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LOCAL BROWNIAN PROPERTY OF THE NARROW WEDGE SOLUTION OF THE KPZ EQUATION

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Abstract

Let $\mathscr{H}(t,x)$ be the Hopf-Cole solution at time t of the Kardar-Parisi-Zhang (KPZ) equation starting with narrow wedge initial condition, i.e. the logarithm of the solution of the multiplicative stochastic heat equation starting from a Dirac delta. Also let $\mathscr{H}^{\mathrm{eq}}(t,x)$ be the solution at time t of the KPZ equation with the same noise, but with initial condition given by a standard two-sided Brownian motion, so that $\mathscr{H}^{\mathrm{eq}}(t,x)-\mathscr{H}^{\mathrm{eq}}(0,x)$ is itself distributed as a standard two-sided Brownian motion. We provide a simple proof of the following fact: for fixed t, $\mathscr{H}(t,x)-\big(\mathscr{H}^{\mathrm{eq}}(t,x)-\mathscr{H}^{\mathrm{eq}}(t,0)\big)$ is locally of finite variation. Using the same ideas we also show that if the KPZ equation is started with a two-sided Brownian motion plus a Lipschitz function then the solution stays in this class for all time.

1 Introduction and statement of the results

The *KPZ equation*

$$\partial_t\mathscr{H}=-\frac{1}{2}\big(\partial_x\mathscr{H}\big)^2+\frac{1}{2}\partial_x^2\mathscr{H}+\dot{\mathscr{W}},\tag{1.1}$$

was introduced by [9] as a model of randomly growing interfaces. Here $\dot{\mathscr{W}}(t,x)$ is Gaussian space-time white noise, $\mathbb{E}\big(\dot{\mathscr{W}}(t,x)\mathscr{W}(s,y)\big)=\delta_{s=t}\delta_{x=y}$ (see Section 1.4 of [2] for a precise definition). It is expected that the one-dimensional KPZ equation appears as the weak asymptotic limit of a large class of stochastic interacting particle systems/growth models, including directed random polymers, stochastic Hamilton-Jacobi-Bellman equations, stochastically perturbed reaction-diffusion equations, stochastic Burgers equations and interacting particle models, and it is in fact rigourously known to describe the fluctuations in weakly asymmetric exclusion processes [3, 2, 5] and the partition function in directed polymer models [2, 1, 11]. All these models belong to the so-called *KPZ universality class*, which is associated with unusual fluctuations of order $t^{1/3}$ at time t on a spatial scale of $t^{2/3}$. We refer the reader to the reviews [4, 12] for more details and background on the KPZ equation and universality class.

As stated the KPZ equation (1.1) is ill-posed due to the non-linear term. To make sense of it we follow the approach of [3]. Observe that if we let $\mathscr{Z}(t,x) = \exp(-\mathscr{H}(t,x))$ then, formally, $\mathscr{Z}(t,x)$ solves the (linear) stochastic heat equation with multiplicative noise

$$\partial_t \mathscr{Z} = \frac{1}{2}\partial_x^2 \mathscr{Z} - \mathscr{Z}\dot{\mathscr{W}}. \tag{1.2}$$

Therefore we simply define the solutions $\mathscr{H}(t,x)$ of (1.1) via the Hopf-Cole transformation

$$\mathscr{H}(t,x) = -\log(\mathscr{Z}(t,x)), \tag{1.3}$$

where $\mathscr{Z}(t,x)$ is the (well-defined) solution of the stochastic PDE (1.2). In a remarkable recent development, M. Hairer [8] has proposed a way to make sense of the KPZ equation directly. The resulting solutions coincide with the Hopf-Cole solutions.

One of the most interesting properties of the KPZ equation is the preservation of Brownian initial data. In particular, if one starts the equation with a standard two-sided Brownian motion, one sees at time t a new Brownian motion with the same diffusivity, but with a (random) height shift (the new Brownian motion will of course be coupled to the starting one in a highly non-trivial way). Furthermore, any initial data, however smooth, will immediately become locally Brownian. This can be understood in many ways. One is that one expects the local quadratic variation to be the same as that of the equilibrium solutions, for any positive time, for arbitrarily nice initial data. Another is that one expects that the solution at time t can be written as a standard two-sided Brownian motion $\mathscr{B}(x)$ plus a more regular object. One would naturally like to take this Brownian motion $\mathscr{B}(x)$ to be the solution of the equation starting from a two-sided Brownian motion. In other words, one would like to couple all solutions to the equilibrium one. For a large class of initial data, Hairer [8] has shown that the solution can be written as a Brownian motion plus a function in $\mathscr{C}^{\frac{3}{2}-}$. Unfortunately, the Brownian motion used is a solution of the Langevin equation obtained by linearizing KPZ, as opposed to the equilibrium solution of KPZ itself. Moreover, not all initial data can be handled by the methods of [8] because they require that certain auxiliary objects be integrable against heat kernels in space and time. The singularity at time 0 rules out one of the most important cases, which is the narrow wedge initial data.

Our main interest will be this last case: the initial condition for KPZ given by starting the stochastic heat equation (1.2) with initial data

$$\mathscr{Z}(0,x) = \delta_{x=0}. \tag{1.4}$$

This defines $\mathscr{H}(t,x)$ for every $t > 0$ via (1.3), but one should not think in terms of the initial data for $\mathscr{H}$, since the delta function does not have a well-defined logarithm. This *narrow wedge* initial data is very basic. For example, it is the one that approximates the free energy of point-to-point polymers.

For this initial data, if we define $\mathscr{A}_t(x)$ by

$$\mathscr{H}(t,x) = \frac{x^2}{2t} + \log(\sqrt{2\pi t}) + \frac{t}{24} - 2^{-1/3}t^{1/3}\mathscr{A}_t(2^{-1/3}t^{2/3}x)$$

then, properly rescaled, $\mathscr{A}_t(x)$ converges to a Gaussian process as $t \to 0$ and it is conjectured that it converges to the Airy$_2$ process as $t \to \infty$ (see Conjecture 1.5 in [2]). $\mathscr{A}_t(x)$ is therefore referred to as the *crossover Airy$_2$ process*, and interpolates between the KPZ and Edwards-Wilkinson [6] universality classes, the last one associated with the stochastic heat equation with additive noise and hence Gaussian statistics.

We will denote by $\mathscr{H}^{\mathrm{eq}}(t,x)$ the solution of the KPZ equation (1.1) started with initial condition

$$\mathscr{H}^{\mathrm{eq}}(0,x) = \mathscr{B}(x), \tag{1.5}$$

where $\mathscr{B}(x)$ is a two-sided standard Brownian motion. We recall that this initial condition is such that, for each fixed $t \geq 0$, $\mathscr{H}^{\mathrm{eq}}(t,x) - \mathscr{H}^{\mathrm{eq}}(t,0)$ is itself a two-sided standard Brownian motion (in space), see Proposition B.2 in [3]. In this equation we use the same white noise as in the earlier solution starting with Dirac mass (1.4). The two solutions exist, are unique, and are coupled for all time.

Our main result is the following:

Theorem 1. *Fix $t > 0$ and let $\mathscr{H}(t,x)$ and $\mathscr{H}^{\mathrm{eq}}(t,x)$ be the Hopf-Cole solutions of the KPZ equation* (1.1) *with respect to the same white noise and with initial conditions given by* (1.4) *and* (1.5). *Then $\mathscr{H}(t,x) - \big(\mathscr{H}^{\mathrm{eq}}(t,x) - \mathscr{H}^{\mathrm{eq}}(t,0)\big)$ is a finite variation process.*

Our initial data is special, but it is in some sense the furthest possible from equilibrium, and it has the benefit of a surprisingly simple proof.

We remark that if $\mathscr{H}(t,x)$ is started with initial condition given by a two-sided Brownian motion plus a Lipschitz function then it is easy to show using our coupling method that it remains a two-sided Brownian motion plus a Lipschitz function for all $t > 0$. This follows from the results of Hairer [8], but the proof there is much more involved. The precise statement is given next, its short proof uses the same ideas as the proof of Theorem 1 and is given in Section 2.

Theorem 2. *Let $\mathscr{H}^{\mathrm{eq}}(t,x)$ and $\mathscr{H}(t,x)$ be the Hopf-Cole solutions of the KPZ equation* (1.1) *with respect to the same white noise and with initial conditions given respectively by* (1.5) *and*

$$\mathscr{H}(0,x) = \mathscr{B}(x) + \varphi(x),$$

where $\mathscr{B}(x)$ is the same two-sided standard Brownian motion as in (1.5) *and φ is a Lipschitz function. Then, for every fixed $t \geq 0$, $\mathscr{H}(t,x) - \big(\mathscr{H}^{\mathrm{eq}}(t,x) - \mathscr{H}^{\mathrm{eq}}(t,0)\big)$ is almost surely a Lipschitz function (with the same Lipschitz constant as φ). In particular, the law of $\mathscr{H}(t,x)$ in a finite interval has finite relative entropy with respect to the law of $\mathscr{B}(x)$ in that interval.*

2 Proofs

The proofs of Theorems 1 and 2 rely on considering the weakly asymmetric simple exclusion process, which provides a microscopic model for the KPZ process. The *simple exclusion process* with parameters $p, q \in [0,1]$ (such that $p + q = 1$) is a $\{0,1\}^{\mathbb{Z}}$-valued continuous time Markov process, where 1's are thought of as particles and 0's as holes. The dynamics of the process are as follows: each particle has an independent exponential clock with parameter 1; when the clock rings, the particle attempts a jump, trying to go one step to the right with probability p and one step to the left with probability q; if there is a particle at the chosen destination, the jump is supressed and the clock is reset. We refer the reader to [10] for a rigorous construction of this process. We will be interested in the case $q > p$, known as the *asymmetric simple exclusion process*

(ASEP). More precisely, we will be interested in the *weakly asymmetric simple exclusion process* (WASEP), where we introduce a parameter in the model and let the asymmetry $q-p$ go to 0 with the parameter.
Given any configuration $\eta \in \{0,1\}^{\mathbb{Z}}$ for the exclusion process we will denote by $\widehat{\eta} \in \{-1,1\}$ the configuration given by $\widehat{\eta}(x) = 2\eta(x) - 1$ for each $x \in \mathbb{Z}$. To any simple exclusion process η_t we can associate the *height function* $h(t,\cdot)\colon \mathbb{R} \longrightarrow \mathbb{Z}$ in the following manner:

$$h(t,x) = \begin{cases} 2N(t) + \sum_{0<y\le x} \widehat{\eta}_t(y) & \text{if } x > 0, \\ 2N(t) & \text{if } x = 0, \\ 2N(t) - \sum_{x<y\le 0} \widehat{\eta}_t(y) & \text{if } x < 0, \end{cases} \tag{2.1}$$

where $N(t)$ is the net number of particles which crossed from the site 1 to the site 0 up to time t.
It is straightforward to check that the simple exclusion process can be recovered from the height function by

$$\widehat{\eta}_t(x) = h(t,x) - h(t,x-1). \tag{2.2}$$

Next we introduce the scaling parameter $\varepsilon > 0$, which should be thought of as going to 0, and consider WASEP with asymmetry $\varepsilon^{1/2}$, that is,

$$p - q = \varepsilon^{1/2}, \qquad p = \tfrac{1}{2} - \tfrac{1}{2}\varepsilon^{1/2}, \qquad q = \tfrac{1}{2} + \tfrac{1}{2}\varepsilon^{1/2}.$$

We will denote by η_t^ε the resulting WASEP, which we start with the step initial condition

$$\eta_0(x) = \mathbf{1}_{x\ge 0}. \tag{2.3}$$

To η_t^ε we associate the *height function* $h_\varepsilon(t,x)$ via (2.1). Our main tool will be the convergence of a suitably rescaled version of h_ε to the solution of the KPZ equation with initial condition (1.4).
The convergence of the height function was proved by [2] by performing a microscopic Hopf-Cole transform analogous to (1.3), an idea introduced originally by [7] and further developed in [3].
Let

$$\begin{aligned} \gamma_\varepsilon = \tfrac{1}{2}\varepsilon^{-1/2}, \qquad \lambda_\varepsilon = \tfrac{1}{2}\log\left(\tfrac{p}{q}\right) = \varepsilon^{1/2} + \tfrac{1}{3}\varepsilon^{3/2} + O(\varepsilon^{5/2}), \\ \nu_\varepsilon = p + q - 2\sqrt{pq} = \tfrac{1}{2}\varepsilon^{1/2} + \tfrac{1}{8}\varepsilon^{3/2} + O(\varepsilon^{5/2}). \end{aligned} \tag{2.4}$$

The *Hopf-Cole transformed height function* is given by

$$Z_\varepsilon(t,x) = \gamma_\varepsilon \exp\Big(-\lambda_\varepsilon h_\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}x) + \nu_\varepsilon t \Big). \tag{2.5}$$

Observe that, with this definition, $Z_\varepsilon(0,x) \to \delta_{x=0}$ as $\varepsilon \to 0$ as discussed in Section 1.2 of [2].
We regard the process $Z_\varepsilon(t,x)$ as taking values in the space $D([0,\infty), D_u(\mathbb{R}))$, where $D_u(\mathbb{R})$ refers to right-continuous paths with left limits with the topology of uniform convergence on compact sets, which we endow with the Skorohod topology. The following result corresponds to Theorem 1.14 of [2]:

Theorem 3. *The family of processes* $(Z_\varepsilon)_{\varepsilon>0}$ *converges in distribution in* $D([0,\infty), D_u(\mathbb{R}))$ *as* $\varepsilon \to 0$ *to the* $C([0,\infty), C(\mathbb{R}))$*-valued process* $\mathscr{Z}$ *given by the solution of the stochastic heat equation* (1.2) *with initial condition* $\mathscr{Z}(0,x) = \delta_{x=0}$.

We recall that the exclusion process is *attractive*, which for our purposes means that two copies η_t^1 and η_t^2 of the process with initial conditions $\eta_0^1 \leq \eta_0^2$ (which just means $\eta_0^1(x) \leq \eta_0^2(x)$ for all x) can be coupled in such a way that $\eta_t^1 \leq \eta_t^2$ for all $t > 0$. We will refer to this coupling as the *basic coupling* and refer the reader to [10] for more details.

We will denote by η_t^{eq} a copy of WASEP in equilibrium, started with a product measure with density $\frac{1}{2}$, and by h^{eq} the associated height function. To prove Theorem 1 we will couple η_t^ε with η_t^{eq} using the basic coupling. The key result will be an estimate on the number of discrepancies between the two processes at time $\varepsilon^{-2}t$ in a window of size $O(\varepsilon^{-1})$, Proposition 2.1 below.

Let $\eta_t^{\min}$ and $\eta_t^{\max}$ denote copies of WASEP started with initial conditions

$$\eta_0^{\min} = \eta_0^\varepsilon \wedge \eta_0^{\mathrm{eq}} \qquad \text{and} \qquad \eta_0^{\max} = \eta_0^\varepsilon(x) \vee \eta_0^{\mathrm{eq}},$$

where the minimum and maximum are meant sitewise. Observe that $\eta_0^{\min}$ corresponds to starting with no particles on the negative half-line and a product measure of density $\frac{1}{2}$ on the positive half-line, while $\eta_0^{\max}$ corresponds to starting with a product measure of density $\frac{1}{2}$ on the negative half-line and all sites occupied on the positive half-line. We will denote by $h_\varepsilon^{\min}$ and $h_\varepsilon^{\max}$ the height functions associated respectively to these two processes.

Let $Z_\varepsilon^{\min}$, $Z_\varepsilon^{\max}$ and $Z_\varepsilon^{\mathrm{eq}}$ be the Hole-Copf transformed height functions associated to the corresponding initial conditions, which are defined in the same way as Z_ε in (2.5) with the scaling (2.4) except that $\gamma_\varepsilon = 1$. The proof in [2] of Theorem 3 can be adapted without difficulty (see [5] for the details) to show that that $Z_\varepsilon^{\min}$ and $Z_\varepsilon^{\max}$ converge in distribution in $D([0,\infty), D_u(\mathbb{R}))$ respectively to the solutions $\mathscr{Z}^{\min}(t,x)$ and $\mathscr{Z}^{\max}(t,x)$ of the stochastic heat equation (1.2) with initial data $\mathscr{Z}^{\min}(0,x) = \exp\big(-\mathscr{B}(x)\big)\mathbf{1}_{x\geq 0}$ and $\mathscr{Z}^{\max}(0,x) = \exp\big(-\mathscr{B}(-x)\big)\mathbf{1}_{x<0}$, where $\mathscr{B}(x)$ is a standard one-sided Brownian motion. We define $\mathscr{H}^{\min}(t,x) = -\log(\mathscr{Z}^{\min}(t,x))$ and $\mathscr{H}^{\max}(t,x) = -\log(\mathscr{Z}^{\max}(t,x))$.

Given any of the height functions h with the different initial conditions we are considering, we will denote by $\tilde{h}_\varepsilon$ its rescaled version

$$\tilde{h}_\varepsilon(t,x) = \varepsilon^{1/2} h(\varepsilon^{-2}t, \varepsilon^{-1}x).$$

Proposition 2.1. *Assume η_t^ε is started with the step initial condition* (2.3) *and fix $a < b$ and $t > 0$. Then, under the basic coupling,*

$$\varepsilon^{1/2} \sum_{x\in[a\varepsilon^{-1}, b\varepsilon^{-1}]\cap\mathbb{Z}} \left|\eta^\varepsilon_{\varepsilon^{-2}t}(x) - \eta^{\mathrm{eq}}_{\varepsilon^{-2}t}(x)\right| \leq \frac{1}{2}\left[\tilde{h}_\varepsilon^{\max}(t,b) - \tilde{h}_\varepsilon^{\max}(t,a)\right] - \frac{1}{2}\left[\tilde{h}_\varepsilon^{\min}(t,b) - \tilde{h}_\varepsilon^{\min}(t,a)\right]$$

almost surely.

Proof. We construct the four processes η_t^ε, η_t^{eq}, $\eta_t^{\min}$ and $\eta_t^{\max}$ together under the basic coupling, so attractiveness implies that

$$\eta_t^{\min} \leq \eta_t^\varepsilon \wedge \eta_t^{\mathrm{eq}} \leq \eta_t^\varepsilon \vee \eta_t^{\mathrm{eq}} \leq \eta_t^{\max}$$

for all $t > 0$. Using this we get by (2.2) that

$$\sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \left|\eta^{\varepsilon}_{\varepsilon^{-2}t}(x) - \eta^{\mathrm{eq}}_{\varepsilon^{-2}t}(x)\right| \tag{2.6}$$

$$= \sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \left[\eta^{\varepsilon}_{\varepsilon^{-2}t}(x) \vee \eta^{\mathrm{eq}}_{\varepsilon^{-2}t}(x) - \eta^{\varepsilon}_{\varepsilon^{-2}t}(x) \wedge \eta^{\mathrm{eq}}_{\varepsilon^{-2}t}(x)\right] \tag{2.7}$$

$$\leq \sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \left[\eta^{\max}_{\varepsilon^{-2}t}(x) - \eta^{\min}_{\varepsilon^{-2}t}(x)\right] = \frac{1}{2}\sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \left[\widehat{\eta}^{\max}_{\varepsilon^{-2}t}(x) - \widehat{\eta}^{\min}_{\varepsilon^{-2}t}(x)\right] \tag{2.8}$$

$$= \frac{1}{2}\sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \Big[\Big(h^{\max}_{\varepsilon}(\varepsilon^{-2}t,x) - h^{\max}_{\varepsilon}(\varepsilon^{-2}t,x-1)\Big) \tag{2.9}$$

$$- \Big(h^{\min}_{\varepsilon}(\varepsilon^{-2}t,x) - h^{\min}_{\varepsilon}(\varepsilon^{-2}t,x-1)\Big)\Big] \tag{2.10}$$

$$= \frac{\varepsilon^{-1/2}}{2}\left[\tilde{h}^{\max}_{\varepsilon}(t,b) - \tilde{h}^{\max}_{\varepsilon}(t,a)\right] - \frac{\varepsilon^{-1/2}}{2}\left[\tilde{h}^{\min}_{\varepsilon}(t,b) - \tilde{h}^{\min}_{\varepsilon}(t,a)\right]. \tag{2.11}$$

Multiplying by $\varepsilon^{1/2}$ we obtain the desired bound. □

Proof of Theorem 1. Fix a finite interval $I = [a,b]$ and let $\mathrm{TV}_I(f)$ denote the total variation of f in I:

$$\mathrm{TV}_I(f) = \sup_{a=x_0<x_1<\cdots<x_n=b,\, n\in\mathbb{N}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Then clearly

$$\mathrm{TV}_I(\tilde{h}_\varepsilon(t,\cdot) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,\cdot)) = 2\varepsilon^{1/2}\sum_{x\in[a\varepsilon^{-1},b\varepsilon^{-1}]\cap\mathbb{Z}} \left|\eta^{\varepsilon}_{\varepsilon^{-2}t}(x) - \eta^{\mathrm{eq}}_{\varepsilon^{-2}t}(x)\right|,$$

so by Proposition 2.1 we get

$$\mathrm{TV}_I(\tilde{h}_\varepsilon(t,\cdot) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,\cdot)) \leq \left[\tilde{h}^{\max}_{\varepsilon}(t,b) - \tilde{h}^{\max}_{\varepsilon}(t,a)\right] - \left[\tilde{h}^{\min}_{\varepsilon}(t,b) - \tilde{h}^{\min}_{\varepsilon}(t,a)\right].$$

On the other hand

$$\tilde{h}_\varepsilon(t,x) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,x) = -\varepsilon^{1/2}\lambda_\varepsilon^{-1}\left[\log(Z_\varepsilon(t,x)) - \log(Z^{\mathrm{eq}}_\varepsilon(t,x))\right] + \varepsilon^{-1/2}\lambda_\varepsilon^{-1}\log(\gamma_\varepsilon).$$

Thus by Theorem 3 and Theorem 2.3 in [3] we get that $\tilde{h}_\varepsilon(t,x) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,x) - \varepsilon^{-1/2}\lambda_\varepsilon^{-1}\log(\gamma_\varepsilon)$ converges in distribution to $\mathscr{H}(t,x) - \mathscr{H}^{\mathrm{eq}}(t,x)$ on the interval I. Note that this requires a very minor extension of the results of [2] and [3], namely that the processes $\tilde{h}_\varepsilon$ and $\tilde{h}^{\mathrm{eq}}_\varepsilon$ built from exclusion processes running with the same background Poisson processes converge jointly to $\mathscr{H}$ and $\mathscr{H}^{\mathrm{eq}}$. There are no issues involved in extending Theorem 3 and Theorem 2.3 in [3] to this situation and therefore we omit the details.

By the lower semicontinuity of TV_I we deduce that

$$\mathbb{P}\Big(\mathrm{TV}_I\big(\mathscr{H}(t,\cdot) - [\mathscr{H}^{\mathrm{eq}}(t,\cdot) - \mathscr{H}^{\mathrm{eq}}(t,0)]\big) > K\Big) = \mathbb{P}\Big(\mathrm{TV}_I\big(\mathscr{H}(t,\cdot) - \mathscr{H}^{\mathrm{eq}}(t,\cdot)\big) > K\Big) \tag{2.12}$$

$$\leq \limsup_{\varepsilon\to 0}\mathbb{P}\Big(\mathrm{TV}_I\Big(\tilde{h}_\varepsilon(t,\cdot) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,\cdot) - \varepsilon^{-1/2}\lambda_\varepsilon^{-1}\log(\gamma_\varepsilon)\Big) > K\Big) \tag{2.13}$$

$$= \limsup_{\varepsilon\to 0}\mathbb{P}\Big(\mathrm{TV}_I\Big(\tilde{h}_\varepsilon(t,\cdot) - \tilde{h}^{\mathrm{eq}}_\varepsilon(t,\cdot)\Big) > K\Big) \tag{2.14}$$

$$\leq \limsup_{\varepsilon\to 0}\mathbb{P}\Big(\big(\tilde{h}^{\max}_{\varepsilon}(t,b) - \tilde{h}^{\max}_{\varepsilon}(t,a)\big) - \big(\tilde{h}^{\min}_{\varepsilon}(t,b) - \tilde{h}^{\min}_{\varepsilon}(t,a)\big) > K\Big). \tag{2.15}$$

To finish the proof of Theorem 1 we observe that the quantity inside the last probability above equals

$$-\varepsilon^{1/2}\lambda_\varepsilon^{-1}\left[\log(Z_\varepsilon^{\max}(t,b))-\log(Z_\varepsilon^{\max}(t,a))-\log(Z_\varepsilon^{\min}(t,b))+\log(Z_\varepsilon^{\min}(t,a))\right].$$

Note the key point that the additive constants in (2.5) cancel. Using the convergence of $Z_\varepsilon^{\min}$ and $Z_\varepsilon^{\max}$ to solutions of the stochastic heat equation discussed before Proposition 2.1, the above converges in distribution to $\left[\mathscr{H}^{\max}(t,b)-\mathscr{H}^{\max}(t,a)\right]-\left[\mathscr{H}^{\min}(t,b)-\mathscr{H}^{\min}(t,a)\right]$. Since the last random variable is finite we deduce that

$$\lim_{K\to\infty}\mathbb{P}\big(\mathrm{TV}_I\big(\mathscr{H}(t,\cdot)-\left[\mathscr{H}^{\mathrm{eq}}(t,\cdot)-\mathscr{H}^{\mathrm{eq}}(t,0)\right]\big)>K\big)=0. \qquad \square$$

Proof of Theorem 2. Let η_t be a copy of WASEP started with product measure with density profile given by

$$\mathbb{P}(\eta_0(x+1)=1)=\frac{1}{2}+\frac{1}{2}\varepsilon^{-1/2}\left(\varphi(\varepsilon x)-\varphi(\varepsilon(x-1))\right).$$

Then Theorem 2.3 of [3] implies that $\tilde{h}_\varepsilon(t,x)$ converges in distribution in $D([0,\infty),D_u(\mathbb{R}))$ as $\varepsilon\to 0$ to $\mathscr{H}(t,x)$ (with initial condition as in the statement of the theorem). Now denote by M the Lipschitz constant of φ and let $\tilde{h}_\varepsilon^+$ and $\tilde{h}_\varepsilon^-$ denote the rescaled height functions corresponding to WASEP started respectively with product measures of densities $\frac{1}{2}(1+\varepsilon^{1/2}M)$ and $\frac{1}{2}(1-\varepsilon^{1/2}M)$. Coupling the initial conditions in the natural way and using the basic coupling and attractiveness, it is clear that $\tilde{h}_\varepsilon^-(t,x)\le\tilde{h}_\varepsilon(t,x)\le\tilde{h}_\varepsilon^+(t,x)$ for all $t>0$. On the other hand, since product measures are invariant for WASEP, $\tilde{h}_\varepsilon^\pm(t,x)$ converges in distribution to $\mathscr{H}^{\mathrm{eq}}(t,x)\pm Mx$, and as before this convergence can be achieved jointly for $\tilde{h}_\varepsilon$, $\tilde{h}_\varepsilon^+$ and $\tilde{h}_\varepsilon^-$. Therefore, given any $a<b$, $\mathscr{H}^{\mathrm{eq}}(t,x)-Mx\le\mathscr{H}(t,x)\le\mathscr{H}^{\mathrm{eq}}(t,x)+Mx$ almost surely for every $x\in[a,b]$, and the result follows. $\square$

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