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SURVIVAL AND EXTINCTION OF CARING DOUBLE-BRANCHING ANNIHILATING RANDOM WALK

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Abstract

Branching annihilating random walk (BARW) is a generic term for a class of interacting particle systems on $\mathbb{Z}^d$ in which, as time evolves, particles execute random walks, produce offspring (on neighbouring sites) and (instantaneously) disappear when they meet other particles. Much of the interest in such models stems from the fact that they typically lack a monotonicity property called *attractiveness*, which in general makes them exceptionally hard to analyse and in particular highly sensitive in their qualitative long-time behaviour to even slight alterations of the branching and annihilation mechanisms.

In this short note, we introduce so-called *caring* double-branching annihilating random walk (cD-BARW) on $\mathbb{Z}$, and investigate its long-time behaviour. It turns out that it either allows survival with positive probability if the branching rate is greater than $1/2$, or a.s. extinction if the branching rate is smaller than $1/3$ and (additionally) branchings are only admitted for particles which have at least one neighbouring particle (so-called 'cooperative branching'). Further, we show a.s. extinction for all branching rates for a variant of this model, where branching is only allowed if offspring can be placed at odd distance between each other.

It is the latter (extinction-type) results which seem remarkable, since they appear to hint at a general extinction result for a non-trivial parameter range in the so-called 'parity-preserving universality class', suggesting the existence of a 'true' phase transition. The rigorous proof of such a non-trivial phase transition remains a particularly challenging open problem.

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1 Introduction

In this note, we consider one-dimensional Branching Annihilating Random Walks (BARW) with *instant* annihilation which are *parity preserving*. That is, if started in an even number of particles, then the total number of particles will remain even for all time. More specifically, we will make sure that this property holds by the assumption that at each given branching event, the number of newly created particles equals two. In this case, we speak of a *double-branching* annihilating random walk, short DBARW.

Such processes have been investigated frequently in the physics literature, see e.g. [CT96] and [CT98] (and [SV10] for a nice overview), where they are considered as examples of elements of the so-called 'parity-preserving universality class'. They also received some attention in the mathematics literature, see for example [SS08], [BEM07], and [SV10], where the interest often stems from their connection (via duality) with models from population biology.

We begin with introducing the classical *symmetric* double branching annihilating random walk considered by Sudbury in [S90]. For $i, j \in \mathbb{Z}$, we write $i \sim j$ iff $|i - j| = 1$. We consider particle configurations $\mathbf{x}:=\{x_i : i \in \mathbb{Z},\ x_i \in \{0,1\}\}$ in the space $E := \{0,1\}^{\mathbb{Z}}$, where 1 indicates the presence, 0 the absence of a particle at site i. We denote the total number of ones in the configuration $\mathbf{x}$ by

$$|\mathbf{x}| := \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{x_i=1\}}.$$

Definition 1.1 (symmetric DBARW)**.** *Let* $\mathbf{x}$ *be a particle configuration in* E *such that* $|\mathbf{x}|$ *is finite. Let* $\alpha \in [0,1]$*. The strong Markov process* $\mathbf{X}=\{X_i(t), i \in \mathbb{Z}\}$ *with values in* E*, starting in* $\mathbf{X}(0) = \mathbf{x}$*, and dynamics given by the transitions*

$$\begin{cases} X_i \mapsto X_i - 1 \\ X_j \mapsto X_j + 1 \ (mod\ 2) \end{cases} \qquad \textit{(migration) at rate } \frac{\alpha}{2} X_i \mathbf{1}_{\{i\sim j\}},$$

$$\begin{cases} X_{i+1} \mapsto X_{i+1} + 1 \ (mod\ 2) \\ X_{i-1} \mapsto X_{i-1} + 1 \ (mod\ 2) \end{cases} \qquad \textit{(branching) at rate } (1-\alpha)X_i,$$

is called symmetric Double-Branching Annihilating Random Walk *with instant annihilation, branching rate* $1 - \alpha$ *and migration rate* α*. We denote this process by* sDBARW(α)*.*

In [S90], Sudbury shows that, for all $\alpha \in (0,1]$, this process, started in a configuration with an even number of particles, dies out a.s. in finite time. In particular, this model does not exhibit a 'true' phase transition in α between extinction und survival regimes.

Here, a.s. extinction relies on a beautiful 'interface duality' with another interacting particle system. Indeed, the boundaries between regions of zeroes and ones in a one-dimensional voter model with 'swapping' (where the types at neighbouring sites are exchanged with each other) follow precisely the above symmetric double-branching annihilating random walk with instant annihilation and parameter α, if the voting rate is $\alpha/2$ and the swapping rate for each given neighbouring pair is $1 - \alpha$. If we start from an even number of such boundaries, our 'interface dual' (the swapping voter model) starts with a finite number of ones and, since the number of ones is a positive martingale, as observed by Sudbury in [S90], it is not hard to see that it will eventually converge to zero. In other words the symmetric double-branching annihilating random walk with instant annihilation dies out for any $\alpha \in (0,1]$. A related martingale argument will be an important tool later in this paper.

As mentioned earlier, the long-time behaviour of DBARW models turns out to be very sensitive with respect to slight alterations of the branching mechanism. For example, Sturm and Swart introduce in [SS08] the *asymmetric double branching annihilating random walk* (aDBARW):

Definition 1.2 (asymmetric DBARW)**.** *Let* $\mathbf{x}$ *be a particle configuration in E such that* $|\mathbf{x}|$ *is finite. Let* $\alpha \in [0,1]$*. The strong Markov process* $\mathbf{X}=\{X_i(t), i \in \mathbb{Z}\}$ *with values in E, starting in* $\mathbf{X}(0) = \mathbf{x}$*, and dynamics given by the transitions*

$$\begin{cases} X_i \mapsto X_i - 1 \\ X_j \mapsto X_j + 1 \ (mod\ 2) \end{cases} \qquad \text{(migration) at rate } \frac{\alpha}{2} X_i \mathbf{1}_{\{i\sim j\}},$$

$$\begin{cases} X_{i+1} \mapsto X_{i+1} + 1 \ (mod\ 2) \\ X_{i+2} \mapsto X_{i+2} + 1 \ (mod\ 2) \end{cases} \qquad \text{(branch right) at rate } \frac{1-\alpha}{2} X_i,$$

$$\begin{cases} X_{i-1} \mapsto X_{i-1} + 1 \ (mod\ 2) \\ X_{i-2} \mapsto X_{i-2} + 1 \ (mod\ 2) \end{cases} \qquad \text{(branch left) at rate } \frac{1-\alpha}{2} X_i,$$

is called asymmetric Double-Branching Annihilating Random Walk *with instant annihilation, branching rate* $1-\alpha$ *and migration rate* α*. We denote this process by* aDBARW(α)*.*

The only difference to the sDBARW(α) of Sudbury is that now the two offspring particles are either placed both to the nearest and next-nearest neighbouring sites to the left or both to the nearest and next-nearest neighbouring sites to the right (each with probability 1/2).
However, Sturm and Swart in [SS08] are able to show that for α sufficiently small, by comparison with oriented percolation, this systems *survives* for all time with positive probability. Further, recent simulations by Swart and Vrbenský [SV10] suggest that the region of survival for this model and a further asymmetric variant is rather substantial, and that there is a phase transition separating survival from extinction at a parameter α close to 1/2.
Given these simulations, together with claims from the physics literature, one might be tempted to conjecture that there is a non-trivial critical value of α below which the aDBARW will die out, but above which it will survive with positive probability, i.e. there exists a true *phase-transition*. However, as pointed out by J.T. Cox, *'(...) a formidable difficulty in answering this question is the lack of a monotonicity property called attractiveness (...)'*[2]. The lack of monotonicity is typical for cancellative spin-flip system, see e.g. Griffeath [G79].
Unfortunately, the mathematical tools to prove such a phase transition seem currently out of reach. Further (and maybe even worse), in some cases it seems hard to develop an intuition which might explain the drastic changes in the qualitative long-time behaviour produced by such slight changes in the branching mechanism.

An intermediate goal is to identify models in the 'parity preserving universality class', for which it is possible to find non-trivial parameter values separating areas of extinction and survival. In the present paper, we introduce and investigate a new class of DBARW models, which we call 'caring'. The results below for the 'cooperative' version of the caring model seem to be the first ones in the literature in which both rigorous survival and extinction for such a parity preserving system can be shown. Further, we show a.s. extinction for all $\alpha \in (0,1]$, similar to Sudbury's result, for a variant of caring DBARW that places new particles at odd distance of each other. Finally, we aim to provide at least some intuition about why in some models extinction is certain, and in others survival is possible, by discussing mechanisms leading to or preventing *local* extinction.

[2] Cf. J.T. Cox' Math Review of [BG85], MR0772192

Remark 1.3. So far, we have only considered alterations to the *branching mechanism* of DBARW. Note that one might also drop the assumption of *instant* annihilation, and replace it with *delayed* annihilation, where two particles may coexist for an exponential time at the same site. Such systems, which allow multiple (in fact unbounded) occupancy, seem harder to analyse. See [BEM07] for some results on symmetric double-branching annihilating random walk with delayed annihilation. In contrast to the symmetric model with instant annihilation, here survival is possible. Results on extinction still remain elusive.

2 The 'caring' model and main results

We now introduce our main object of study, namely a version of DBARW which we call *caring* in the sense that newly born offspring particles will always be placed on the nearest vacant sites, so that they are safe from instant annihilation. More precisely, in each branching event, two new particles will be created, where one of them will be placed at the nearest unoccupied site to the left and the other one on the nearest unoccupied site to the right.

Intuitively, this should render survival more likely, since annihilation now only takes place during walk-steps. Indeed, the proof of a survival result is simple. However, more interestingly, if in this model we allow only for *cooperative* branchings, that is, branchings only for particles which have at least one occupied neighbouring site, it is also possible to prove an extinction result, where the parameter region ensuring extinction is rather substantial.

Let $\mathbf{x}$ be a non-trivial particle configuration in E with $|\mathbf{x}| < \infty$. Pick $j \in \mathbb{Z}$ such that $x_j = 1$. Then, we set

$$l_j(\mathbf{x}) := \max\{k < j : x_k = 0\} \quad \text{and} \quad r_j(\mathbf{x}) := \min\{k > j : x_k = 0\},$$

the positions of the nearest vacant sites to the left and the right of j in $\mathbf{x}$.

Definition 2.1 (caring DBARW)**.** *Let* $\mathbf{x}$ *be a particle configuration in* E *such that* $|\mathbf{x}| < \infty$. *Let* $\alpha \in [0,1]$. *The strong Markov process* $\mathbf{X}=\{X_i(t), i \in \mathbb{Z}\}$ *with values* E*, starting in* $\mathbf{X}(0) = \mathbf{x}$*, and dynamics given by the transitions*

$$\begin{cases} X_i \mapsto X_i - 1 \\ X_j \mapsto X_j + 1 \ (mod\ 2) \end{cases} \qquad \textit{(migration) at rate } \frac{\alpha}{2} X_i \mathbf{1}_{\{i \sim j\}},$$

$$\begin{cases} X_{l_i} \mapsto X_{l_i} + 1 \\ X_{r_i} \mapsto X_{r_i} + 1 \end{cases} \qquad \textit{(careful branching) at rate } (1-\alpha) X_i$$

is called caring Double-Branching Annihilating Random Walk *with instant annihilation branching rate* $(1-\alpha)$ *and migration rate* α*. We denote it by* cDBARW(α).

The 'cooperative' version of cDBARW is defined as follows:

Definition 2.2 (cooperative caring DBARW)**.** *The* DBARW *with symmetric migration at rate* α *and branching given by*

$$\begin{cases} X_{l_i} \mapsto X_{l_i} + 1 \\ X_{r_i} \mapsto X_{r_i} + 1 \end{cases} \qquad \textit{(cooperative caring branching) at rate } (1-\alpha) X_i \mathbf{1}_{\{r(i)-l(i) \geq 3\}},$$

is called cooperative caring Double-Branching Annihilating Random Walk *with parameter* α, ccDBARW(α).

Remark 2.3. Due to the long range branching mechanism, unlike sDBARW or aDBARW, it is not obvious that caring DBARW can be defined for starting configurations with an infinite number of particles.

Our first result is the following theorem on survival and extinction of ccDBARW.

Theorem 2.4. *Consider* ccDBARW(α) *on* $\mathbb{Z}$, *with finite starting configuration* $\mathbf{x}$.

(a) If $\alpha < 1/2$, *then* ccDBARW(α) *survives with positive probability.*

(b) If $\alpha > 2/3$ *and* $|\mathbf{x}|$ *is even, then* ccDBARW(α) *dies out almost surely.*

(c) If $\alpha > 2/3$ *and* $|\mathbf{x}|$ *is odd, then* ccDBARW(α) *almost surely in finite time reaches a state consisting of one isolated particle.*

The proof of this theorem will be given in Section 3 below, by analysing the behaviour inside blocks of particles of a fixed length.

Remark 2.5. Note that the state consisting of one isolated particle is absorbing, in the sense that a single particle still migrates, but the number of particles does not change anymore, since in the "cooperative" model neither branching nor annihilation are possible.

Remark 2.6. We can think of ccDBARW as the 'interface' process of a 'block-flipping voter model' which we will introduce in section 3. Part (c) of the above theorem then implies interface tightness (see [SV10] for a definition) of this model, since it shows that almost surely in finite time there will be a sole isolated interface, and this state will be absorbing due to the cooperative nature of the branching.

To state our result for cDBARW, we introduce 'odd' and 'even' versions of this model, which refer to restricting branchings to particles which belong to odd resp. even blocks.

Definition 2.7 (odd caring DBARW)**.** *The* DBARW *with symmetric migration at rate* α *and branching given by*

$$\begin{cases} X_{l_i} \mapsto X_{l_i} + 1 \\ X_{r_i} \mapsto X_{r_i} + 1 \end{cases} \qquad \text{(odd caring branching) at rate } (1-\alpha)X_i \mathbf{1}_{\{r(i)-l(i)=0 \text{ mod } 2\}},$$

is called odd caring Double-Branching Annihilating Random Walk *with parameter* α, *in short* ocDBARW(α).

Definition 2.8 (even caring DBARW)**.** *The* DBARW *with symmetric migration at rate* α *and branching given by*

$$\begin{cases} X_{l_i} \mapsto X_{l_i} + 1 \\ X_{r_i} \mapsto X_{r_i} + 1 \end{cases} \qquad \text{(even caring branching) at rate } (1-\alpha)X_i \mathbf{1}_{\{r(i)-l(i)=1 \text{ mod } 2\}},$$

is called even caring Double-Branching Annihilating Random Walk *with parameter* α, *or* ecDBARW(α).

Theorem 2.9. *Consider* cDBARW(α) *on* $\mathbb{Z}$, *with finite starting configuration* $\mathbf{x}$.

(a) If $\alpha < 1/2$, *then* cDBARW(α) *survives with positive probability.*

(b) If $\alpha > 2/3$ *and* $|\mathbf{x}|$ *is even, then* ecDBARW(α) *dies out almost surely.*

(c) If $\alpha > 0$ *and* $|\mathbf{x}|$ *is even, then* ocDBARW(α) *dies out almost surely.*

Parts (a) and (b) are proved analogously to Theorem 2.4. The rather surprising fact that odd cDBARW dies out almost surely for all positive α follows by a variant of Sudbury's martingale argument, using 'interface duality' with the block-flipping voter model (Definition 3.2). One might hope to be able to combine the two extinction results for even and odd branchings to obtain almost sure extinction of cDBARW for suffinciently large α. However, since the extinction result for the odd and for the even case rely on entirely different, and at first sight incompatible, techniques, it is not straightforward to combine them, and thus the extinction result for cDBARW without the assumption of cooperative branching remains open.

3 Proofs of the main results

Proof of Theorem 2.4. (a) Fix $0 < \alpha < 1/2$, and a starting configuration $\mathbf{X}(0)$ consisting of finitely many particles. We let

$$n(t) := |\mathbf{X}(t)| = \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{X_i(t)=1\}}$$

denote the number of particles at time $t \geq 0$, and

$$s(t) := \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{X_i(t)=1, r_i - l_i = 2\}}$$

the number of isolated particles at time t. Note that at any branching event, $n(t)$ increases by 2, while during a migration event, $n(t)$ either stays constant or decreases by 2, depending on whether the neighbouring site is occupied or not. A branching events happens in the system at rate $(1-\alpha)(n(t)-s(t))$, while migration takes place at rate $\alpha n(t)$. Every branching event increases the number of particles by 2. Thus we have that $n(t)$ increases to $n(t)+2$ at rate $(1-\alpha)(n(t)-s(t))$. At rate $\alpha(n(t)-s(t))$, an event occurs that may result in a reduction of $n(t)$ by two. This is because annihilation can only happen subsequent to a migration event, but not every migration event will necessarily lead to annihilation. If we choose $\alpha < 1/2$, this implies that we can couple $\{n(t)\}$ with a supercritical branching process, such that $n(t)$ is bounded below at any time t by this branching process that survives with positive probability. Hence $\mathbb{P}_{\mathbf{X}_0}\{n(t) > 0 \; \forall t\} > 0$.

(b) To prove the extinction result for ccDBARW, it is useful to consider blocks of particles. We say that a particle at site i belongs to a *block* of length $k \in \mathbb{N}$ in configuration $\mathbf{x}$, if $r_i(\mathbf{x}) - l_i(\mathbf{x}) = k+1$, and denote by $M_k(t)$ the number of blocks of length k at time t. We have

$$n(t) = \sum_{k=1}^{\infty} k M_k(t).$$

Since the branchings are cooperative and caring, we can analyze the dynamics of the whole particle configuration by considering the distinct blocks. The next branching or migration step may affect the block size as follows (compare Figure 1):

(A) It increases by 2 due to internal branching at rate $(1-\alpha)k$ (this might make the block merge with a neighbouring block, but the total number of particles will still increase by 2).

(B) It decreases by 2 at rate $(k-1)\alpha$, that is, if migration takes place inside the block in a way that makes it split into two new blocks (possibly of length 0) whose total number of particles is $k-2$.

(C) It remains unchanged, if migration takes place at the boundary in direction away from the block (which happens at rate α), that is, the block splits into two blocks of size 1 and $k-1$, respectively. (This might again increase the number of particles in a neighbouring block, but will not change $n(t)$).

(D) It increases by one if a particle from another block migrates and merges with our block, but then this decreases the size of that block, thus leaves the total number of particles unchanged (happens at rate at most α).

(E) It may increase by any number m due to a branching event in a neighbouring block of size at least k at distance one. However, the total number of particles will still increase by 2. In fact, this case corresponds to case (A) taking place in a block adjacent to the one we consider.

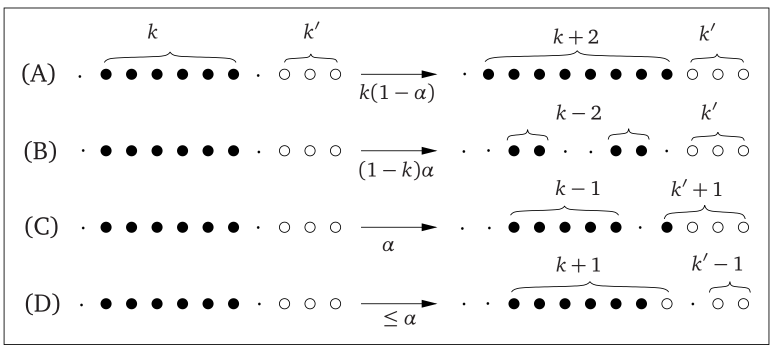

Figure 1: Possible transitions inside a block

Summarizing the considerations in (A)-(E) above, we see that branching and annihilation events in the process at time t happen at a rate which can be expressed in terms of the block sizes $M_k(t)$ at that time. We define

$$\beta(t) := (1-\alpha)\sum_{k=2}^{\infty} kM_k(t)$$

and

$$\lambda(t) := \alpha\sum_{k=2}^{\infty}(k-1)M_k(t).$$

Then (A)-(E) imply that $\beta(t)$ is the rate of a branching event at time t conditional on $M_k(t), k \geq 1$, and $\lambda(t)$ the rate of an annihilation event:

$$\{n(t) \mapsto n(t)+2\} \quad \text{at rate } \beta(t)$$

and

$$\{n(t) \mapsto n(t)-2\} \quad \text{at rate } \lambda(t).$$

We see that $\beta(t) = 0$ if and only if $\lambda(t) = 0$, and that this happens exactly at the times when there is no block of size at least 2. Otherwise, $\sum_{k\geq 2} M_k(t) > 0$, and the ratio of the rates is

$$\frac{\lambda(t)}{\beta(t)} = \frac{\alpha}{1-\alpha}\left(1 - \frac{\sum_{k=2}^{\infty} M_k(t)}{\sum_{k=2}^{\infty} kM_k(t)}\right). \tag{1}$$

In this case,

$$\frac{\sum_{k=2}^{\infty} M_k(t)}{\sum_{k=2}^{\infty} kM_k(t)} \in (0,1/2],$$

thus we have

$$\frac{\lambda(t)}{\beta(t)} \geq \frac{\alpha}{2(1-\alpha)}.$$

This means that if we choose $\alpha > 2/3$, we have that $\lambda(t) \geq \beta(t)$ at all times t, and thus $n(t)$ obtains a negative drift as long as there is at least one block of size at least 2. If there are only single particles left, their number remains constant until two of them become nearest neighbours (which, due to the recurrence of simple random walk in dimension 1, will happen a.s.), at which occasion we again have a negative drift in the number of particles. This proves part *(b)*.
(c) Like in the proof of *(b)*, we obtain a negative drift if $\alpha > 2/3$ as long as there are blocks of size at least 2, and otherwise the number of particles remains constant. Note that the expected time until two out of three or more particles meet is finite, so the single-particle state will even be reached in finite time. □

Remark 3.1. If we allow branchings also for isolated particles, the rate of branching events becomes

$$\hat{\beta}(t) := (1-\alpha)\sum_{k=1}^{\infty} kM_k(t)$$

and so equation (1), if $n(t) \neq 0$, is replaced by

$$\frac{\lambda(t)}{\hat{\beta}(t)} = \frac{\alpha}{1-\alpha}\left(1 - \frac{\sum_{k=1}^{\infty} M_k(t)}{\sum_{k=1}^{\infty} kM_k(t)}\right).$$

But then $\frac{\sum_{k=1}^{\infty} M_k(t)}{\sum_{k=1}^{\infty} kM_k(t)}$ is bounded from above only by the trivial bound 1, which yields extinction only for $\alpha = 1$. To obtain a hypothetical sharp critical α, more precise estimates would be needed, and since $M_k(t)$ depends on the particle configuration in a very subtle way, this does not seem to be easy.

Proof of Theorem 2.9. (a) Survival for cDBARW(α) if $\alpha < 1/2$ is proved in the same way as for ccDBARW. For cDBARW, branchings happen at rate $(1-\alpha)n(t)$, and annihilation events at rate at most $\alpha n(t)$, so the above proof goes through.
(b) For ecDBARW, the ratio of the rates at which annihilation and branching events occur, given that they are non-zero, satisfies

$$\begin{aligned}\frac{\lambda(t)}{\beta(t)} &= \frac{\alpha}{1-\alpha}\frac{\sum_{k=1}^{\infty}(2k-1)M_{2k}(t) + \sum_{k=1}^{\infty} 2kM_{2k+1}(t)}{\sum_{k=1}^{\infty} 2kM_{2k}(t)} \\ &\geq \frac{\alpha}{1-\alpha}\left(1 - \frac{\sum_{k=1}^{\infty} M_{2k}(t)}{\sum_{k=1}^{\infty} 2kM_{2k}(t)}\right).\end{aligned}$$

Then the same argument as before for ccDBARW applies.
(c) To prove extinction of ocDBARW, we introduce the interface process of cDBARW, which is a voter model with block-flips, where now the blocks correspond to sequences of consecutive alternating zeroes and ones.

Let $\eta = \{\eta_i : i \in \mathbb{Z}\} \in E$. For $i \in \mathbb{Z}$, we define $\eta^{i,i-1}$ to be the configuration obtained from η by changing η_i to η_{i-1}, that is,

$$\eta_j^{i,i-1} := \begin{cases} \eta_j & \text{if } j \neq i, \\ \eta_{i-1} & \text{else,} \end{cases}$$

and similarly, define

$$\eta_j^{i,i+1} := \begin{cases} \eta_j & \text{if } j \neq i, \\ \eta_{i+1} & \text{else.} \end{cases}$$

Finally, let $\eta^{i,[n]}$ be the configuration that is obtained from η by flipping the states of *all* sites $j \in [i, i+n]$, i.e.

$$\eta_j^{i,[n]} := \begin{cases} \eta_j + 1 \bmod 2 & \text{if } j \in [i, i+n], \\ \eta_j & \text{else.} \end{cases}$$

We say that η exhibits an *interface* at $(i, i+1)$, if either ($\eta_i = 0$ and $\eta_{i+1} = 1$) or ($\eta_i = 1$ and $\eta_{i+1} = 0$).

Definition 3.2 (block-flipping voter model)**.** *Let $\eta \in E$ denote a configuration with finitely many ones and an even number of interfaces. Let $\alpha \in [0,1]$. We denote by $\eta = \{\eta(t), t \geq 0\}$ the strong Markov process with values $\eta(t) \in E$, starting in $\eta(0) = \eta$, described by the following dynamics:*

$$\eta \mapsto \eta^{i,i-1} \text{ (voting from left) at rate } \frac{\alpha}{2}\mathbf{1}_{\{\eta_i \neq \eta_{i-1}\}},$$

$$\eta \mapsto \eta^{i,i+1} \text{ (voting from right) at rate } \frac{\alpha}{2}\mathbf{1}_{\{\eta_i \neq \eta_{i+1}\}},$$

$$\eta \mapsto \eta^{i,[n]} \text{ (block flip) at rate } n(1-\alpha)\mathbf{1}_{\{\eta_i(t) \neq \eta_{i+1}(t)\, \forall j \in [i,i+n]\}} \times \mathbf{1}_{\{\eta_{i-1} = \eta_i,\, \eta_{i+n} = \eta_{i+n+1}\}}.$$

We call this process block-flipping voter model*.*

Note that flips can only occur to whole consecutive sequences of interfaces, i.e. blocks. It is straightforward to check that cDBARW(α) appears as the *interface process* of the block-flipping voter model with parameter α in the same sense as sDBARW describes the interface process of the usual swapping voter model (see [S90], Section 10): Particles in the cDBARW are located precisely at the interfaces between zeroes and ones in the corresponding block-flipping voter model. Voting steps correspond to migration, while block flips correspond to 'caring' branching. We now use this duality to prove extinction of ocDBARW. Let

$$\xi(t) := |\eta(t)| = \sum_{i \in \mathbb{Z}} \mathbf{1}_{\{\eta_i(t) = 1\}}$$

be the number of ones in the block flipping voter model at time $t \geq 0$. Note that if we only allow for *odd* branchings, starting from a finite initial condition, this means that for the corresponding voter model we only allow for flips of blocks of an *even* length, which means that any such flip leaves the number of ones unaffected, and therefore ξ unchanged. We now verify the result by means of a martingale argument (an alternative would be to argue as in [S90] via the graphical representation, which in our case however would have to be defined with some care, since our branching mechanism is not of bounded range).

Define by $\mathscr{F}_t = \sigma\{\eta(s), s \leq t\}, t \geq 0$, the canonical filtration associated with $\{\eta(t)\}$. Note that, for each $t \geq 0$, $\xi(t)$ is a measurable function of $\eta(t)$ and hence $\mathscr{F}_t$-adapted. Recall that we only consider *finite* initial states $\eta(0) \in E$ (i.e. configurations with finitely many ones). Hence, for such $\eta(0)$,

$$\tau := \inf\{s > 0 : \eta(s) \neq \eta(0)\}$$

is strictly positive and finite a.s. At the stopping time τ, we have either

$$\eta(\tau) = \eta^{i,j}(0), \quad \text{for some } i \sim j,$$

(voting event) or

$$\eta(\tau) = \eta^{i,[2n-1]}(0), \quad \text{for some } i \in \mathbb{Z}, n > 0,$$

(swapping event). Note that in the latter case, $\xi(\tau) = \xi(0)$, since the swap is of even length, and that in the first case we have either $\xi(\tau) = \xi(0) - 1$ or $\xi(\tau) = \xi(0) + 1$, each with probability $1/2$. For $\zeta, \zeta' \in E$, ζ finite, denote by

$$p(\zeta, \zeta') := \mathbb{P}\{\eta(\tau) = \zeta' | \eta(0) = \zeta\} =: \mathbb{P}^{\zeta}\{\eta(\tau) = \zeta'\} \tag{2}$$

the corresponding probability weights. Let now $\tau_0 := 0$, and

$$\tau_{k+1} := \inf\{t > \tau_k : \eta(t) \neq \eta(\tau_k)\}, \quad k \geq 1.$$

The τ_k are stopping times for the Markov process $\{\eta(t)\}$ which are finite almost surely as long as $\xi(t) \neq 0$, and which satisfy $\tau_{k-1} < \tau_k$ a.s. We define

$$\hat{\mathscr{F}}_k := \mathscr{F}_{\tau_k}, \quad \hat{\eta}(k) := \eta(\tau_k), \quad k \geq 0,$$

noting that $\{\hat{\eta}(k)\}$ is a discrete-time $\{\hat{\mathscr{F}}_k\}$-Markov process, absorbed in 0, with transition probabilities $p(\cdot, \cdot)$ as in (2) above. Finally, we set

$$\hat{\xi}(k) := \xi(\tau_k), \quad k \geq 0.$$

Note that $\{\hat{\xi}(k)\}$ is $\{\hat{\mathscr{F}}_k\}$-adapted. It is now easy to see that $\{\hat{\xi}(k)\}$ is an $\{\hat{\mathscr{F}}_k\}$-martingale (in fact, $\xi(k) = |\eta(k)|$ is a *harmonic function* for $p(\cdot, \cdot)$ on E). Indeed, we have by the above considerations and the strong Markov property of $\{\eta_t\}$, for $k \geq 1$,

$$\begin{aligned}
\mathbb{E}\Big[\hat{\xi}(k+1)\Big|\hat{\mathscr{F}}_k\Big] &= \mathbb{E}\big[\xi(\tau_{k+1})\big|\mathscr{F}_{\tau_k}\big] = \mathbb{E}^{\eta(\tau_k)}\big[\xi(\tau_{k+1})\big] \\
&= \sum_{\zeta' \in E} |\zeta'|\, p\big(\eta(\tau_k), \zeta'\big) \\
&= \hat{\xi}(k), \ \text{ a.s.}
\end{aligned}$$

Hence, $\{\hat{\xi}(k)\}$ is a non-negative martingale absorbed in 0, and therefore converges to 0 almost surely, which proves the result. □

Remark 3.3. If we take ocDBARW with $\alpha = 0$ (i.e., no migration) started from an even initial condition, this process will not die out: In finite time, a.s., it will be absorbed in a configuration consisting only of blocks of even length, where no further branching is possible.

4 Local extinction mechanisms

A possible heuristic explanation for the extinction of this model, despite the 'caring' branching mechanism, might be given in terms of the behaviour just before *local* extinction. Suppose two otherwise isolated particles come close to each other until they are neighbours. Then we see local extinction if these particles meet.
A way to guarantee survival with positive probability might be to increase the probability that such a 'dangerous' configuration of two adjoint (but otherwise isolated) particles will be left through a 'safer' state (e.g. if the two particles walk away from each other, or produce new offspring) to be sufficiently close to one.

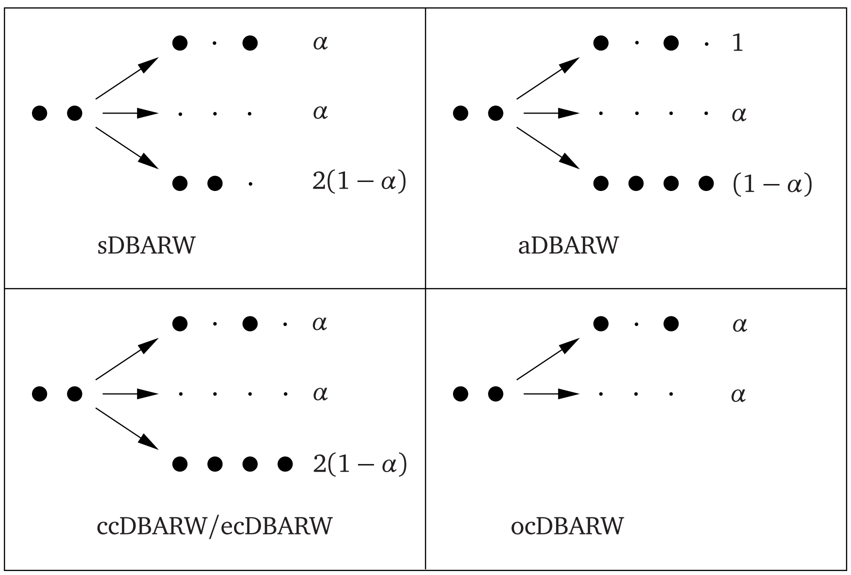

Figure 2: 'Dangerous' configurations and rates of leaving them

Figure 2 shows that both for cDBARW and for aDBARW, this is always possible by decreasing α, whereas the probability of leaving the 'dangerous configuration' without extinction remains bounded away from 1 for sDBARW and *odd* cDBARW: Indeed, consider for example sDBARW. Any of the two neighbouring particles walks away (in the 'good' direction) with rate $\alpha/2$, migrates in the 'bad' direction which leads to local extinction with the same rate $\alpha/2$ and branches symmetrically with rate $1-\alpha$, which leaves the local configuration unchanged (it is just shifted by one unit, we assume that there are no more particles close by). The total rates with which the configuration of two neighbouring particles changes is represented in Figure 2. Thus the probability of moving into a safer configuration in the next step – let us denote it by p_s – is the same as the probability for extinction in the next step (denoted by p_e), in fact,

$$p_s = p_e = \frac{\alpha}{\alpha + \alpha + 2(1-\alpha)} = \frac{\alpha}{2} \leq \frac{1}{2}.$$

For aDBARW on the other hand, branching has a different effect on this particular configuration: With rate $(1-\alpha)/2$, a particle branches in the direction of the second particle, thus producing the same configuration (the first exit configuration in Figure 2) as for a 'good' migration step (rate $\alpha/2$

for each particle), and with rate $(1-\alpha)/2$ branching takes place in the other direction, producing additional particles. Thus

$$p_s = \frac{1+1-\alpha}{2} = 1 - \frac{\alpha}{2}, \quad p_e = \frac{\alpha}{2}.$$

Hence, for aDBARW (and similarly for cDBARW) we can make p_s arbitrarily close to one by choosing α small enough, while this is not possible for sDBARW where it is always bounded above by $1/2$.

This consideration, although somewhat speculative, might yield a general pattern to predict whether survival is possible or not for a given variant of DBARW. Indeed, the idea that extinction of the whole system is governed by the local extinction events we just discussed, is supported by the fact that the conjectured critical value for aDBARW is $\alpha_c = 1/2$ (see [SS08]), which is precisely the value of α for which the probability of increasing the number of particles in the next step is the same as the probability of (local) extinction in the next step (compare Figure 2).

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References

[BEM07] Blath, J.; Etheridge, A.M.; Meredith, M.: Coexistence in competing populations and branching annihilating random walk. *Ann. Appl. Probab.* 17, no. 5/6, 1474–1507, (2007). MR2358631

[BG85] Bramson, M.; Gray, L.: The survival of branching annihilating random walk. *Z. Wahrsch. Verw. Gebiete* 68, no. 4, 447–460, (1985). MR0772192

[CT96] Cardy, J.; Täuber, U.: Theory of branching and annihilating random walk. *Phys. Rev. Lett* 77, 4780–4783, (1996).

[CT98] Cardy, J.; Täuber, U.: Field theory of branching and annihilating random walk. *J. Statist. Phys.* 90, 1–56, (1998). MR1611125

[G79] Griffeath, D.: *Additive and Cancellative Interacting Particle Systems.* Lecture Notes in Mathematics, Springer, (1979). MR0538077

[S90] Sudbury, A.: The branching annihilating process: an interacting particle system. *Ann. Probab.* 18, no. 2, 581–601, (1990). MR1055421

[SS08] Sturm, A.; Swart, J.: Voter models with heterozygosity selection. *Ann. Appl. Probab.* 18, no. 1, 59–99, (2008). MR2380891

[SV10] Swart, J.; Vrbenský, K.: Numerical analysis of the rebellious voter model. *J. Stat. Phys.* 140, no. 5 873–899 (2010). MR2673338