb{M}^I$, by Theorem 3 of [13], we find $\delta > 0$ such that $\mathbf{P}(z_\varepsilon((\underline{R}, \underline{U})^{\mathscr{X}}) < \delta) < \varepsilon$ and $z_\varepsilon((\underline{R}, \underline{U})^{\mathscr{X}})$ does not have an atom at δ. For $A := \{(\underline{r}, \underline{u}) : z_\varepsilon(\underline{r}, \underline{u}) < \delta\}$ we have $\partial A \subseteq \{(\underline{r}, \underline{u}) : z_\varepsilon(\underline{r}, \underline{u}) = \delta\}$ and it follows $\mathbf{P}((\underline{R}, \underline{U})^{\mathscr{X}} \in \partial A) = 0$. By the Portmanteau Theorem,

$$
\begin{aligned}
\mathbf{P}(z_\varepsilon((\underline{R}, \underline{U})^{\mathscr{X}_n}) < \delta) &= \mathbf{P}((\underline{R}, \underline{U})^{\mathscr{X}_n} \in A) \xrightarrow{n \to \infty} \mathbf{P}((\underline{R}, \underline{U})^{\mathscr{X}} \in A) \\
&= \mathbf{P}(z_\varepsilon((\underline{R}, \underline{U})^{\mathscr{X}}) < \delta) < \varepsilon.
\end{aligned}
\tag{40}
$$

This shows (b).

In order to obtain (35) for $i = 2$, note that $\nu_* \nu^{\mathscr{X}_n} \in \mathscr{M}_1(I)$ is the first moment measure of the distribution of the $\mathscr{M}_1(I)$-valued random variable $\pi_2(\mathscr{X}_n)$ and recall that tightness in $\mathscr{M}_1(\mathscr{M}_1(I))$ is implied by tightness of the first moment measure. By (37), we find that for $g \in V_I$

$$
\mathbf{E}[g(\nu((\underline{R}, \underline{U})^{\mathscr{X}_n}))] \xrightarrow{n \to \infty} \mathbf{E}[g(\nu((\underline{R}, \underline{U})^{\mathscr{X}}))],
\tag{41}
$$

so $\nu((\underline{R}, \underline{U})^{\mathscr{X}_n}) \xrightarrow{n \to \infty} \nu((\underline{R}, \underline{U})^{\mathscr{X}})$ and, in particular, (35) holds for $i = 2$.

Acknowledgments

References

[1] L. Addario-Berry, N. Broutin, and C. Goldschmidt. The continuum limit of critical random graphs. *Probab. Theory Relat. Fields*, online first, 2010.

[2] D. Aldous. The continuum random tree III. *Ann. Probab.*, 21(1):248–289, 1993. MR1207226

[3] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry, graduate studies in mathematics. *AMS, Boston, MA*, 33, 2001. MR1835418

[4] D. Dawson. Measure-valued Markov processes. In P.L. Hennequin, editor, *École d'Été de Probabilités de Saint-Flour XXI–1991*, volume 1541 of *Lecture Notes in Mathematics*, pages 1–260, Berlin, 1993. Springer. MR1242575

[5] A. Depperschmidt, A. Greven, and P. Pfaffelhuber. Tree-valued Fleming-Viot dynamics with mutation and selection. *Preprint*, 2011.

[6] J.-S. Dhersin, L. Decreusefond, P. Moyal, and V. C. Tran. Large graph limit for an infection process in random network with heterogeneous connectivity. *Preprint*, 2011.

[7] A. Dress. Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: A note on combinatorical properties of metric spaces. *Adv. Math.*, 53:321–402, 1984. MR0753872

[8] R. Durrett. *Random graph dynamics*. Cambridge University Press, 2007. MR2271734

[9] S. Ethier and T.G. Kurtz. *Markov Processes. Characterization and Convergence*. John Wiley, New York, 1986. MR0838085

[10] S. Evans. Kingman's coalescent as a random metric space. In *Stochastic Models: Proceedings of the International Conference on Stochastic Models in Honour of Professor Donald A. Dawson, Ottawa, Canada, June 10-13, 1998 (L.G Gorostiza and B.G. Ivanoff eds.)*, Canad. Math. Soc., 2000. MR1775475

[11] S. Evans, J. Pitman, and A. Winter. Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Relat. Fields*, 134(1):81–126, 2006. MR2221786

[12] S. Evans and A. Winter. Subtree prune and re-graft: A reversible real-tree valued Markov chain. *Ann. Probab.*, 34(3):918–961, 2006. MR2243874

[13] A. Greven, P. Pfaffelhuber, and A. Winter. Convergence in distribution of random metric measure spaces (The Λ-coalescent measure tree). *Probab. Theory Relat. Fields*, 145(1):285–322, 2009. MR2520129

[14] A. Greven, P. Pfaffelhuber, and A. Winter. Tree-valued resampling dynamics (martingale problems and applications). *Submitted*, 2010.

[15] A. Greven, R. Sun, and A. Winter. Limit genealogies of interacting Fleming-Viot processes on $\mathbb{Z}^1$. *Preprint*, 2011.

[16] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. MR1699320

[17] J.-F. LeGall.   Random trees and applications.   *Probability surveys*, 2:245–311, 2005. MR2203728

[18] J.-F. LeGall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169:621–670, 2007. MR2336042

[19] S. Piotrowiak.   *Dynamics of Genealogical Trees for Type- and State-dependent Resampling Models*.   PhD thesis, Department Mathematik, Erlangen-Nürnberg University, 2011. http://www.opus.ub.uni-erlangen.de/opus/volltexte/2011/2260/.

[20] R. van der Hofstad. Percolation and random graphs. Kendall, Wilfrid S. (ed.) et al., New perspectives in stochastic geometry. Oxford University Press. 173-247, 2010. MR2654679