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WHICH DISTRIBUTIONS HAVE THE MATSUMOTO-YOR PROPERTY?

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Abstract

For four types of functions $\xi :]0,\infty[\to]0,\infty[$, we characterize the law of two independent and positive r.v.'s X and Y such that $U := \xi(X+Y)$ and $V := \xi(X) - \xi(X+Y)$ are independent. The case $\xi(x) = 1/x$ has been treated by Letac and Wesołowski (2000). As for the three other cases, under the weak assumption that X and Y have density functions whose logarithm is locally integrable, we prove that the distribution of (X,Y) is unique. This leads to Kummer, gamma and beta distributions. This improves the result obtained in [1] where more regularity was required from the densities.

1 Introduction

Consider a decreasing and bijective function $\xi :]0,\infty[\to]0,\infty[$. If X, Y are non-Dirac, positive and independent random variables with law μ_X and μ_Y respectively, we say that the triplet (ξ, μ_X, μ_Y) has the *Matsumoto-Yor property* if the r.v.'s :

$$U := \xi(X+Y), \quad V := \xi(X) - \xi(X+Y) \tag{1.1}$$

are independent.
Given ξ, let $\mathscr{D}(\xi)$ denote the set of all possible laws of (X,Y) such that (ξ, μ_X, μ_Y) has the Matsumoto-Yor property. Define $\mathscr{D}_{log}(\xi)$ (resp. $\mathscr{D}_2(\xi)$) as the subset of $\mathscr{D}(\xi)$ so that X and Y have densities whose logarithms are locally integrable over $]0,\infty[$ (resp. the densities of X and Y are of class C^2).
It is convenient to introduce $e_\alpha(x) = (e^{\alpha x} - 1)/\alpha$ for $\alpha > 0$ and $e_0(x) = x$ (this notation has been wisely given by a referee). Let $\alpha, \beta \geq 0$ and $\delta > 0$. For any $x > 0$, define $y = h(\alpha, \beta, \delta)(x)$ as the unique $y > 0$ so that $e_\alpha(x)e_\beta(y) = \delta$. Under additional assumptions, it has been proved in [1] that

$\mathcal{D}_2(\xi)$ is not empty if and only if $\xi \in \mathcal{F}$, where $\mathcal{F} := \{h(\alpha,\beta,\delta),\ \alpha,\beta \geq 0, \delta > 0\}$. Obviously $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$ where $\mathcal{F}_1 := \{\delta/x,\ \delta > 0\}$, $\mathcal{F}_2 := \{h(\alpha,0,\delta),\ \alpha,\delta > 0\}$, $\mathcal{F}_3 := \{h(0,\beta,\delta),\ \beta,\delta > 0\}$ and $\mathcal{F}_4 := \{h(\alpha,\beta,\delta),\ \alpha,\beta,\delta > 0\}$. Moreover $\mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ admit a more explicit form : $\mathcal{F}_2 = \{\delta f(\alpha x);\ \alpha,\delta > 0\}$, $\mathcal{F}_3 = \{\delta g(\beta x);\ \beta,\delta > 0\}$ and $\mathcal{F}_4 = \{\beta f_\alpha(\beta x)/;\ \alpha,\beta,\delta > 0\}$, with

$$f(x) := h(1,0,1)(x) = \frac{1}{e^x - 1},\ \ x > 0, \tag{1.2}$$

$$g(x) = f^{-1}(x) = h(0,1,1)(x) = \log\left(\frac{1+x}{x}\right),\ \ x > 0, \tag{1.3}$$

$$f_\delta(x) := h(1,1,\delta)(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right),\ \ x > 0 \quad (\delta > 0). \tag{1.4}$$

1) For $p,a,b > 0$, consider the gamma distribution $\gamma(\lambda,a)(dx) = \dfrac{a^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-ax} \mathbf{1}_{(0,\infty)}(x)dx$ and the generalized inverse gaussian distribution GIG(p,a,b) with density proportional to $x^{p-1}e^{-\frac{1}{2}(a^2x^{-1}+b^2x)}\mathbf{1}_{(0,\infty)}(x)$. The first class $\mathcal{F}_1$ corresponds to the case considered in [4] and [3]. More precisely, for $a = b$ it has been proved in [4] that if X and Y are two independent r.v.'s, $X \sim$ GIG$(-p,a,b)$ and $Y \sim \gamma(p, b^2/2)$, then $U := \dfrac{1}{X+Y}$ and $V := \dfrac{1}{X} - \dfrac{1}{X+Y}$ are independent and $U \sim$ GIG$(-p,b,a)$, $V \sim \gamma(p, a^2/2)$. Subsequently, it has been shown in [3] that, if $\xi(x) = 1/x$, then $\mathcal{D}(\xi)$ equals $\{\text{GIG}(-p,a,b) \otimes \gamma(p,b^2/2),\ p,a,b > 0\}$.
2) The aim of this paper is to prove that, for $\xi \in \mathcal{F}_i$, $2 \leq i \leq 4$, $\mathcal{D}_{log}(\xi) = \mathcal{D}_2(\xi)$. (see Theorems 2.1 and 2.3). This is a first step towards the proof of the following conjecture : $\mathcal{D}(\xi) = \mathcal{D}_2(\xi)$. Let us recall the description of $\mathcal{D}_2(\xi)$ given in [1] where $\xi \in \mathcal{F}_i$, $2 \leq i \leq 4$.
a) $\mathcal{D}_2(f)$ is the family of distributions $p_X(x)dx \otimes p_Y(y)dy$, with

$$p_X(x) = C^* e^{-(a+b)x}(1-e^{-x})^{-b-1} \exp\left(-c\frac{e^{-x}}{1-e^{-x}}\right)\mathbf{1}_{\{x>0\}} \tag{1.5}$$

$$p_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}} \tag{1.6}$$

where $a,b,c > 0$ and C^* is the normalizing constant (in the sequel, C^* stands for the unique positive constant so that the related function is a density).
Moreover, $\mathcal{D}_2(f^{-1}) = \{K^{(2)}(a,b,c) \otimes \gamma(b,c),\ a,b,c > 0\}$, with $K^{(2)}(a,b,c)$ the *Kummer distribution of type 2* :

$$K^{(2)}(a,b,c)(dx) := C^* x^{a-1}(1+x)^{-a-b}e^{-cx}\mathbf{1}_{(0,\infty)}(x)dx,\ a,c > 0,\ b \in \mathbb{R}. \tag{1.7}$$

b) $\mathcal{D}_2(f_\delta)$ is the family of distributions $p_X(x)dx \otimes p_Y(y)dy$, with

$$p_X(x) = C^* e^{-(a+b)x}(\delta e^{-x} + 1 - e^{-x})^{-\lambda-b}(1-e^{-x})^{\lambda-1}\mathbf{1}_{\{x>0\}}, \tag{1.8}$$

$$p_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}}. \tag{1.9}$$

The corresponding laws of U and V are

$$p_U(u) = C^* e^{-u(\lambda+b)}(1-e^{-u})^{a-1}(1+(\delta-1)e^{-u})^{-a-b}\mathbf{1}_{\{u>0\}}, \tag{1.10}$$

$$p_V(v) = \frac{\Gamma(\lambda+b)}{\Gamma(\lambda)\Gamma(b)}e^{-\lambda v}(1-e^{-v})^{b-1}\mathbf{1}_{\{v>0\}}, \tag{1.11}$$

where $a, b > 0$ and $\lambda \in \mathbb{R}$.
3) Note that $\lim_{\delta \to 0} f_\delta(\delta x) = g(x)$. This permits to make a link between $\mathscr{D}_2(g)$ and $\{\mathscr{D}_2(f_\delta);\ \delta > 0\}$ (see Proposition 2.5 below).
We state the results in Section 2 and prove them in Section 3.

2 The results

1) Consider (U, V) defined by (1.1) with $\xi = f$. Then

$$(U,V) := \left(\frac{1}{e^{X+Y}-1}, \frac{1}{e^X - 1} - \frac{1}{e^{X+Y}-1} \right) \tag{2.1}$$

and $(X, Y) = \big(g(U+V), g(U) - g(U+V)\big)$, i.e.

$$(X,Y) = \left(\log\left(1 + \frac{1}{U+V}\right), \log\left(1+\frac{1}{U}\right) - \log\left(1+\frac{1}{U+V}\right) \right). \tag{2.2}$$

Theorem 2.1. *Let (X, Y) be a couple of independent positive r.v.'s with densities p_X and resp. p_Y. It is supposed that p_X and p_Y are positive and that $\log p_X$ and $\log p_Y$ are locally integrable over $]0, \infty[$. Suppose that U and V defined by (2.1) are independent, then the densities of X and Y are given by (1.5), resp. (1.6). Moreover, $U \sim K^{(2)}(a, b, c)$ and $V \sim \gamma(b, c)$.*

Remark 2.2.

1. Keeping the notation given in the Introduction, Theorem 2.1 means that $\mathscr{D}_{log}(f) = \big\{p_X(x)dx \otimes p_Y(y)dy,\ a, b, c > 0\big\}$.

2. It is clear that $(X, Y) = \big(\log(1 + 1/V'), -\log U'\big)$ with :

$$U' := \frac{1 + \frac{1}{U+V}}{1 + \frac{1}{U}}, \quad V' = U + V. \tag{2.3}$$

   It is easy to verify that $\log p_X$ and $\log p_Y$ are locally integrable on $]0, \infty[$ if and only if $\log p_{U'}$ and $\log p_{V'}$ are locally integrable on $]0, 1[$ and $]0, \infty[$ respectively. Using usual calculations, the formulation of Matsumoto-Yor independence property related to U, V, U' and V' is the following : suppose that U and V are independent and U' and V' defined by (2.3) are independent, then $U' \sim \text{Beta}(a, b)$ and $V' \sim K^{(2)}(a+b, -b, c)$ where $a, b, c > 0$ and $\text{Beta}(a,b)(dx) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbf{1}_{\{0<x<1\}} dx$. Moreover $U \sim K^{(2)}(a, b, c)$ and $V \sim \gamma(b, c)$. This gives a characterization of the Kummer distributions. Observe that we retrieve by the way the following convolution formula mentioned in [2] :

$$K^{(2)}(a, b, c) * \gamma(b, c) = K^{(2)}(a+b, -b, c). \tag{2.4}$$

2) Let us deal with $\xi = f_\delta$.

Theorem 2.3. *Let (X, Y) be a couple of independent positive r.v.'s with densities p_X and resp. p_Y such that p_X and p_Y are positive, $\log p_X$ and $\log p_Y$ are locally integrable over $]0, \infty[$. If $U = f_\delta(X+Y)$ and $V = f_\delta(X) - f_\delta(X+Y)$ are independent, then the densities of Y and X are given by (1.8) and resp. (1.9). Moreover (1.10) and (1.11) are the densities of U and resp. V.*

It is interesting to introduce new r.v.'s X', Y', U' and V' taking their values in $]0,1[$, via $X' := e^{-X}$, $Y' := e^{-Y}$, $U' := e^{-U}$ and $V' := e^{-V}$, see Subsection 2.2 in [1] for interpretations and details. We easily deduce from definitions :

$$(U',V') = \left(\frac{1-X'Y'}{1+(\delta-1)X'Y'}, \frac{1-X'}{1+(\delta-1)X'} \frac{1+(\delta-1)X'Y'}{1-X'Y'} \right). \tag{2.5}$$

Note that $\log p_X$ and $\log p_Y$ are locally integrable on $]0,\infty[$ if and only if $\log p_{X'}$ and $\log p_{Y'}$ are locally integrable on $]0,1[$. In this new setting Theorem 2.3 takes the following form.

Theorem 2.4. *If X', Y' are independent, if U', V' are independent and if the densities $p_{X'}$, $p_{Y'}$ are such that $\log p_{X'}$ and $\log p_{Y'}$ are locally integrable over $]0,1[$, then $X' \sim \beta_\delta(a+b, \lambda; -\lambda-b)$, and $Y' \sim \mathrm{Beta}(a,b)$. Moreover, $U' \sim \beta_\delta(\lambda+b, a; -a-b)$, $V' \sim \mathrm{Beta}(\lambda, b)$, where*

$$\beta_\alpha(a,b;c)(dx) = kx^{a-1}(1-x)^{b-1}(\alpha x + 1 - x)^c \mathbf{1}_{(0,1)}(x)dx \tag{2.6}$$

$a, b, \lambda > 0$ and k is the normalizing constant.

Let X_δ and Y_δ be two independent r.v.'s with densities given by (1.8) and (1.9) respectively, with $\lambda = a_0$, $b = b_0$ and $a = c_0/\delta$.

Proposition 2.5. *1) $\big(X_\delta/\delta, Y_\delta/\delta\big)$ converges in distribution as $\delta \to 0$ to (X^*, Y^*) where X^* and Y^* are independent, $X^* \sim K^{(2)}(a_0, b_0, c_0)$ and $Y^* \sim \gamma(b_0, c_0)$.*
2) Let $U_\delta := f_\delta(X_\delta + Y_\delta)$ and $V_\delta := f_\delta(X_\delta) - f_\delta(X_\delta + Y_\delta)$. Then (U_δ, V_δ) converges in distribution as $\delta \to 0$ to (U^*, V^*). Furthermore, (U^*, V^*) and $\big(g(X^*+Y^*), g(X^*) - g(X^*+Y^*)\big)$ have the same distribution. In particular U^* and V^* are independent with densities the right-hand side of (1.5) and (1.6) respectively, with $a = a_0$, $b = b_0$ and $c = c_0$.

3 Proofs

3.1 Proof of Theorem 2.1

Suppose that X and Y are independent, that the functions $\log p_X$ and $\log p_Y$ are locally integrable over $]0,\infty[$, and that U and V are independent. Our approach is direct and is based on the calculation of the densities of (U,V) and (X,Y) using (2.1) and (2.2). This leads to two functional equations involving p_X, p_Y, p_U and p_V.

Lemma 3.1. *Let us introduce : $h := -\dfrac{p_X}{p_Y f'}$, $k := -\dfrac{p_U}{p_V g'}$, $F := \log k$, $\alpha := \log p_V$, $\beta := \log p_Y$ and $H(r) := \log\left(h\Big(\log\big(1+\dfrac{2}{r}\big)\Big)\right)$, where $r > 0$. Then the following functional equations hold :*

$$H(s) - H(t) = \alpha\left(\frac{s(s+2)}{2(s+t+2)}\right) - \alpha\left(\frac{t(t+2)}{2(s+t+2)}\right), \quad s,\ t > 0. \tag{3.7}$$

$$F(u) - F(v) = \beta\big(g(u) - g(u+v)\big) - \beta\big(g(v) - g(u+v)\big), \quad u,\ v > 0. \tag{3.8}$$

Proof of Lemma 3.1
Under the above assumptions, the density function of (U,V) is

$$p_U(u)p_V(v) = p_X(g(u+v))\,p_Y(g(u) - g(u+v))\,g'(u+v)g'(u) \tag{3.9}$$

and the one of (X,Y) is :

$$p_X(z)p_Y(w) = p_U(f(z+w))\,p_V(f(z)-f(z+w))\,f'(z+w)f'(z), \quad z,w>0. \tag{3.10}$$

Replacing (z,w) by (w,z) in (3.10) leads to :

$$p_X(w)p_Y(z) = p_U\big(f(z+w)\big)\,p_V\big(f(w)-f(z+w)\big)\,f'(z+w)f'(w), \quad z,w>0. \tag{3.11}$$

First, we divide the left-hand side of (3.10) by the left-hand side of (3.11) and second, we take the logarithm, we obtain :

$$\log h(z) - \log h(w) = \log\Big(p_V\big(f(z)-f(z+w)\big)\Big) - \log\Big(p_V\big(f(w)-f(z+w)\big)\Big). \tag{3.12}$$

Set: $z = \log\left(1+\frac{2}{s}\right)$ and $w = \log\left(1+\frac{2}{t}\right)$. Note that $z,w>0 \Leftrightarrow s,t>0$. Then the left-hand side of (3.12) is $H(s)-H(t)$. According to the definition of f, we have successively :

$$f(z) = \frac{1}{e^z-1} = \frac{s}{2}, \quad f(w) = \frac{t}{2}, \quad f(z+w) = \frac{st}{2(s+t+2)}.$$

Therefore,

$$f(z)-f(z+w) = \frac{s(s+2)}{2(s+t+2)}, \quad f(w)-f(z+w) = \frac{t(t+2)}{2(s+t+2)}.$$

Thus, the right-hand side of (3.12) is $\alpha\left(\frac{s(s+2)}{2(s+t+2)}\right) - \alpha\left(\frac{t(t+2)}{2(s+t+2)}\right)$ and we get (3.7). Relation (3.8) can be proved similarly using (3.9).

□

Lemma 3.2. *The functions H, α, F and β are of class C^1.*

The proof of Lemma 3.2 is postponed in a special subsection 3.2 devoted to this problem. Denote $\phi(s,t) := \big(g(s)-g(s+t), g(t)-g(s+t)\big)$. Then (3.8) can be written as :

$$F(s)-F(t) = \beta\big(\phi_1(s,t)\big) - \beta\big(\phi_2(s,t)\big) \tag{3.13}$$

where $\phi = (\phi_1,\phi_2)$. Similarly, according to (3.12), there exists ϕ so that $\log h(s) - \log h(t) = \alpha\big(\phi_1(s,t)\big) - \alpha\big(\phi_2(s,t)\big)$. This leads us to consider functional equations of the type :

$$G(s)-G(t) = \theta\big(\phi_1(s,t)\big) - \theta\big(\phi_2(s,t)\big), \quad s,t>0.$$

and the goal is to give sufficient conditions so that θ and G are of class C^1. Theorem 2.3 will be proved using the above approach.

Lemma 3.3. *There exists $N<0$, $L\in\mathbb{R}$ and $M>-1$ such that*

$$p_V(s) = e^{2Ns}s^M e^L, \quad s>0. \tag{3.14}$$

Consequently, $V \sim \gamma(M+1,-2N)$.

Proof of Lemma 3.3 According to Lemma 3.2, the functions H and α are C^1. Let us differentiate (3.7) with respect to s. With the notation

$$\hat{s} := \frac{s(s+2)}{2(s+t+2)}, \quad \hat{t} := \frac{t(t+2)}{2(s+t+2)}$$

we get

$$H'(s) = \alpha'(\hat{s})\left[\frac{s+1}{s+t+2} - \frac{s(s+2)}{2(s+t+2)^2}\right] + \alpha'(\hat{t})\frac{t(t+2)}{2(s+t+2)^2}. \tag{3.15}$$

a) Let $t \to 0$ in (3.15).
We have $\hat{s} \to s/2$ and $\hat{t} \to 0$ as $t \to 0$. Writing

$$\alpha'(\hat{t})\frac{t(t+2)}{2(s+t+2)^2} = \hat{t}\alpha'(\hat{t})\frac{1}{s+t+2}$$

we deduce from (3.15) that

$$M := \lim_{u\to 0} u\alpha'(u) \tag{3.16}$$

exists and

$$H'(s) = \frac{1}{2}\alpha'\left(\frac{s}{2}\right) + \frac{M}{s+2}. \tag{3.17}$$

b) Let us now take $t \to \infty$ in (3.15).
It is clear that $\lim_{t\to\infty} \hat{s} = 0$. Therefore, taking the limit $t \to \infty$ in (3.15) and using identities

$$\alpha'(\hat{s})\frac{s+1}{s+t+2} = \left(\hat{s}\alpha'(\hat{s})\right)\frac{2(s+1)}{s(s+2)}$$

$$\alpha'(\hat{s})\frac{s(s+2)}{2(s+t+2)^2} = \left(\hat{s}\alpha'(\hat{s})\right)\frac{1}{s+t+2}$$

we can conclude that :

$$2N := \lim_{u\to\infty} \alpha'(u) \tag{3.18}$$

exists and

$$H'(s) = N + \frac{M}{s} + \frac{M}{s+2}, \quad s > 0. \tag{3.19}$$

Combining the above identity and (3.17) leads to :

$$\alpha'(s) = 2N + \frac{M}{s}, \quad s > 0. \tag{3.20}$$

Recall that $\alpha = \log(p_V)$ and p_V is a density function. Then, (3.14) follows from integration of (3.20).

□

Lemma 3.4. *There exists $A > 0$, $C \in \mathbb{R}$ and $B < 0$ such that*

$$p_Y(v) = e^C(1-e^{-v})^A e^{Bv}, \quad v > 0. \tag{3.21}$$

Proof of Lemma 3.4 Taking the u-derivative in (3.8) we get :

$$F'(u) = (g'(u) - g'(u+v))\beta'(g(u) - g(u+v)) + g'(u+v)\beta'\big(g(v) - g(u+v)\big), \quad u, v > 0. \tag{3.22}$$

a) Let $v \to \infty$ in (3.22).
It is clear that (1.3), $\lim_{x\to\infty} g'(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$ imply:

$$\lim_{v\to\infty} (g'(u) - g'(u+v))\beta'\big(g(u) - g(u+v)\big) = g'(u)\beta'(g(u)).$$

Let us rewrite the terms in the right-hand side of (3.22), we have

$$g(v) - g(u+v) = \log\left(1 + \frac{u}{v(u+v+1)}\right), \quad u, v > 0, \tag{3.23}$$

$$g'(u+v)\beta'\big(g(v) - g(u+v)\big) = A_1 A_2$$

where

$$A_1 = \log\left(1 + \frac{u}{v(u+v+1)}\right)\beta'\left(\log\left(1 + \frac{u}{v(u+v+1)}\right)\right)$$

$$A_2 = -\frac{1}{(u+v)(u+v+1)\log\left(1 + \frac{u}{v(u+v+1)}\right)}.$$

Consequently, $\lim_{v\to\infty} A_2 = -\frac{1}{u}$. Next, taking $v \to \infty$ in (3.22) we deduce that

$$A := \lim_{x\to 0} x\beta'(x) \tag{3.24}$$

exists and

$$F'(u) = g'(u)\beta'(g(u)) - \frac{A}{u}, \quad u > 0. \tag{3.25}$$

b) Let $v \to 0$ in (3.22). Obviously,

$$\big(g'(u) - g'(u+v)\big)\beta'\big(g(u) - g(u+v)\big) = A_3 A_4$$

where

$$A_3 = \big(g(u) - g(u+v)\big)\beta'\big(g(u) - g(u+v)\big), \quad A_4 = \frac{g'(u) - g'(u+v)}{g(u) - g(u+v)}.$$

As $\lim_{v\to 0}\big(g(u) - g(u+v)\big) = 0$, (3.24) implies that $\lim_{v\to 0} A_3 = A$.
A direct calculation shows that :

$$g'(u) - g'(u+v) = -\frac{v(v+2u+1)}{u(u+1)(u+v)(u+v+1)}$$

then, $\lim_{v\to 0} A_4 = -\frac{1+2u}{u(1+u)}$.
From (3.23), we have $\lim_{v\to 0}\big(g(v) - g(u+v)\big) = \infty$, then, taking $v \to 0$ in (3.22) and using the above result imply that $B := \lim_{x\to\infty} \beta'(x)$ exists and

$$F'(u) = -\frac{1+2u}{u(1+u)}A + Bg'(u), \quad u > 0. \tag{3.26}$$

c) We now determine β. Using both (3.25) and (3.26) we get :

$$\beta'(g(u)) = -\frac{A}{(1+u)g'(u)} + B, \quad u > 0. \tag{3.27}$$

Setting $v = g(u)$, we have $u = f(v)$, $g'(u) = \frac{1}{f'(v)}$ and

$$\beta'(v) = B - \frac{Af'(v)}{1+f(v)} = B + \frac{A}{e^v - 1}, \quad v > 0.$$

Recall that $\beta = \log p_Y$ and p_Y is a density function. Then integrating the previous identity gives directly (3.21).

□

Lemma 3.5. *The density functions of U and X are respectively given by :*

$$p_U(u) = e^{L+J} u^{-A-B+M-1}(u+1)^{B-A-1} e^{2Nu} \quad u > 0, \tag{3.28}$$

$$p_X(x) = e^{C+I} 2^{2M} \exp\left(\frac{2N}{e^x - 1}\right) e^{(B-M-1)x}(1-e^{-x})^{A-2M-2}, \quad x > 0, \tag{3.29}$$

($I, J \in \mathbb{R}$ and $N < 0$) and this proves that U follows the Kummer distribution $K^{(2)}(-A-B+M, 2A-M+1, -2N)$.

Proof of Lemma 3.5
a) It is clear that using the definition of g, (3.26) may be written as

$$F'(u) = -(A+B)\frac{1}{u} + (B-A)\frac{1}{u+1}.$$

As a consequence,

$$F(u) = -(A+B)\log u + (B-A)\log(u+1) + J$$

where J is a constant. Thus, $k(u) = e^{F(u)} = u^{-A-B}(u+1)^{B-A}e^J$. Identity (3.28) follows from the definition of k (cf Lemma (3.1)) and (3.14).
b) Integrating (3.19) leads to

$$H(s) = Ns + M\log s + M\log(s+2) + I, \quad s > 0, \tag{3.30}$$

where I is a constant. Recall that $H(s) := \log\Big(h\Big(\log\big(1+\frac{2}{s}\big)\Big)\Big)$, therefore

$$h\left(\log\left(1+\frac{2}{s}\right)\right) = e^I e^{Ns} s^M (s+2)^M, \quad s > 0. \tag{3.31}$$

Setting $x = \log(1+\frac{2}{s})$ we have $s = \frac{2}{e^x-1}$ and

$$h(x) = e^I \exp\left(\frac{2N}{e^x-1}\right)\left(\frac{2}{e^x-1}\right)^M \left(\frac{2}{e^x-1}+2\right)^M, \quad x > 0. \tag{3.32}$$

As a result, by the definition of h (cf Lemma (3.1)) and (3.21)) we easily obtain (3.29).

□

Plugging the expressions of the four densities (3.14), (3.21), (3.28) and (3.29) in (3.10) we obtain $A = M$. This ends the proof of Theorem 2.1, with $A = M = b-1$, $B = -a$ and $N = -c/2$.

□

3.2 Auxiliary results

In this section we give a theoretical setting which allows to prove that the pairs (H,α) and (F,β) (resp. (F_δ,β)) introduced in Lemma 3.1 (resp. relation (3.39)) are of class C^1.

Lemma 3.6. *Let $\xi :]0,\infty[\to]0,\infty[$ be a bijection of class C^1 such that $\xi'(x) < 0$ and consider $\phi(u,v) := \big(\xi(u)-\xi(u+v),\ \xi(v)-\xi(u+v)\big)$. Denote $J_\phi(u,v)$ the determinant of the Jacobian of ϕ. Then*

$$J_\phi(u,v) = -\xi'(u)\xi'(v)\xi'(u+v)\big(\xi_1(u+v)-\xi_1(u)-\xi_1(v)\big), \quad u,v>0 \tag{3.33}$$

where $\xi_1(x) := -1/\xi'(x)$.

Proof of Lemma 3.6 The proof is straightforward.

□

Lemma 3.7. *Consider $\phi = (\phi_1,\phi_2) : (0,\infty)^2 \to (0,\infty)^2$ as in Lemma 3.6, satisfying moreover $J_\phi(u,v) \neq 0$ for any $u,v>0$ and :*

$$G(s)-G(t) = \theta\big(\phi_1(s,t)\big) - \theta\big(\phi_2(s,t)\big), \quad s,t>0. \tag{3.34}$$

where either θ or $G : (0,\infty)\to(0,\infty)$ is a locally integrable function over $(0,\infty)$.
Then G and θ are C^1.

Proof of Lemma 3.7
1) Let $y_0>0$. Let $0<t_0<\xi^{-1}(y_0)$ and $s_0 := \xi^{-1}\big(\xi(t_0)-y_0\big)-t_0$. Then $s_0>0$, $\phi_2(s_0,t_0)=y_0$ and $J_\phi(s_0,t_0)\neq 0$. Let us introduce $x_0 := \phi_1(s_0,t_0)$. Then, $\phi(s_0,t_0)=(x_0,y_0)$. Using the theorem of implicit functions, there exists $\psi=(\psi_1,\psi_2)$ of class C^1 such that $\psi(x_0,y_0)=(s_0,t_0)$ and :

$$\phi\circ\psi(x,y) = (x,y), \quad (x,y)\in V(x_0)\times V(y_0) \tag{3.35}$$

where $V(x_0)$ and $V(y_0)$ are some neighborhoods of x_0 and resp. y_0.
In particular, for any $y\in V(y_0)$ and $i=1,2$, the map $\psi_{i,y} : x\mapsto \psi_i(x,y)$ is C^1 and bijective on $V(x_0)$.
2) It is clear that (3.35) and (3.34) imply :

$$G\big(\psi_1(x,y)\big) - G\big(\psi_2(x,y)\big) = \theta(x)-\theta(y), \quad (x,y)\in V(x_0)\times V(y_0). \tag{3.36}$$

Consider any $x_1<x_0<x_2$ in $V(x_0)$ and suppose that G is locally integrable. We can integrate (3.36) over (x_1,x_2) with respect to x:

$$\int_{x_1}^{x_2} G\big(\psi_1(x,y)\big)\,dx - \int_{x_1}^{x_2} G\big(\psi_2(x,y)\big)\,dx = \int_{x_1}^{x_2}\theta(x)\,dx - (x_2-x_1)\theta(y). \tag{3.37}$$

After the change of variable $s=\psi_1(x,y)$ in the first integral and the change of variable $t=\psi_2(x,y)$ in the second integral, we get :

$$\int_{\psi_1(x_1,y)}^{\psi_1(x_2,y)} G(s)\big(\psi_{1,y}^{-1}\big)'(s)\,ds \quad - \quad \int_{\psi_2(x_1,y)}^{\psi_2(x_2,y)} G(t)\big(\psi_{2,y}^{-1}\big)'(t)\,dt$$
$$= \int_{x_1}^{x_2}\theta(x)\,dx - (x_2-x_1)\theta(y). \tag{3.38}$$

Taking absolute values in (3.36) implies that all the previous integrals are finite. The case where θ is locally integrable can be handled similarly.
Since the left-hand side of (3.38) is continuous in y (because ψ is C^1), the function θ is continuous. From the continuity of θ and (3.34) we deduce that G is continuous (because ϕ is continuous). Consequently, the left-hand side of (3.38) is a C^1 function in y, hence θ is C^1. We deduce, again from (3.34), that G is C^1. Our approach has been inspired by the one of [5].

□

From now on, we consider the particular cases $\xi = f$ and $\xi = g$. We have :

$$\xi_1(u+v) - \xi_1(u) - \xi_1(v) = \begin{cases} (1-e^{-u})(1-e^{-v})(e^{u+v}+1) & \text{if } \xi = f \\ 2uv & \text{if } \xi = g. \end{cases}$$

Therefore $J_\xi(u,v) \neq 0$.
Using the definition of h, we have $\log h = \log p_X - \log p_Y - \log(-f')$. Thus $\log h$ is locally integrable (f' is continuous). Then, relations (3.8), (3.12), Lemmas 3.6 and 3.7 imply Lemma 3.2.

3.3 Proof of Theorem 2.3

Let us assume that X and Y are independent and U and V are independent. Since f_δ is equal to its inverse then $X = f_\delta(U+V)$ and $Y = f_\delta(U) - f_\delta(U+V)$. Reasoning as in the proof of Lemma 3.1, we easily get :

$$F_\delta(u) - F_\delta(v) = \beta\big(f_\delta(u) - f_\delta(u+v)\big) - \beta\big(f_\delta(v) - f_\delta(u+v)\big), \quad u, v > 0. \tag{3.39}$$

where $k := -\dfrac{p_U}{p_V f_\delta'}$, $F_\delta := \log k$ and $\beta := \log p_Y$.
With $\xi := f_\delta$, we get :

$$\xi_1(u+v) - \xi_1(u) - \xi_1(v) = \frac{1}{\delta}(1-e^{-u})(1-e^{-v})(e^{u+v}+1-\delta).$$

Consequently, we deduce from (3.39) and Lemma 3.7 that β and F_δ are of class C^1. Next, taking the u-derivative in (3.39) leads to :

$$F_\delta'(u) = (f_\delta'(u) - f_\delta'(u+v))\beta'\big(f_\delta(u) - f_\delta(u+v)\big) + f_\delta'(u+v)\beta'\big(f_\delta(v) - f_\delta(u+v)\big), \quad u, v > 0. \tag{3.40}$$

The sequel of the proof of Theorem 2.3 is very similar to the one of Theorem 2.1. For this reason we just mention the main steps, without any details.
a) Letting $v \to \infty$ in (3.40) implies that $A := \lim_{x\to 0} x\beta'(x)$ exists and

$$F_\delta'(u) = f_\delta'(u)\beta'(f_\delta(u)) - \frac{A}{e^u - 1}, \quad u > 0. \tag{3.41}$$

b) Taking $v \to 0$ in (3.40) leads to

$$F_\delta'(u) = A\frac{1-\delta-e^{2u}}{(e^u-1)(e^u-1+\delta)} + Bf_\delta'(u), \quad u > 0. \tag{3.42}$$

where B is the real number defined as $B := \lim_{x\to\infty} \beta'(x)$.

c) Combining (3.41) with (3.42) we deduce :

$$f'_\delta(u)\beta'(f_\delta(u)) = -A\frac{e^u}{e^u - 1 + \delta} + Bf'_\delta(u).$$

Integrating the above relation and setting $v = f_\delta(u)$ we obtain :

$$\beta(v) = -A\log\left(\frac{\delta e^v}{e^v - 1}\right) + Bv + C, \quad v > 0 \tag{3.43}$$

where $C > 0$. Recall that $\beta = \log p_Y$, then

$$p_Y(v) = e^C\delta^{-A}(1 - e^{-v})^A e^{Bv}, \quad v > 0 \quad (A > -1, B < 0). \tag{3.44}$$

Note that $(X, Y) = \big(f_\delta(U + V), f_\delta(U) - f_\delta(U + V)\big)$ and recall that X, Y are independent and U, V are independent. Applying (3.44) with Y instead of V gives :

$$p_V(v) = e^{C'}\delta^{-A'}(1 - e^{-v})^{A'} e^{B'v} \quad v > 0 \quad (A' > -1, B' < 0, C' \in \mathbb{R}).$$

Then the computation of the densities of U and X is straightforward as in the proof of Theorem 2.1 and we have the desired result with $A = b - 1$, $B' = -\lambda$, $B = -a$, $A' = b - 1$.

□

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