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Central limit theorems for the L^2 norm of increments of local times of Lévy processes*

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Abstract

Let $X = \{X_t, t \in R_+\}$ be a symmetric Lévy process with local time $\{L_t^x\,;\,(x,t) \in R^1 \times R_+^1\}$. When the Lévy exponent $\psi(\lambda)$ is regularly varying at zero with index $1 < \beta \le 2$, and satisfies some additional regularity conditions,

$$\lim_{t\to\infty} \frac{\int_{-\infty}^{\infty} (L_t^{x+1} - L_t^x)^2\,dx - E\left(\int_{-\infty}^{\infty} (L_t^{x+1} - L_t^x)^2\,dx\right)}{t\sqrt{\psi^{-1}(1/t)}}$$
$$\stackrel{\mathcal{L}}{=} (8c_{\psi,1})^{1/2} \left(\int_{-\infty}^{\infty} \left(L_{\beta,1}^x\right)^2\,dx\right)^{1/2} \eta,$$

where $L_{\beta,1} = \{L_{\beta,1}^x\,;\,x \in R^1\}$ denotes the local time, at time 1, of a symmetric stable process with index β, η is a normal random variable with mean zero and variance one that is independent of $L_{\beta,1}$, and $c_{\psi,1}$ is a known constant that depends on ψ.

When the Lévy exponent $\psi(\lambda)$ is regularly varying at infinity with index $1 < \beta \le 2$ and satisfies some additional regularity conditions

$$\lim_{h\to 0} \sqrt{h\psi^2(1/h)} \left\{\int_{-\infty}^{\infty} (L_1^{x+h} - L_1^x)^2\,dx - E\left(\int_{-\infty}^{\infty} (L_1^{x+h} - L_1^x)^2\,dx\right)\right\}$$
$$\stackrel{\mathcal{L}}{=} (8c_{\beta,1})^{1/2}\ \eta\ \left(\int_{-\infty}^{\infty} (L_1^x)^2\,dx\right)^{1/2},$$

where η is a normal random variable with mean zero and variance one that is independent of $\{L_1^x, x \in R^1\}$, and $c_{\beta,1}$ is a known constant.

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1 Introduction

The earliest result we know about the asymptotic behavior in time, of increments of local times in the spatial variable, is due to Dobrushin, [6]. Let $\{S_n\,;\,n=0,1,2,\ldots\}$ be a simple random walk on Z^1 and let $\ell_n^x=\sum_{j=1}^n 1_{\{S_j=x\}}$ denote its local time. Dobrushin shows that

$$\lim_{n\to\infty}\frac{\ell_n^1-\ell_n^0}{n^{1/4}}\stackrel{\mathcal{L}}{=}(2|Z|)^{1/2}\,\eta, \tag{1.1}$$

where Z and η are independent normal random variables with mean zero and variance one. Two aspects of this result are relevant to this paper. One is that it is a result about fluctuations, since ℓ_n^0 grows like $n^{1/2}$, (see for example [16, (10.1), (9.13)]). The other is that the right-hand side of (1.1) is not a standard normal random variable, but is the product of a standard normal random variable and an independent random variable. Extensions of (1.1) to the local time of Brownian motion and other processes can be found in Révész, [16, (11.10), (12.17), (12.19)], Marcus and Rosen, [13, 14], Rosen, [17] and Yor, [19].

One of the motivations for considering increments of local times is interest in the Hamiltonian for the critical attractive random polymer in one dimension, [8, 9],

$$H_n=\sum_{x\in Z^1}\left(\ell_n^{x+1}-\ell_n^x\right)^2, \tag{1.2}$$

where ℓ_n^x is as defined above. Clearly, this is the square of the ℓ^2 norm of the increments of the local time at time n.

We began our study of expressions like (1.2) in [4], with X. Chen and W. Li, by considering the continuous version of this problem for the local times of Brownian motion.

Let $\{L_t^x\,;\,(x,t)\in R^1\times R^1_+\}$ denote the local times of Brownian motion. In [4] we show that

$$\lim_{t\to\infty}\frac{\int_{-\infty}^{\infty}(L_t^{x+1}-L_t^x)^2\,dx-4t}{t^{3/4}}\stackrel{\mathcal{L}}{=}(64/3)^{1/2}\left(\int_{-\infty}^{\infty}\left(L_1^x\right)^2\,dx\right)^{1/2}\eta, \tag{1.3}$$

where η is a normal random variable with mean zero and variance one that is independent of $\{L_1^x,x\in R^1\}$.

The proof of this result in [4] makes extensive use of the scaling property of Brownian motion. A different proof in [18] uses stochastic integrals and a theorem of Papanicolaou, Stroock, and Varadhan, [15, Chapter XIII]. Neither of these approaches can be used to extend (1.3) to general Lévy processes. In this paper we use the method of moments.

Let $X=\{X_t,t\in R_+\}$ be a symmetric Lévy process with characteristic function

$$E\left(e^{i\lambda X_t}\right)=e^{-\psi(\lambda)t} \tag{1.4}$$

and local time which we continue to denote by $\{L_t^x\,;\,(x,t)\in R^1\times R^1_+\}$. The behavior of a suitably scaled version of $\int_{\infty}^{\infty}(L_t^{x+1}-L_t^x)^2\,dx$ as t goes to infinity depends primarily on the behavior of $\psi(\lambda)$ as λ goes to 0. This is not surprising, since large time properties of X, such as transience and recurrence, depend on the behavior of $\psi(\lambda)$ as λ goes to 0; (see [1, Chapter 1, Theorem 17], which shows, in particular, that the processes we consider are recurrent).

We assume that $\psi(\lambda)$ satisfies the following conditions:

1. $\psi(\lambda)$ is regularly varying at 0 with index $1<\beta\le 2$; (1.5)

2. $$\int_{-\infty}^{\infty}\frac{1}{1+\psi(\lambda)}\,d\lambda<\infty; \tag{1.6}$$

3. ψ is twice differentiable almost everywhere, and there exist constants $D_1, D_2 < \infty$ such that for $0 < \lambda \leq 1$

$$\lambda|\psi'(\lambda)| \leq D_1\psi(\lambda) \quad \text{and} \quad \lambda^2|\psi''(\lambda)| \leq D_2\psi(\lambda) \tag{1.7}$$

and

$$\int_1^\infty \frac{|\psi'(\lambda)|}{\psi^2(\lambda)}\,d\lambda < \infty, \quad \int_1^\infty \frac{|\psi'(\lambda)|^2}{\psi^2(\lambda)}\,d\lambda < \infty, \quad \int_1^\infty \frac{|\psi''(\lambda)|}{\psi(\lambda)}\,d\lambda < \infty. \tag{1.8}$$

(Condition 1. is substantive. Condition 2. is the necessary and sufficient condition for a symmetric Lévy process to have a local time. The criteria in Condition 3. are rather weak.)

We prove the following theorem:

Theorem 1.1 *Let $\{L_t^x\,;\,(x,t) \in R^1 \times R_+^1\}$ be the local time of a symmetric Lévy process X, with Lévy exponent $\psi(\lambda)$, that is regularly varying at zero with index $1 < \beta \leq 2$ and satisfies (1.6)–(1.8). Then*

$$\lim_{t\to\infty} \frac{\int_{-\infty}^\infty (L_t^{x+1} - L_t^x)^2\,dx - E\left(\int_{-\infty}^\infty (L_t^{x+1} - L_t^x)^2\,dx\right)}{t\sqrt{\psi^{-1}(1/t)}} \tag{1.9}$$

$$\stackrel{\mathcal{L}}{=} (8c_{\psi,1})^{1/2}\left(\int_{-\infty}^\infty \left(L_{\beta,1}^x\right)^2\,dx\right)^{1/2}\eta,$$

where $L_{\beta,1} = \{L_{\beta,1}^x\,;\,x \in R^1\}$ is the local time, at time 1, of a symmetric stable process of index β, η and $L_{\beta,1}$ are independent, and

$$c_{\psi,1} = \frac{16}{\pi}\int_0^\infty \frac{\sin^4 p/2}{\psi^2(p)}\,dp. \tag{1.10}$$

(Since ψ is regularly varying at zero, it is asymptotic to a monotonic function at zero. We define ψ^{-1} as the inverse of this function.)

It follows from Lemma 3.2, in this paper, that

$$E\left(\int_{-\infty}^\infty (L_t^{x+1} - L_t^x)^2\,dx\right) = 4c_{\psi,0}t + o\left(t\sqrt{\psi^{-1}(1/t)}\right), \tag{1.11}$$

where

$$c_{\psi,0} = \frac{2}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{\psi(p)}\,dp. \tag{1.12}$$

Therefore, we can replace the mean in (1.9) by $4c_{\psi,0}t$.

In Remark 2.5, we evaluate the constants and make the necessary changes to verify that when X is Brownian motion, (1.9) along with (1.11), is the same as (1.3).

Also note that by Lemma 2.4 $\int_{-\infty}^\infty (L_t^x)^2\,dx$ grows like $t^2\psi^{-1}(1/t)$, therefore (1.9) is also a fluctuation result.

One can use the scaling relationship for the local times of β stable processes,

$$\{L_{\beta,t/\delta^\beta}^x\,;\,(x,t) \in R^1 \times R_+^1\} \stackrel{\mathcal{L}}{=} \{\delta^{-(\beta-1)}L_{\beta,t}^{\delta x}\,;\,(x,t) \in R^1 \times R_+^1\}, \tag{1.13}$$

in Theorem 1.1 to get a central limit theorem for the L^2 modulus of continuity of local times of symmetric stable processes:

$$\lim_{h\to 0} \frac{\int_{-\infty}^\infty (L_{\beta,1}^{x+h} - L_{\beta,1}^x)^2\,dx - 4c_{\psi,0}h^{\beta-1}}{h^{(2\beta-1)/2}} \tag{1.14}$$

$$\stackrel{\mathcal{L}}{=} (8c_{\psi,1})^{1/2}\left(\int_{-\infty}^\infty \left(L_{\beta,1}^x\right)^2\,dx\right)^{1/2}\eta,$$

where $\psi(\lambda)=|\lambda|^\beta$.

We were intrigued to obtain this result under much more general hypotheses, similar to those in Theorem 1.1. We assume that

1. $\psi(\lambda)$ is regularly varying at infinity with index $1<\beta\leq 2$; (1.15)

2. ψ is twice differentiable almost surely and there exist constants $D_1, D_2 < \infty$ such that for all $\lambda\geq 1$

$$\lambda|\psi'(\lambda)|\leq D_1\psi(\lambda) \quad \text{and} \quad \lambda^2|\psi''(\lambda)|\leq D_2\psi(\lambda) \tag{1.16}$$

and

$$\int_0^1 (\psi'(\lambda))^2\,d\lambda<\infty, \quad \int_0^1 |\psi''(\lambda)|\,d\lambda<\infty. \tag{1.17}$$

3.

$$\int_0^1 \frac{\psi(\lambda)}{\lambda}\,d\lambda<\infty. \tag{1.18}$$

We obtain the following theorem:

Theorem 1.2 *Let $\{L_t^x\,;\,(x,t)\in R^1\times R^1_+\}$ be the local time of the symmetric Lévy process X with Lévy exponent $\psi(\lambda)$ that satisfies (1.15)–(1.18). Then*

$$\begin{aligned}\lim_{h\to 0}\sqrt{h\psi^2(1/h)}\left\{\int_{-\infty}^{\infty}(L_1^{x+h}-L_1^x)^2\,dx-E\left(\int_{-\infty}^{\infty}(L_1^{x+h}-L_1^x)^2\,dx\right)\right\}&\\ \stackrel{\mathcal{L}}{=}(8c_{\beta,1})^{1/2}\left(\int_{-\infty}^{\infty}(L_1^x)^2\,dx\right)^{1/2}\eta,,&\end{aligned} \tag{1.19}$$

where

$$c_{\beta,1}=\frac{16}{\pi}\int_0^\infty \frac{\sin^4 p/2}{p^{2\beta}}\,dp. \tag{1.20}$$

For symmetric stable processes we can give the mean explicitly and get (1.14).

The conditions in (1.16)–(1.18) are very general. Only the regularly varying condition, (1.15), is restrictive. However, it is not surprising that Theorem 1.2 depends on the behavior of $\psi(\lambda)$ as λ goes to infinity, since the behavior of $\psi(\lambda)$ as λ goes to infinity controls the small jumps of X.

A key ingredient in much of our work on sample path properties of local times is the Eisenbaum Isomorphism Theorem which allows us transfer results on Gaussian processes to the local times of related symmetric Markov processes. Unfortunately this approach, which works so well for many almost sure results, is ineffective for weak limits. Instead we obtain both Theorems 1.1 and 1.2 using the method of moments. At first thought one might think that the proofs would be similar, but they are not. It turns out that estimating the size of very small increments and the behavior of a fixed increment as time goes to infinity requires very different sets of inequalities.

The proofs are quite detailed and very long. In all they require 111 pages. In discussions with the referees and the editor we decided to write a brief paper, this one, that states the main theorems and gives an overview of their proofs, and to relegate the meat of the proofs to an Appendix, which a highly motivated reader may wish to tackle.

In Section 2 we show how the proof of Theorem 1.1 follows from four basic lemmas. In Section 3 we discuss the main ingredients needed to prove them. Similarly, in Section 4 we show how the proof of Theorem 1.2 follows from a different set of lemmas. The details of the proofs are contained in the Appendix to this paper. The sections of the

Appendix are labeled by letters A-K. A reference in this paper to equation (D.4), for example, means an equation in the Appendix, Section D. Similarly, for references to lemmas, theorems, etc.

We would like to point out that the main ideas in the proof Theorem 1.1 can also be used to obtain a central limit theorem for the Hamiltonian of the critical attractive random polymer in one dimension, which is usually written as

$$H_n = 2\sum_{i,j=1}^{n} 1_{\{S_i=S_j\}} - \sum_{i,j=1}^{n} 1_{\{|S_i-S_j|=1\}}. \tag{1.21}$$

It is easy to see that this is the same as (1.2). We can show that

$$\lim_{n\to\infty} \frac{H_n - 4n}{n^{3/4}} \stackrel{\mathcal{L}}{\Longrightarrow} (12)^{1/2} \left(\int_{-\infty}^{\infty} (L_1^x)^2 \, dx \right)^{1/2} \eta, \tag{1.22}$$

where $\{L_1^x, x \in R^1\}$ is the local time of Brownian motion at time 1.

More generally, we can find a version of Theorem 1.1 for the local times of a large class of symmetric random walks. Let S_n be a 1-dimensional symmetric random walk. Assume that S_n is in the domain of attraction of a symmetric stable process $\{X(t), t \in R_+\}$ of index $1 < \beta \le 2$, or equivalenty, that

$$\lim_{n\to\infty} \frac{S_n}{b(n)} = X(1), \tag{1.23}$$

where $b(x)$ is a regularly varying function at infinity with index $1/\beta$. Let

$$\phi(\lambda) = E\left(e^{i\lambda S_1}\right). \tag{1.24}$$

Therefore, for some $\delta > 0$

$$\phi(\lambda) = e^{-\psi(\lambda)}, \qquad |\lambda| \le \delta, \tag{1.25}$$

where $\psi(\lambda)$ is regularly varying at zero with index β. (See, e.g., [10, Proposition 2.3].)

Assume for simplicity that S_n is strongly aperiodic. It follows from this that for some $\gamma > 0$

$$|\phi(\lambda)| \le e^{-\gamma}, \qquad |\lambda| \ge \delta. \tag{1.26}$$

Assume also that $\phi(\lambda)$ is twice continuously differentiable for $\lambda \ne 0$, $\phi'(0) = 0$, and for some $\delta > 0$ there exist constants $D_1, D_2 < \infty$ such that for $0 < \lambda \le \delta$

$$\lambda|\psi'(\lambda)| \le D_1\psi(\lambda), \quad \lambda^2|\psi''(\lambda)| \le D_2\psi(\lambda). \tag{1.27}$$

Let

$$L_n^x = \sum_{i=1}^{n} 1_{\{S_i=x\}}. \tag{1.28}$$

and

$$c_{\phi,1} = \frac{16}{\pi} \int_0^{\pi} \frac{\sin^2 p/2}{(1-\phi(p))^2} \, dp. \tag{1.29}$$

Theorem 1.3 *Let $\{L_n^x\,;\,(x,n) \in Z^1 \times Z_+^1\}$ be the local times of a symmetric random walk S_n that satisfies (1.25)–(1.27). Then*

$$\lim_{n\to\infty} \frac{\sum_{x\in Z^1}(L_n^{x+1}-L_n^x)^2 - E\left(\sum_{x\in Z^1}(L_n^{x+1}-L_n^x)^2\right)}{n/\sqrt{b(n)}} \tag{1.30}$$
$$\stackrel{\mathcal{L}}{=} (8c_{\phi,1})^{1/2} \left(\int_{-\infty}^{\infty} \left(L_{\beta,1}^x\right)^2 dx \right)^{1/2} \eta,$$

where $L_{\beta,\cdot}^x$ and η are independent.

We do not give a proof this theorem. It follows along the lines of the proof of Theorem 1.1.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 is long and difficult. In order to make it easier to follow we first present the main steps of the proof heuristically. We then restate them, precisely, in a series of lemmas and show how Theorem 1.1 follows from these lemmas. In Section 3 we 'prove' these lemmas using several other fundamental results that are proved in the Appendix, Sections A–E.

As usual, let θ_t denote time translation of the path ω, so that $\theta_t\omega(r) = \omega(t+r)$. Let

$$I_{j,k,t} := \int (L_t^{x+1} - L_t^x) \circ \theta_{jt} \; (L_t^{x+1} - L_t^x) \circ \theta_{kt} \; dx \tag{2.1}$$

and

$$\alpha_{j,k,t} := \int L_t^x \circ \theta_{jt} \; L_t^x \circ \theta_{kt} \; dx. \tag{2.2}$$

(An integral sign without limits is to be read as $\int_{-\infty}^{\infty}$.)

For any integer l set

$$\widetilde{I}_{l,t} := \sum_{\substack{j,k=0\\ j<k}}^{l-1} I_{j,k,t/l}. \tag{2.3}$$

Using the additivity property of local times we can write

$$L_t^x = \sum_{j=0}^{l-1} L_{t/l}^x \circ \theta_{jt/l}, \tag{2.4}$$

so that

$$\int (L_t^{x+1} - L_t^x)^2 \; dx = \sum_{j,k=0}^{l-1} I_{j,k,t/l} = 2\widetilde{I}_{l,t} + \sum_{j=0}^{l-1} I_{j,j,t/l}. \tag{2.5}$$

Consequently

$$\begin{aligned} &\int (L_t^{x+1} - L_t^x)^2 \, dx - E\left(\int (L_t^{x+1} - L_t^x)^2 \, dx\right) \\ &\quad = 2\widetilde{I}_{l,t} + \left\{ \sum_{j=0}^{l-1} I_{j,j,t/l} - E\left(\int (L_t^{x+1} - L_t^x)^2 \, dx\right)\right\}. \end{aligned} \tag{2.6}$$

Similarly we set

$$\widetilde{\alpha}_{l,t} = \sum_{\substack{j,k=0\\ j<k}}^{l-1} \alpha_{j,k,t/l}, \tag{2.7}$$

and write

$$\alpha_t := \int (L_t^x)^2 \; dx = \sum_{j,k=0}^{l-1} \alpha_{j,k,t/l} = 2\widetilde{\alpha}_{l,t} + \sum_{j=0}^{l-1} \alpha_{j,j,t/l}. \tag{2.8}$$

The main steps in the proof of Theorem 1.1 are to show that:

1. The 'off-diagonal' terms $\widetilde{I}_{l,t}$ and $\sqrt{\widetilde{\alpha}_{l,t}}$ are comparable asymptoticly as $t \to \infty$.
2. The diagonal term $\sum_{j=0}^{l-1} \alpha_{j,j,t/l}$ is negligible, as $t \to \infty$, compared to the terms in 1.

3. The diagonal term $\sum_{j=0}^{l-1} I_{j,j,t/l}$ is such that

$$\sum_{j=0}^{l-1} I_{j,j,t/l} - E\left(\int (L_t^{x+1} - L_t^x)^2\,dx\right) \tag{2.9}$$

is negligible, as $t \to \infty$, compared to the terms in 1.

We now explain the precise meaning of these statements, and show how they imply Theorem 1.1.

The precise meaning of step 1. is given by the following lemma. (Lemmas 2.1–2.3 are proved in Section 3.)

Lemma 2.1 *Under the hypotheses of Theorem 1.1, for each m, with $l = l(t) = [\log t]^q$, for any $q > 0$,*

$$E\left(\left(\widetilde{I}_{l,t}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}\,(4c_{\psi,1})^n\,E\left\{(\widetilde{\alpha}_{l,t})^n\right\} + o((t^2\psi^{-1}(1/t))^n) & \text{if } m = 2n \\ O((t^2\psi^{-1}(1/t))^{m/2}t^{-\epsilon}) & \textit{otherwise.} \end{cases} \tag{2.10}$$

This lemma is the crux of the proof of Theorem 1.1. We note that even though the summands $I_{j,k,t/l}$ of $\widetilde{I}_{l,t}$ are not independent, the fact that $j \neq k$ provides enough structure for a proof.

The precise meaning of step 2. is given by the next lemma.

Lemma 2.2 *Under the hypotheses of Theorem 1.1, for each n, with $l = l(t) = [\log t]^q$, for any $q > 0$,*

$$\lim_{t\to\infty} \frac{|E(2\widetilde{\alpha}_{l,t})^n - E(\alpha_t)^n|}{(t^2\psi^{-1}(1/t))^n} = 0. \tag{2.11}$$

Lastly, the precise meaning of step 3. is given by the next lemma.

Lemma 2.3 *Under the hypotheses of Theorem 1.1, with $l = l(t) = [\log t]^q$, for q sufficiently large,*

$$\lim_{t\to\infty} \frac{\sum_{j=0}^{l-1}\left(I_{j,j,t/l} - E\left(\int (L_t^{x+1} - L_t^x)^2\,dx\right)\right)}{t\sqrt{\psi^{-1}(1/t)}} = 0 \quad \text{in } L^2. \tag{2.12}$$

We also need to know the limiting behavior of the moments of α_t. It is given by the next lemma which is proved in the Appendix, Section D.

Lemma 2.4 *Under the hypotheses of Theorem 1.1, for each n,*

$$\lim_{t\to\infty} E\left\{\left(\frac{\alpha_t}{t^2\psi^{-1}(1/t)}\right)^n\right\} = E\left\{(\alpha_{\beta,1})^n\right\}. \tag{2.13}$$

Proof of Theorem 1.1 In (2.10) replace $\widetilde{I}_{l,t}$ by $2\widetilde{I}_{l,t}$ and $(4c_{\psi,1})^n\,E\left\{(\widetilde{\alpha}_{l,t})^n\right\}$ by $(8c_{\psi,1})^n\,E\left\{(2\widetilde{\alpha}_{l,t})^n\right\}$. Set

$$E\left\{(2\widetilde{\alpha}_{l,t})^n\right\} = E(\alpha_t)^n + E\left\{(2\widetilde{\alpha}_{l,t})^n\right\} - E(\alpha_t)^n. \tag{2.14}$$

Then use Lemmas 2.2 and 2.4 to see that for each integer m

$$\lim_{t\to\infty} E\left(\left(\frac{2\widetilde{I}_{l,t}}{t\sqrt{\psi^{-1}(1/t)}}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}\,(8c_{\psi,1})^n\,E\left\{(\alpha_{\beta,1})^n\right\} & \text{if } m = 2n \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

Note that the right-hand side of (2.15) is the $2n$–th moment of $(8c_{\psi,1})^{1/2}$ $\sqrt{\alpha_{\beta,1}}\,\eta$ when $\alpha_{\beta,1}$ and η are independent. Furthermore, it follows from [5, (6.12)] that

$$E\left(\alpha_{\beta,1}\right)^n \leq C^n((2n)!)^{1/(2\beta)}. \tag{2.16}$$

Consequently, since $\sqrt{(2n)!} \leq 2^n n!$

$$E\left((8c_{\psi,1})^{1/2}\sqrt{\alpha_{\beta,1}}\,\eta\right)^m \leq C^m (m!)^{(\beta+1)/(2\beta)}. \tag{2.17}$$

This implies that $(8c_{\psi,1})^{1/2}\sqrt{\alpha_{\beta,1}}\,\eta$ is determined by its moments; (see [7, p. 227-228]). Therefore, by the method of moments, [2, Theorem 30.2]), it follows from (2.15) that

$$\lim_{t\to\infty}\frac{2\widetilde{I}_{l,t}}{t\sqrt{\psi^{-1}(1/t)}} \stackrel{\mathcal{L}}{\Longrightarrow} (8c_{\psi,1})^{1/2}\sqrt{\alpha_{\beta,1}}\;\eta. \tag{2.18}$$

Theorem 1.1 then follows from (2.6), (2.18) and Lemma 2.3.

Remark 2.5 For Brownian motion $\psi(p) = p^2/2$. We have

$$c_{p^2/2,0} \;=\; \frac{4}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{p^2}\,dp = 1; \tag{2.19}$$

$$c_{p^2/2,1} \;=\; \frac{64}{\pi}\int_0^\infty \frac{\sin^4(p/2)}{p^4}\,dp = \frac{8}{3}. \tag{2.20}$$

Also $L^x_{2,1} \stackrel{\mathcal{L}}{=} (1/2)L^x_2 \stackrel{\mathcal{L}}{=} (1/\sqrt{2})L_1^{x/\sqrt{2}}$ where $\{L^x_t\}$ denotes the local time of Brownian motion, so that

$$\left(\int_{-\infty}^{\infty}\left(L^x_{2,1}\right)^2\,dx\right)^{1/2} = \frac{1}{2^{1/4}}\left(\int_{-\infty}^{\infty}\left(L^x_1\right)^2\,dx\right)^{1/2}. \tag{2.21}$$

Since $\sqrt{\psi^{-1}(1/t)} = 2^{1/4}/t^{1/4}$, we see that (1.9) and (1.11) imply (1.3).

3 Partial proofs of Lemmas 2.1–2.3

The following key lemma is proved in the Appendix, Section B, in which it is restated as Lemma B.1. The terms $I_{j,k,t}$ and $\alpha_{j,k,t}$ are defined in (2.1) and (2.2).

Lemma 3.1 *Let $m_{j,k}$, $0 \leq j < k \leq K$ be positive integers with $\sum_{j,k=0,j<k}^K$ $m_{j,k} = m$. If all the integers $m_{j,k}$ are even, then for some $\epsilon > 0$*

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K \left(I_{j,k,t}\right)^{m_{j,k}}\right) \tag{3.1}$$

$$= \prod_{\substack{j,k=0\\ j<k}}^K \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)}\left(4c_{\psi,1}\right)^{n_{j,k}} E\left(\prod_{\substack{j,k=0\\ j<k}}^K \left(\alpha_{j,k,t}\right)^{n_{j,k}}\right) + O\left(t^{(2-1/\beta)m/2-\epsilon}\right),$$

where $n_{j,k} = m_{j,k}/2$.

If any of the $m_{j,k}$ are odd, then

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K \left(I_{j,k,t}\right)^{m_{j,k}}\right) = O\left(t^{(2-1/\beta)\frac{m}{2}-\epsilon}\right). \tag{3.2}$$

In (3.1) and (3.2) the error terms may depend on m, but not on the individual terms $m_{j,k}$.

Proof of Lemma 2.1 Using the multinomial theorem on the sum in (2.3) we have

$$E\left(\left(\widetilde{I}_{l,t}\right)^m\right) = \sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) E\left(\prod_{\substack{j,k=0\\ j<k}}^{l-1} \left(I_{j,k,t/l}\right)^{m_{j,k}}\right), \tag{3.3}$$

where

$$\mathcal{M} = \left\{ \widetilde{m} = \{m_{j,k}, 0\le j<k\le l-1\} \,\middle|\, \sum_{\substack{j,k=0\\ j<k}}^{l-1} m_{j,k} = m \right\}.$$

We now use Lemma 3.1, with t replaced by t/l to compute the expectation on the right-hand side of (3.3). We get that when all the $m_{j,k}$ are even, there exists an $\epsilon > 0$ such that

$$\begin{aligned} &E\left(\left(\widetilde{I}_{l,t}\right)^m\right) \qquad (3.4)\\ &= \sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) \prod_{\substack{j,k=0\\ j<k}}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)} \left(4c_{\psi,1}\right)^{n_{j,k}} E \prod_{\substack{j,k=0\\ j<k}}^{l-1} \left(\alpha_{j,k,t/l}\right)^{n_{j,k}} \\ &\qquad + O(l^m (t^2\psi^{-1}(1/t))^n t^{-\epsilon}). \end{aligned}$$

(Recall that when all the $m_{j,k}$ are even, $m_{j,k} = 2n_{j,k}$ for all j and k and $n = m/2$.) Here we use the fact that

$$\sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) = l^m \tag{3.5}$$

to compute the error term. It also follows from Lemma 3.1 that

$$E\left(\left(\widetilde{I}_{l,t}\right)^m\right) = O(l^m (t^2\psi^{-1}(1/t))^{m/2} t^{-\epsilon}) \tag{3.6}$$

if any of the $m_{j,k}$ are odd. (Lemma 3.1 is for a fixed partition of m. In (3.4) and (3.6) we include the factor l^m, to account for the number of possible partitions.) Recall that $l = [\log t]^q$ for some $q > 0$.

When $m_{j,k} = 2n_{j,k}$ for all j and k,

$$\left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) \prod_{\substack{j,k=0\\ j<k}}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)} = \frac{(2n)!}{2^n n!} \frac{n!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(n_{j,k}!)}. \tag{3.7}$$

Using this in (3.4) we get

$$\begin{aligned} &E\left(\left(\widetilde{I}_{l,t}\right)^m\right) \qquad (3.8)\\ &= \frac{(2n)!}{2^n n!} \left(4c_{\psi,1}\right)^n \sum_{\mathcal{N}} \left(\frac{n!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1} n_{j,k}!}\right) E\left\{\prod_{\substack{j,k=0\\ j<k}}^{l-1} \left(\alpha_{j,k,t/l}\right)^{n_{j,k}}\right\} \\ &\qquad + O(l^m (t^2\psi^{-1}(1/t))^n t^{-\epsilon}).,\end{aligned}$$

where $\mathcal{N}$ is defined similarly to $\mathcal{M}$. Using the multinomial theorem as in (3.3) we see that the sum in (3.8) is equal to $E\left\{(\widetilde{\alpha}_{l,t})^n\right\}$, which completes the proof of (2.10). □

Proof of Lemma 2.2 By the Kac Moment Formula; (see Theorem F.1),

$$E\{(\alpha_t)^n\} = E\left(\left(\int (L_t^x)^2\,dx\right)^n\right) \tag{3.9}$$
$$= 2^n \sum_\pi \int\int_{\{\sum_{i=1}^{2n} r_i \le t\}} \prod_{i=1}^{2n} p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^{2n} dr_i \prod_{i=1}^{n} dx_i,$$

where the sum runs over all maps $\pi : [1,2n] \mapsto [1,n]$ with $|\pi^{-1}(i)| = 2$ for each i. The factor 2^n comes from the fact that $|\pi^{-1}(i)| = 2$ for each i.

It is not difficult to see that we can find a subset $J = \{i_1, \dots, i_n\} \subseteq [1,2n]$, such that each of $x_1, \dots, x_n$ can be written as a linear combination of $y_j := x_{\pi(i_j)} - x_{\pi(i_j-1)}$, $j = 1, \dots, n$. For $i \in J^c$ we use the bound $p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \le p_{r_i}(0)$, then change variables and integrate out the y_j, to see that

$$\begin{aligned}
&\int \left(\prod_{i=1}^{2n} \int_0^t p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)})\,dr_i\right) \prod_{i=1}^{n} dx_i \\
&\le C\left(\int_0^t p_r(0)\,dr\right)^n \int \left(\prod_{i\in J} \int_0^t p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)})\,dr_i\right) \prod_{i=1}^{n} dx_i \\
&= Cu^n(0,t)\left(\prod_{i\in J} \int \int_0^t p_{r_i}(y_i)\,dr_i\,dy_i\right) \\
&= Cu^n(0,t)\left(\int u(x,t)\,dx\right)^n \le C\left(t^2\psi^{-1}(1/t)\right)^n,
\end{aligned} \tag{3.10}$$

where we use (A.5) and (A.8) for the last line. This shows that

$$\|\,\alpha_t\|_n \le Ct^2\psi^{-1}(1/t), \tag{3.11}$$

for all t sufficiently large, where C depends only on n, and where $\|\cdot\|_n := (E(\,\cdot\,)^n)^{1/n}$.

It follows from (3.11) that for l sufficiently large,

$$\begin{aligned}
\Big|\|2\widetilde{\alpha}_{l,t}\|_n - \|\alpha_t\|_n\Big| &\le \|2\widetilde{\alpha}_{l,t} - \alpha_t\|_n = \|\sum_{j=0}^{l-1} \alpha_{j,j,t/l}\|_n \\
&\le l\|\,\alpha_{0,0,t/l}\|_n = l\|\,\alpha_{t/l}\|_n \\
&\le Ct^2\,\frac{\psi^{-1}(l/t)}{l}.
\end{aligned} \tag{3.12}$$

We next show that when $l = l(t) = [\log t]^q$ for any $q > 0$,

$$\lim_{t\to\infty} \frac{\Big|\|2\widetilde{\alpha}_{l,t}\|_n - \|\alpha_t\|_n\Big|}{t^2\psi^{-1}(1/t)} = 0. \tag{3.13}$$

This follows from (3.12) since

$$\lim_{t\to\infty} \frac{\psi^{-1}(l/t)}{l\psi^{-1}(1/t)} = 0. \tag{3.14}$$

To obtain (3.14) we use [3, Theorem 1.5.6, (iii)] to see that for all $\delta > 0$, there exists a t_0, such that for all $t \ge t_0$

$$\frac{\psi^{-1}(l/t)}{\psi^{-1}(1/t)} \le l^{(1/\beta)+\delta}. \tag{3.15}$$

Obviously, we pick δ such that $(1/\beta) + \delta < 1$.

The statement in (2.11) follows from (3.13). □

For the proof of Lemma 2.3 we need the following lemma which is proved in the Appendix, Section E.

Lemma 3.2 *Under the hypotheses of Theorem 1.1*

$$E\left(\int (L_t^{x+1}-L_t^x)^2\,dx\right) = 4c_{\psi,0}t + O\left(g(t)\right) \tag{3.16}$$

as $t\to\infty$, where

$$g(t) = \begin{cases} t^2\left(\psi^{-1}(1/t)\right)^3 & 3/2<\beta\le 2 \\ L(t) & \beta = 3/2 \\ C & 1<\beta<3/2 \end{cases} \tag{3.17}$$

and $L(\,\cdot\,)$ is slowly varying at infinity. Also

$$\mathit{Var}\left(\int (L_t^{x+1}-L_t^x)^2\,dx\right) \le Ct^2\psi^{-1}(1/t)\log t. \tag{3.18}$$

Proof of Lemma 2.3 We prove this lemma by showing that

$$\lim_{t\to\infty}\frac{\sum_{j=0}^{l-1}E(I_{j,j,t/l}) - E\left(\int (L_t^{x+1}-L_t^x)^2\,dx\right)}{t(\psi^{-1}(1/t))^{1/2}} = 0 \tag{3.19}$$

and

$$\lim_{t\to\infty}\frac{\sum_{j=0}^{l-1}\left(I_{j,j,t/l}-E(I_{j,j,t/l})\right)}{t(\psi^{-1}(1/t))^{1/2}} = 0 \tag{3.20}$$

in L^2, where $l=l(t)=[\log t]^q$, for some q sufficiently large.

Set

$$\phi(t) = t^2\psi^{-1}(1/t). \tag{3.21}$$

It follows from (3.15) that

$$\frac{l\phi(t/l)}{\phi(t)} = \frac{\psi^{-1}(l/t)}{l\psi^{-1}(1/t)} \le l^{(1/\beta)+\delta-1}, \tag{3.22}$$

for all $\delta>0$. Recall $l=[\log t]^q$. We choose a δ and $q<\infty$ such that

$$\frac{l(t)\phi(t/l(t))}{\phi(t)} = O\left(\frac{1}{\log^2 t}\right), \tag{3.23}$$

as $t\to\infty$.

We see from (3.16) that

$$\begin{aligned}\lim_{t\to\infty}&\frac{\sum_{j=0}^{l-1}E(I_{j,j,t/l}) - E\left(\int (L_t^{x+1}-L_t^x)^2\,dx\right)}{t(\psi^{-1}(1/t))^{1/2}} \\ &= \lim_{t\to\infty}\frac{l(t)O\left(g(t/l(t))\right)+O\left(g(t)\right)}{t(\psi^{-1}(1/t))^{1/2}} = 0.\end{aligned} \tag{3.24}$$

The last equality follows from the fact that $t(\psi^{-1}(1/t))^{1/2}$ is regularly varying as $t\to\infty$ with index $1-1/(2\beta)>1/2$, since $\beta>1$, whereas $g(t)$ is regularly varying as $t\to\infty$ with index $(2-3/\beta)^+\le 1/2$ since, $\beta\le 2$, and $l(t)$ is slowly varying.

Since $I_{j,j,t/l}$ are independent and identically distributed, we obtain (3.20) by showing that

$$\lim_{t\to\infty} l(t)\text{Var}\left(\frac{I_{j,j,t/l}}{t(\psi^{-1}(1/t))^{1/2}}\right) = 0. \tag{3.25}$$

Using (3.18) and (3.21) we see that

$$l(t)\text{Var}\left(\frac{I_{j,j,t/l}}{t(\psi^{-1}(1/t))^{1/2}}\right) = O\left(\frac{l(t)\phi(t/l(t))}{\phi(t)}\log t\right). \tag{3.26}$$

as $t\to\infty$. Thus (3.25) follows from (3.23). □

4 Proof of Theorem 1.2

We follow the same procedure in the proof of Theorem 1.2 that we used in the proof of Theorem 1.1. We first present the main steps of the proof heuristically and then restate them, precisely, in a series of lemmas and show how Theorem 1.2 follows from the lemmas. The lemmas are proved in the Appendix, Sections G through K.

Let

$$J_{j,k,l,h} := \int (L^{x+h}_{1/l} - L^x_{1/l})\circ\theta_{j/l}\ (L^{x+h}_{1/l} - L^x_{1/l})\circ\theta_{k/l}\ \ dx \tag{4.1}$$

and

$$\widetilde{J}_{l,h} := \sum_{\substack{j,k=0\\ j<k}}^{l-1} J_{j,k,l,h}. \tag{4.2}$$

Using the additivity property of local time we can write

$$L^x_1 = \sum_{j=0}^{l-1} L^x_{1/l}\circ\theta_{j/l}. \tag{4.3}$$

so that

$$\int (L^{x+h}_1 - L^x_1)\ (L^{x+h}_1 - L^x_1)\ dx = \sum_{j,k=0}^{l-1} J_{j,k,l,h} = 2\widetilde{J}_{l,h} + \sum_{j=0}^{l-1} J_{j,j,l,h}. \tag{4.4}$$

Consequently

$$\begin{aligned}&\int (L^{x+h}_1 - L^x_1)^2\,dx - E\int (L^{x+h}_1 - L^x_1)^2\,dx \\ &\quad = 2\widetilde{J}_{l,h} + \sum_{j=0}^{l-1}\left(J_{j,j,l,h} - E\left(\int (L^{x+h}_1 - L^x_1)^2\,dx\right)\right).\end{aligned} \tag{4.5}$$

Similarly, let

$$\widetilde{\alpha}_l := \sum_{\substack{j,k=0\\ j<k}}^{l-1} \alpha_{j,k,1/l}, \tag{4.6}$$

where, as in (2.2)

$$\alpha_{j,k,1/l} = \int L^x_{1/l}\circ\theta_{j/l}\ L^x_{1/l}\circ\theta_{k/l}\ \ dx. \tag{4.7}$$

Recall that in (2.8) we defined

$$\alpha_t = \int_{-\infty}^{\infty} (L^x_t)^2\,dx. \tag{4.8}$$

Therefore,

$$\alpha_1 = \int L_1^x \; L_1^x \; dx = \sum_{j,k=0}^{l-1} \alpha_{j,k,1/l} = 2\widetilde{\alpha}_l + \sum_{j=0}^{l-1} \alpha_{j,j,1/l}. \tag{4.9}$$

In what follows we take l to be a function of h such that $\lim_{h\to 0} l(h) = \infty$.

The main steps in the proof of Theorem 1.2 are to show that:

1. The 'off-diagonal' terms $\widetilde{J}_{l,h}$ and $\sqrt{\widetilde{\alpha}_l}$ are comparable, asymptoticly as $h \to 0$.
2. The diagonal term $\sum_{j=0}^{l-1} \alpha_{j,j,1/l}$ is negligible, as $h \to 0$, compared to the terms in 1.
3. The diagonal term $\sum_{j=0}^{l-1} J_{j,j,l,h}$ is such that

$$\sum_{j=0}^{l-1} J_{j,j,l,h} - E\left(\int (L_1^{x+h} - L_1^x)^2 \, dx \right) \tag{4.10}$$

   is negligible, as $h \to 0$, compared to the terms in 1.

We now explain the precise meaning of these statements, and show how they imply Theorem 1.2.

The precise meaning of step 1. is given by the following lemma. (Lemmas 4.1–4.3 are proved in the Appendix, Section I.)

Lemma 4.1 *Under the hypotheses of Theorem 1.2 and with $l = l(h) = [\log 1/h]$,*

$$E\left(\left(\widetilde{J}_{l,h} \right)^m \right) = \begin{cases} \dfrac{(2n)!}{2^n n!} \left(4c_{\psi,h,1}\right)^n E\left\{ (\widetilde{\alpha}_l)^n \right\} + o((h\psi^2(1/h))^{-n}) & m = 2n \\ O(h^\epsilon (h\psi^2(1/h))^{-n}) & \textit{otherwise}, \end{cases} \tag{4.11}$$

for some $\epsilon > 0$, where

$$c_{\psi,h,1} := \int \left(\int \left(\Delta^h \Delta^{-h} \, p_s(x) \right) \, ds \right)^2 \, dx. \tag{4.12}$$

This lemma is the crux of the proof of Theorem 1.2. Although the appearance of (4.11) and (2.10) might seem similar, the proof of this lemma is very different from the proof of of Lemma 2.1.

The precise meaning of step 2. is given by the next lemma.

Lemma 4.2 *Under the hypotheses of Theorem 1.2, for each n, with $l = l(h) = [\log 1/h]^q$, for any $q > 0$,*

$$\lim_{h\to 0} E(\widetilde{\alpha}_l)^n = E(\alpha_1/2)^n. \tag{4.13}$$

The precise meaning of step 3. is given by the next lemma.

Lemma 4.3 *Under the hypotheses of Theorem 1.2, with $l = l(h) = [\log 1/h]^q$, for q sufficiently large,*

$$\lim_{h\to 0} \sqrt{h\psi^2(1/h)} \sum_{j=0}^{l-1} \left(J_{j,j,l,h} - E\left(\int (L_1^{x+h} - L_1^x)^2 \, dx \right) \right) = 0. \tag{4.14}$$

in L^2.

Finally, to prove Theorem 1.2, we also need the following limit.

Lemma 4.4 *Under the hypotheses of Theorem 1.2,*

$$\lim_{h\to 0} h\psi^2(1/h)c_{\psi,h,1} = c_{\beta,1} \tag{4.15}$$

Lemma 4.4 is proved in the Appendix, Section J.

Proof of Theorem 1.2 Let $l = [\log 1/h]^q$, for some q sufficiently large. In (4.11) we replace $\widetilde{J}_{l,h}$ by $2\widetilde{J}_{l,h}$ and $(4c_{\psi,h,1})^n E\{(\widetilde{\alpha}_l)^n\}$ by $(8c_{\psi,h,1})^n E\{(2\widetilde{\alpha}_l)^n\}$ and write

$$E\{(2\widetilde{\alpha}_l)^n\} = E\{(\alpha_1)^n\} - E\{(2\widetilde{\alpha}_l)^n\} + E\{(\alpha_1)^n\}. \tag{4.16}$$

It follows from Lemmas 4.2 and 4.4 that for each m

$$\lim_{h\to 0} E\left(\left(2\sqrt{h\psi^2(1/h)}\widetilde{J}_{l,h}\right)^m\right) = \begin{cases} \dfrac{(2n)!}{2^n n!}(8c_{\beta,1})^n E\{(\alpha_1)^n\} & \text{if } m = 2n \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

We now show that the distribution of $(8c_{\beta,1})^{1/2}\sqrt{\alpha_1}\,\eta$ is determined by its moments. It follows from [5, (6.12)] that for the β-stable process, with $\beta > 1$,

$$E\left\{\left(\int (L_1^x)^2\,dx\right)^n\right\} \le C^n((2n)!)^{1/(2\beta)}. \tag{4.18}$$

(This was used in (2.16)). When ψ is regularly varying at infinity with index β, for all $\epsilon > 0$, there exists a constant $D = D_\epsilon$ such that

$$\int_0^\infty e^{-s\psi(p)}\,dp \le C\left(1 + \int_1^\infty e^{-sDp^{\beta-\epsilon}}\,dp\right). \tag{4.19}$$

Using this, and the same proof as in [5], one can show that (4.18) holds, with β replaced by $\beta-\epsilon$ for any $\epsilon > 0$, for any Lévy process with Lévy exponent $\psi(\lambda)$ which is regularly varying at infinity with index β. Consequently

$$E\{(\alpha_1)^n\} \le C^n((2n)!)^{1/(2(\beta-\epsilon))}, \tag{4.20}$$

for any $\epsilon > 0$. As in the paragraph containing (2.16), this implies that the weak limit $(8c_{\beta,1})^{1/2}\sqrt{\alpha_1}\,\eta$ is determined by its moments; (see [7, p. 227-228]). Therefore, by the method of moments, [2, Theorem 30.2]), it follows from (4.17) that

$$\lim_{h\to 0} 2\sqrt{h\psi^2(1/h)}\widetilde{J}_{l,h} \overset{\mathcal{L}}{\Longrightarrow} (8c_{\beta,1})^{1/2}\sqrt{\alpha_1}\;\eta. \tag{4.21}$$

Theorem 1.2 now follows from (4.5), (4.21) and Lemma 4.3 . □

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Appendix

When we say, ‘the paper’ we mean the paper to which this is an appendix. Sections B–F contain the proofs of many of the fundamental lemmas needed to give complete proofs of Lemmas 2.1–2.4, which are used in Section 3 of the paper to prove Theorem 1.1. The most critical ingredient in the proof of these lemmas is Lemma B.1, which is the same as Lemma 3.1 in the paper. To prove Lemma B.1 we need estimates on the asymptotic behavior of fixed differences of the transition probability densities for the Lévy processes under consideration.

Sections G–K provide the details for the proof of Theorem 1.2. We discuss this in greater detail when we get to them.

In this Appendix, references such as (3.2) are to Section 3 of ‘the paper’. Similarly, all citations are to the References at the end of ‘the paper’.

A Estimates for the asymptotic behavior of fixed differences of the transition probability densities of certain Lévy processes

Let $p_s(x)$ denote the density of the symmetric Lévy process X with Lévy exponent $\psi(\lambda)$ as described in (1.4). Let Δ_x^γ denote the finite difference operator on the variable x, i.e.

$$\Delta_x^\gamma f(x) = f(x+\gamma) - f(x). \tag{A.1}$$

We write Δ^γ for Δ_x^γ when the variable x is clear.

Let

$$u(x,t) := \int_0^t p_s(x)\, ds \tag{A.2}$$

$$v_\gamma(x,t) := \int_0^t |\Delta^\gamma p_s(x)|\, ds \tag{A.3}$$

$$w_\gamma(x,t) := \int_0^t |\Delta^\gamma \Delta^{-\gamma} p_s(x)|\, ds \tag{A.4}$$

We also write $v(x,t)$ for $v_1(x,t)$ and $w(x,t)$ for $w_1(x,t)$.

The Lemmas in this section are proved in Section C.

Lemma A.1 *Under the hypotheses of Theorem 1.1, for all t sufficiently large*

$$\sup_{x\in R^1} u(x,t) \le Ct\psi^{-1}(1/t); \tag{A.5}$$

$$\sup_{x\in R^1} v(x,t) \le C\log t; \tag{A.6}$$

$$\sup_{x\in R^1} w(x,t) \le C, \tag{A.7}$$

and

$$\int u(x,t)\, dx = t; \tag{A.8}$$

$$\int v(x,t)\, dx \le Ct\left(\psi^{-1}(1/t)\right)\log t; \tag{A.9}$$

$$\int w(x,t)\, dx \le C(\log t)^2; \tag{A.10}$$

$$\int w^2(x,t)\, dx \le C\log t; \tag{A.11}$$

$$\int_{|x|\ge u} w^2(x,t)\, dx \le C\frac{(\log t)^2}{u}. \tag{A.12}$$

Lemma A.2 *Under the hypotheses of Theorem 1.1, for all t sufficiently large*

$$|\Delta^1 p_t(0)| \le C\left(\psi^{-1}(1/t)\right)^3, \tag{A.13}$$

and

$$\int_0^{2t}\int_0^{2t} |\Delta^1 p_{r+s}(0)|\, dr\, ds \le C\left(t^2\left(\psi^{-1}(1/t)\right)^3 + L(t) + 1\right), \tag{A.14}$$

where $L(t)$ is a slowly varying function at infinity.

Lemma A.3 *Under the hypotheses of Theorem 1.1,*

$$\int_0^\infty \Delta^1\ p_s(0)\,ds \quad = \quad -c_{\psi,0}; \tag{A.15}$$

$$\int \left(\int_0^\infty \Delta^1\Delta^{-1}\ p_s(x)\,ds\right)^2 dx \quad = \quad c_{\psi,1}; \tag{A.16}$$

and

$$\int \left(\int_0^t \Delta^1\Delta^{-1}\,p_s(x)\,ds\right)^2 dx = c_{\psi,1} + O(t^{-1/3}), \tag{A.17}$$

as $t \to \infty$.

B Moments of increments of local times.

We use the method of moments to prove Theorem 1.1. In the next lemma we calculate the moments that we need. The terms $I_{j,k,t}$ and $\alpha_{j,k,t}$ are defined in (2.1) and (2.2).

Lemma B.1 *Let* $m_{j,k}$*,* $0 \le j < k \le K$ *be positive integers with* $\sum_{j,k=0,j<k}^K m_{j,k} = m$*. If all the integers* $m_{j,k}$ *are even, then for some* $\epsilon > 0$

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K (I_{j,k,t})^{m_{j,k}}\right) \tag{B.1}$$

$$= \prod_{\substack{j,k=0\\ j<k}}^K \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)}\,(4c_{\psi,1})^{n_{j,k}}\, E\left(\prod_{\substack{j,k=0\\ j<k}}^K (\alpha_{j,k,t})^{n_{j,k}}\right) + O\left(t^{(2-1/\beta)m/2-\epsilon}\right),$$

where $n_{j,k} = m_{j,k}/2$*.*

If any of the $m_{j,k}$ *are odd, then*

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K (I_{j,k,t})^{m_{j,k}}\right) = O\left(t^{(2-1/\beta)\frac{m}{2}-\epsilon}\right). \tag{B.2}$$

In (B.1) and (B.2) the error terms may depend on m*, but not on the individual terms* $m_{j,k}$*.*

Proof We can write

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K (I_{j,k,t})^{m_{j,k}}\right) \tag{B.3}$$

$$= E\left(\prod_{\substack{j,k=0\\ j<k}}^K \prod_{i=1}^{m_{j,k}} \left(\int (\Delta^1_{x_{j,k,i}} L_t^{x_{j,k,i}}\circ\theta_{jt})\ (\Delta^1_{x_{j,k,i}} L_t^{x_{j,k,i}}\circ\theta_{kt})\ dx_{j,k,i}\right)\right)$$

$$= \int \left\{\prod_{\substack{j,k=0\\ j<k}}^K \prod_{i=1}^{m_{j,k}} \Delta^{1,j}_{x_{j,k,i}}\Delta^{1,k}_{x_{j,k,i}}\right\} E\left(\prod_{\substack{j,k=0\\ j<k}}^K \prod_{i=1}^{m_{j,k}} \left((L_t^{x_{j,k,i}}\circ\theta_{jt})\ (L_t^{x_{j,k,i}}\circ\theta_{kt})\right)\right)$$

$$\prod_{\substack{j,k=0\\ j<k}}^{K}\prod_{i=1}^{m_{j,k}} dx_{j,k,i},$$

where the notation $\Delta^{1,j}_{x_{j,k,i}}$ indicates that we apply the difference operator $\Delta^1_{x_{j,k,i}}$ to $L_t^{x_{j,k,i}}\circ\theta_{jt}$. Note that there are $2m$ applications of the difference operator Δ^1.

Consider

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}\prod_{i=1}^{m_{j,k}}\left((L_t^{x_{j,k,i}}\circ\theta_{jt})\ (L_t^{x_{j,k,i}}\circ\theta_{kt})\right)\right). \tag{B.4}$$

We collect all the factors containing θ_{lt} and write

$$\begin{aligned} &E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}\prod_{i=1}^{m_{j,k}}\left((L_t^{x_{j,k,i}}\circ\theta_{jt})\ (L_t^{x_{j,k,i}}\circ\theta_{kt})\right)\right) \\ &\quad= E\left(\prod_{l=0}^{K}\left\{\left(\prod_{j=0}^{l-1}\prod_{i=1}^{m_{j,l}}L_t^{x_{j,l,i}}\right)\left(\prod_{k=l+1}^{K}\prod_{i=1}^{m_{l,k}}L_t^{x_{l,k,i}}\right)\right\}\circ\theta_{lt}\right) \\ &\quad= E\left(\prod_{l=0}^{K}H_l\circ\theta_{lt}\right),\end{aligned} \tag{B.5}$$

where

$$H_l=\left(\prod_{j=0}^{l-1}\prod_{i=1}^{m_{j,l}}L_t^{x_{j,l,i}}\right)\left(\prod_{k=l+1}^{K}\prod_{i=1}^{m_{l,k}}L_t^{x_{l,k,i}}\right). \tag{B.6}$$

By the Markov property

$$E\left(\prod_{l=0}^{K}H_l\circ\theta_{lt}\right)=E\left(H_0\,E^{X_t}\left(\prod_{l=1}^{K}H_l\circ\theta_{(l-1)t}\right)\right). \tag{B.7}$$

Let

$$m_l=\sum_{k=l+1}^{K}m_{l,k}+\sum_{j=0}^{l-1}m_{j,l},\qquad l=0,\dots,K-1, \tag{B.8}$$

and note that m_l is the number of local time factors in H_l.

Let

$$f(y)=E^y\left(\prod_{l=1}^{K}H_l\circ\theta_{(l-1)t}\right). \tag{B.9}$$

It follows from Kac's Moment Formula, Theorem F.1, that for any $z\in R^1$

$$\begin{aligned} &E^z\left(\prod_{l=0}^{K}H_l\circ\theta_{lt}\right) \\ &\quad= E^z\left(H_0\,f(X_t)\right) \\ &\quad= \sum_{\pi_0}\int_{\{\sum_{q=1}^{m_0}r_{0,q}\le t\}}p_{r_{0,1}}(x_{\pi_0(1)}-z)\prod_{q=2}^{m_0}p_{r_{0,q}}(x_{\pi_0(q)}-x_{\pi_0(q-1)}) \\ &\qquad\qquad\left(\int p_{(t-\sum_{q=1}^{m_0}r_{0,q})}(y-x_{\pi_0(m_0)})f(y)\,dy\right)\prod_{q=1}^{m_0}dr_{0,q},\end{aligned} \tag{B.10}$$

where the sum runs over all bijections π_0 from $[1, m_0]$ to

$$I_0 = \bigcup_{k=1}^{K} \{(0,k,i),\ 1 \le i \le m_{0,k}\}. \tag{B.11}$$

Clearly, I_0 is the set of subscripts of the terms $x_\cdot$ appearing in the local time factors in H_0.

By the Markov property

$$\begin{aligned} f(y) &= E^y\left(H_1 E^{X_{2t}}\left(\prod_{l=2}^{K} H_l \circ \theta_{(l-2)t}\right)\right) \\ &=: E^y\left(H_1 g(X_{2t})\right). \end{aligned} \tag{B.12}$$

Therefore, by (B.7)–(B.12), for any $z' \in R^1$

$$\begin{aligned} & E^{z'}\left(\prod_{l=0}^{K} H_l \circ \theta_{lt}\right) \qquad (B.13) \\ &= E^{z'}\left(H_0 E^{X_t}\left(H_1\, g(X_{2t})\right)\right) \\ &= \sum_{\pi_0} \int_{\{\sum_{q=1}^{m_0} r_{0,q} \le t\}} p_{r_{0,1}}(x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)} - x_{\pi_0(q-1)}) \\ &\qquad \left(\int p_{(t-\sum_{q=1}^{m_0} r_{0,q})}(y - x_{\pi_0(m_0)}) E^y\left(H_1\, g(X_{2t})\right)\, dy\right) \prod_{q=1}^{m_0} dr_{0,q} \\ &= \sum_{\pi_0} \int_{\{\sum_{q=1}^{m_0} r_{0,q} \le t\}} p_{r_{0,1}}(x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)} - x_{\pi_0(q-1)}) \\ &\qquad p_{(t-\sum_{q=1}^{m_0} r_{0,q})}(y - x_{\pi_0(m_0)}) \\ &\qquad \sum_{\pi_1} \int_{\{\sum_{q=1}^{m_1} r_{1,q} \le t\}} p_{r_{1,1}}(x_{\pi_1(1)} - y) \prod_{q=2}^{m_1} p_{r_{1,q}}(x_{\pi_1(q)} - x_{\pi_1(q-1)}) \\ &\qquad \left(\int p_{(t-\sum_{q=1}^{m_1} r_{1,q})}(y' - x_{\pi_1(m_1)}) g(y')\, dy'\right) \prod_{q=1}^{m_1} dr_{1,q}\, dy \prod_{q=1}^{m_0} dr_{0,q} \end{aligned}$$

where the second sum runs over all bijections π_1 from $[1, m_1]$ to

$$I_1 = \{(0,1,i),\ 1 \le i \le m_{0,1}\} \bigcup_{k=2}^{K} \{(1,k,i),\ 1 \le i \le m_{1,k}\} \tag{B.14}$$

As above, I_1 is the set of subscripts of the terms $x_\cdot$ appearing in the local time factors in H_1.

We now use the Chapman-Kolmogorov equation to integrate with respect to y to get

$$\begin{aligned} & E^{z'}\left(H_0 E^{X_t}\left(H_1\, g(X_t)\right)\right) \qquad (B.15) \\ &= \sum_{\pi_0, \pi_1} \int_{\{\sum_{q=1}^{m_0} r_{0,q} \le t\}} p_{r_{0,1}}(x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)} - x_{\pi_0(q-1)}) \\ &\qquad \int_{\{\sum_{q=1}^{m_1} r_{1,q} \le t\}} p_{(t-\sum_{q=1}^{m_0} r_{0,q}) + r_{1,1}}(x_{\pi_1(1)} - x_{\pi_0(m_0)}) \\ &\qquad\qquad \prod_{q=2}^{m_1} p_{r_{1,q}}(x_{\pi_1(q)} - x_{\pi_1(q-1)}) \end{aligned}$$

$$\left(\int p_{(t-\sum_{q=1}^{m_1} r_{1,q})}(y'-x_{\pi_1(m_1)})g(y')\,dy'\right)\prod_{q=1}^{m_1} dr_{1,q}\prod_{q=1}^{m_0} dr_{0,q}.$$

Iterating this procedure, and recalling (B.5) we see that

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}\prod_{i=1}^{m_{j,k}}\left((L_t^{x_{j,k,i}}\circ\theta_{jt})\ (L_t^{x_{j,k,i}}\circ\theta_{kt})\right)\right) \tag{B.16}$$
$$=\sum_{\pi_0,\ldots,\pi_K}\prod_{l=0}^{K}\int_{\{\sum_{q=1}^{m_l} r_{l,q}\le t\}} p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})$$
$$\prod_{q=2}^{m_l} p_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\prod_{q=1}^{m_l} dr_{l,q},$$

where $\pi_{-1}(m_{-1}):=0$ and $1-\sum_{q=1}^{m_{-1}} r_{-1,q}:=0$. In (B.16) the sum runs over all $\pi_0,\ldots,\pi_K$ such that each π_l is a bijection from $[1,m_l]$ to

$$I_l=\bigcup_{j=0}^{l-1}\{(j,l,i),\,1\le i\le m_{j,l}\}\bigcup_{k=l+1}^{K}\{(l,k,i),\,1\le i\le m_{l,k}\}. \tag{B.17}$$

As in the observations about I_0 and I_1, we see that I_l is the set of subscripts of the terms $x_\cdot$ terms appearing in the local time factors in H_l. Since there are $2m$ local time factors we have that $\sum_{l=0}^{K} m_l=2m$.

We now use (B.16) in (B.3) and continue to develop an expression for the left-hand side of (B.3). Let $\mathcal{B}$ to denote the set of $(K+1)$–tuples, $\pi=(\pi_0,\ldots,\pi_K)$, of bijections described in (B.17). Clearly

$$|\mathcal{B}|=\prod_{l=0}^{K} m_l!\le(2m)!. \tag{B.18}$$

Also, similarly to the way we obtain the first equality in (B.5), we see that

$$\prod_{\substack{j,k=0\\ j<k}}^{K}\prod_{i=1}^{m_{j,k}}\Delta^{1,j}_{x_{j,k,i}}\Delta^{1,k}_{x_{j,k,i}}=\prod_{l=0}^{K}\prod_{q=1}^{m_l}\Delta^{1,l}_{x_{\pi_l(q)}}. \tag{B.19}$$

Consequently

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}(I_{j,k,t})^{m_{j,k}}\right)=\sum_{\pi_0,\ldots,\pi_K}\int\widetilde{\mathcal{T}}_t(x;\,\pi)\prod_{j,k,i} dx_{j,k,i} \tag{B.20}$$

where we take the product over $\{0\le j<k\le K,\,1\le i\le m_{j,k}\}$, $\pi\in\mathcal{B}$ and

$$\widetilde{\mathcal{T}}_t(x;\,\pi) \tag{B.21}$$
$$=\prod_{l=0}^{K}\prod_{q=1}^{m_l}\Delta^{1}_{x_{\pi_l(q)}}\int_{\{\sum_{q=1}^{m_l} r_{l,q}\le t\}} p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})$$
$$\prod_{q=2}^{m_l} p_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\prod_{q=1}^{m_l} dr_{l,q}.$$

We continue to rewrite the right-hand side of (B.20).

In (B.21), each difference operators, say Δ^1_u is applied to the product of two terms, say $p_{\cdot}(u-a)\,p_{\cdot}(u-b)$, using the product rule for difference operators we see that

$$\begin{aligned}&\Delta^1_u\{p_{\cdot}(u-a)p_{\cdot}(u-b)\} \\ &\quad=\Delta^1_u\, p_{\cdot}(u-a)p_{\cdot}(u+1-b)+p_{\cdot}(u-a)\Delta^1_u\, p_{\cdot}(u-b).\end{aligned} \tag{B.22}$$

Consider an example of how the term $\Delta^1_a\Delta^1_u p_{\cdot}(u-a)$ may appear. It could be by the application

$$\Delta^1_a\left(\Delta^1_u\, p_{\cdot}(u-a)p_{\cdot}(v-a)\right), \tag{B.23}$$

in which we take account of the two terms to which Δ^1_a is applied. Using the product rule in (B.22) we see that (B.23)

$$=\left(\Delta^1_a\Delta^1_u\, p_{\cdot}(u-a)\right)p_{\cdot}(v-(a+1))+\Delta^1_u\, p_{\cdot}(u-a)\Delta^1_a p_{\cdot}(v-a). \tag{B.24}$$

Consider one more example

$$\begin{aligned}&\Delta^1_a\left(\Delta^1_u\, p_{\cdot}(u-a)\Delta^1_v\, p_{\cdot}(v-a)\right) \\ &\quad=\left(\Delta^1_a\Delta^1_u\, p_{\cdot}(u-a)\right)\Delta^1_v\, p_{\cdot}(v-(a+1)) \\ &\qquad\qquad+\Delta^1_u\, p_{\cdot}(u-a)\Delta^1_a\Delta^1_v p_{\cdot}(v-a).\end{aligned} \tag{B.25}$$

Note that in both examples the arguments of probability densities with two difference operators applied to it does not contain a 1. This is true in general because the difference formula, (B.22), does not add a 1 to the argument of a term to which a difference operator is applied. Otherwise we may have a ± 1 added to the arguments of probability densities to which one difference operator is applied, as in (B.25), or to the arguments of probability densities to which no difference operator is applied, as in (B.24).

Based on the argument of the preceding paragraph we write (B.21) in the form

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}(I_{j,k,t})^{m_{j,k}}\right)=\sum_a\sum_{\pi_0,\ldots,\pi_K}\int\mathcal{T}'_t(x;\,\pi,a)\prod_{j,k,i}dx_{j,k,i}, \tag{B.26}$$

where

$$\begin{aligned}\mathcal{T}'_t(x;\,\pi,a)=\prod_{l=0}^{K}\int_{\mathcal{R}_l}&\left(\left(\Delta^1_{x_{\pi_l(1)}}\right)^{a_1(l,1)}\left(\Delta^1_{x_{\pi_{l-1}(m_{l-1})}}\right)^{a_2(l,1)}\right. \\ &\left.\qquad p^{\sharp}_{(t-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})\right) \\ \prod_{q=2}^{m_l}&\left(\left(\Delta^1_{x_{\pi_l(q)}}\right)^{a_1(l,q)}\left(\Delta^1_{x_{\pi_l(q-1)}}\right)^{a_2(l,q)}p^{\sharp}_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\right)\prod_{q=1}^{m_l}dr_{l,q}.\end{aligned} \tag{B.27}$$

In (B.27) $\mathcal{R}_l=\{\sum_{q=1}^{m_l}r_{l,q}\le t\}$. In (B.26) the first sum is taken over all

$$a=(a_1,a_2)\,:\,\{(l,q),\,0\le l\le K,\,1\le q\le m_l\}\mapsto\{0,1\}\times\{0,1\} \tag{B.28}$$

with the restriction that for each triple j,k,i, there are exactly two factors of the form $\Delta^1_{x_{j,k,i}}$, each of which is applied to one of the terms $p^{\sharp}_{r_{\cdot}}(\cdot)$ that contains $x_{j,k,i}$ in its argument. This condition can be stated more formally by saying that for each l and $q=1,\ldots,m_l-1$, if $\pi_l(q)=(j,k,i)$, then $\{a_1(l,q),a_2(l,q+1)\}=\{0,1\}$ and if $q=m_l$ then $\{a_1(l,m_l),a_2(l+1,1)\}=\{0,1\}$. (Note that when we write $\{a_1(l,q),a_2(l,q+1)\}=\{0,1\}$ we mean as two sets, so, according to what a is, we may have $a_1(l,q)=1$ and $a_2(l,q+1)=0$

or $a_1(l,q)=0$ and $a_2(l,q+1)=1$ and similarly for $\{a_1(l,m_l),a_2(l+1,1)\}$.) Also, in (B.27) we define $(\Delta^1_{x_i})^0=1$ and $(\Delta^1_0)=1$.

In (B.27), $p^{\sharp}_{r_\cdot}(z)$ can take any of the values $p_{r_\cdot}(z)$, $p_{r_\cdot}(z+1)$ or $p_{r_\cdot}(z-1)$. (We must consider all three possibilities, as explained in the paragraph containing (B.22),) Finally, it is important to emphasize that in (B.27) each of the difference operators is applied to only one of the terms $p^{\sharp}_{r_\cdot}(\cdot)$.

Rather than (B.26), we first analyze

$$\sum_a \sum_{\pi_0,\ldots,\pi_K} \int \mathcal{T}_t(x;\,\pi,a) \prod_{j,k,i} dx_{j,k,i}, \tag{B.29}$$

where

$$\begin{aligned}\mathcal{T}_t(x;\,\pi,a) = \prod_{l=0}^{K} \int_{\mathcal{R}_l} \Bigg(&\left(\Delta^1_{x_{\pi_l(1)}}\right)^{a_1(l,1)} \left(\Delta^1_{x_{\pi_{l-1}(m_{l-1})}}\right)^{a_2(l,1)} \\ &\qquad p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})\Bigg) \\ \prod_{q=2}^{m_l}\left(\left(\Delta^1_{x_{\pi_l(q)}}\right)^{a_1(l,q)} \left(\Delta^1_{x_{\pi_l(q-1)}}\right)^{a_2(l,q)} p_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\right) &\prod_{q=1}^{m_l} dr_{l,q}.\end{aligned} \tag{B.30}$$

The difference between $\mathcal{T}_t(x;\,\pi,a)$ and $\mathcal{T}'_t(x;\,\pi,a)$ is that in the former we replace $p^\sharp$ by p. It is easier to analyze (B.29) than (B.26). At the conclusion of this proof we show that both (B.29) and (B.26) have the same asymptotic limit as t goes to infinity.

We first obtain (B.1). Let $m=2n$, since $m_{j,k}=2n_{j,k}$, $m_l=2n_l$ for some integer n_l. (Recall (B.8)). To begin we consider the case in which $a=e$, where

$$e(l,2q)=(1,1) \quad \text{and} \quad e(l,2q-1)=(0,0) \qquad \forall q. \tag{B.31}$$

When $a=e$ we have

$$\begin{aligned}\mathcal{T}_t(x;\,\pi,e) = \prod_{l=0}^{K} \int_{\mathcal{R}_l} & p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})}) \\ &\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\pi_l(2q-1)}-x_{\pi_l(2q-2)}) \\ &\prod_{q=1}^{n_l} \Delta^1\Delta^{-1}\, p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)}) \prod_{q=1}^{m_l} dr_{l,q}.\end{aligned} \tag{B.32}$$

Here we use the following notation: $\Delta^1 p(u-v)=p(u-v+1)-p(u-v)$, i.e., when Δ^1 has no subscript, the difference operator is applied to the whole argument of the function. In this notation,

$$\Delta^1_u\Delta^1_v p(u-v) = \Delta^1\Delta^{-1}p(u-v). \tag{B.33}$$

Consider the multigraph G_π with vertices $\{(j,k,i),\, 0\le j<k\le K,\, 1\le i\le m_{j,k}\}$. Assign an edge between the vertices $\pi_l(2q-1)$ and $\pi_l(2q)$ for each $0\le l\le K$ and $1\le q\le n_l$. Each vertex is connected to two edges. To see this suppose that $\pi_l(2q)=\{(j,k,i)\}$, with $j=l$ and $k=l'\neq l$, then there is a unique q' such that $\pi_{l'}(2q')$ or $\pi_{l'}(2q'-1)$ is equal to $\{(j,k,i)\}$. Therefore all the vertices lie in some cycle. Assume that there are S cycles. We denote them by C_s, $s=1,\ldots,S$. Clearly, it is possible to have cycles of order two, in which case two vertices are connected by two edges.

It is important to note that the graph G_π does not assign edges between $\pi_l(2q)$ and $\pi_l(2q+1)$, although these vertices may be connected by the edge assigned between $\pi_{l'}(2q'-1)$ and $\pi_{l'}(2q')$ for some l' and q'.

We proceed to estimate (B.30) by breaking the calculation into two cases: when $a=e$ and all the cycles of G_π are of order two; when $a=e$ and not all the cycles of G_π are of order two or when $a\neq e$.

B.1 a = e, with all cycles of order two

Let $\mathcal{P}=\{(\gamma_{2v-1},\gamma_{2v})\,,\,1\le v\le n\}$ be a pairing of the m vertices

$$\{(j,k,i),\,0\le j<k\le K,\,1\le i\le m_{j,k}\}$$

of G_π, that satisfies the following special property: whenever (j,k,i) and (j',k',i') are paired together, $j=j'$ and $k=k'$. Equivalently,

$$\mathcal{P}=\bigcup_{\substack{j,k=0\\ j<k}}^{K}\mathcal{P}_{j,k} \tag{B.34}$$

where each $\mathcal{P}_{j,k}$ is a pairing of the $m_{j,k}$ vertices

$$\{(j,k,i),\,1\le i\le m_{j,k}\}.$$

We refer to such a pairing $\mathcal{P}$ as a special pairing and denote the set of special pairings by $\mathcal{S}$.

Given a special pairing $\mathcal{P}\in\mathcal{S}$, let π be such that for each $0\le l\le K$ and $1\le q\le n_l$,

$$\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2v-1},\gamma_{2v}\} \tag{B.35}$$

for some, necessarily unique, $1\le v\le n_l$. In this case we say that π is compatible with the pairing $\mathcal{P}$ and write this as $\pi\sim\mathcal{P}$. (Recall that when we write $\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2v-1},\gamma_{2v}\}$, we mean as two sets, so, according to what π_l is, we may have $\pi_l(2q-1)=\gamma_{2v-1}$ and $\pi_l(2q)=\gamma_{2v}$ or $\pi_l(2q-1)=\gamma_{2v}$ and $\pi_l(2q)=\gamma_{2v-1}$.) Clearly

$$|\mathcal{S}|\le\frac{(2n)!}{2^n n!} \tag{B.36}$$

the number of pairings of $m=2n$ objects.

Let $\pi\in\mathcal{B}$ be such that G_π consists of cycles of order two. It is easy to see that $\pi\sim\mathcal{P}$ for some $\mathcal{P}\in\mathcal{S}$. To see this note that if $\{(j,k,i),(j',k',i')\}$ form a cycle of order two, there must exist l and l' with $l\neq l'$ and q and q' such that both $\{(j,k,i),(j',k',i')\}=\{\pi_l(2q-1),\pi_l(2q)\}$ and $\{(j,k,i),(j',k',i')\}=\{\pi_{l'}(2q'-1),\pi_{l'}(2q')\}$. This implies that $j=j'$, $k=k'$ and $\{j,k\}=\{l,l'\}$. Furthermore, by (B.35) we have

$$\{\pi_l(2q-1),\pi_l(2q)\}=\{\pi_{l'}(2q'-1),\pi_{l'}(2q')\}=\{\gamma_{2v-1},\gamma_{2v}\} \tag{B.37}$$

When $\pi\sim\mathcal{P}$ and all cycles are of order two we can write

$$\begin{aligned}&\prod_{l=0}^{K}\prod_{q=1}^{n_l}\Delta^1\Delta^{-1}p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)}) \qquad\qquad (B.38)\\ &\quad=\prod_{v=1}^{n}\Delta^1\Delta^{-1}p_{r_{2\nu}}(x_{\gamma_{2v}}-x_{\gamma_{2v-1}})\Delta^1\Delta^{-1}p_{r'_{2\nu}}(x_{\gamma_{2v}}-x_{\gamma_{2v-1}}),\end{aligned}$$

where $r_{2\nu}$ and $r'_{2\nu}$ are the rearranged indices $r_{l,2q}$ and $r_{l',2q'}$. We also use the fact that $\sum_{l=0}^{K} n_l = 2n$.

For use in (B.44) below we note that

$$\begin{aligned}\int_0^t \int_0^t |\Delta^1\Delta^{-1}\, p_{r_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})|\, |\Delta^1\Delta^{-1}\, p_{r'_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})|\, dr_{2\nu}\, dr'_{2\nu} \\ = \left(\int_0^t |\Delta^1\Delta^{-1}\, p_r(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})|\, dr\right)^2 = w^2(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}, t), \qquad \text{(B.39)}\end{aligned}$$

(see (C.7).)

We want to estimate the integrals in (B.29). However, it is difficult to integrate $\mathcal{T}_t(x;\,\pi,e)$ directly, because the variables,

$$\begin{aligned}\{x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})},\, x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)},\, x_{\pi_l(2q)} - x_{\pi_l(2q-1)};\\ l \in [0,K],\, q \in [1,n_l]\},\end{aligned}$$

are not independent. We begin the estimation by showing that over much of the domain of integration, the integral is negligible, asymptotically, as $t \to \infty$. To begin, we write

$$1 = \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \le t^{(\beta-1)/(4\beta)}\}} + 1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \ge t^{(\beta-1)/(4\beta)}\}}\right) \qquad \text{(B.40)}$$

and expand it as a sum of 2^n terms and use it to write

$$\begin{aligned}&\int \mathcal{T}_t(x;\,\pi,e) \prod_{j,k,i} dx_{j,k,i} \qquad \text{(B.41)}\\ &= \int \prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \le t^{(\beta-1)/(4\beta)}\}}\right) \mathcal{T}_t(x;\,\pi,e) \prod_{j,k,i} dx_{j,k,i} + E_{1,t}.\end{aligned}$$

We now show that

$$E_{1,t} = O\left(t^{-(\beta-1)/(5\beta)}\left(t^2\psi^{-1}(1/t)\right)^n\right). \qquad \text{(B.42)}$$

Note that every term in $E_{1,t}$ can be written in the form

$$B_t(\pi,e,D) := \int \prod_{v=1}^{n} 1_{D_v}\, \mathcal{T}_t(x;\,\pi,e) \prod_{j,k,i} dx_{j,k,i} \qquad \text{(B.43)}$$

where each D_v is either $\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \le t^{(\beta-1)/(4\beta)}\}$ or $\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \ge t^{(\beta-1)/(4\beta)}\}$, and at least one of the D_v is of the second type.

Consider (B.43) and the representation of $\mathcal{T}_t(x;\,\pi,e)$ in (B.32). We take absolute values in the integrand in (B.32) and take all the integrals with $r_{\cdot}$ between 0 and t and use (B.39) to get

$$\begin{aligned}|B_t(\pi,e,D)| \quad \le \quad &\int \prod_{v=1}^{n} 1_{D_v} w^2(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}, t) \prod_{l=0}^{K} u(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}, t)\\ &\prod_{q=2}^{n_l} u(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}, t) \prod_{j,k,i} dx_{j,k,i}. \qquad \text{(B.44)}\end{aligned}$$

We now take

$$\{x_{\gamma_{2v}} - x_{\gamma_{2v-1}},\, v = 1,\dots,n\} \qquad \text{(B.45)}$$

and an additional n variables from the $2n$ arguments of the u terms,

$$\cup_{l=0}^{K}\{x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})},x_{\pi_l(2q-1)}-x_{\pi_l(2q-2)},\,q=2,\ldots,n_l\} \tag{B.46}$$

so that the chosen $2n$ variables generate the space spanned by the 2n variables $\{x_{j,k,i}\}$. There are n variables in (B.46) that are not used. We bound the functions u of these variables by their sup norm, which by (C.5) is bounded by $Ct\psi^{-1}(1/t)$. Then we make a change of variables and get that

$$\begin{aligned}|B_t(\pi,e,D)| &\leq C\left(t\psi^{-1}(1/t)\right)^n\int\prod_{v=1}^{n}1_{D_v}w^2(y_v,t)\prod_{v=n+1}^{2n}u(y_v,t))\prod_{v=1}^{2n}dy_v\\ &\leq C\left(t^2\psi^{-1}(1/t)\right)^n\int\prod_{v=1}^{n}1_{D_v}w^2(y_v,t))\prod_{v=1}^{n}dy_v,\\ &= O\left(t^{-(\beta-1)/(5\beta)}\left(t^2\psi^{-1}(1/t)\right)^n\right).\end{aligned} \tag{B.47}$$

Here we use (A.8) to see that the integral of a u term is t. Then we use (A.11) and (A.12) to obtain (B.42). (Note that it is because at least one of the D_v is of the second type that we can use (A.12).)

We now study

$$\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\leq t^{(\beta-1)/(4\beta)}\}}\right)\mathcal{T}_t(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i}. \tag{B.48}$$

We identify the relationships in (B.37) by setting $v=\sigma_l(q)$ so that

$$\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2\sigma_l(q)-1},\gamma_{2\sigma_l(q)}\}, \tag{B.49}$$

for each $0\leq l\leq K$ and $1\leq q\leq n_l$. We use both (B.37) and (B.49) in what follows.

We now make a change of variables that, eventually, enables us to make the arguments of the u terms and the w terms independent. For $q\geq 2$ we write

$$\begin{aligned}&p_{r_{l,2q-1}}(x_{\pi_l(2q-1)}-x_{\pi_l(2q-2)})\\ &\quad=p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})+\Delta^{h_{l,q}}p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}}),\end{aligned} \tag{B.50}$$

where $h_{l,q}=(x_{\pi_l(2q-1)}-x_{\gamma_{2\sigma_l(q)-1}})+(x_{\gamma_{2\sigma_l(q-1)-1}}-x_{\pi_l(2q-2)})$. When $q=1$ we can make a similar decomposition

$$\begin{aligned}&p_{(t-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})\\ &\quad=p_{(t-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\\ &\qquad+\Delta^{h_{l,1}}p_{(1-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}),\end{aligned} \tag{B.51}$$

where $h_{l,1}=(x_{\pi_l(1)}-x_{\gamma_{2\sigma_l(1)-1}})+(x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}-x_{\pi_{l-1}(m_{l-1})})$. Note that because of the presence of the term $\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\leq t^{(\beta-1)/(4\beta)}\}}\right)$ in the integral in (B.48) we need only be concerned with values of $|h_{l,q}|\leq 2t^{(\beta-1)/(4\beta)}$, $0\leq l\leq K$ and $1\leq q\leq n_l$.

For $q=1,\ldots,n_l$, $l=0\ldots,K$, we substitute (B.50) and (B.51) into the term $\mathcal{T}_t(x;\,\pi,e)$ in (B.48), (see also (B.32)), and expand the products so that we can write (B.48) as a sum of $2^{\sum_{l=0}^{K}n_l}$ terms, which we write as

$$\begin{aligned}&\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\leq t^{(\beta-1)/(4\beta)}\}}\right)\mathcal{T}_t(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i}\\ &\quad=\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\leq t^{(\beta-1)/(4\beta)}\}}\right)\mathcal{T}_{t,1}(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i}+E_{2,t},\end{aligned} \tag{B.52}$$

where

$$\mathcal{T}_{t,1}(x;\pi,e)=\prod_{l=0}^{K}\int_{\mathcal{R}_l} p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})$$
$$\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}}) \tag{B.53}$$
$$\prod_{q=1}^{n_l}\Delta^1\Delta^{-1}p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)})\prod_{q=1}^{m_l}dr_{l,q}.$$

Using (B.38) we can rewrite this as

$$\mathcal{T}_{t,1}(x;\pi,e) \tag{B.54}$$
$$=\int_{\mathcal{R}_0\times\cdots\times\mathcal{R}_K}\left(\prod_{l=0}^{K}p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right.$$
$$\left.\prod_{q=2}^{n_l}p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\right)$$
$$\left(\prod_{v=1}^{n}\Delta^1\Delta^{-1}p_{r_{2\nu}}(x_{\gamma_{2v}}-x_{\gamma_{2v-1}})\Delta^1\Delta^{-1}p_{r'_{2\nu}}(x_{\gamma_{2v}}-x_{\gamma_{2v-1}})\right)$$
$$\prod_{l=0}^{K}\prod_{q=1}^{m_l}dr_{l,q},$$

where $r_{2\nu}$ and $r'_{2\nu}$ are the rearranged indices $r_{l,2q}$ and $r_{l',2q'}$.

The usefulness of the representations in (B.50) and (B.51) is now apparent. Since the variables $x_{\gamma_{2v}}$, $v=1,\ldots,n$, occur only in the last line of (B.54), we make the change of variables $x_{\gamma_{2v}}-x_{\gamma_{2v-1}}\to x_{\gamma_{2v}}$ and $x_{\gamma_{2v-1}}\to x_{\gamma_{2v-1}}$ and get that

$$\int\mathcal{T}_{t,1}(x;\pi,e)\prod_{j,k,i}dx_{j,k,i} \tag{B.55}$$
$$=\int\int_{\mathcal{R}_0\times\cdots\times\mathcal{R}_K}\left(\prod_{l=0}^{K}p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right.$$
$$\left.\prod_{q=2}^{n_l}p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\right)$$
$$\left(\prod_{v=1}^{n}\Delta^1\Delta^{-1}p_{r_{2\nu}}(x_{\gamma_{2v}})\Delta^1\Delta^{-1}p_{r'_{2\nu}}(x_{\gamma_{2v}})\right)\prod_{l=0}^{K}\prod_{q=1}^{m_l}dr_{l,q}\prod_{j,k,i}dx_{j,k,i}.$$

Now, since the variables $x_{\gamma_{2v}}$, $v=1,\ldots,n$ occur only in the last line of (B.55) and the variables $x_{\gamma_{2v-1}}$, $v=1,\ldots,n$ occur only in the second and third lines of (B.55), we can write (B.55) as

$$\int\mathcal{T}_{t,1}(x;\pi,e)\prod_{j,k,i}dx_{j,k,i} \tag{B.56}$$
$$=\int_{\mathcal{R}_0\times\cdots\times\mathcal{R}_K}\int\left(\prod_{l=0}^{K}p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right.$$
$$\left.\prod_{q=2}^{n_l}p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\right)\prod_{v=1}^{n}dx_{\gamma_{2v-1}}$$

$$\left(\prod_{v=1}^{n}\int \Delta^1\Delta^{-1}\,p_{r_{2\nu}}(x_{\gamma_{2v}})\Delta^1\Delta^{-1}\,p_{r'_{2\nu}}(x_{\gamma_{2v}})\,dx_{\gamma_{2v}}\right)\prod_{l=0}^{K}\prod_{q=1}^{m_l}dr_{l,q}.$$

Note that we also use Fubini's Theorem, which is justified since the absolute value of the integrand is integrable, (as we point out in the argument preceding (B.44)). (In the rest of this section use Fubini's Theorem frequently for integrals like (B.56) without repeating the explanation about why it is justified.)

We now show that

$$E_{2,t}=O\left(t^{-(\beta-1)/(3\beta)}\left(t^2\psi^{-1}(1/t)\right)^n\right). \tag{B.57}$$

To see this note that the terms in $E_{2,t}$ are of the form

$$\begin{aligned}&\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\le t^{(\beta-1)/(4\beta)}\}}\right) \qquad\qquad \text{(B.58)}\\ &\qquad\prod_{l=0}^{K}\int_{\mathcal{R}_l}\widetilde{p}_{(t-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\\ &\qquad\quad\prod_{q=2}^{n_l}\widetilde{p}_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\\ &\qquad\qquad\prod_{q=1}^{n_l}\Delta^1\Delta^{-1}\,p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)})\prod_{q=1}^{m_l}dr_{l,q}\prod_{j,k,i}dx_{j,k,i},\end{aligned}$$

where $\widetilde{p}_{r_{l,2q-1}}$ is either $p_{r_{l,2q-1}}$ or $\Delta^{h_{l,q}}p_{r_{l,2q-1}}$. Furthermore, at least one of the terms $\widetilde{p}_{r_{l,2q-1}}$ is of the form $\Delta^{h_{l,q}}p_{r_{l,2q-1}}$.

As in the transition from (B.43) to (B.44) we bound the absolute value of (B.58) by

$$\begin{aligned}&\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\le t^{(\beta-1)/(4\beta)}\}}\right)w^2(x_{\gamma_{2v}}-x_{\gamma_{2v-1}},t) \qquad\qquad \text{(B.59)}\\ &\quad\prod_{l=0}^{K}\widetilde{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}},t)\prod_{q=2}^{n_l}\widetilde{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}},t)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\prod_{j,k,i}dx_{j,k,i},\end{aligned}$$

where each $\widetilde{u}(\,\cdot\,,t)$ is either of the form $u(\,\cdot\,,t)$ or $v_{h_{l,q}}(\,\cdot\,,t)$; (see (A.3)).

We need to introduce the following notation and estimates. The next lemma is proved in Section C. Let

$$v_*(x,t):=\left(\log t\wedge\frac{t\psi^{-1}(1/t)}{|x|}\wedge t\frac{1+\log^+x}{x^2}\right), \tag{B.60}$$

Lemma B.2 *Under the hypotheses of Theorem 1.1, for all t sufficiently large,*

$$v_{h_{l,q}}(x,t)\ \le\ Ch_{l,q}^2v_*(x,t) \tag{B.61}$$

$$\sup_{x\in R^1}v_*(x,t)\ \le\ \log t \tag{B.62}$$

$$\int v_*(x,t)\,dx\ \le\ Ct\left(\psi^{-1}(1/t)\right)\log t. \tag{B.63}$$

Proof of Lemma B.1 continued: We have J terms of the type $v_{h_{l,q}}(\,\cdot\,,t)$, for some $J\geq 1$. It follows from (B.61) and and the fact that $|h_{l,q}|\leq 2t^{(\beta-1)/(4\beta)}$, that we can bound the integral in (B.59) by

$$Ct^{J(\beta-1)/(2\beta)}\int\prod_{v=1}^{n}w^2(x_{\gamma_{2v}}-x_{\gamma_{2v-1}},t) \tag{B.64}$$
$$\prod_{l=0}^{K}\widetilde{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}},t)\prod_{q=2}^{n_l}\widetilde{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}},t)$$
$$\prod_{j,k,i}dx_{j,k,i},$$

where $\widetilde{u}(\,\cdot\,,t)$ is either $u(\,\cdot\,,t)$ or $v_*(\,\cdot\,,t)$, and we have precisely J of the latter.

Since the variables $x_{\gamma_{2\nu}}$, $\nu=1,\ldots,n$, occur only in the w terms in (B.64) and the variables $x_{\gamma_{2v-1}}$, $v=1,\ldots,n$ occur only in the $\widetilde{u}$ terms in (B.64) , (refer to the change of variables arguments in (B.55) and (B.56)), we can write (B.64) as

$$Ct^{t^{J(\beta-1)/(2\beta)}}\int\Bigg(\prod_{l=0}^{K}\widetilde{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}},t) \tag{B.65}$$
$$\prod_{q=2}^{n_l}\widetilde{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}},t)\Bigg)\prod_{v=1}^{n}dx_{\gamma_{2v-1}}\prod_{v=1}^{n}w^2(x_{\gamma_{2v}},t)\prod_{v=1}^{n}dx_{\gamma_{2v}}$$
$$\leq Ct^{t^{J(\beta-1)/(2\beta)}}(\log t)^n\int\Bigg(\prod_{l=0}^{K}\widetilde{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}},t)$$
$$\prod_{q=2}^{n_l}\widetilde{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}},t)\Bigg)\prod_{v=1}^{n}dx_{\gamma_{2v-1}}$$

where the last inequality uses (A.11).

As we have been doing we extract a linearly independent set of variables from the arguments of the $\widetilde{u}$ terms. The other $\widetilde{u}$ terms we bound by their supremum. Then we make a change of variables and integrate the remaining $\widetilde{u}$ terms.

Compare (A.5) with (B.62). Replacing the sup of a u term by the sup of a v_* term reduces the upper bound by a factor of $1/(t\psi^{-1}(1/t))$, (neglecting the factor of $\log t$ which is irrelevant.) On the other hand, considering (A.8) and (B.63), we see that replacing the integral of a u term by the integral of a v_* term reduces the upper bound by a factor of $\psi^{-1}(1/t)$, (again neglecting the factor of $\log t$.) Counting the initial factor of $t^{J(\beta-1)/(2\beta)}$ we have a reduction with is at least

$$\left(t^{(\beta-1)/(2\beta)}(t\psi^{-1}(1/t))^{-1}\right)^J=o\left(\left(t^{-(\beta-1)/(3\beta)}\right)^J\right) \tag{B.66}$$

for all $\epsilon>0$. Since $J\geq 1$, we get (B.57).

Analogous to (B.41) we note that

$$\int\mathcal{T}_{t,1}(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i} \tag{B.67}$$
$$=\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\leq t^{(\beta-1)/(4\beta)}\}}\right)\mathcal{T}_{t,1}(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i}+\widetilde{E}_{1,t},$$

where $\widetilde{E}_{1,t}=O\left(t^{-(\beta-1)/(5\beta)}\left(t^2\psi^{-1}(1/t)\right)^n\right)$. The proof of (B.67) is the same as the proof of (B.42).

Since ψ is regularly varying with index $\beta > 1$ we see that there exists an $\epsilon(\beta) := \epsilon > 0$ such that

$$E_{1,t} + E_{2,t} + \widetilde{E}_{2,t} = O\left(t^{(2-1/\beta)n-\epsilon}\right). \tag{B.68}$$

Therefore, it follows from (B.41), (B.52) and (B.67) that

$$\begin{aligned}\int \mathcal{T}_t(x;\,\pi,e) \prod_{j,k,i} dx_{j,k,i} \qquad\qquad (B.69)\\ = \int \mathcal{T}_{t,1}(x;\,\pi,e) \prod_{j,k,i} dx_{j,k,i} + O\left(t^{(2-1/\beta)n-\epsilon}\right).\end{aligned}$$

We now obtain a sharp estimate , (asymptotically as $t \to \infty$), of the second integral in (B.69) that leads to the (B.1). Let $\widetilde{\mathcal{R}}_l(s) = \{\sum_{q=1}^{n_l} r_{l,2q-1}$
$\leq t - s\}$ and $\widetilde{\sigma}_l(q) := \gamma_{2\sigma_l(q)-1}$. We define

$$\begin{aligned}&F_t(\widetilde{\sigma}, s_0, \dots, s_K) \qquad\qquad (B.70)\\ &= \int \Bigg(\int_{\widetilde{\mathcal{R}}_0(s_0)\times\cdots\times\widetilde{\mathcal{R}}_K(s_K)} \prod_{l=0}^{K} p_{(t-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1})+r_{l,1}}\\ &\qquad (x_{\widetilde{\sigma}_l(1)} - x_{\widetilde{\sigma}_{l-1}(n_{l-1})}) \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\widetilde{\sigma}_l(q)} - x_{\widetilde{\sigma}_l(q-1)}) \prod_{q=1}^{n_l} dr_{l,2q-1} \Bigg)\, dx,\end{aligned}$$

where $(t - \sum_{q=1}^{n_{-1}} r_{-1,2q-1} - s_{-1}) := 0$ and $\widetilde{\sigma}_{-1}(n_{-1}) := 0$. Here the generic term dx indicates integration with respect to all the variables $x_\cdot$ that appear in the integrand.

Since $\widetilde{\sigma}_l(q) = \gamma_{2\sigma_l(q)-1}$ we can also write (B.70) as

$$\begin{aligned}&F_t(\widetilde{\sigma}, s_0, \dots, s_K) \qquad\qquad (B.71)\\ &= \int \Bigg(\int_{\widetilde{\mathcal{R}}_0(s_0)\times\cdots\times\widetilde{\mathcal{R}}_K(s_K)} \prod_{l=0}^{K} p_{(t-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1})+r_{l,1}}\\ &\qquad (x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}) \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}})\\ &\qquad\qquad \prod_{q=1}^{n_l} dr_{l,2q-1} \Bigg)\, dx,\end{aligned}$$

with $x_{\gamma_{2\sigma_{-1}(n_{-1})-1}} := 0$.

Consider (B.71). By extending the time integration we have

$$\begin{aligned}&F_t(\widetilde{\sigma}, s_0, \dots, s_K) \qquad\qquad (B.72)\\ &\leq \int \prod_{l=0}^{K} u(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\\ &\qquad (\prod_{q=2}^{n_l} u_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}})\, dx.\end{aligned}$$

Note that there are n different $x_\cdot$ variables, each one of which appears twice. Therefore, by an argument similar to the one in the paragraph containing (B.47), we see that

$$F_t(\widetilde{\sigma}, s_0, \dots, s_K) \leq C\left(t^2\psi^{-1}(1/t)\right)^n, \tag{B.73}$$

for some constant depending only on $m = 2n$.

Let $\widehat{\mathcal{R}}_l = \{\sum_{q=1}^{n_l} r_{l,2q} \le t\}$, $l = 0, \dots, K$. We break up the integration over $\mathcal{R}_0 \times \cdots \times \mathcal{R}_K$ in (B.56) into integration over $\widetilde{\mathcal{R}}_0 \times \cdots \times \widetilde{\mathcal{R}}_K$ and $\widehat{\mathcal{R}}_0 \times \cdots \times \widehat{\mathcal{R}}_K$; (see (B.71)). If one carefully examines the time indices in (B.30) and (B.70) and uses Fubini's Theorem, one sees that

$$\begin{aligned}
&\int \mathcal{T}_t(x;\pi,e)\prod_{j,k,i} dx_{j,k,i} \qquad\qquad (B.74)\\
&\quad= \int_{\widehat{\mathcal{R}}_0\times\cdots\times\widehat{\mathcal{R}}_K} F_t(\widetilde{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \dots, \sum_{q=1}^{n_K} r_{K,2q})\\
&\qquad\quad \prod_{i=1}^n \left(\int \left(\Delta^1\Delta^{-1}\, p_{r_i}(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'.
\end{aligned}$$

The variables $\{r_i, r_i' \,|\, i = 1,\dots,n\}$ are simply a relabeling of the variables $\{r_{l,2q} \,|\, 0 \le l \le K,\, 1 \le q \le n_l\}$. (The exact form of this relabeling does not matter in what follows.) Here, as always, we set $p_r(x) = 0$, if $r \le 0$.

By Parseval's Theorem

$$\begin{aligned}
&\int \left(\Delta^1\Delta^{-1}\, p_r(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r'}(x)\right)\,dx \qquad\qquad (B.75)\\
&\quad= \frac{1}{2\pi}\int |2 - e^{ip} - e^{-ip}|^2 e^{-r\psi(p)} e^{-r'\psi(p)}\,dp\\
&\quad= \frac{16}{\pi}\int_0^\infty \sin^4(p/2) e^{-r\psi(p)} e^{-r'\psi(p)}\,dp \ge 0.
\end{aligned}$$

Using this, (B.73) and Fubini's Theorem, we see that

$$\begin{aligned}
&\int_{\left(\widehat{\mathcal{R}}_0\times\cdots\times\widehat{\mathcal{R}}_K\right)\cap([0,\sqrt{t}]^{2n})^c} F_t(\widetilde{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \dots, \sum_{q=1}^{n_K} r_{K,2q}) \qquad\qquad (B.76)\\
&\qquad \prod_{i=1}^n \left(\int \left(\Delta^1\Delta^{-1}\, p_{r_i}(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'\\
&\le C\left(t^2\psi^{-1}(1/t)\right)^n\\
&\qquad \int_{([0,\sqrt{t}]^{2n})^c}\prod_{i=1}^n \left(\int \left(\Delta^1\Delta^{-1}\, p_{r_i}(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'\\
&\le C\left(t^2\psi^{-1}(1/t)\right)^n \left(\int\left(\int (\Delta^1\Delta^{-1}\, p_r(x))\,dr\right)^2 dx\right)^{n-1}\\
&\qquad \int\left\{\int_0^\infty\int_{\sqrt{t}}^\infty \left(\Delta^1\Delta^{-1}\, p_{r_i}(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r_i'}(x)\right)\,dr_i\,dr_i'\right\}dx\\
&\le C\left(t^2\psi^{-1}(1/t)\right)^n\\
&\qquad \int\left\{\int_0^\infty\int_{\sqrt{t}}^\infty \left(\Delta^1\Delta^{-1}\, p_{r_i}(x)\right)\left(\Delta^1\Delta^{-1}\, p_{r_i'}(x)\right)\,dr_i\,dr_i'\right\}dx,
\end{aligned}$$

by (A.11). By (A.16) and (A.17) the integral in the final line of (B.76)

$$\le c_{\psi,1} - \int\left(\int_0^{\sqrt{t}} \Delta^1\Delta^{-1}\; p_s(x)\,ds\right)^2 dx \le O\left(t^{-1/6}\right). \qquad (B.77)$$

Therefore the first integral in (B.76) is $O(t^{(2-1/\beta)n-\epsilon})$, for some $\epsilon > 0$.

Since $\left(\widehat{\mathcal{R}}_0\times\cdots\times\widehat{\mathcal{R}}_K\right)\supseteq[0,\sqrt{t}]^{2n}$, for $2n\sqrt{t}\le t$, it follows from (B.74) and the preceding sentence, that

$$\int \mathcal{T}_t(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i} \tag{B.78}$$
$$=\int_{[0,\sqrt{t}]^{2n}}F_t(\widetilde{\sigma},\sum_{q=1}^{n_0}r_{0,2q}\,,\dots,\sum_{q=1}^{n_K}r_{K,2q})\prod_{i=1}^{n}\bigg(\int\left(\Delta^1\Delta^{-1}\,p_{r_i}(x)\right)$$
$$\left(\Delta^1\Delta^{-1}\,p_{r_i'}(x)\right)\,dx\bigg)\prod_{l=0}^{K}\prod_{q=1}^{n_l}dr_{l,2q}+O(t^{(2-1/\beta)n-\epsilon})$$

We use the next lemma which is proved in Subsection B.3.

Lemma B.3 *Under the hypotheses of Theorem 1.1, for any fixed m and $s_0,\dots,s_K\le m\sqrt{t}$ and $1<\beta\le 2$, there exists an $\epsilon>0$ such that for all $t>0$, sufficiently large,*

$$|F_t(\widetilde{\sigma},s_0,\dots,s_K)-F_t(\widetilde{\sigma},0,\dots,0)|\le C\left(t^2\psi^{-1}(1/t)\right)^{n-\epsilon}. \tag{B.79}$$

Proof of Lemma B.1 continued: It follows from (B.78) and Lemmas B.3 and A.3, that

$$\int \mathcal{T}_t(x;\,\pi,e)\prod_{j,k,i}dx_{j,k,i} \tag{B.80}$$
$$=F_t(\widetilde{\sigma},0,\dots,0)\int_{[0,\sqrt{t}]^{2n}}\prod_{i=1}^{n}\bigg(\int\left(\Delta^1\Delta^{-1}\,p_{r_i}(x)\right)$$
$$\left(\Delta^1\Delta^{-1}\,p_{r_i'}(x)\right)\,dx\bigg)\prod_{l=0}^{K}\prod_{q=1}^{n_l}dr_{l,2q}+O\left(t^{(2-1/\beta)n-\epsilon}\right)$$
$$=(c_{\psi,1})^n\,F_t(\widetilde{\sigma},0,\dots,0)+O\left(t^{(2-1/\beta)n-\epsilon}\right),$$

for some $\epsilon>0$.

Consider the mappings $\widetilde{\sigma}_l$ that are used in (B.70). Recall that $\sigma_l(q)$ is defined by the relationship $\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2\sigma_l(q)-1},\gamma_{2\sigma_l(q)}\}$. Therefore, since $\widetilde{\sigma}_l(q)=\gamma_{2\sigma_l(q)-1}$ we can have that either $\widetilde{\sigma}_l(q)=\pi_l(2q-1)$ or $\widetilde{\sigma}_l(q)=\pi_l(2q)$. However, since the terms $\widetilde{\sigma}_l(q)$ are subscripts of the terms x, all of which are integrated, it is more convenient to define $\widetilde{\sigma}_l$ differently.

Recall that $\mathcal{P}$, (see (B.34)), is a union of pairings $\mathcal{P}_{j,k}$ of the $m_{j,k}$ vertices

$$\{(j,k,i),\,1\le i\le m_{j,k}\}.$$

Each $\mathcal{P}_{j,k}$ consists of $n_{j,k}$ pairs, that can ordered arbitrarily. Consider one such ordering. If $\{\gamma_{2\sigma_l(q)-1},$
$\gamma_{2\sigma_l(q)}\}$ is the i-th pair in $\mathcal{P}_{j,k}$, we set $\widetilde{\sigma}_l(q)=(j,k,i)$. (Necessarily, l will be either j or k, as we point out in the paragraph containing (B.37)). Thus, each $\widetilde{\sigma}_l$ is a bijection from $[1,n_l]$ to

$$\widetilde{I}_l=\bigcup_{k=l+1}^{K}\{(l,k,i),\,1\le i\le n_{l,k}\}\bigcup_{j=0}^{l-1}\{(j,l,i),\,1\le i\le n_{j,l}\}. \tag{B.81}$$

Let $\widetilde{\mathcal{B}}$ denote the set of $K+1$ tuples, $\widetilde{\sigma}=(\widetilde{\sigma}_0,\dots,\widetilde{\sigma}_K)$ of such bijections. Note that with this definition of $\widetilde{\sigma}_l(q)$, (B.70) remains unchanged since we have simply renamed the variables of integration.

By interchanging the elements in any of the $2n$ pairs $\{\pi_l(2q-1), \pi_l(2q)\}$ we obtain a new $\pi' \sim \mathcal{P}$. In fact we obtain 2^{2n} different permutations π, in this way, all of which are compatible with $\mathcal{P}$, and all of which give the same $\widetilde{\sigma}$ in (B.70). Furthermore, by permuting the pairs $\{\pi_l(2q-1), \pi_l(2q)\}$, $1 \leq q \leq n_l$, for each l, we get all the possible permutation $\widetilde{\pi} \sim \mathcal{P}$, and these give all possible mappings $\widetilde{\sigma} \in \widetilde{\mathcal{B}}$. Note that $|\widetilde{\mathcal{B}}| = \prod_{l=0}^{K} n_l! \leq (2n)!$.

We now use the notation introduced in the paragraph containing (B.81), and the fact that there are 2^{2n} permutations that are compatible with $\mathcal{P}$, to see that

$$\sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_t(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{B.82}$$
$$= (4c_{\psi,1})^n \sum_{\widetilde{\sigma} \in \widetilde{\mathcal{B}}} F_t(\widetilde{\sigma}, 0, \ldots, 0) + O\left(t^{(2-1/\beta)n-\epsilon}\right).$$

Since $|\widetilde{\mathcal{B}}| \leq (2n)!$, we see that the error term only depends on m. Consider (B.82) and the definition of $F_t(\widetilde{\sigma}, 0, \ldots, 0)$ in (B.70) and use (B.16), with $m_{j,k}$ replaced by $n_{j,k}$, to see that

$$\sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_t(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{B.83}$$
$$= (4c_{\psi,1})^n \, E\left(\prod_{\substack{j,k=0 \\ j<k}}^{K} (\alpha_{j,k,t})^{n_{j,k}} \right) + O\left(t^{(2-1/\beta)n-\epsilon}\right);$$

($\alpha_{j,k,t}$ is defined in (2.2)).

Recall the definition of $\mathcal{S}$, the set of special pairings, given in the first paragraph of this subsection. Since there are $\frac{(2n_{j,k})!}{2^{n_{j,k}} n_{j,k}!}$ pairings of the $2n_{j,k}$ elements $\{1, \ldots, m_{j,k}\}$, (recall that $m_{j,k} = 2n_{j,k}$), we see that when we sum over all the special pairings we get

$$\sum_{\mathcal{P} \in \mathcal{S}} \sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_t(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{B.84}$$
$$= \prod_{\substack{j,k=0 \\ j<k}}^{K} \frac{(2n_{j,k})!}{2^{n_{j,k}} n_{j,k}!} (4c_{\psi,1})^{n_{j,k}} \, E\left\{ \prod_{\substack{j,k=0 \\ j<k}}^{K} (\alpha_{j,k,t})^{n_{j,k}} \right\} + O\left(t^{(2-1/\beta)n-\epsilon}\right).$$

It follows from (B.36) that the error term, still, only depends on m.

The right-hand side of (B.84) is precisely the desired expression in (B.1). Therefore, to complete the proof of Lemma B.1, we show that for all the other possible values of a, the integral in (B.26) can be absorbed in the error term.

B.2 $\mathrm{a} = \mathrm{e}$ but not all cycles are of order two or $\mathrm{a} \neq \mathrm{e}$

We show that when $a = e$ but not all cycles are of order two or when $a \neq e$

$$\left| \int \mathcal{T}_t(x;\, \pi, a) \prod_{j,k,i} dx_{j,k,i} \right| = O\left(\left(t^2 \psi^{-1}(1/t)\right)^{\frac{m}{2}} t^{-\epsilon} \right), \tag{B.85}$$

for some $\epsilon = \epsilon_\beta > 0$. In this subsection we do not assume that m is even.

Consider the basic formula (B.30). Since we only need an upper bound, we take absolute values in the integrand and extend all the time integrals to $[0, t]$, as we have done several times above. We refer to this integral as the extended integral. We take the time integrals and get an upper bound for (B.30) involving the terms u, v and w. As we have done several times above, we choose m of the u, v and w terms with arguments that span R^m. We then bound the remaining u, v and w terms and then make a change of variables and integrate the u, v and w terms with the chosen arguments. Since we want to find the smallest possible upper bound for the extended integral, it is obvious that we first integrate as many of the w terms as possible, since such integrals are effectively bounded. (We continue to ignore slowly varying functions of t). We then try to integrate as many of the v terms as possible.

In order to do this efficiently, we divide the v and w terms into sets. As we construct the sets of v and w terms, we also choose a subset $\mathcal{I}$ of the v and w terms with arguments that are linearly independent. The cardinality of this subset is a lower bound on the number of v and w terms that we can integrate.

This is how we divide the v and w terms into sets. For each π and a we define a multigraph $G_{\pi,a}$ with vertices $\{(j,k,i),\, 0\le j<k\le K,\, 1\le i\le m_{j,k}\}$, and an edge between the vertices $\pi_l(q-1)$ and $\pi_l(q)$ whenever $(a_1(l,q),a_2(l,q))=(1,1)$, $l=0,\dots K$, $2\le q\le m_l$, and an edge between the vertices $\pi_l(1)$ and $\pi_{l-1}(m_l)$, whenever $(a_1(l,1),a_2(l,1))=(1,1)$, $1\le l\le K$.

This graph divides the w terms into cycles and chains. Assume that there are S cycles. We denote them by $C_s=\{\phi_{s,1},\dots,\phi_{s,l(s)}\}$, written in cyclic order, where the cycle length $l(s)=|C_s|\ge 1$ and $1\le s\le S$. For each $1\le s\le S$ we take the set of $l(s)$ terms

$$\mathcal{G}_s^{\text{cycle}}=\{w(x_{\phi_{s,2}}-x_{\phi_{s,1}}),\cdots,w(x_{\phi_{s,l(s)}}-x_{\phi_{s,l(s)-1}}),w(x_{\phi_{s,1}}-x_{\phi_{s,l(s)}})\}. \tag{B.86}$$

Let

$$y_{\phi_{s,i}}=x_{\phi_{s,i}}-x_{\phi_{s,i-1}},\qquad i=2,\dots,l(s). \tag{B.87}$$

It is easy to see that $\{y_{\phi_{s,i}}\,|i=2,\dots,l(s)\}$, are linearly independent. We put the corresponding w terms, $w(x_{\phi_{s,2}}-x_{\phi_{s,1}}),\cdots,w(x_{\phi_{s,l(s)}}-x_{\phi_{s,l(s)-1}})$ into $\mathcal{I}$. (On the other hand, since

$$\sum_{i=2}^{l(s)}y_{\phi_{s,i}}=-(x_{\phi_{s,1}}-x_{\phi_{s,l(s)}}), \tag{B.88}$$

we see that we can only extract $l(s)-1$ linearly independent variables from the $l(s)$ arguments of w for a given s.)

A cycle of length 1 consists of a single point $\phi_{s,1}=\phi_{l(s),1}$ in the graph, so in this case

$$\mathcal{G}_s^{\text{cycle}}=\{w(0)\}. \tag{B.89}$$

We explain below how this can occur. Obviously, $w(0)$ is not put into $\mathcal{I}$.

Next, suppose there are S' chains. We denote them by $C'_s=\{\phi'_{s,1},\dots,\phi'_{s,l'(s)}\}$, written in order, where $l'(s)=|C'_s|\ge 2$ and $1\le s\le S'$. Note that there are $l'(s)-1$, w terms corresponding to C'_s. Then for each $1\le s\le S'$ we form the set of $l'(s)+1$ terms

$$\begin{aligned}\mathcal{G}_s^{\text{chain}}=\{v(x_{\phi'_{s,1}}-x_{a(s)}),w(x_{\phi'_{s,2}}-x_{\phi'_{s,1}}),\cdots,\\ \cdots,w(x_{\phi'_{s,l(s)}}-x_{\phi'_{s,l(s)-1}}),v(x_{b(s)}-x_{\phi'_{s,l(s)}})\}\end{aligned} \tag{B.90}$$

where $v(x_{\phi'_{s,1}}-x_{a(s)})$ is the unique v term associated with $\Delta^1_{x_{\phi'_{s,1}}}$, and similarly, $v(x_{b(s)}-x_{\phi'_{s,l(s)}})$ is the unique v term associated with $\Delta^1_{x_{\phi'_{s,l(s)}}}$. (This deserves further clarification. There may be other v terms containing the variable $x_{\phi'_{s,1}}$ in the extended integral, but there is only one v term of the form

$$\int_0^t \left|\Delta^1_{x_{\phi'_{s,1}}} p_s(x_{\phi'_{s,1}} - u)\right| ds, \tag{B.91}$$

where u is some other $x_\cdot$ variable which we denote by $x_{a(s)}$. This is because one operator $\Delta^1_{x_{\phi'_{s,1}}}$ is associated with $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}})$ and there are precisely two operators $\Delta^1_{x_{\phi'_{s,1}}}$ in (B.30)).

It is easy to see that variables $y_{\phi'_{s,i}} = x_{\phi'_{s,i}} - x_{\phi'_{s,i-1}}$, $i = 2, \ldots, l(s)$, are linearly independent. We put the w terms, $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}}), \cdots, w(x_{\phi'_{s,l(s)}} - x_{\phi'_{s,l(s)-1}})$ into $\mathcal{I}$. We leave the v terms in $\mathcal{G}^{\text{chain}}_s$ out of $\mathcal{I}$.

At this stage we emphasize that the terms we have put in $\mathcal{I}$ from all cycles and chains have linearly independent arguments. If fact, the set of x's appearing in the different chains and the cycles are disjoint. This is obvious for the cycles and the interior of the chains since there are exactly two difference operators Δ^x_h for each x. It also must be true for the endpoints of the chains, since if this is not the case they could be made into larger chains or cycles.

For the same reason, if a v term involving $\Delta^h_{x'}$ is not in any of the sets of chains, then x' will not appear in the arguments of the terms that are put in $\mathcal{I}$ from all the cycles and chains.

Suppose, after considering the w terms and the v terms associated with the chains of w terms, that there are p pairs of v terms left, each pair corresponding to difference operators $\Delta^1_{z_j}$, $j = 1, \ldots, p$; (p may be 0). Let

$$\mathcal{Z} := \{z_1, \ldots, z_p\} \tag{B.92}$$

A typical v term is of the form

$$v^{(j)}(z_j - u_{j'}) := v(z_j - u_{j'}) = \int_0^t |\Delta^h_{z_j} p_t(z_j - u_{j'})| \, dt, \tag{B.93}$$

where $u_{j'}$ is some $x_\cdot$ term. We use the superscript (j) is to keep track of the fact that this v term is associated with the difference operator $\Delta^1_{z_j}$. We distinguish between the variables z_j and $u_{j'}$ by referring to z_j as a marked variable. Note that if $u_{j'}$ is also in $\mathcal{Z}$, say $u_{j'} = z_k$, then $u_{j'}$ is also a marked variable but in a different v term. (In this case, in $v^{(k)}(z_k - u_{k'})$, where $u_{k'}$ is some other $x_\cdot$ variable.)

Thus $\mathcal{Z}$ is the collection of marked variables. Consider the corresponding terms

$$v^{(j)}(z_j - u_j) \quad \text{and} \quad v^{(j)}(z_j - v_j), \qquad j = 1, \ldots, p \tag{B.94}$$

where u_j and v_j represent whatever terms $x_\cdot$ and $x'_\cdot$ are coupled with the two variables z_j.

There may be some j for which u_j and v_j in (B.94) are both in $\mathcal{Z}$. Choose such a j. Suppose $u_j = v_j = z_k$. We set

$$\begin{aligned} \mathcal{G}^{\mathcal{Z},1}_j = \{&v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_k), \\ &v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k)\} \end{aligned} \tag{B.95}$$

and put $v^{(j)}(z_j - z_k)$ into $\mathcal{I}$. Here u_k and v_k are whatever two variables appear with the two marked variables z_k.

On the other hand, suppose u_j and v_j are both in $\mathcal{Z}$ but $u_j = z_k$ and $v_j = z_l$ with $k \neq l$. We set

$$\begin{aligned} \mathcal{G}^{\mathcal{Z},2}_j = \{&v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_l), \\ &v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k), v^{(l)}(z_l - u_l), v^{(l)}(z_l - v_l)\} \end{aligned} \tag{B.96}$$

and put both $v^{(j)}(z_j - z_k)$ and $v^{(j)}(z_j - z_l)$ into $\mathcal{I}$.

We then turn to the elements in $\mathcal{Z}$ which have not yet appeared in the arguments of the terms that have been put into $\mathcal{I}$. If there is another j' for which $u_{j'}$ and $v_{j'}$ are both in $\mathcal{Z}$, choose such a j' and proceed as above. If there are no longer any such elements in $\mathcal{Z}$, choose some remaining element, say, z_i. Set

$$\mathcal{G}_i^{\mathcal{Z},3} = \{v^{(i)}(z_i - u_i), v^{(i)}(z_i - v_i)\} \tag{B.97}$$

and if $u_i \notin \mathcal{Z}$, place $v^{(i)}(z_i - u_i)$ into $\mathcal{I}$. If $u_i \in \mathcal{Z}$, so that $v_i \notin \mathcal{Z}$, place $v^{(i)}(z_i - v_i)$ into $\mathcal{I}$.

We continue until we have exhausted $\mathcal{Z}$.

The v and w terms in $\mathcal{I}$ have linearly independent arguments. We choose an additional $\mathcal{I}' = m - |\mathcal{I}|$ terms from the remaining u, v and w terms so that the arguments of the m terms are a spanning set of R^m. We bound the remaining terms by their supremum. We then make a change of variables and integrate separately each of the m terms in $\mathcal{I} \cup \mathcal{I}'$. Our goal is to integrate as few u terms as necessary.

Let S_1 denote the number of cycles of length 1. The number of w terms in $\mathcal{I}$ from cycles is

$$\sum_{s=1}^{S} (l(s) - 1) \geq \frac{1}{2} \sum_{s=1}^{S} l(s) - \frac{S_1}{2}. \tag{B.98}$$

The number of w terms in $\mathcal{I}$ from chains is

$$\sum_{s=1}^{S'} (l'(s) - 1) \geq \frac{1}{2} \sum_{s=1}^{S'} (l'(s) - 1) + \frac{S'}{2}. \tag{B.99}$$

The number of w terms may be less than m. (In general it is, but we see below that it is possible that the number of w terms may be equal to m.) Suppose there are ρ terms of type w. Then the number of v terms must be $2(m-\rho)$, and consequently, the number of u terms must be ρ.

We note that

$$\rho = \sum_{s=1}^{S} l(s) + \sum_{s=1}^{S'} (l'(s) - 1). \tag{B.100}$$

Since the total number of w terms is ρ, we see from (B.98) and (B.99) that the number of w terms in $\mathcal{I}$ is at least

$$\frac{\rho}{2} + \frac{S'}{2} - \frac{S_1}{2}. \tag{B.101}$$

This shows that for a given ρ the the number of w terms in $\mathcal{I}$ is minimized when their are no chains.

We now turn to the number of integrated v terms. Since the total number of v terms is $2(m-\rho)$, and there are also two v terms in each set $\mathcal{G}_s^{\text{chain}}$ we see that

$$2(m - \rho) = \sum_{i,j} |\mathcal{G}_j^{\mathcal{Z},i}| + 2S'. \tag{B.102}$$

It is easily seen that we place in $\mathcal{I}$ at least $1/4$ the number of v terms in the sets $\mathcal{G}_j^{\mathcal{Z},i}$ for all i, j. Consequently, the number of v terms with arguments in $\mathcal{I}$ is at least

$$\frac{1}{4} \sum_{i,j} |\mathcal{G}_j^{\mathcal{Z},i}| = \frac{m}{2} - \frac{\rho}{2} - \frac{S'}{2}. \tag{B.103}$$

Combined with (B.101) we see that the number of w and v terms in $\mathcal{I}$ is at least

$$\left(\frac{m}{2} - \frac{\rho}{2} - \frac{S'}{2}\right) + \left(\frac{\rho}{2} + \frac{S'}{2} - \frac{S_1}{2}\right) = \frac{m}{2} - \frac{S_1}{2}. \tag{B.104}$$

Since $\frac{m}{2}-\frac{S_1}{2}$ is an integer, it is at least

$$\frac{m}{2}-\frac{S_1}{2}+\frac{\bar{1}}{2}, \tag{B.105}$$

where $\bar{1}=0$, if $m-S_1$ is even, and $\bar{1}=1$, if $m-S_1$ is odd.

Suppose that $\rho\geq\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$. Then, since there are m terms that are integrated, the upper bound of the extended integral will be greatest if we integrate $\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$ terms of the form u, and bound the remaining u terms by their supremum. (Note that $\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$ is also an integer.) This gives a bound for the u terms of

$$t^{\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2}\left(t\psi^{-1}(1/t)\right)^{\rho-\frac{m}{2}-\frac{S_1}{2}+\bar{1}/2}. \tag{B.106}$$

(We ignore slowly varying function of t.)

Note that when we integrate $\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$ terms of the form u, we only integrate $\frac{m}{2}-\frac{S_1}{2}+\bar{1}/2$ terms of the form v and w. By Lemma A.1 integrated v terms are much larger than integrated u terms. What is the maximum number of v terms that can be integrated?

The maximum number of v terms that can be integrated occurs when all the w terms are in cycles of length 1 or 2, in which case $(\rho-S_1)/2$ terms of the form w are integrated. This is easy to see, since in this case the right-hand side of (B.98) is realized. (We point out in the paragraph containing (B.101) that to minimize the number of w terms that are integrated there should be no chains.)

We are left with $\frac{m}{2}-\frac{\rho}{2}+\bar{1}/2$ terms of the form v that are integrated. Since, by (A.6), the supremum of the v terms are effectively bounded this gives a contribution from all v terms of $\left(t\psi^{-1}(1/t)\right)^{\frac{m}{2}-\rho/2+\bar{1}/2}$. Combining the bounds for u and v terms we obtain

$$\begin{aligned} & t^{\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2}\left(t\psi^{-1}(1/t)\right)^{\rho/2-\frac{S_1}{2}+\bar{1}} \\ &\quad=\left(t^2\psi^{-1}(1/t)\right)^{\frac{m}{2}}\left(t\psi^{-1}(1/t)\right)^{-(\frac{m}{2}-\rho/2)}\left(\psi^{-1}(1/t)\right)^{-\frac{S_1}{2}} \\ &\qquad\qquad\left(t\left(\psi^{-1}(1/t)\right)^2\right)^{\bar{1}/2}. \end{aligned} \tag{B.107}$$

It follows from [11, (4.77)] that

$$t(\psi^{-1}(1/t))^2\leq C\qquad\forall\, t\geq t_0. \tag{B.108}$$

Therefore, when $\rho\geq\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$, (B.107) is bounded by

$$C\left(t^2\psi^{-1}(1/t)\right)^{\frac{m}{2}}\left(t\psi^{-1}(1/t)\right)^{-(\frac{m}{2}-\rho/2)}\left(\psi^{-1}(1/t)\right)^{-\frac{S_1}{2}}. \tag{B.109}$$

On the other hand, when $\rho<\frac{m}{2}+\frac{S_1}{2}-\bar{1}/2$, we get the largest upper bound for the extended integral when we integrate all ρ of the u terms. As above, to get the most v terms integrated, we only integrate $(\rho-S_1)/2$ terms of the form w. Consequently, since m terms are integrated, $m-\frac{3\rho}{2}+\frac{S_1}{2}$ of the v terms are integrated. (The remaining v terms are bounded by their supremum, which is effectively bounded.) Combining the bounds for u and v terms we obtain

$$\begin{aligned} & t^{\rho}\left(t\psi^{-1}(1/t)\right)^{m-\frac{3\rho}{2}+\frac{S_1}{2}} \\ &\quad=\left(t^2\psi^{-1}(1/t)\right)^{\frac{m}{2}}\left(t\left(\psi^{-1}(1/t)\right)^3\right)^{-\rho/2}\left(\psi^{-1}(1/t)\right)^{\frac{m}{2}}\left(t\psi^{-1}(1/t)\right)^{\frac{S_1}{2}}. \end{aligned} \tag{B.110}$$

We now show that we obtain (B.85) when $S_1 = 0$. Consider the case when $\rho < \frac{m}{2} - \bar{1}/2$ and refer to (B.110). Note that

$$\begin{aligned}\left(t\left(\psi^{-1}(1/t)\right)^3\right)^{-\rho/2}\left(\psi^{-1}(1/t)\right)^{\frac{m}{2}} &= t^{(3/\beta-1)\rho/2-m/(2\beta)}L(t)\\ &< t^{(3/\beta-1)m/4-m/(2\beta)}L(t)\\ &= t^{-(\beta-1)m/(4\beta)}L(t),\end{aligned} \tag{B.111}$$

where $L(t)$ is slowly varying at infinity and we use the facts that $(3/\beta)-1 > 0$ and $\rho < \frac{m}{2}$.

Now we consider the case when $\rho \geq \frac{m}{2} - \bar{1}/2$. In dealing with (B.109) we also have that $\rho \leq m-1$, since we arrived at this inequality by assuming that all cycles are of order two, but are excluding the case when the graph $G_{\pi,a}$ consists solely of cycles of order two. Therefore $\frac{m}{2} - (\rho/2)$ in (B.109) is strictly positive. This observation and (B.111) gives (B.85) when $S_1 = 0$.

We now eliminate the restriction that $S_1 = 0$. This requires additional work since the estimates on the right-hand side of (B.107) and (B.110) are larger in this case. Actually we show that the bounds in (B.107) and (B.110) that we obtained when $S_1 = 0$, remain the same when $S_1 \neq 0$.

The only way there can be cycles of length one is in terms of the type

$$\Delta^1\Delta^{-1}p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}) \tag{B.112}$$

when $\gamma_{2\sigma_l(1)-1} = \gamma_{2\sigma_{l-1}(n_{l-1})-1}$. In this case

$$\int_0^t |\Delta^1\Delta^{-1}p_s(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})|\,ds \quad = \quad w(0,t). \tag{B.113}$$

Note that

$$|\Delta^1\Delta^{-1}p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(0)| = 2|\Delta^1 p_{(t-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(0)|. \tag{B.114}$$

This is how we bound the right-hand side of (B.85) when $G_{\pi,a}$ contains cycles of length one. We return to the basic formulas (B.29) and (B.30). We obtain an upper bound for (B.30) by taking the absolute value of the integrand. However, we do not, initially extend the region of integration with respect to time. Instead we proceed as follows: Let l' be the largest value of l for which $\gamma_{2\sigma_l(1)-1} = \gamma_{2\sigma_{l-1}(n_{l-1})-1}$. We extend the integral with respect to $r_{l,q}$ to $[0,t]$ for all $l > l'$, and also for $l = l'$ and $q > 1$, and bound these integrals with terms of the form $u(\,\cdot\,,t)$, $v(\,\cdot\,,t)$ and $w(\,\cdot\,,t)$. We then consider the integral of the term in (B.114) with respect to $r_{l',1}$.

Clearly

$$\int_0^t |\Delta^1 p_{(t-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q})+r_{l',1}}(0)|\,dr_{l',1} \leq \int_{t-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}}^{2t-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}} |\Delta^1 p_s(0)|\,ds \tag{B.115}$$

If $\sum_{q=1}^{m_{l-1}} r_{l-1,q} \leq t/2$, we use (A.13) to bound the left-hand side of (B.115) by

$$\int_{t/2}^{2t} |\Delta^1 p_r(0)|\,dr \leq Ct\left(\psi^{-1}(1/t)\right)^3 \leq C\psi^{-1}(1/t), \tag{B.116}$$

by (B.108).

Suppose, on the other hand, that $\sum_{q=1}^{m_{l-1}} r_{l-1,q} \geq t/2$. Then for some q we have $r_{l-1,q} \geq t/2m$. We do two things. We bound the contribution of

$$\left|\left(\left(\Delta^1_{x_{\pi_{l-1}(q)}}\right)^{a_1(l-1,q)}\left(\Delta^1_{x_{\pi_{l-1}(q-1)}}\right)^{a_2(l-1,q)} p_{r_{l-1,q}}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)})\right)\right| \tag{B.117}$$

by its supremum over $t/2m \leq r_{l-1,q} \leq t$.

To express we use the notation

$$\overline{u}(x,t) = \sup_{t/2m \leq r \leq t} u(x,r), \qquad \overline{v}(x,t) = \sup_{t/2m \leq r \leq t} v(x,r), \tag{B.118}$$

and

$$\overline{w}(x,t) = \sup_{t/2m \leq r \leq t} w(x,r), \tag{B.119}$$

so that the bound of (B.117) may be

$$\overline{u}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t), \qquad \overline{v}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t) \tag{B.120}$$

or

$$\overline{w}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t)$$

according to whether there are no, one or two difference operators.

The terms in (B.120) no longer depend on $r_{l-1,q}$ therefore we can integrate (B.114) with respect to both $r_{l,1}$ and $r_{l-1,q}$ and use (A.14) get

$$\int_0^{2t}\int_0^{2t} |\Delta^1 p_{r+s}(0)|\, dr\, ds \leq C\left(t^2\left(\psi^{-1}(1/t)\right)^3 + L(t) + 1\right), \tag{B.121}$$

where $L(t)$ is a slowly varying function at infinity.

Consider how (B.117) contributes to the bounds in (B.107) and (B.110). If there are no difference operators they would ultimately contribute either

$$\sup_x u(x,t) \qquad \text{or} \qquad \int u(x,t)\, dx \tag{B.122}$$

Now because of the bound in (B.120) we get a contribution of

$$\sup_x \overline{u}(x,t) \qquad \text{or} \qquad \int \overline{u}(x,t)\, dx \tag{B.123}$$

The following table summarizes results from Lemmas A.1 and C.6. It shows that each term in (B.122) is smaller than the corresponding term in (B.123) by a factor of Ct^{-1}. Up to factors of $\log t$ the same diminution, or more, occurs when we compare the two functions of $v(x,t)$ with those of $\overline{v}(x,t)$ and the two functions of $w(x,t)$ with those of $\overline{w}(x,t)$.

$f(x,t)$	$\sup_x f(x,t) \leq$	$\int f(x,t)\, dx \leq$
$u(x,t)$	$Ct\psi^{-1}(1/t)$	t
$\overline{u}(x,t))$	$C\psi^{-1}(1/t)$	C
$v(x,t)$	$C\log t$	$Ct\psi^{-1}(1/t)\log t$
$\overline{v}(x,t)$	$C\left(\psi^{-1}(1/t)\right)^2 \leq C/t$	$C\psi^{-1}(1/t)\log t$
$w(x,t)$	C	$C(\log t)^2$
$\overline{w}(x,t)$	$C\left(\psi^{-1}(1/t)\right)^3$	$C\left(\psi^{-1}(1/t)\right)^2 \leq C/t$

To read the table note that the second line states that $\sup_x u(x,t) \leq C\psi^{-1}(1/t)$ and $\int u(x,t)\,dx \leq t$, and similarly for the remaining lines.

Combined with (B.121) we see that we have reduced the bounds in (B.107) and (B.110) by a factor of

$$C\left(t\left(\psi^{-1}(1/t)\right)^3 + \frac{L(t)+1}{t}\right) \leq C\psi^{-1}(1/t), \tag{B.124}$$

where for the last inequality we use (B.108), as we do in (B.116).

We apply a similar procedure for each l in decreasing order, with one exception. If $r_{l-1,q} = r_{l-1,1}$, i.e., $q = 1$ in this case and we are also in the (B.114) with l is replaced by $l-1$, we skip this term because this it has already been modified. We then proceed to deal with remaining terms as we did when we assumed that there were no cycles of length one.

Consequently, if there are S_1 cycles of length 1 we have diminished the bounds in (B.107) and (B.110) by a factor of at least $C\left(\psi^{-1}(1/t)\right)^{\frac{S_1}{2}}$, if S_1 is even and by a factor of at least $C\left(\psi^{-1}(1/t)\right)^{\frac{S_1}{2}+\frac{1}{2}}$, if S_1 is odd.

In the case of (B.107) we are precisely in the case we considered when $S_1 = 0$, which gives (B.85). In the case of (B.110) the final factor is now $\left(t(\psi^{-1}(1/t))^2\right)^{\frac{S_1}{2}}$, which is bounded by a constant by (B.108). Thus we are again in the case we considered when $S_1 = 0$, which also gives (B.85).

It follows from (B.84) and (B.85) that when m is even

$$\begin{aligned}
&\sum_a \sum_{\pi_0,\ldots,\pi_K} \int \mathcal{T}_t(x;\,\pi,a) \prod_{j,k,i} dx_{j,k,i} \qquad\qquad \text{(B.125)}\\
&\quad = \prod_{\substack{j,k=0\\ j<k}}^{K} \frac{(2n_{j,k})!}{2^{n_{j,k}} n_{j,k}!} \left(4c_{\psi,1}\right)^{n_{j,k}} E\left\{\prod_{\substack{j,k=0\\ j<k}}^{K} \left(\alpha_{j,k,t}\right)^{n_{j,k}}\right\} + O\left(t^{(2-1/\beta)n-\epsilon}\right).
\end{aligned}$$

We now show that we get the same estimates when $\mathcal{T}_t(x;\,\pi,a)$ is replaced by $\mathcal{T}'_t(x;\,\pi,a)$; (see (B.27) and (B.30)).

We point out, in the paragraph containing (B.22) that terms of the form $\Delta^1\Delta^{-1}p^{\sharp}_{\cdot}$ in (B.27) are always of the form $\Delta^1\Delta^{-1}p_{\cdot}$. Therefore, in showing that (B.26) and (B.29) have the same asymptotic behavior as $t \to \infty$ we need only consider how the proof of (B.125) must be modified when the arguments of the density functions with one or no difference operators applied is effected by adding ± 1.

It is easy to see that the presence of these terms has no effect on the integrals that are $O\left(\left(t^2\psi^{-1}(1/t)\right)^n t^{-\epsilon}\right)$ as $t \to 0$. This is because in evaluating these expressions we either integrate over all of R^1 or else use bounds that hold on all of R^1. Since terms with one difference operator only occur in these estimations, we no longer need to be concerned with them.

Consider the terms with no difference operators applied to them, now denoted by $p^{\sharp}$. So, for example, instead of $F(\widetilde{\sigma},0,\ldots,0)$ on the right-hand side of (B.80), we now have

$$\begin{aligned}
&\int\Bigg(\int_{\widetilde{\mathcal{R}}_0(0)\times\cdots\times\widetilde{\mathcal{R}}_K(0)} \prod_{l=0}^{K} p^{\sharp}_{\left(1-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1}\right)+r_{l,1}} \qquad\qquad \text{(B.126)}\\
&\quad \left(x_{\widetilde{\sigma}_l(1)} - x_{\widetilde{\sigma}_{l-1}(n_{l-1})}\right) \prod_{q=2}^{n_l} p^{\sharp}_{r_{l,2q-1}}\left(x_{\widetilde{\sigma}_l(q)} - x_{\widetilde{\sigma}_l(q-1)}\right) \prod_{q=1}^{n_l} dr_{l,2q-1}\Bigg)\,dx.
\end{aligned}$$

Suppose that $p^\sharp_r(y_{\sigma(i)} - y_{\sigma(i-1)}) = p_r(y_{\sigma(i)} - y_{\sigma(i-1)} \pm 1)$. We write this term as

$$p^\sharp_r(y_{\sigma(i)} - y_{\sigma(i-1)}) = p_r(y_{\sigma(i)} - y_{\sigma(i-1)}) + \Delta^{\pm 1} p_r(y_{\sigma(i)} - y_{\sigma(i-1)}). \tag{B.127}$$

Substituting all such terms into (B.126) and expanding we get (B.125) and many other terms with at least one $p_r(y_{\sigma(i)} - y_{\sigma(i-1)})$ replaced by $\Delta^{\pm 1} p_r(y_{\sigma(i)} - y_{\sigma(i-1)})$.

Substitute (B.127) into (B.126) and write it as the sum of 2^m terms. One term, which contains no difference operator, is the term we analyzed when we replaced $p^\sharp$ by p. All the other terms contain at least one difference operator. It is easy to see that all these other terms are $O\left(\left(t^2\psi^{-1}p_t(x,y)1/t\right)^n t^{-\epsilon}\right)$, for some $\epsilon > 0$.

By (B.73) the term with no difference operators is bounded by $C((t^2 \psi^{-1}(1/t))^n)$. This bound is obtained by extending the integrals to $[0,t]$ in (B.72) and integrating or bounding the resulting terms $u(\,\cdot\,,t)$. We are in a situation similar to the one considered in the paragraph containing (B.122). Each difference operator in the other terms replaces a $u(\,\cdot\,,t)$ term by a $v(\,\cdot\,,t)$ term. By Lemma A.1 each replacement reduces $C((t^2 \psi^{-1}(1/t))^n)$ by a factor of at least $(t\psi^{-1}(1/t))^{-1}$. Therefore, the replacement of p by $p^\sharp$ doesn't change (B.125) when m is even.

We now obtain (B.2). In Subsection B.2 we do not require that m is even. Therefore, (B.2) follows from (B.85) unless $G_{\pi,a}$ consists solely of cycles of order two and there are no terms with a single difference operator. Therefore, (B.2) follows from (B.85) when m is odd unless we are in the situation covered in Subsection B.1. This also holds when when $p_\cdot$ is replaced by $p^\sharp_\cdot$ for the reasons given in the case when m is even.

However, if any of the $m_{j,k}$ are odd we can not be in the situation covered in Subsection B.1. Consider the multigraph G_π described in the paragraph following (B.33), with vertices $\{(j,k,i),\, 0 \le j < k \le K,\, 1 \le i \le m_{j,k}\}$, and an edge between the vertices $\pi_l(2q-1)$ and $\pi_l(2q)$ for each $0 \le l \le K$ and $1 \le q \le n_l$. Each vertex is connected to two edges. Suppose that $\{(j,k,i)\} = \pi_l(2q)$, with $j = l$ and $k = l' \neq l$. Then there is a unique q' such that $\pi_{l'}(2q')$ or $\pi_{l'}(2q'-1)$ is also equal to $\{(j,k,i)\}$.

Suppose $\pi_{l'}(2q') = \{(j,k,i)\}$ and consider $\pi_l(2q-1)$ and $\pi_{l'}(2q'-1)$ Suppose that $\pi_l(2q-1) = \{(j,k',i')\}$ for some k' and $\pi_{l'}(2q'-1) = \{(j',k,i'')\}$ for some j'. In order that G_π consist of cycles or order two, we must have $(j,k',i') = (j',k,i'')$, in particular, $j' = j, k' = k$, (but, of course, $i \neq i'$). This shows that for G_π to consist of cycles or order two $m_{j,k}$ must be even for each j,k.

This completes the proof of Lemma B.1. □

B.3 Proof of Lemma B.3

For any $A \subseteq [0,3t]^n$ we set

$$\begin{aligned} F_A = \int \Bigg\{ \int_A \prod_{l=0}^K p_{r_{l,1}}(x_{\widetilde{\sigma}_l(1)} - x_{\widetilde{\sigma}_{l-1}(n_{l-1})}) \\ \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\widetilde{\sigma}_l(q)} - x_{\widetilde{\sigma}_l(q-1)}) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q-1} \Bigg\} \prod_{q=1}^{n_l} dx_{\widetilde{\sigma}_l(q)}. \end{aligned} \tag{B.128}$$

Rather than bound the time integral by that over $[0,3t]^n$ as we have in the past, we have to be more careful.

It follows from (B.70), paying special attention to the time variable of $p_\cdot$ in the second line, that

$$F(\sigma, s_0, \ldots, s_K) = F_{A_{s_0,\ldots,s_K}} \tag{B.129}$$

where

$$\begin{aligned} A_{s_0,\ldots,s_K} &= \left\{ r\in R_+^n \,\middle|\, \sum_{\lambda=0}^{l-1}(t-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1}-s_\lambda)\le \sum_{q=1}^{n_l} r_{l,2q-1} \right. \\ &\qquad \left. \le \sum_{\lambda=0}^{l-1}(t-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1}-s_\lambda)+(t-s_l);\ l=0,1,\ldots,K \right\}. \end{aligned} \tag{B.130}$$

In particular

$$\begin{aligned} A_{0,\ldots,0} &= \left\{ r\in [0,3t]^n \,\middle|\, \sum_{\lambda=0}^{l-1}(t-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})\le \sum_{q=1}^{n_l} r_{l,2q-1} \right. \\ &\qquad \left. \le \sum_{\lambda=0}^{l-1}(t-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})+t;\ l=0,1,\ldots,K \right\}. \end{aligned} \tag{B.131}$$

Let $\phi_l(r)=\sum_{\lambda=0}^{l}(t-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})$. We have

$$\begin{aligned} & A_{s_0,\ldots,s_K}\,\Delta\, A_{0,\ldots,0} \\ &\subseteq \bigcup_{l=1}^{K}\left\{ r\in[0,3t]^n \,\middle|\, \phi_{l-1}(r)-\sum_{\lambda=0}^{l-1}s_\lambda \le \sum_{q=1}^{n_l} r_{l,2q-1}\le \phi_{l-1}(r)\right\} \\ &\quad \bigcup_{l=0}^{K}\left\{ r\in[0,3t]^n \,\middle|\, \phi_{l-1}(r)+t-\sum_{\lambda=0}^{l}s_\lambda \le \sum_{q=1}^{n_l} r_{l,2q-1}\le \phi_{l-1}(r)+t\right\}. \\ &= \bigcup_{l=1}^{K}\mathcal{A}_l\cup\mathcal{B}_l \end{aligned} \tag{B.132}$$

where, setting $\bar\phi_{l-1}(r)=\phi_{l-1}(r)-\sum_{q=1}^{n_l-1} r_{l,2q-1}$ we can write

$$\mathcal{A}_l=\left\{ r\in[0,3t]^n \,\middle|\, \bar\phi_{l-1}(r)-\sum_{\lambda=0}^{l-1}s_\lambda \le\, r_{l,2n_l-1}\le \bar\phi_{l-1}(r)\right\} \tag{B.133}$$

and

$$\mathcal{B}_l=\left\{ r\in[0,3t]^n \,\middle|\, \bar\phi_{l-1}(r)+t-\sum_{\lambda=0}^{l-1}s_\lambda \le\, r_{l,2n_l-1}\le \bar\phi_{l-1}(r)+t\right\}. \tag{B.134}$$

(The first union in (B.132) are the points in $A_{s_0,\ldots,s_K}$ that are not in $A_{0,\ldots,0}$ and the second union are the points in $A_{0,\ldots,0}$ that are not in $A_{s_0,\ldots,s_K}$.)

Note that each time $r_{l,2n_l-1}$ is contained in an interval of length $2(K+1)n\sqrt{t}$.

We bound each $F_{\mathcal{A}_l}$ and $F_{\mathcal{B}_l}$ as in (B.73) except that we only integrate with respect to $r_{l,2n_l-1}$ over $\mathcal{A}_l$ or $\mathcal{B}_l$. Therefore, instead of getting a bound of $u(x,t)$ or $\int u(x,t)\,dx$ from this term we get a smaller bound.

To see this, for fixed $a,b\ge 0$, let

$$u_{a,b}(x)=\int_a^{a+b} p_s(x)\,ds. \tag{B.135}$$

Clearly

$$\int u_{a,b}(x)\,dx=\int_a^{a+b}1\,ds=b. \tag{B.136}$$

In addition, by (C.5),

$$\sup_x u_{a,b}(x) = \sup_x \int_a^{a+b} \int e^{ipx} e^{-s\psi(p)}\, dp\, ds \tag{B.137}$$
$$= \int_a^{a+b} \int e^{-s\psi(p)}\, dp\, ds \le \int_0^b \int e^{-s\psi(p)}\, dp\, ds \le Cb\psi^{-1}(b).$$

Using (B.135)–(B.137) with $b = C\sqrt{t}$, and Lemma A.1 we see that the bound in (B.73) is reduced by a factor of at least $(t\psi^{-1}(1/t))^{-(\frac{1}{2}-\epsilon')}$ for any $\epsilon' > 0$. $\square$

C Proofs of Lemmas A.1–A.3 and B.2

Since the Lévy processes, X, that we are concerned with satisfy (1.6), it follows from the Riemann Lebesgue Lemma that they have transition probability density functions, which we designate as $p_s(\,\cdot\,)$. Taking the inverse Fourier transform of the characteristic function X_s, and using the symmetry of ψ, we see that

$$\begin{aligned} p_s(x) &= \frac{1}{2\pi}\int e^{ipx}\, e^{-s\psi(p)}\, dp \\ &= \frac{1}{\pi}\int_0^\infty \cos(px)\, e^{-s\psi(p)}\, dp. \end{aligned} \tag{C.1}$$

Our basic hypothesis is that $\psi(\lambda)$ is regularly varying at 0 with index $1 < \beta \le 2$. Therefore $\psi(\,\cdot\,)$ is asymptotic to an increasing function near zero. Considering the way we use $\psi(\,\cdot\,)$ in the estimates below, we can assume that $\psi(\lambda)$ is strictly increasing for $0 \le \lambda \le \lambda_0$, for some $\lambda_0 > 0$, and that $\psi^{-1}(\lambda)$ is well defined for $0 \le \lambda \le \lambda_0$. Actually, we are really interested in $\psi^{-1}(1/s)$ as $s \to \infty$. Therefore, there exists an s_0 such that $\psi^{-1}(1/s)$, as a function of s is regularly varying with index $-1/\beta$ for $s \ge s_0$.

The next two lemmas give fundamental estimates that are used in the proofs of the lemmas in Section A.

Lemma C.1 *Let X be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at 0 with index $1 < \beta \le 2$ and satisfies (1.6)–(1.8). Then for all $\gamma \ge 1$ and for all s sufficiently large and all $x \in R^1$,*

$$p_s(x) \le C\left(\psi^{-1}(1/s) \wedge \frac{1}{\psi^{-1}(1/s)x^2}\right); \tag{C.2}$$

$$|\Delta^\gamma p_s(x)| \le C\gamma^2\left(\left(\psi^{-1}(1/s)\right)^2 \wedge \frac{1+\log^+|x|}{x^2}\right); \tag{C.3}$$

$$|\Delta^\gamma\Delta^{-\gamma} p_s(x)| \le C\gamma^2\left(\left(\psi^{-1}(1/s)\right)^3 \wedge \frac{\psi^{-1}(1/s)}{x^2}\right). \tag{C.4}$$

Lemma C.2 *Let X be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at 0 with index $1 < \beta \le 2$ and satisfies (1.6)–(1.8). Then for all t sufficiently large and all $x \in R^1$*

$$u(x,t) := \int_0^t p_s(x)\, ds \le C\left(t\psi^{-1}(1/t) \wedge \frac{t(1+\log^+|x|)}{|x|}\right), \tag{C.5}$$

$$v_\gamma(x,t) := \int_0^t |\Delta^\gamma\, p_s(x)|\, ds \le C\gamma^2\left(\log t \wedge \frac{t\psi^{-1}(1/t)}{|x|} \wedge t\frac{1+\log^+|x|}{x^2}\right) \tag{C.6}$$

and

$$w_\gamma(x,t) := \int_0^t |\Delta^\gamma\Delta^{-\gamma}\, p_s(x)|\, ds \le C\gamma^2\left(1 \wedge \frac{\log t}{|x|} \wedge \frac{t\psi^{-1}(1/t)}{|x|^2}\right). \tag{C.7}$$

We use the following lemma repeatedly.

Lemma C.3 *For all* $p \in R^1$ *and* $s, q > 0$*,*

$$e^{-s\psi(p)} \leq \frac{C}{s^q \psi^q(p)}. \tag{C.8}$$

Proof This is elementary since for all $q > 0$

$$e^{-s\psi(p)} \leq \frac{\sup_{s \geq 0} s^q \psi^q(p) e^{-s\psi(p)}}{s^q \psi^q(p)}. \tag{C.9}$$

□

Proof of Lemma C.1 We first note that by (C.8) with $q = 1$, and (1.6)

$$\begin{aligned}
&\int_0^\infty e^{-s\psi(p)}\, dp \qquad\qquad (C.10)\\
&\leq \left(\int_0^{\psi^{-1}(1/s)} e^{-s\psi(p)}\, dp + \int_{\psi^{-1}(1/s)}^1 e^{-s\psi(p)}\, dp + \int_1^\infty e^{-s\psi(p)}\, dp \right) \\
&\leq \left(\psi^{-1}(1/s) + \frac{1}{s} \int_{\psi^{-1}(1/s)}^1 \frac{1}{\psi(p)}\, dp + \frac{1}{s} \int_1^\infty \frac{1}{\psi(p)}\, dp \right) \\
&\leq C \left(\psi^{-1}(1/s) + \frac{1}{s} \right) \leq C \psi^{-1}(1/s),
\end{aligned}$$

for all s sufficiently large. Therefore, it follows from (C.1), that for all s sufficiently large

$$p_s(x) \leq C \left(\psi^{-1}(1/s) \right). \tag{C.11}$$

By integration by parts

$$\begin{aligned}
p_s(x) &= \frac{1}{\pi x} \int_0^\infty e^{-s\psi(p)}\, d(\sin px) \qquad (C.12)\\
&= -\frac{1}{\pi x} \int_0^\infty \sin px \left(\frac{d}{dp} e^{-s\psi(p)} \right) dp \\
&= -\frac{1}{\pi x^2} \int_0^\infty \cos px \left(\frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp.
\end{aligned}$$

where the last line uses the fact that $\psi'(0) = 0$, which follows from (1.5) and the first inequality in (1.7).

We have

$$\frac{d^2}{dp^2} e^{-s\psi(p)} = \left(s^2 (\psi'(p))^2 - s\psi''(p) \right) e^{-s\psi(p)} \tag{C.13}$$

By (C.8) and (1.7) for $p \leq 1$

$$s^2 (\psi'(p))^2 e^{-s\psi(p)} \leq C \frac{s(\psi'(p))^2}{\psi(p)} \leq C \frac{s\, \psi(p)}{p^2} \tag{C.14}$$

and

$$s|\psi''(p)| e^{-s\psi(p)} \leq C \frac{s\, \psi(p)}{p^2}. \tag{C.15}$$

Therefore, for all s sufficiently large

$$\begin{aligned} \left| \int_0^{\psi^{-1}(1/s)} \cos px \left(\frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| &\leq Cs \int_0^{\psi^{-1}(1/s)} \frac{\psi(p)}{p^2} \, dp \\ &\leq \frac{C}{\psi^{-1}(1/s)}. \end{aligned} \tag{C.16}$$

By (C.8), (C.13) and (1.7), for $p \leq 1$

$$\left| \frac{d^2}{dp^2} e^{-s\psi(p)} \right| \leq C \left\{ \left(\frac{\psi'(p)}{\psi(p)} \right)^2 + \frac{|\psi''(p)|}{\psi(p)} \right\} \leq \frac{C}{p^2}. \tag{C.17}$$

Therefore, for all s sufficiently large

$$\begin{aligned} \left| \int_{\psi^{-1}(1/s)}^1 \cos px \left(\frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| &\leq C \left| \int_{\psi^{-1}(1/s)}^1 \frac{1}{p^2} \, dp \right. \\ &\leq \frac{C}{\psi^{-1}(1/s)}. \end{aligned} \tag{C.18}$$

By (C.17) and (1.8)

$$\left| \int_1^\infty \cos px \left(\frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| \leq C \int_1^\infty \left\{ \left(\frac{\psi'(p)}{\psi(p)} \right)^2 + \frac{|\psi''(p)|}{\psi(p)} \right\} dp \leq C. \tag{C.19}$$

Using (C.11), (C.12), (C.16), (C.18) and (C.19) we get (C.2).

We now obtain (C.4).

$$\begin{aligned} \Delta^\gamma \Delta^{-\gamma} p_s(x) &= 2p_s(x) - p_s(x+\gamma) - p_s(x-\gamma) \\ &= \frac{4}{\pi} \int_0^\infty \cos(px) \sin^2(p\gamma/2) \, e^{-s\psi(p)} \, dp. \end{aligned} \tag{C.20}$$

Therefore, by (C.8)

$$\begin{aligned} &|\Delta^\gamma \Delta^{-\gamma} p_s(x)| \\ &\leq C \int_0^\infty \sin^2(p\gamma/2) \, e^{-s\psi(p)} \, dp \\ &\leq C\gamma^2 \left(\int_0^{\psi^{-1}(1/s)} p^2 \, dp + \frac{1}{s^3} \int_{\psi^{-1}(1/s)}^1 \frac{p^2}{\psi^3(p)} \, dp + \frac{1}{s^3} \int_1^\infty \frac{p^2}{\psi^3(p)} \, dp \right) \\ &\leq C\gamma^2 \left(\psi^{-1}(1/s) \right)^3. \end{aligned} \tag{C.21}$$

We next show that

$$\Delta^\gamma \Delta^{-\gamma} p_s(x) = \frac{8}{\pi} \frac{K_\gamma(s,x)}{x^2} \tag{C.22}$$

where

$$K_\gamma(s,x) := \int_0^\infty \sin^2(px/2) \left(\sin^2(p\gamma/2) \, e^{-s\psi(p)} \right)'' \, dp. \tag{C.23}$$

To get this we integrate by parts in (C.20),

$$\begin{aligned} &\int_0^\infty \cos px \sin^2(p\gamma/2) \, e^{-s\psi(p)} \, dp \\ &\quad = \frac{1}{x} \int_0^\infty \sin^2(p\gamma/2) \, e^{-s\psi(p)} \, d(\sin px) \end{aligned} \tag{C.24}$$

$$
\begin{aligned}
&= -\frac{1}{x}\int_0^\infty \sin px \left(\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)' \, dp \\
&= -\frac{1}{x}\int_0^\infty \left(\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)' \, d\left(\int_0^p \sin rx\, dr\right) \\
&= -\frac{1}{x^2}\int_0^\infty \left(\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)' \, d\,(1-\cos px) \\
&= \frac{2}{x^2}\int_0^\infty \sin^2(px/2) \left(\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)'' \, dp.
\end{aligned}
$$

which gives (C.22).

Let $g(p) = e^{-s\psi(p)}$ and note that

$$
\left(2\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)' = \gamma g(p)\sin(p\gamma) + 2g'(p)\sin^2(p\gamma/2) \tag{C.25}
$$

and

$$
\left(2\sin^2(p/2)\, e^{-s\psi(p)}\right)'' = \gamma^2 g(p)\cos(p\gamma) + 2\gamma g'(p)\sin(p\gamma) + 2g''(p)\sin^2(p\gamma/2). \tag{C.26}
$$

Substituting (C.26) in (C.23) we write $K_\gamma(s,x) = I + II + III$.

Note that

$$
\begin{aligned}
|I| &= \gamma^2\left|\int_0^\infty \cos(p\gamma)\sin^2(px/2)e^{-s\psi(p)}\, dp\right| \\
&\le \gamma^2\int_0^\infty e^{-s\psi(p)}\, dp \le C\gamma^2\psi^{-1}(1/s)
\end{aligned} \tag{C.27}
$$

by (C.10).

By (C.26)

$$
\begin{aligned}
|II| &= 2\gamma\left|\int_0^\infty \sin(p\gamma)\sin^2(px/2)g'(p)\, dp\right| \\
&\le C\gamma^2\int_0^{\psi^{-1}(1/s)} ps|\psi'(p)|\, e^{-s\psi(p)}\, dp \\
&\quad + C\gamma^2\int_{\psi^{-1}(1/s)}^1 ps|\psi'(p)|\, e^{-s\psi(p)}\, dp \\
&\quad + C\gamma\int_1^\infty s|\psi'(p)|\, e^{-s\psi(p)}\, dp
\end{aligned} \tag{C.28}
$$

By (1.7) and (C.8) the first of these last three integrals

$$
\le C\int_0^{\psi^{-1}(1/s)} s\psi(p)\, e^{-s\psi(p)}\, dp \le C\int_0^{\psi^{-1}(1/s)} dp \le C\psi^{-1}(1/s). \tag{C.29}
$$

By (1.7) and (C.8) the second of the last three integrals in (C.28)

$$
\le \frac{C}{s}\int_{\psi^{-1}(1/s)}^1 \frac{s^2\psi^2(p)}{\psi(p)}\, e^{-s\psi(p)}\, dp \le \frac{C}{s}\int_{\psi^{-1}(1/s)}^1 \frac{dp}{\psi(p)} \le C\psi^{-1}(1/s). \tag{C.30}
$$

By (1.8) and (C.8) the third of the last three integrals in (C.28)

$$
\frac{1}{s}\int_1^\infty s^2\psi^2(p)\frac{|\psi'(p)|}{\psi^2(p)}\, e^{-s\psi(p)}\, dp \le \frac{1}{s}\int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)}\, dp \le \frac{C}{s}. \tag{C.31}
$$

Since $1/s < \psi^{-1}(1/s)$ for all s sufficiently large, and $\gamma \ge 1$, we see that

$$
|II| \le C\gamma^2\psi^{-1}(1/s) \qquad \forall\, x \in R^1. \tag{C.32}
$$

Similarly,

$$\begin{aligned}
|III| &= 2\Big|\int_0^\infty \sin^2(p\gamma)\sin^2(px/2)g''(p)\,dp\Big| \qquad \text{(C.33)}\\
&\le C\gamma^2\int_0^{\psi^{-1}(1/s)} p^2\left(s|\psi''(p)|+s^2|\psi'(p)|^2\right)e^{-s\psi(p)}\,dp\\
&\quad +C\gamma^2\int_{\psi^{-1}(1/s)}^1 p^2\left(s|\psi''(p)|+s^2|\psi'(p)|^2\right)e^{-s\psi(p)}\,dp\\
&\quad +C\int_1^\infty \left(s|\psi''(p)|+s^2|\psi'(p)|^2\right)e^{-s\psi(p)}\,dp\\
&\le C\gamma^2\int_0^{\psi^{-1}(1/s)} \left(s\psi(p)+s^2\psi^2(p)\right)e^{-s\psi(p)}\,dp\\
&\quad +\frac{C\gamma^2}{s}\int_{\psi^{-1}(1/s)}^1 \frac{1}{\psi(p)}\left(s^2\psi^2(p)+s^3\psi^3(p)\right)e^{-s\psi(p)}\,dp\\
&\quad +\frac{C}{s}\int_1^\infty \left(s^2\psi^2(p)\frac{|\psi''(p)|}{\psi^2(p)}+s^3\psi^3(p)\frac{|\psi'(p)|^2}{\psi^3(p)}\right)e^{-s\psi(p)}\,dp\\
&\le C\gamma^2\psi^{-1}(1/s).
\end{aligned}$$

Note that $\lim_{\lambda\to\infty}\psi(\lambda)=\infty$, see e.g. [11, Lemma 4.2.2], so that (1.8) implies that

$$\int_1^\infty \frac{|\psi'(\lambda)|^2}{\psi^3(\lambda)}\,d\lambda<\infty,\qquad \int_1^\infty \frac{|\psi''(\lambda)|}{\psi^2(\lambda)}\,d\lambda<\infty. \qquad \text{(C.34)}$$

We use this to bound the next to last line in (C.33).

Combining (C.21), (C.22), (C.27), (C.32), and (C.33) we get (C.4).

We now obtain (C.3). Note that

$$\begin{aligned}
\Delta^\gamma p_s(x) &= p_s(x+\gamma)-p_s(x) \qquad \text{(C.35)}\\
&= \frac{1}{\pi}\int_0^\infty (\cos p(x+\gamma)-\cos px)\; e^{-s\psi(p)}\,dp\\
&= -\frac{2}{\pi}\int_0^\infty \cos(px)\sin^2(p\gamma/2)e^{-s\psi(p)}\\
&\qquad -\frac{1}{\pi}\int_0^\infty \sin(px)\sin(p\gamma)\,e^{-s\psi(p)}\,dp
\end{aligned}$$

Thus

$$\Delta^\gamma p_s(x)=-\frac{1}{2}\Delta^\gamma\Delta^{-\gamma}p_s(x)-\frac{1}{\pi}\int_0^\infty \sin(px)\sin(p\gamma)\,e^{-s\psi(p)}\,dp. \qquad \text{(C.36)}$$

The second order difference is bounded in (C.4). We deal with the second integral which is bounded by

$$\begin{aligned}
&\int_0^\infty |\sin(p\gamma)|\,e^{-s\psi(p)}\,dp \qquad \text{(C.37)}\\
&\le \gamma\left(\int_0^{\psi^{-1}(1/s)} p\,dp+\int_{\psi^{-1}(1/s)}^1 pe^{-s\psi(p)}\,dp+\int_1^\infty e^{-s\psi(p)}\,dp\right)\\
&\le C\gamma\left(\left(\psi^{-1}(1/s)\right)^2+\frac{1}{s^2}\int_{\psi^{-1}(1/s)}^1 \frac{p}{\psi^2(p)}\,dp+\frac{1}{s^2}\int_1^\infty \frac{1}{\psi^2(p)}\,dp\right)\\
&\le C\gamma\left(\psi^{-1}(1/s)\right)^2.
\end{aligned}$$

This gives us the first bound in (C.3). To obtain the second bound we integrate by parts twice to get

$$\begin{aligned}
&\int_0^\infty \sin(px)\sin(p\gamma)\, e^{-s\psi(p)}\, dp \qquad (C.38)\\
&= -\frac{1}{x}\int_0^\infty \sin(p\gamma)\, e^{-s\psi(p)}\, d(\cos px)\\
&= \frac{1}{x}\int_0^\infty \cos(px)\left(\sin(p\gamma)\, e^{-s\psi(p)}\right)'\, dp\\
&= \frac{1}{x^2}\int_0^\infty \left(\sin(p\gamma)\, e^{-s\psi(p)}\right)'\, d\left(\sin px\right)\\
&= -\frac{1}{x^2}\int_0^\infty \sin(px)\left(\sin(p\gamma)\, e^{-s\psi(p)}\right)''\, dp.\\
&:= \frac{G}{x^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(\sin(p\gamma)\, e^{-s\psi(p)}\right)'' = &\left(-\gamma^2\sin(p\gamma) - 2s\gamma\cos(p\gamma)\,\psi'(p)\right.\\
&\left. - \sin(p\gamma)(s\,\psi''(p) - s^2(\psi'(p))^2\right) e^{-s\psi(p)},
\end{aligned} \tag{C.39}$$

we can write

$$G = G_1 + G_2 + G_3, \tag{C.40}$$

where

$$\begin{aligned}
|G_1| &= \gamma^2\Big|\int_0^\infty \sin(px)\sin(p\gamma)\, e^{-s\psi(p)}\, dp\Big|\\
&\le C\gamma^2\left(\psi^{-1}(1/s)\right),
\end{aligned} \tag{C.41}$$

for all s sufficiently large, by (C.10).

Using (C.8), (1.7) and (1.8), we see that

$$\begin{aligned}
|G_2| &= 2\gamma\Big|\int_0^\infty \sin px\cos(p\gamma)\left(\psi'(p)\, se^{-s\psi(p)}\right)\, dp\\
&\le C\gamma\int_0^1 |\sin px|\,|\psi'(p)|se^{-s\psi(p)}\, dp + C\gamma\int_1^\infty |\psi'(p)|se^{-s\psi(p)}\, dp\\
&\le C\gamma\int_0^1 \frac{|\sin px|}{p}\, dp + \frac{C\gamma}{s}\int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)}\, dp\\
&\le C\gamma\left(1+\log^+ x + (1/s)\right),
\end{aligned} \tag{C.42}$$

where we use

$$\int_0^1 \frac{|\sin px|}{p}\, dp = \int_0^{|x|} \frac{|\sin p|}{p}\, dp \le C\gamma\left(1+\log^+|x|\right). \tag{C.43}$$

Therefore, for s sufficiently large

$$|G_2| \le C\gamma\left(1+\log^+|x|\right). \tag{C.44}$$

Similarly,

$$|G_3| = \Big|\int_0^\infty \sin px\ \sin p\gamma\left(s\psi''(p) - s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, dp\Big|$$

$$
\begin{aligned}
&\leq C\gamma \int_0^1 |\sin px|\, p \left(s|\psi''(p)| + s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, dp \\
&\qquad + C\int_1^\infty \left(s|\psi''(p)| + s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, dp \\
&\leq C\gamma \int_0^1 |\sin px| \left(\frac{s\psi(p)|}{p} + \frac{s^2\psi^2(p)}{p}\right) e^{-s\psi(p)}\, dp \\
&\qquad + C\int_1^\infty \left(\frac{|\psi''(p)|}{\psi(p)} + \frac{(\psi'(p))^2}{\psi^2(p)}\right) dp \\
&\leq C\gamma \left(\int_0^1 \frac{|\sin px|}{p}\, dp + C\right) \leq C\gamma \left(1 + \log^+ |x|\right).
\end{aligned} \tag{C.45}
$$

Combining (C.41)–(C.45) we get the second bound in (C.3). □

We use the next two lemmas in the proof of Lemma C.2.

Lemma C.4 *Let X be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at zero with index $1 < \beta \leq 2$ and satisfies (1.6). Then for any $r \geq 0$ and $t > 0$,*

$$
\int_0^t s^r e^{-s\psi(p)}\, ds \leq C_k \left(t \wedge \frac{1}{\psi(p)}\right)^{r+1}; \tag{C.46}
$$

for all $t \geq 0$, where $C_k < \infty$, is a constant depending on k. Furthermore, for any $r \geq 0$ and all t sufficiently large,

$$
\int_0^\infty \psi^r(p) \int_0^t s^r e^{-s\psi(p)}\, ds\, dp \leq Ct\psi^{-1}(1/t). \tag{C.47}
$$

Proof The first part of the bound in the first inequality in (C.46) comes from taking $e^{-s\psi(p)} \leq 1$; the second from letting $t = \infty$.

Since

$$
\psi^r(p) s^r e^{-s\psi(p)} = 2^r \psi^r(p) \left(\frac{s}{2}\right)^r e^{-s\psi(p)/2} e^{-s\psi(p)/2}, \tag{C.48}
$$

it follows from (C.8) and (C.10) that

$$
\int_0^\infty \psi^r(p) s^r e^{-s\psi(p)}\, dp \leq C \int_0^\infty e^{-s\psi(p)/2}\, dp \leq C\psi^{-1}(1/s) \tag{C.49}
$$

for all s sufficiently large. On the other hand for any fixed t_0,

$$
\int_0^\infty \int_0^{t_0} e^{-s\psi(p)/2}\, ds\, dp = 2\int_0^\infty \frac{1 - e^{-t_0\psi(p)/2}}{\psi(p)}\, dp < \infty, \tag{C.50}
$$

by (1.6). Putting these two together, and using the fact that $\psi^{-1}(1/t)$ is regularly varying at infinity, gives (C.47). □

Lemma C.5 *Under the hypotheses of Theorem 1.1, for $r = 0, 1, \ldots$*

$$
\begin{aligned}
\int_0^1 \frac{|\sin px|}{p} \psi^r(p) \left(t \wedge \frac{1}{\psi(p)}\right)^{r+1} dp &\leq Ct(1 + \log^+ |x|); && \text{(C.51)} \\
\int_0^1 \psi^r(p) \left(t \wedge \frac{1}{\psi(p)}\right)^{r+1} dp &\leq Ct\psi^{-1}(1/t); && \text{(C.52)} \\
\int_0^t \int_0^\infty |\sin p\gamma|\, e^{-s\psi(p)}\, dp\, ds &\leq C\gamma \log t, && \text{(C.53)}
\end{aligned}
$$

for all t sufficiently large.

Proof We first note that for $r = 0, 1, \ldots$

$$\psi^{r+1}(p)\left(t \wedge \frac{1}{\psi(p)}\right)^{r+2} \leq \psi^{r}(p)\left(t \wedge \frac{1}{\psi(p)}\right)^{r+1} \tag{C.54}$$

So we need only prove (C.51) and (C.52) for $r = 0$. In this case, (C.51) follows immediately from (C.43).

For (C.52) we have

$$\begin{aligned}
&\int_0^1 \left(t \wedge \frac{1}{\psi(p)}\right) dp \\
&\qquad \leq t \int_0^{\psi^{-1}(1/t)} dp + \int_{\psi^{-1}(1/t)}^1 \frac{1}{\psi(p)}\, dp \leq C t \psi^{-1}(1/t),
\end{aligned}$$

for all t sufficiently large.

By (C.50) and (C.37), there exists a t_0 such that for all $t \geq t_0$,

$$\begin{aligned}
&\int_0^t \int_0^\infty |\sin p\gamma|\, e^{-s\psi(p)}\, dp\, ds \qquad\qquad (C.55)\\
&\quad \leq \int_0^{t_0} \int_0^\infty e^{-s\psi(p)}\, dp\, ds + \int_{t_0}^t \int_0^\infty |\sin p\gamma|\, e^{-s\psi(p)}\, dp\, ds \\
&\quad \leq C + C\gamma \int_{t_0}^t \left(\psi^{-1}(1/s)\right)^2\, ds \leq C + C\gamma \log t,
\end{aligned}$$

where for the last bound we use (B.108). (This bound can not be smaller since we may have $\psi(p) = p^2$.) □

Proof of Lemma C.2 For the first bound in (C.5) we use (C.1) and (C.47) with $r = 0$ to get

$$\int_0^t p_s(x)\, ds \leq \frac{1}{\pi} \int_0^\infty \int_0^t e^{-s\psi(p)}\, ds\, dp = O(t\psi^{-1}(1/t)), \tag{C.56}$$

as $t \to \infty$. For the second bound in (C.5) we use (1.7), (C.46) and (C.51), to see that

$$\begin{aligned}
\int_0^t p_s(x)\, ds &= \frac{1}{\pi} \int_0^\infty \cos px \int_0^t e^{-s\psi(p)}\, ds\, dp \qquad\qquad (C.57)\\
&= \left| \frac{1}{\pi x} \int_0^\infty \int_0^t e^{-s\psi(p)}\, ds\, d(\sin px) \right| \\
&\leq \frac{1}{\pi |x|} \int_0^\infty |\sin px| \left| \frac{d}{dp} \int_0^t e^{-s\psi(p)}\, ds \right| dp \\
&\leq \frac{C}{|x|} \int_0^\infty |\sin px| |\psi'(p)| \left(t \wedge \frac{1}{\psi(p)}\right)^2 dp \\
&\leq \frac{C}{|x|} \left(\int_0^1 \frac{|\sin px|}{p} \psi(p) \left(t \wedge \frac{1}{\psi(p)}\right)^2 dp \right. \\
&\qquad\qquad \left. + \int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)}\, dp \right) \\
&\leq C \frac{t(1 + \log^+ |x|)}{|x|}.
\end{aligned}$$

Thus we get (C.5).

We next obtain (C.7). Consider (C.20). For $\gamma > 1$

$$\begin{aligned}
&\int_0^\infty \frac{\sin^2(p\gamma/2)}{\psi(p)}\,dp \\
&\quad\le C\gamma^2 \int_0^{1/\gamma} \frac{p^2}{\psi(p)}\,dp + \int_{1/\gamma}^1 \frac{1}{\psi(p)}\,dp + \int_1^\infty \frac{1}{\psi(p)}\,dp \\
&\quad\le C\left(\frac{1}{\gamma\psi(1/\gamma)} + 1\right),
\end{aligned} \tag{C.58}$$

and for $\gamma = 1$ the integral is a constant. It follows from this and (B.108) that

$$\sup_{x\in R^1} \int_0^\infty |\Delta^\gamma \Delta^{-\gamma} p_s(x)|\,ds \le C\gamma^2. \tag{C.59}$$

This gives the first bound in (C.7).

To obtain the third bound in (C.7), consider (C.22)–(C.26). By (C.47) with $r = 0$, we have

$$\begin{aligned}
\int_0^t |I|\,ds &= \gamma^2 \int_0^t \Big| \int_0^\infty \cos p\gamma \sin^2(px/2) e^{-s\psi(p)}\,dp\Big|\,ds \\
&\le \gamma^2 \int_0^\infty \left(\int_0^t e^{-s\psi(p)}\,ds\right)\,dp \le C\gamma^2 t\psi^{-1}(1/t),
\end{aligned} \tag{C.60}$$

for all t sufficiently large. Using (1.7). (1.8) and (C.47) with $r = 1$ we get

$$\begin{aligned}
\int_0^t |II|\,ds &= 2\gamma \int_0^t \Big|\int_0^\infty \sin p\gamma \sin^2(px/2) g'(p)\,dp\Big|\,ds \\
&\le 2\gamma \int_0^1 |\sin(p\gamma)\,\psi'(p)| \left(\int_0^t se^{-s\psi(p)}\,ds\right)\,dp \\
&\qquad + 2\gamma \int_1^\infty |\sin(p\gamma)\,\psi'(p)| \left(\int_0^t se^{-s\psi(p)}\,ds\right)\,dp \\
&\le C\gamma^2 \int_0^1 \psi(p) \left(\int_0^t se^{-s\psi(p)}\,ds\right)\,dp \\
&\qquad + C\gamma \int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)}\,dp \\
&\le C\gamma^2 \left(t\psi^{-1}(1/t)\right) + C\gamma \int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)}\,dp \\
&\le C\gamma^2 \left(t\psi^{-1}(1/t)\right) + C\gamma.
\end{aligned} \tag{C.61}$$

Similarly,

$$\begin{aligned}
\int_0^t |III|\,ds &= 2\int_0^t \Big|\int_0^\infty \sin^2(p\gamma/2)\sin^2(px/2) g''(p)\,dp\Big|\,ds \\
&\le C\gamma^2 \int_0^1 p^2 \left(\int_0^t \left(s|\psi''(p)| + s^2|\psi'(p)|^2\right) e^{-s\psi(p)}\,ds\right)\,dp \\
&\qquad + C\int_1^\infty \left(\int_0^\infty \left(s|\psi''(p)| + s^2|\psi'(p)|^2\right) e^{-s\psi(p)}\,ds\right)\,dp \\
&\le C\gamma^2 \int_0^1 \left(\int_0^t \left(s\psi(p) + s^2\psi^2(p)\right) e^{-s\psi(p)}\,ds\right)\,dp \\
&\qquad + C\int_1^\infty \left(\frac{\psi''(p)}{\psi^2(p)} + \frac{(\psi'(p))^2}{\psi^3(p)}\right)\,dp \\
&\le C\gamma^2 t\psi^{-1}(1/t) + C.
\end{aligned} \tag{C.62}$$

Combining (C.60)-(C.62) with (C.22) we get the third bound in (C.7).

To get the second bound in (C.7) we use the third integral in (C.24) to see that

$$\Delta^{\gamma}\Delta^{-\gamma}p_s(x) = -\frac{4}{\pi}\frac{L(s,x)}{x} \tag{C.63}$$

where

$$L(s,x) = \int_0^\infty \sin px \left(\sin^2(p\gamma/2)\, e^{-s\psi(p)}\right)'\, dp. \tag{C.64}$$

Using (C.25) and (1.7) we see that

$$\begin{aligned} &\int_0^t |L|\, ds \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (C.65)\\ &\quad \le C\gamma \int_0^t \int_0^\infty |\sin p\gamma| g(p)\, dp\, ds + C\int_0^t \int_0^\infty \sin^2(p\gamma/2)|g'(p)|\, dp\, ds. \end{aligned}$$

By (C.53) the first term on the right-hand side is bounded by $C\gamma^2\log t$. For the second term we note that

$$\begin{aligned} &\int_0^t \int_0^\infty \sin^2(p\gamma/2)|g'(p)|\, dp\, ds \qquad\qquad (C.66)\\ &\quad \le C\gamma^2 \int_0^1 p^2|\psi'(p)| \int_0^t se^{-s\psi(p)}\, ds\, dp \\ &\qquad\qquad + C\int_1^\infty |\psi'(p)| \int_0^t se^{-s\psi(p)}\, ds\, dp \\ &\quad = IV + V. \end{aligned}$$

By (C.46)

$$\begin{aligned} IV &\le C\gamma^2 \int_0^1 p\psi(p)\left(t^2 \wedge \frac{1}{\psi^2(p)}\right) dp \qquad\qquad (C.67)\\ &\le Ct^2\gamma^2 \int_0^{\psi^{-1}(1/t)} p\psi(p)\, dp + C\gamma^2 \int_{\psi^{-1}(1/t)}^1 \frac{p}{\psi(p)}\, dp \\ &\le C\gamma^2 t(\psi^{-1}(1/t))^2 + C\gamma^2 \int_{\psi^{-1}(1/t)}^1 \frac{1}{p}\, dp \\ &\le C\gamma^2 \log t, \end{aligned}$$

where we use (B.108) which implies that $p/\psi(p) \le C/p$ for $p \in [0,1]$. The integral $V \le C$ by (C.46) and (1.8). Using all the material from (C.63) to this point we get the second bound in (C.7). This completes the proof of (C.7).

Using (C.36), (C.59), (C.7) and (C.53) we get the first bound in (C.6).

We now obtain the third bound in (C.6). Considering (C.36) and (C.7), it suffices to show that

$$\int_0^t \left| \int_0^\infty \sin(px)\sin(p\gamma)\, e^{-s\psi(p)}\, dp\right| ds \le Ct\gamma^2 \frac{1+\log^+|x|}{x^2}. \tag{C.68}$$

Consider (C.38)–(C.40). We have

$$\begin{aligned} \int_0^t |G_1|\, ds &= \gamma^2 \int_0^t \left| \int_0^\infty \sin px \sin p\gamma\, e^{-s\psi(p)}\, dp\right| ds \qquad (C.69)\\ &\le \gamma^2 \int_0^t \int_0^\infty e^{-s\psi(p)}\, dp \le C\gamma^2 t\psi^{-1}(1/t), \end{aligned}$$

by (C.47).

Using (C.51) and (1.8), we see that

$$\begin{aligned}\int_0^t |G_2|\,ds &= 2\gamma\int_0^t\Big|\int_0^\infty \sin px\cos p\gamma\left(\psi'(p)\,se^{-s\psi(p)}\right)\,dp\Big|\,ds \\ &\leq 2\gamma\int_0^1 |\sin px||\psi'(p)|\left(\int_0^t se^{-s\psi(p)}\,ds\right)\,dp+2\gamma\int_1^\infty\frac{|\psi'(p)|}{\psi^2(p)}\,dp \\ &\leq C\gamma\int_0^1\frac{|\sin px|}{p}\psi(p)\left(t\wedge\frac{1}{\psi(p)}\right)^2\,dp+C\gamma \\ &\leq C\gamma t\left(1+\log^+|x|\right).\end{aligned} \tag{C.70}$$

Similarly,

$$\begin{aligned}&\int_0^t |G_3|\,ds \\ &=\int_0^t\Big|\int_0^\infty \sin px\;\;\sin p\gamma\left(s\psi''(p)-s^2(\psi'(p))^2\right)e^{-s\psi(p)}\,dp\Big|\,ds \\ &\leq C\gamma\int_0^1|\sin px|\,p\left(\int_0^t\left(s|\psi''(p)|+s^2(\psi'(p))^2\right)e^{-s\psi(p)}\,ds\right)\,dp \\ &\quad +C\int_1^\infty\left(\int_0^\infty\left(s|\psi''(p)|+s^2(\psi'(p))^2\right)e^{-s\psi(p)}\,ds\right)\,dp \\ &\leq C\gamma\int_0^1\frac{|\sin px|}{p}\psi(p)\left(t\wedge\frac{1}{\psi(p)}\right)^2\,dp \\ &\quad\quad +C\gamma\int_0^1\frac{|\sin px|}{p}\psi^2(p)\left(t\wedge\frac{1}{\psi(p)}\right)^3\,dp \\ &\quad\quad +C\int_1^\infty\frac{|\psi''(p)|}{\psi^2(p)}\,dp+C\int_1^\infty\frac{(\psi'(p))^2}{\psi^3(p)}\,dp \\ &\leq C\gamma\left(1+\log^+|x|\right)t.\end{aligned} \tag{C.71}$$

This completes the proof of (C.68) and gives us the third bound in (C.6)

The second bound in (C.6) follows from the third line of (C.38) and the observation that

$$\begin{aligned}&\left|\int_0^t\int_0^\infty\cos px\left(\sin(p\gamma)\,e^{-s\psi(p)}\right)'\,dp\right| \\ &\quad\leq\gamma\int_0^\infty\int_0^t e^{-s\psi(p)}\,ds\,dp+\int_0^\infty\int_0^t|\sin p\gamma||\psi'(p)|se^{-s\psi(p)}\,ds\,dp \\ &\quad\leq C\gamma(t\psi^{-1}(1/t))+\gamma\int_0^1\psi(p)\left(t\wedge\frac{1}{\psi(p)}\right)^2\,dp \\ &\quad\quad\quad +\int_1^\infty\frac{|\psi'(p)|}{\psi^2(p)}\,dp \\ &\quad\leq C(1\vee\gamma)(t\psi^{-1}(1/t)).\end{aligned} \tag{C.72}$$

In this chain of inequalities we use (C.47), (1.7), (1.8) and (C.52). □

Proof of Lemma A.1 The inequalities in (A.5)–(A.7) follow immediately from Lemma C.2.

The inequality in (A.8) is trivial, since $p_s(x)$ is a probability density for all $s>0$.

To obtain (A.9) we use (C.6) with $\gamma = 1$ to get

$$\begin{aligned}
&\int_0^\infty v(x,t)\,dx \qquad\qquad (C.73)\\
&\le C\left(\log t\int_0^{t\psi^{-1}(1/t)}\,dx + t\psi^{-1}(1/t)\int_{t\psi^{-1}(1/t)}^{2c/\psi^{-1}(1/t)}\frac{1}{x}\,dx\right.\\
&\qquad\left.+Ct\int_{2c/\psi^{-1}(1/t)}^\infty \frac{\log x}{x^2}\,dx\right)\\
&\le C(t\psi^{-1}(1/t)\log t),
\end{aligned}$$

for all t sufficiently large. Note that when $\beta < 2$ it is clear that $t\psi^{-1}(1/t) < 1/\psi^{-1}(1/t)$ for all t sufficiently large. In general we use (B.108) with c representing the constant.

For (A.11) we use (C.7) to see that

$$\begin{aligned}
&\int_0^\infty \left(\int_0^t |\Delta^1\Delta^{-1}\,p_s(x)|\,ds\right)^2\,dx \qquad\qquad (C.74)\\
&\le C\int_0^{\log t}\left(\int_0^t |\Delta^1\Delta^{-1}\,p_s(x)|\,ds\right)^2\,dx\\
&\qquad +2\int_{\log t}^\infty \left(\int_0^t |\Delta^1\Delta^{-1}\,p_s(x)|\,ds\right)^2\,dx\\
&\le C\log t + C\int_{\log t}^\infty \frac{(\log t)^2}{x^2}\,dx \le C\log t.
\end{aligned}$$

A similar argument gives (A.12) since

$$\int_u^\infty \left(\int_0^t |\Delta^1\Delta^{-1}\,p_s(x)|\,ds\right)^2\,dx \le C\int_u^\infty \frac{(\log t)^2}{x^2}\,dx = C\frac{(\log t)^2}{u}. \qquad (C.75)$$

Finally, to obtain (A.10) we use (C.7) to get

$$\begin{aligned}
&\int_0^\infty\int_0^t \left|\Delta^1\Delta^{-1}\,p_s(x)\right|\,ds\,dx \qquad\qquad (C.76)\\
&= \int_0^1\int_0^t \left|\Delta^1\Delta^{-1}\,p_s(x)\right|\,ds\,dx + \int_1^{t\psi^{-1}(1/t)}\int_0^t \left|\Delta^1\Delta^{-1}\,p_s(x)\right|\,ds\,dx\\
&\qquad + \int_{t\psi^{-1}(1/t)}^\infty\int_0^t \left|\Delta^1\Delta^{-1}\,p_s(x)\right|\,ds\,dx\\
&\le C\int_0^1 1\,dx + C\log t\int_1^{t\psi^{-1}(1/t)}\frac{1}{|x|}\,dx\\
&\qquad +C\int_{t\psi^{-1}(1/t)}^\infty \frac{t\psi^{-1}(1/t)}{|x|^2}\,dx\\
&\le C + C(\log t)^2 + C.
\end{aligned}$$

□

We use the next lemma in the proof of Lemma A.2.

Lemma C.6 *Under the hypotheses of Theorem 1.1, for all t sufficiently large and all $x \in R^1$*

$$\overline{u}(x,t) := \sup_{\delta t\le s\le t} p_s(x) \quad\le\quad C\left(\psi^{-1}(1/t)\wedge\frac{1}{\psi^{-1}(1/t)x^2}\right); \qquad (C.77)$$

$$\overline{v}(x,t) := \sup_{\delta t\le s\le t}|\Delta^1 p_s(x)| \quad\le\quad C\left(\left(\psi^{-1}(1/t)\right)^2\wedge\frac{1+\log^+|x|}{x^2}\right); \tag{C.78}$$

$$\overline{w}(x,t) := \sup_{\delta t\le s\le t}|\Delta^1\Delta^{-1} p_s(x)| \quad\le\quad C\left(\left(\psi^{-1}(1/t)\right)^3\wedge\frac{\psi^{-1}(1/t)}{x^2}\right). \tag{C.79}$$

In addition

$$\int \overline{u}(x,t)\,dx \quad\le\quad C; \tag{C.80}$$

$$\int \overline{v}(x,t)\,dx \quad\le\quad C\psi^{-1}(1/t)\log t; \tag{C.81}$$

$$\int \overline{w}(x,t)\,dx \quad\le\quad C\left(\psi^{-1}(1/t)\right)^2\le\frac{C}{t}. \tag{C.82}$$

Proof By (C.2)

$$\sup_{\delta t\le s\le t} p_s(x)\le C\left(\psi^{-1}(1/\delta t)\wedge\frac{1}{\psi^{-1}(1/t)x^2}\right), \tag{C.83}$$

and by the regular variation property $\psi^{-1}(1/\delta t)\le C\psi^{-1}(1/t)$. (The constant depends on δ but that doesn't matter.) The inequalities in (C.78) and (C.79) follow similarly from (C.3) and (C.4).

The inequalities in (C.80)–(C.82) follow easily from (C.77)–(C.79). For (C.80) we write

$$\int \overline{u}(x,t)\,dx\le C\int_0^a\psi^{-1}(1/t)\,dx+\int_a^\infty\frac{1}{\psi^{-1}(1/t)x^2}\,dx, \tag{C.84}$$

where $a=1/\psi^{-1}(1/t)$. For (C.81) and (C.82) we proceed similarly with $a=1/\psi^{-1}(1/t)$ in both cases. □

Proof of Lemma A.2 The inequality in (A.13) follows from (C.36) and (C.4).

To obtain (A.14) we write

$$\begin{aligned}&\int_0^{2t}\int_0^{2t}|\Delta^1 p_{r+s}(0)|\,dr\,ds \qquad\qquad (C.85)\\ &\qquad=\int_0^{2t}u|\Delta^1 p_u(0)|\,du+\int_{2t}^{4t}(4t-u)|\Delta^1 p_u(0)|\,du.\end{aligned}$$

By (C.36) and (C.20)

$$\begin{aligned}\int_0^{2t}u|\Delta^1 p_u(0)|\,du &\le \frac{2}{\pi}\int_0^{2t}u\int_0^\infty\sin^2(p/2)\,e^{-u\psi(p)}\,dp\,du \qquad (C.86)\\ &= \frac{2}{\pi}\int_0^\infty\sin^2(p/2)\int_0^{2t}u\,e^{-u\psi(p)}\,du\,dp.\end{aligned}$$

In addition

$$\int_0^{2t}u\,e^{-u\psi(p)}\,du\le\left(\frac{1}{\psi^2(p)}\left(1-e^{-2t\psi(p)}\right)\wedge\frac{Ct}{\psi(p)}\right), \tag{C.87}$$

where, for the final inequality we use Lemma C.3. Consequently, for all t sufficiently large,

$$\begin{aligned}&\int_0^\infty\sin^2(p/2)\int_0^{2t}u\,e^{-u\psi(p)}\,du\,dp \qquad\qquad (C.88)\\ &\le Ct\int_0^{\psi^{-1}(1/t)}\frac{p^2}{\psi(p)}\,dp+\frac14\int_{\psi^{-1}(1/t)}^1\frac{p^2}{\psi^2(p)}+\int_1^\infty\frac{1}{\psi^2(p)}\,dp\\ &\le C\left(t^2\left(\psi^{-1}(1/t)\right)^3+1\right)+\frac14\int_{\psi^{-1}(1/t)}^1\frac{p^2}{\psi^2(p)}\,dp.\end{aligned}$$

Note that

$$\int_{\psi^{-1}(1/t)}^{1} \frac{p^2}{\psi^2(p)}\,dp \le \begin{cases} Ct^2\left(\psi^{-1}(1/t)\right)^3 & \text{if } \beta > 3/2 \\ L(t) & \text{if } \beta = 3/2 \\ C & \text{if } \beta < 3/2, \end{cases}$$

where $L(t)$ is a slowly varying function at infinity. Therefore

$$\int_0^{2t} u|\Delta^1 p_u(0)|\,du \le C\left(t^2\left(\psi^{-1}(1/t)\right)^3 + L(t) + 1\right). \tag{C.89}$$

In addition

$$\begin{aligned} \int_{2t}^{4t} (4t-u)|\Delta^1 p_u(0)|\,du &\le Ct\int_{2t}^{4t} |\Delta^1 p_u(0)|\,du \\ &= Ct\int_0^{2t} |\Delta^1 p_{v+2t}(0)|\,dv \end{aligned} \tag{C.90}$$

Note that by (C.36), (C.20) and (C.79)

$$\begin{aligned} |\Delta^1 p_{v+2t}(0)| &= \frac{1}{2}|\Delta^1\Delta^{-1} p_{v+2t}(0)| \\ &\le \frac{1}{2}|\Delta^1\Delta^{-1} p_{2t}(0)| \le C\left(\psi^{-1}(1/t)\right)^3. \end{aligned} \tag{C.91}$$

Here we also use the fact that $\Delta^1\Delta^{-1}p_s(0)$ is decreasing in s, and the regular variation of ψ. Consequently

$$\int_{2t}^{4t} (4t-u)|\Delta^1 p_u(0)|\,du \le Ct^2\left(\psi^{-1}(1/t)\right)^3. \tag{C.92}$$

Thus we obtain (A.14). □

Proof of Lemma A.3 The equality in (A.15) follows easily from (C.1).

The equality in (A.16) follows from (B.75) integrated with respect to r and r'.

For (A.17) we use Parseval's Theorem, (see (B.75)) to get

$$\int\left(\int_0^t \Delta^1\Delta^{-1}\; p_s(x)\,ds\right)^2 dx = \frac{16}{\pi}\int_0^\infty \frac{\sin^4(p/2)}{\psi^2(p)}\left(1-e^{-t\psi(p)}\right)^2 dp. \tag{C.93}$$

To complete the proof of (A.17) we note that by (C.8)

$$\int_0^\infty \frac{\sin^4(p/2)}{\psi^2(p)} e^{-t\psi(p)}\,dp \le \frac{C}{t^{1/3}}\int_0^\infty \frac{\sin^4(p/2)}{\psi^{7/3}(p)}\,dp \le \frac{C}{t^{1/3}}. \tag{C.94}$$

□

Proof of Lemma B.2 The first inequality is given in (C.6). The second inequality follows from the definition of v_* in (B.60). For (B.63) we note that in the proof of (A.9), on page 53, we are actually integrating $v_*(x,t)$. □

D Proof of Lemma 2.4

Set

$$h = \psi^{-1}(1/t), \qquad \text{so that} \qquad \psi(h) = 1/t. \tag{D.1}$$

Proof of Lemma 2.4 By the Kac Moment Formula, (see (3.9)),

$$\begin{aligned}
&E\left(\left(\int (L_t^x)^2\,dx\right)^n\right) \qquad\qquad (D.2)\\
&= 2^n \sum_\pi \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le t\}} \prod_{i=1}^{2n} p_{s_i}(y_{\pi(i)} - y_{\pi(i-1)})\,ds_i\right) \prod_{i=1}^n dy_i\\
&= 2^n \sum_\pi \frac{1}{(2\pi)^{2n}} \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le t\}} \prod_{i=1}^{2n} \int e^{ip_i(y_{\pi(i)}-y_{\pi(i-1)})} e^{-s_i\psi(p_i)}\,dp_i\,ds_i\right)\\
&\qquad \prod_{i=1}^n dy_i\\
&= 2^n \left(\frac{t}{2\pi}\right)^{2n} \sum_\pi \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int e^{ip_i(y_{\pi(i)}-y_{\pi(i-1)})} e^{-s_i t\psi(p_i)}\,dp_i\,ds_i\right)\\
&\qquad \prod_{i=1}^n dy_i.
\end{aligned}$$

Here the sum in the second line runs over all maps π of $\{1,\ldots,2n\}$ into $\{1,\ldots,n\}$, such that $|\pi^{-1}(j)| = 2$ for each $1 \le j \le n$, and we set $\pi(0) = 0$. Thus, by (D.1) and many changes of variables

$$\begin{aligned}
&(2\pi)^{2n}(t^2\psi^{-1}(1/t))^{-n} E\left(\left(\int (L_t^x)^2\,dx\right)^n\right) \qquad\qquad (D.3)\\
&= (2\pi)^{2n} t^{-2n} h^{-n} 2^n E\left(\left(\int (L_t^x)^2\,dx\right)^n\right)\\
&= h^{-n} 2^n \sum_\pi \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int e^{ip_i(y_{\pi(i)}-y_{\pi(i-1)})} e^{-s_i t\psi(p_i)}\,dp_i\,ds_i\right) \prod_{i=1}^n dy_i\\
&= h^{n} 2^n \sum_\pi \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int e^{ip_i h(y_{\pi(i)}-y_{\pi(i-1)})} e^{-s_i t\psi(p_i h)}\,dp_i\,ds_i\right) \prod_{i=1}^n dy_i\\
&= 2^n \sum_\pi \int \left(\int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int e^{ip_i(y_{\pi(i)}-y_{\pi(i-1)})} e^{-s_i \psi(p_i h)/\psi(h)}\,dp_i\,ds_i\right) \prod_{i=1}^n dy_i.
\end{aligned}$$

Using the regular variation of ψ at zero the proof follows once we justify interchanging the limit and the integrals.

For σ fixed let

$$\begin{aligned}
f_h(y) &= \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int e^{ip_i y_i} e^{-s_i\psi(p_i h)/\psi(h)}\,dp_i\,ds_i \qquad (D.4)\\
&= 2^{2n} \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int_0^\infty \cos p_i y_i\, e^{-s_i\psi(p_i h)/\psi(h)}\,dp_i\,ds_i
\end{aligned}$$

Considering (D.3) it suffices to show that for each fixed $y = (y_1,\ldots,y_n)$

$$\lim_{h\to 0} f_h(y) = 2^{2n} \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int_0^\infty \cos p_i y_i\, e^{-s_i p_i^\beta}\,dp_i\,ds_i. \tag{D.5}$$

and $f_h(y)$ is bounded and integrable in y, uniformly in $h \le h_0$, for some $h_0 > 0$, sufficiently small. In fact we show that

$$\sup_{h\le h_0} |f_h(y)| \le C \prod_{i=1}^{2n} \left(1 \wedge \frac{1}{y_i^2}\right). \tag{D.6}$$

We first obtain (D.5). For M large, write

$$1 = \prod_{i=1}^{2n} \left(1_{\{0\le p_i \le M\}} + 1_{\{p_i \ge M\}}\right) \tag{D.7}$$

and

$$f_h(y) = 2^{2n} \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int_0^M \cos p_i y_i \, e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i + G_h. \tag{D.8}$$

Here G_h is a sum of many terms, in each of which $p_i \ge M$, for at least one $1 \le i \le 2n$. Suppose there are k terms with $p_i \ge M$. We bound these terms by

$$2^{2n} \left(\int_0^1 \int_0^M e^{-s\psi(ph)/\psi(h)} \, dp \, ds \right)^{2n-k} \left(\int_0^1 \int_M^\infty e^{-s\psi(ph)/\psi(h)} \, dp \, ds \right)^k. \tag{D.9}$$

By (1.5), for any $\epsilon > 0$, (see also [3, Theorem 1.5.6]),

$$\begin{aligned} &\int_0^1 \int_0^M e^{-s\psi(ph)/\psi(h)} \, dp \, ds \\ &\qquad \le 1 + \int_0^1 \int_1^M e^{-s\psi(ph)/\psi(h)} \, dp \, ds \\ &\qquad \le 1 + \int_0^1 \int_1^M e^{-sCp^{\beta-\epsilon}} \, dp \, ds, \end{aligned} \tag{D.10}$$

which is bounded by a constant independent of M. Using the regular variation of ψ at zero, we have

$$\begin{aligned} &\int_0^1 \int_M^\infty e^{-s\psi(ph)/\psi(h)} \, dp \, ds \\ &\qquad \le \psi(h) \int_M^\infty \frac{1}{\psi(hp)} \, dp = \frac{\psi(h)}{h} \int_{hM}^\infty \frac{1}{\psi(s)} \, ds \\ &\qquad = \frac{\psi(h)}{h} \int_{hM}^1 \frac{1}{\psi(s)} \, ds + \frac{\psi(h)}{h} \int_1^\infty \frac{1}{\psi(s)} \, ds \\ &\qquad \le C \frac{\psi(h) M}{\psi(hM)} + C \frac{\psi(h)}{h}, \end{aligned} \tag{D.11}$$

for all h sufficiently small. Therefore, as in (D.10),

$$\limsup_{h\to 0} \int_0^1 \int_M^\infty e^{-s\psi(ph)/\psi(h)} \, dp \, ds \le \frac{C}{M^{\beta-1}}. \tag{D.12}$$

Thus

$$\limsup_{h\to 0} |G_h| \le \frac{C}{M^{\beta-1}}. \tag{D.13}$$

Now consider the integral in (D.8). By the regular variation of ψ at zero and the Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{h\to 0} \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int_0^M \cos p_i y_i \, e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i \\ &\qquad = \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i=1}^{2n} \int_0^M e^{ip_i y_i} \cos p_i y_i \, e^{-s_i p_i^\beta} \, dp_i \, ds_i. \end{aligned} \tag{D.14}$$

Thus we get (D.5).

We show below that for any $J \subseteq \{1, \ldots, 2n\}$ we have

$$\sup_{h \le h_0} |f_h(y)| \le C \prod_{i \in J} \left(\frac{1}{y_i^2} \right). \tag{D.15}$$

In particular, (D.15) also holds when J is the empty set, so that $\sup_{h \le h_0} |f_h(y)| \le C$. Using this it is easy to see that (D.6) holds.

It follows from integrating by parts twice that

$$\int_0^\infty \cos(py)\; e^{-s\psi(ph)/\psi(h)}\, dp = -\frac{1}{y^2} \int_0^\infty \cos(py) \left(e^{-s\psi(ph)/\psi(h)} \right)''\, dp. \tag{D.16}$$

where we use the fact that $\psi'(0) = 0$, which follows from (1.5) and the first inequality in (1.7). Applying this for all $i \in J$ we see that

$$\begin{aligned} f_h(y) \quad &= \quad \prod_{i \in J} \left(\frac{-1}{y_i^2} \right) \int_{\{\sum_{i=1}^{2n} s_i \le 1\}} \prod_{i \in J} \int_0^\infty \cos p_i y_i \; \left(e^{-s_i \psi(p_i h)/\psi(h)} \right)'' \, dp_i \, ds_i \\ &\qquad \prod_{i \in J^c} \int_0^\infty \cos p_i y_i \, e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i. \end{aligned} \tag{D.17}$$

Therefore

$$\begin{aligned} |f_h(y)| \quad &\le \quad \prod_{i \in J} \left(\frac{1}{y_i^2} \right) \left(\int_0^1 \int_0^\infty \left| \left(e^{-s\psi(ph)/\psi(h)} \right)'' \right| dp\, ds \right)^{|J|} \\ &\qquad \left(\int_0^1 \int_0^\infty e^{-s\psi(ph)/\psi(h)}\, dp\, ds \right)^{|J^c|}. \end{aligned} \tag{D.18}$$

It is easily seen that

$$\int_0^1 \int_0^\infty e^{-s\psi(ph)/\psi(h)}\, dp\, ds \quad \le \quad C \sup_h \int \left(1 \wedge \frac{\psi(h)}{\psi(hp)} \right) dp. \tag{D.19}$$

Therefore, for $h \le 1$,

$$\begin{aligned} &\int \left(1 \wedge \frac{\psi(h)}{\psi(hp)} \right) dp \\ &\quad \le \int_0^1 1\, dp + \int_1^{1/h} \frac{\psi(h)}{\psi(hp)}\, dp + \int_{1/h}^\infty \frac{\psi(h)}{\psi(hp)}\, dp \\ &\quad = 1 + \frac{\psi(h)}{h} \left(\int_h^1 \frac{1}{\psi(p)}\, dp + \int_1^\infty \frac{1}{\psi(p)}\, dp \right) \le C. \end{aligned} \tag{D.20}$$

Consequently, to obtain (D.15) we need only show that, for $h \le 1$,

$$\int_0^1 \int_0^\infty \left| \left(e^{-s\psi(ph)/\psi(h)} \right)'' \right| dp\, ds < \infty. \tag{D.21}$$

We have

$$\begin{aligned} &\left| \left(e^{-s\psi(ph)/\psi(h)} \right)'' \right| \\ &\quad \le \left| \frac{h^2 (\psi'(hp))^2}{\psi^2(h)} \right| s^2 e^{-s\psi(ph)/\psi(h)} + \left| \frac{h^2 \psi''(hp))}{\psi(h)} \right| s e^{-s\psi(ph)/\psi(h)}. \end{aligned} \tag{D.22}$$

Using (1.7) and (1.8) we see that, for $h \le 1$,

$$\begin{aligned}
&\int_0^1 \int_0^\infty \left|\frac{h^2(\psi'(hp))^2}{\psi^2(h)}\right| s^2 e^{-s\psi(ph)/\psi(h)}\, dp\, ds \qquad \text{(D.23)}\\
&\le C \int_0^\infty \left|\frac{h^2(\psi'(hp))^2}{\psi^2(h)}\right| \left(1 \wedge \frac{\psi(h)}{\psi(hp)}\right)^3 dp\\
&\le \frac{Ch^2}{\psi^2(h)} \int_0^1 (\psi'(hp))^2\, dp + Ch^2\psi(h) \int_1^\infty \frac{(\psi'(hp))^2}{\psi^3(hp)}\, dp\\
&\le \frac{Ch}{\psi^2(h)} \int_0^h (\psi'(s))^2\, ds + Ch\psi(h) \int_h^\infty \frac{(\psi'(s))^2}{\psi^3(s)}\, ds\\
&\le C\left(\frac{h}{\psi^2(h)} \int_0^h \frac{\psi^2(s)}{s^2}\, ds + h\psi(h)\int_h^1 \frac{1}{s^2\psi(s)}\, ds + C\right)\\
&\le C'.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \int_0^\infty \left|\frac{h^2\psi''(hp))}{\psi(h)}\right| s e^{-s\psi(ph)/\psi(h)}\, dp\, ds \qquad \text{(D.24)}\\
&\le C \int_0^\infty \left|\frac{h^2\psi''(hp))}{\psi(h)}\right| \left(1 \wedge \frac{\psi(h)}{\psi(hp)}\right)^2 dp\\
&\le \frac{Ch^2}{\psi(h)} \int_0^1 \psi''(hp)\, dp + Ch^2\psi(h) \int_1^\infty \frac{\psi''(hp)}{\psi^2(hp)}\, dp\\
&\le \frac{Ch}{\psi(h)} \int_0^h \psi''(s)\, ds + Ch\psi(h) \int_h^\infty \frac{\psi''(s)}{\psi^2(s)}\, ds\\
&\le C\left(\frac{h}{\psi(h)} \int_0^h \frac{\psi(s)}{s^2}\, ds + h\psi(h)\int_h^1 \frac{1}{s^2\psi(s)}\, ds + C\right)\\
&\le C'.
\end{aligned}$$

Thus we obtain (D.21). □

E Estimates for the mean and variance

Proof of Lemma 3.2 By the Kac Moment Formula

$$\begin{aligned}
&E\left(\int (L_t^{x+1} - L_t^x)^2\, dx\right) \qquad \text{(E.1)}\\
&\quad = 2\int\int_{\{\sum_{i=1}^2 r_i \le t\}} \Delta^1 p_{r_1}(x)\Delta^1 p_{r_2}(0)\, dr_1\, dr_2\, dx\\
&\qquad +2\int\int_{\{\sum_{i=1}^2 r_i \le t\}} p_{r_1}(x)\Delta^1\Delta^{-1} p_{r_2}(0)\, dr_1\, dr_2\, dx.
\end{aligned}$$

When we integrate with respect to x we get zero in the first integral and one in the second. Consequently, by (C.1)

$$\begin{aligned}
E\left(\int (L_t^{x+1} - L_t^x)^2\, dx\right) &= 2\int_{\{\sum_{i=1}^2 r_i \le t\}} \Delta^1\Delta^{-1} p_{r_2}(0)\, dr_1\, dr_2 \qquad \text{(E.2)}\\
&= 4\int_0^t (t-r)\left(p_r(0) - p_r(1)\right)\, dr\\
&= \frac{8}{\pi}\int_0^\infty \sin^2 p/2 \int_0^t (t-r)e^{-r\psi(p)}\, dr\, dp.
\end{aligned}$$

Note that

$$\int_0^t (t-r)e^{-r\psi(p)}\,dr = \frac{t}{\psi(p)} - \frac{1-e^{-t\psi(p)}}{\psi^2(p)}. \tag{E.3}$$

By (1.12)

$$\frac{8t}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{\psi(p)}\,dp = 4c_{\psi,0}t. \tag{E.4}$$

Therefore the absolute value of the error term in (3.16) is

$$\frac{8}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{\psi^2(p)}\left(1-e^{-t\psi(p)}\right)\,dp \le \frac{8}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{\psi^2(p)}\left(1\wedge t\psi(p)\right)\,dp. \tag{E.5}$$

We break this last integral into three parts and see that it is bounded by

$$C\left(t\int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)}\,dp + \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)}\,dp + \int_1^\infty \frac{1}{\psi^2(p)}\,dp\right) \tag{E.6}$$

We have

$$t\int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)}\,dp \le Ct^2\left(\psi^{-1}(1/t)\right)^3. \tag{E.7}$$

In addition

$$\int_1^\infty \frac{1}{\psi^2(p)}\,dp \le C. \tag{E.8}$$

If $\beta > 3/2$

$$\int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)}\,dp \le Ct^2\left(\psi^{-1}(1/t)\right)^3. \tag{E.9}$$

If $\beta = 3/2$

$$\int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)}\,dp \le CL(t) \tag{E.10}$$

for some function L that is slowly varying at infinity. If $\beta < 3/2$

$$\int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)}\,dp \le C. \tag{E.11}$$

Using (E.5)–(E.11) we get (3.17).

Let

$$Z = \int (L_t^{x+1} - L_t^x)^2\,dx. \tag{E.12}$$

We get an upper bound for the variance of Z by finding an upper bound for EZ^2 and using (3.16) to estimate $(EZ)^2$. We proceed as in the beginning of the proof of Lemma B.1, however there are enough differences that it is better to repeat some of the arguments.

By the Kac Moment Theorem

$$\begin{aligned} &E\left(\prod_{i=1}^2 \left(\Delta^1_{x_i} L_t^{x_i}\right)\left(\Delta^1_{y_i} L_t^{y_i}\right)\right) \qquad \text{(E.13)}\\ &\quad = \prod_{i=1}^2 \left(\Delta^1_{x_i}\Delta^1_{y_i}\right)\sum_\sigma \int_{\{\sum_{i=1}^4 r_i \le t\}} \prod_{i=1}^4 p_{r_i}(\sigma(i)-\sigma(i-1))\ \prod_{i=1}^4 dr_i \end{aligned}$$

where the sum runs over all bijections $\sigma : [1,4] \mapsto \{x_i, y_i, 1 \le i \le 2\}$ and we take $\sigma(0) = 0$. We rewrite (E.13) so that each $\Delta^1_\cdot$ applies to a single $p_\cdot$ factor and then set $y_i = x_i$ and then integrate with respect to $x_1, \ldots, x_m$ to get

$$E\left(\left(\int (L_t^{x+1} - L_t^x)^2\,dx\right)^2\right) \tag{E.14}$$
$$= 4\sum_{\pi,a}\int\int_{\{\sum_{i=1}^4 r_i \le t\}} \prod_{i=1}^4 \left(\Delta^1_{x_{\pi(i)}}\right)^{a_1(i)} \left(\Delta^1_{x_{\pi(i-1)}}\right)^{a_2(i)}$$
$$p^\sharp_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^4 dr_i \prod_{i=1}^2 dx_i.$$

In (E.14) the sum runs over all maps $\pi : [1,4] \mapsto [1,2]$ with $|\pi^{-1}(i)| = 2$ for each i and over all $a = (a_1, a_2) : [1, \ldots, 4] \mapsto \{0,1\} \times \{0,1\}$ with the property that for each i there are exactly two factors of the form $\Delta^1_{x_i}$. The factor 4 comes from the fact that we can interchange each y_i and x_i, $i = 1,2$. As usual we take $\pi(0) = 0$.

As we did in Section B, we continue the analysis with $p^\sharp$ replaced by p.

Note that in (E.14) it is possible to have 'bound states', that is values of i for which $\pi(i) = \pi(i-1)$. We first consider the terms in (E.14) with two bound states. There are two possible maps. They are $(\pi(1), \pi(2), \pi(3), \pi(4)) = (1,1,2,2)$ and $(\pi(1), \pi(2), \pi(3), \pi(4)) = (2,2,1,1)$. The terms in (E.14) for the map $(\pi(1), \pi(2), \pi(3), \pi(4)) = (1,1,2,2)$ are of the form

$$\prod_{i=1}^4 \left(\Delta^1_{x_{\pi(i)}}\right)^{a_1(i)} \left(\Delta^1_{x_{\pi(i-1)}}\right)^{a_2(i)} p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}), \tag{E.15}$$

where the density terms have the form

$$p_{r_1}(x_1)p_{r_2}(y_1 - x_1)p_{r_3}(x_2 - y_1)p_{r_4}(y_2 - x_2), \tag{E.16}$$

and where $y_i - x_i = 0$, $1 = 1,2$.

The value of the integrals of the terms in (E.15) depend upon how the difference operators are distributed. In many cases the integrals are equal to zero. For example suppose we have

$$\Delta^1_{x_1} p_{r_1}(x_1) \Delta^1_{x_1} p_{r_2}(0) \Delta^1_{x_2} p_{r_3}(x_2 - x_1) \Delta^1_{x_2} p_{r_4}(0), \tag{E.17}$$

which we obtain by setting $y_1 = x_1$. Written out this term is

$$(p_{r_1}(x_1+1) - p_{r_1}(x_1))\,\Delta^1_{x_1} p_{r_2}(0) \tag{E.18}$$
$$(p_{r_3}(x_2 - x_1 + 1) - p_{r_3}(x_2 - x_1))\,\Delta^1_{x_2} p_{r_4}(0)$$

By a change of variables one sees that the integral of this term with respect to x_1 and x_2 is zero.

The only non-zero integrals in (E.15) comes from

$$p_{r_1}(x_1)\Delta^1\Delta^{-1} p_{r_2}(0) p_{r_3}(x_2 - x_1)\Delta^1\Delta^{-1} p_{r_4}(0). \tag{E.19}$$

The integral of this term with respect to x_1 and x_2 is

$$\Delta^1\Delta^{-1} p_{r_2}(0)\Delta^1\Delta^{-1} p_{r_4}(0). \tag{E.20}$$

We get the same contribution when $(\pi(1),\pi(2),\pi(3),\pi(4)) = (2,2,1,1)$. Consequently, the contribution to (E.14) of maps with two bound states is

$$\begin{aligned} & 8\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^1\Delta^{-1}p_{r_2}(0)\,\Delta^1\Delta^{-1}p_{r_4}(0)\prod_{i=1}^4 dr_i \qquad \text{(E.21)}\\ &\quad = 32\int_{\{\sum_{i=1}^4 r_i\le t\}} (p_{r_2}(0)-p_{r_2}(1))\,(p_{r_4}(0)-p_{r_4}(1))\prod_{i=1}^4 dr_i\\ &\quad = 16\int_{\{u+v\le t\}} (t-u-v)^2\,(p_u(0)-p_u(1))\,((p_v(0)-p_v(1))\,du\,dv.\\ &\quad \le 16t^2\left(\int_0^\infty (p_u(0)-p_u(1))\,du\right)^2 = (4c_{\psi,0}t)^2, \end{aligned}$$

(see (A.15)).

We next consider the contribution from terms with exactly one bound state. These come from maps of the form $(\pi(1),\pi(2),\pi(3),\pi(4)) = (1,2,2,1)$ or $(\pi(1),\pi(2),\pi(3),\pi(4)) = (2,1,1,2)$. These terms give non-zero contributions of the form

$$\begin{aligned} Q_2 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^1 p_{r_2}(y-x)\,\Delta_y^1\Delta_y^{-1}p_{r_3}(0)\,\Delta_x^1 p_{r_4}(x-y)\\ &\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(E.22)}\\ &= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^{-1}p_{r_2}(y)\,\Delta^1\Delta^{-1}p_{r_3}(0)\,\Delta^{-1}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy;\\ Q_3 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^1\Delta_y^1 p_{r_2}(y-x)\,p_{r_3}(0)\,\Delta_x^1\Delta_y^h p_{r_4}(x-y)\\ &\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(E.23)}\\ &= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^1\Delta^{-1}p_{r_2}(y)\,p_{r_3}(0)\,\Delta^1\Delta^{-1}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy; \end{aligned}$$

and

$$\begin{aligned} Q_4 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^1\Delta_y^1 p_{r_2}(y-x)\,\Delta_y^1 p_{r_3}(0)\,\Delta_x^h p_{r_4}(x-y)\\ &\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(E.24)}\\ &= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^1\Delta^{-1}p_{r_2}(y)\,\Delta^1 p_{r_3}(0)\,\Delta^{-1}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy. \end{aligned}$$

For further explanation consider Q_2. This arrangement comes from the sequence (x_1,y_2,x_2,y_1). The expression it is equal to comes by making the change of variables, $y-x\to y$ and then integrating with respect to x.

Integrating and using (A.6), (A.7) and (A.9) we see that

$$|Q_2| \;\le\; t\left(\int_0^t |\Delta^1\Delta^{-1}p_s(0)|\,ds\right)\int\left(\int_0^t |\Delta^{-1}p_r(y)|\,dr\right)^2 dy \qquad \text{(E.25)}$$

$$\begin{aligned} &\leq \quad t\, w(0,t) \sup_x v(x,t) \int v(y,t)\, dy \\ &\leq \quad Ct^2 \psi^{-1}(1/t)(\log t)^2. \end{aligned}$$

To obtain a bound for Q_3 we use (A.5) and (A.11) to see that it is bounded in absolute value by

$$\begin{aligned} & t \left(\int_0^t p_s(0)\ ds \right) \int \left(\int_0^t |\Delta^1 \Delta^{-1} p_r(y)|\ dr \right)^2\ dy \\ &\quad = tu(0,t) \int w^2(y,t)\, dy \\ &\quad \leq Ct^2 \psi^{-1}(1/t) \log t. \end{aligned} \tag{E.26}$$

Integrating Q_4 we see that it is bounded in absolute value by

$$\begin{aligned} & t \int_0^t \left| \Delta^1 p_r(0) \right| dr \int \left(\int_0^t \left| \Delta^1 \Delta^{-1} p_r(y) \right|\ dr \int_0^t \left| \Delta^{-1} p_r(y) \right|\ dr \right)\ dy \\ &\quad \leq tv(0,t) \sup_x v(x,t) \int w(y,t)\, dy \\ &\quad \leq Ct (\log t)^3, \end{aligned} \tag{E.27}$$

by (A.6) and (A.10).

Finally, we consider the contribution from terms in (E.14) with no bound states. These have to be from π of the form $(\pi(1), \pi(2), \pi(3), \pi(4))$ $= (1,2,1,2)$ or of the form $(\pi(1), \pi(2), \pi(3), \pi(4)) = (2,1,2,1)$. They give contributions of the form

$$\begin{aligned} & Q_5 \\ &:= \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} p_{r_1}(x) \Delta_x^1 p_{r_2}(y-x)\, \Delta_y^1 \Delta_x^1 p_{r_3}(x-y)\, \Delta_y^1 p_{r_4}(y-x) \\ &\qquad\qquad \prod_{i=1}^4 dr_i\, dx\, dy \\ &= \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} \Delta^{-1} p_{r_2}(y)\, \Delta^1 \Delta^{-1} p_{r_3}(y)\, \Delta^1 p_{r_4}(y) \prod_{i=1}^4 dr_i\, dy \end{aligned} \tag{E.28}$$

and

$$\begin{aligned} & Q_6 \\ &:= \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} p_{r_1}(x) \Delta_x^1 \Delta_y^1 p_{r_2}(y-x)\, p_{r_3}(x-y)\, \Delta_x^h \Delta_y^1 p_{r_4}(x-y) \\ &\qquad\qquad \prod_{i=1}^4 dr_i\, dx\, dy \\ &= \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} \Delta^1 \Delta^{-1} p_{r_2}(y)\, p_{r_3}(y)\, \Delta^1 \Delta^{-1} p_{r_4}(y) \prod_{i=1}^4 dr_i\, dy. \end{aligned} \tag{E.29}$$

Clearly

$$|Q_5| \leq t \int \left(\int_0^t |\Delta^{-1} p_r(y)|\, dr \right) \tag{E.30}$$

$$\left(\int_0^t |\Delta^1 p_r(y)|\,dr\right)\left(\int_0^t |\Delta^1\Delta^{-1} p_r(y)|\,dr\right)\,dy$$
$$\leq t\sup_x v^2(x,t)\int w(y,t)\,dy.$$
$$\leq Ct(\log t)^3,$$

by (A.6) and (A.10).

The term Q_6 is bounded the same way we bounded Q_3 and has the same bound.

We can now obtain an upper bound for the variance. Note that by (3.16)

$$(EZ)^2 = \left(E\left(\int (L_t^{x+1} - L_t^x)^2\,dx\right)\right)^2 = (4c_{\psi,0}t)^2 + O\left(tg(t)\right). \tag{E.31}$$

Therefore, it follows from (E.14) and (E.21) that

$$\text{Var } Z \leq EZ^2 - (EZ)^2 = \sum_{j=2}^{6} |Q_j| + Ctg(t) \tag{E.32}$$

as $t \to \infty$. Thus we see that

$$\text{Var } \ Z \leq C\left(tg(t) + t^2\psi^{-1}(1/t)\log t\right). \tag{E.33}$$

Note that for all t sufficiently large

$$tg(t) \leq (t\psi^{-1}(1/t))^3 \leq Ct^2\psi^{-1}(1/t), \tag{E.34}$$

where we use (B.108). Thus we get (3.18). □

F Kac Moment Formula

We derive the version of the Kac Moment Formula that we have been using.

Let $X = \{X_t, t \in R_+\}$ denote a symmetric Lévy process with continuous local time $L = \{L_t^x\,;\,(x,t) \in R^1 \times R_+\}$. Since L is continuous we have the occupation density formula,

$$\int_0^t g(X_s)\,ds = \int g(x)L_t^x\,dx, \tag{F.1}$$

for all continuous functions g with compact support. (See, e.g. [11, Theorem 3.7.1].)

Let $f(x)$ be a continuous function on R^1 with compact support with $\int f(x)\,dx = 1$. Let $f_{\epsilon,y}(x) := \frac{1}{\epsilon} f\left(\frac{x-y}{\epsilon}\right)$. I.e., $f_{\epsilon,y}(x)$ is an approximate δ-function at x. Set

$$L_{t,\epsilon}^x = \int_0^t f_{\epsilon,x}\left(X_s\right)\,ds. \tag{F.2}$$

It follows from (F.1) that

$$L_t^x = \lim_{\epsilon\to 0} L_{t,\epsilon}^x \qquad \text{a. s.} \tag{F.3}$$

Let $p_t(x,y)$ denote the probability density of X_t.

Theorem F.1 (Kac Moment Formula) *Let $X = \{X_t, t \in R_+\}$ denote a symmetric Lévy process with continuous local time $L = \{L_t^x\,;\,(x,t) \in R^1 \times R_+\}$. For any fixed $0 < t < \infty$, bounded continuous g, and any $x_1, \ldots, x_m, z \in R^1$,*

$$E^z\left(\prod_{i=1}^m L_t^{x_i}\,g(X_t)\right) = \sum_\pi \int_{\{\sum_{j=1}^m r_j \leq t\}} \prod_{j=1}^m p_{r_j}(x_{\pi(j-1)}, x_{\pi(j)}) \tag{F.4}$$
$$\left(\int p_{t-r_m}(x_{\pi(m)}, y)g(y)\,dy\right)\prod_{j=1}^m dr_j,$$

where the sums run over all permutations π of $\{1,\dots,m\}$ and $\pi(0) := 0$ and $x_0 := z$.

Proof Let

$$F_t(x_1,\dots,x_m) = \int_{\{\sum_{j=1}^m r_j \le t\}} \prod_{j=1}^m p_{r_j}(x_{j-1},x_j) \left(\int p_{t-r_m}(x_m,y) g(y)\,dy \right) \prod_{j=1}^m dr_j \tag{F.5}$$

Then

$$\begin{aligned} & E^z\left(\prod_{i=1}^m L^{x_i}_{t,\epsilon}\, g(X_t) \right) && \text{(F.6)} \\ &= \sum_\pi \int_{\{0\le t_{\pi(1)}\le\dots\le t_{\pi(m)}\le t\}} E^z\left(\prod_{j=1}^m f_{\epsilon,x_j}\left(X_{t_{\pi(j)}}\right)\, g(X_t) \right) \prod_{j=1}^m dt_{\pi(j)} \\ &= \sum_\pi \int_{\{0\le t_1\le\dots\le t_m\le t\}} E^z\left(\prod_{j=1}^m f_{\epsilon,x_{\pi(j)}}\left(X_{t_j}\right)\, g(X_t) \right) \prod_{j=1}^m dt_j \\ &= \sum_\pi \int\int_{\{\sum_{j=1}^m r_j\le t\}} \prod_{j=1}^m f_{\epsilon,x_{\pi(j)}}(y_j) p_{r_j}(y_{j-1},y_j) && \text{(F.7)} \\ &\qquad \left(\int p_{t-r_m}(y_m,y) g(y)\,dy \right) \prod_{j=1}^m dr_j\,dy_j \\ &= \sum_\pi \int F_t(y_0,\dots,y_m) \prod_{j=1}^m f_{\epsilon,x_{\pi(j)}}(y_j)\,dy_j \end{aligned}$$

where $y_0 := z$.

Since the integrand in (F.5) is dominated by $(2\pi)^{-m/2}\prod_{j=1}^m r_j^{-1/2}$ it follows from the Dominated Convergence Theorem that $F_t(x_1,\dots,x_m)$ is a continuous function of $(x_1,\dots,x_m)$ for all $0\le t<\infty$ and all m. It then follows immediately from (F.6) and the fact that $\prod_{j=1}^m f_{\epsilon,x_{\pi(j)}}(y_j)$ has compact support that

$$\lim_{\epsilon\to 0} E\left(\prod_{i=1}^m L^{x_i}_{t,\epsilon}\, g(X_t) \right) = \sum_\pi F_t(x_{\pi(0)},x_{\pi(1)},\dots,x_{\pi(m)}). \tag{F.8}$$

A repetition of the above proof shows that $E\left(\left\{ \prod_{i=1}^m L^{x_i}_{t,\epsilon} \right\}^2 \right)$ is bounded uniformly in $\epsilon>0$. This fact and (F.3) show that

$$\lim_{\epsilon\to 0} E\left(\prod_{i=1}^m L^{x_i}_{t,\epsilon}\, g(X_t) \right) = E\left(\prod_{i=1}^m L^{x_i}_{t}\, g(X_t) \right). \tag{F.9}$$

Obviously (F.8) and (F.9) imply (F.4). □

G Estimates for the asymptotic behavior of small differences of the transition probability densities of certain Lévy processes

Sections G–K contain the proofs of the lemmas needed to prove Theorem 1.2. The most critical ingredient in the proof of Theorem 1.2 is Lemma H.1. To prove it we need estimates on the asymptotic behavior of small differences of the transition probability densities for the Lévy processes under consideration.

The proofs of the following lemmas are given in Section J. For notation see the first paragraph of Section A.

Lemma G.1 *Let X be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at infinity with index $1<\beta\le 2$ and satisfies (1.16) and (1.17). Let $p_s(x)$ denote the transition probability density of X. Then*

$$p_s(x) \le C\frac{\psi^{-1}(1/s)\vee 1}{1+x^2}, \qquad \forall\, x\in R^1, s\in(0,1]; \tag{G.1}$$

$$u(x) := \int_0^1 p_s(x)\,ds \le \frac{C}{1+x^2}, \qquad \forall\, x\in R^1; \tag{G.2}$$

$$\int\int_0^t p_s(x)\,ds\,dx = t, \tag{G.3}$$

and for all h sufficiently small

$$\begin{aligned} v(x) := \int_0^1 |\Delta^h\, p_s(x)|\,ds \quad &\le \quad C\left(\frac{1}{h\psi(1/h)}\wedge\frac{h}{|x|}\wedge\frac{h}{x^2}\right) \\ &\le \quad \frac{C}{h\psi(1/h)}\left(\frac{1}{1+x^2}\right), \end{aligned} \tag{G.4}$$

$$\int v(x)\,dx = O\left(h\log 1/h\right), \tag{G.5}$$

and

$$\int v^p(x)\,dx = O\left(\frac{h}{h^{p-1}\psi^{p-1}(1/h)}\right), \quad p\ge 2, \tag{G.6}$$

as $h\to 0$. In addition,

$$w(x) := \int_0^1 |\Delta^h\Delta^{-h}\, p_s(x)|\,ds \le C\left(\frac{1}{h\psi(1/h)}\wedge\frac{1}{\psi(1/h)|x|}\wedge\frac{h^2}{|x|^2}\right); \tag{G.7}$$

$$\int w(x)\,dx = O\left(\frac{\log(1/h)}{\psi(1/h)}\right); \tag{G.8}$$

$$\int w^2(x)\,dx = O\left(\frac{1}{h\psi^2(1/h)}\right); \tag{G.9}$$

$$\int_{|x|\ge u} w^2(x)\,dx \le O\left(\frac{1}{u\psi^2(1/h)}\right), \tag{G.10}$$

as $h\to 0$.

Lemma G.2 *Under the same hypotheses as Lemma G.1,*

$$h\psi^2(1/h)\left(c_{\psi,h,1} - \int\left(\int_0^{\sqrt{h}} \Delta^h\Delta^{-h}\; p_s(x)\,ds\right)^2 dx\right) = O(h^{1/2}). \tag{G.11}$$

Remark G.3 We allow ψ to be regularly varying at infinity with index 2, but note that that because ψ is the Lévy exponent of a symmetric Lévy process

$$\psi(\lambda) = O(\lambda^2) \qquad \text{as } \lambda\to\infty. \tag{G.12}$$

(See, e.g., [11, Lemma 4.2.2] and then include Brownian motion.)

Lemma G.4 *Under the same hypotheses as Lemma G.1,*

$$\sup_{\delta\le r\le 1} p_r(0) \le C\left(\psi^{-1}(1/\delta)\vee 1\right); \tag{G.13}$$

$$\sup_{\delta\le r\le 1} |\Delta^h p_r(0)| \quad\le\quad \frac{C}{\delta^3}h^2; \tag{G.14}$$

and

$$\sup_{\delta\le r\le 1} |\Delta^h\Delta^{-h} p_r(0)| \quad\le\quad \frac{C}{\delta^3}h^2. \tag{G.15}$$

Lemma G.5 *Let $0<\delta<1$, then, under the hypotheses of Theorem 1.2, for*

$$\overline{u}_\delta(x) := \sup_{\delta\le r\le 1} p_r(x), \qquad \overline{v}_\delta(x) := \sup_{\delta\le r\le 1} |\Delta^h p_r(x)|, \tag{G.16}$$

$$\textit{and} \qquad \overline{w}_\delta(x) := \sup_{\delta\le r\le 1} |\Delta^h\Delta^{-h} p_r(x)|,$$

we have

$$\overline{u}_\delta(x) \le C\psi^{-1}(1/\delta)\left(1\wedge\frac{1}{x^2}\right), \tag{G.17}$$

$$\overline{v}_\delta(x) \le \frac{C}{\delta^3}h\left(1\wedge\frac{1}{x^2}\right), \tag{G.18}$$

and

$$\overline{w}_\delta(x) \le \frac{C}{\delta^3}h^2\left(1\wedge\frac{1}{x^2}\right). \tag{G.19}$$

In addition

$$\int \overline{u}_\delta(x)\,dx \le C\psi^{-1}(1/\delta), \qquad \int (\overline{u}_\delta(x))^2\,dx \le C\left(\psi^{-1}(1/\delta)\right)^2, \tag{G.20}$$

$$\int \overline{v}_\delta(x)\,dx \le \frac{C}{\delta^3}h, \qquad \int \overline{v}^2_\delta(x)\,dx \le \frac{C}{\delta^6}h^2, \tag{G.21}$$

$$\int \overline{w}_\delta(x)\,dx \le \frac{C}{\delta^3}h^2, \qquad \int \overline{w}^2_\delta(x)\,dx \le \frac{C}{\delta^6}h^4, \tag{G.22}$$

as $h\to 0$.

H Moments of increments of local times.

Refer to (4.1) and (4.5). To simplify the notation we set

$$J_{j,k,h} := J_{j,k,h,1}, \qquad \alpha_{j,k} := \alpha_{j,k,1}. \tag{H.1}$$

Lemma H.1 *Let $m_{j,k}$, $0\le j<k\le K$ be positive integers with $\sum_{j,k=0,j<k}^K m_{j,k} = m$. If all the integers $m_{j,k}$ are even, then for some $\epsilon>0$*

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^K (J_{j,k,h})^{m_{j,k}}\right) \tag{H.2}$$

$$= \prod_{\substack{j,k=0\\ j<k}}^K \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)}\left(4c_{\psi,h,1}\right)^{n_{j,k}} E\left(\prod_{\substack{j,k=0\\ j<k}}^K (\alpha_{j,k})^{n_{j,k}}\right) + O\left(h^{(2\beta-1)n+\epsilon}\right),$$

where $n_{j,k} = m_{j,k}/2$ *and* $n = m/2$*. If any of the* $m_{j,k}$ *are odd, then*

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} (J_{j,k,h})^{m_{j,k}}\right) = O\left(h^{(2\beta-1)m/2+\epsilon}\right). \tag{H.3}$$

In (H.2) and (H.3) the error terms may depend on m*, but not on the individual terms* $m_{j,k}$*.*

Proof We can write

$$\begin{aligned}
&E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} (J_{j,k,h})^{m_{j,k}}\right) \qquad\qquad (\text{H.4})\\
&= E\left(\prod_{\substack{j,k=0\\ j<k}} \prod_{i=1}^{m_{j,k}} \left(\int (\Delta^h_{x_{j,k,i}} L_1^{x_{j,k,i}} \circ \theta_j)\ (\Delta^h_{x_{j,k,i}} L_1^{x_{j,k,i}} \circ \theta_k)\ dx_{j,k,i}\right)\right)\\
&= \int \left\{\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \Delta^{h,j}_{x_{j,k,i}} \Delta^{h,k}_{x_{j,k,i}}\right\} E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \left((L_1^{x_{j,k,i}} \circ \theta_j)\ (L_1^{x_{j,k,i}} \circ \theta_k)\right)\right)\\
&\qquad\qquad \prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} dx_{j,k,i},
\end{aligned}$$

where the notation $\Delta^{h,j}_{x_{j,k,i}}$ indicates that we apply the difference operator $\Delta^h_{x_{j,k,i}}$ to $L_1^{x_{j,k,i}} \circ \theta_j$. Note that there are $2m$ applications of the difference operator Δ^h.

Consider

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \left((L_1^{x_{j,k,i}} \circ \theta_j)\ (L_1^{x_{j,k,i}} \circ \theta_k)\right)\right). \tag{H.5}$$

We collect all the factors containing θ_l and write

$$\begin{aligned}
&E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \left((L_1^{x_{j,k,i}} \circ \theta_j)\ (L_1^{x_{j,k,i}} \circ \theta_k)\right)\right) \qquad\qquad (\text{H.6})\\
&= E\left(\prod_{l=0}^{K} \left\{\left(\prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_1^{x_{j,l,i}}\right) \left(\prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_1^{x_{l,k,i}}\right)\right\} \circ \theta_l\right)\\
&= E\left(\prod_{l=0}^{K} H_l \circ \theta_l\right),
\end{aligned}$$

where

$$H_l = \left(\prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_1^{x_{j,l,i}}\right) \left(\prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_1^{x_{l,k,i}}\right). \tag{H.7}$$

By the Markov property

$$E\left(\prod_{l=0}^{K} H_l\circ\theta_l\right)=E\left(H_0\,E^{X_1}\left(\prod_{l=1}^{K} H_l\circ\theta_{l-1}\right)\right). \tag{H.8}$$

Let

$$m_l=\sum_{k=l+1}^{K} m_{l,k}+\sum_{j=0}^{l-1} m_{j,l},\qquad l=0,\ldots,K-1, \tag{H.9}$$

and note that m_l is the number of local time factors in H_l.

Let

$$f(y)=E^y\left(\prod_{l=1}^{K} H_l\circ\theta_{l-1}\right). \tag{H.10}$$

It follows from the Kac Moment Formula, Theorem F.1, that for any $z\in R^1$

$$\begin{aligned}
&E^z\left(\prod_{l=0}^{K} H_l\circ\theta_l\right) \qquad\qquad (\text{H.11})\\
&\quad= E^z\left(H_0\,f(X_1)\right)\\
&\quad= \sum_{\pi_0}\int_{\{\sum_{q=1}^{m_0} r_{0,q}\le 1\}} p_{r_{0,1}}(x_{\pi_0(1)}-z)\prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)}-x_{\pi_0(q-1)})\\
&\qquad\qquad \left(\int p_{(1-\sum_{q=1}^{m_0} r_{0,q})}(y-x_{\pi_0(m_0)})f(y)\,dy\right)\prod_{q=1}^{m_0} dr_{0,q},
\end{aligned}$$

where the sum runs over all bijections π_0 from $[1,m_0]$ to

$$\bar I_0=\bigcup_{k=1}^{K}\{(0,k,i),\,1\le i\le m_{0,k}\}. \tag{H.12}$$

Clearly, $\bar I_0$ is the set of subscripts of the terms $x_{\cdot}$ appearing in the local time factors in H_0.

By the Markov property

$$\begin{aligned}
f(y) &= E^y\left(H_1E^{X_2}\left(\prod_{l=2}^{K} H_l\circ\theta_{l-2}\right)\right) \qquad (\text{H.13})\\
&:= E^y\left(H_1 g(X_2)\right).
\end{aligned}$$

Therefore, by (H.8)–(H.13), for any $z'\in R^1$

$$\begin{aligned}
&E^{z'}\left(\prod_{l=0}^{K} H_l\circ\theta_l\right) \qquad\qquad (\text{H.14})\\
&= E^{z'}\left(H_0E^{X_1}\left(H_1\,g(X_2)\right)\right)\\
&= \sum_{\pi_0}\int_{\{\sum_{q=1}^{m_0} r_{0,q}\le 1\}} p_{r_{0,1}}(x_{\pi_0(1)}-z')\prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)}-x_{\pi_0(q-1)})\\
&\qquad \left(\int p_{(1-\sum_{q=1}^{m_0} r_{0,q})}(y-x_{\pi_0(m_0)})E^y\left(H_1\,g(X_2)\right)\,dy\right)\prod_{q=1}^{m_0} dr_{0,q}\\
&= \sum_{\pi_0}\int_{\{\sum_{q=1}^{m_0} r_{0,q}\le 1\}} p_{r_{0,1}}(x_{\pi_0(1)}-z')\prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)}-x_{\pi_0(q-1)})
\end{aligned}$$

$$
\begin{aligned}
&p_{(1-\sum_{q=1}^{m_0} r_{0,q})}(y - x_{\pi_0(m_0)}) \\
&\sum_{\pi_1} \int_{\{\sum_{q=1}^{m_1} r_{1,q} \le 1\}} p_{r_{1,1}}(x_{\pi_1(1)} - y) \prod_{q=2}^{m_1} p_{r_{1,q}}(x_{\pi_1(q)} - x_{\pi_1(q-1)}) \\
&\left(\int p_{(1-\sum_{q=1}^{m_1} r_{1,q})}(y' - x_{\pi_1(m_1)}) g(y')\, dy' \right) \prod_{q=1}^{m_1} dr_{1,q}\, dy \prod_{q=1}^{m_0} dr_{0,q}
\end{aligned}
$$

where the second sum runs over all bijections π_1 from $[1, m_1]$ to

$$
\bar{I}_1 = \{(0,1,i),\ 1 \le i \le m_{0,1}\} \bigcup_{k=2}^{K} \{(1,k,i),\ 1 \le i \le m_{1,k}\} \tag{H.15}
$$

As above, $\bar{I}_1$ is the set of subscripts of the terms $x_\cdot$ appearing in the local time factors in H_1.

We now use the Chapman-Kolmogorov equation to integrate with respect to y to get

$$
\begin{aligned}
&E^{z'}\left(H_0 E^{X_1}\left(H_1\, g(X_1)\right)\right) \qquad\qquad \text{(H.16)}\\
&= \sum_{\pi_0,\pi_1} \int_{\{\sum_{q=1}^{m_0} r_{0,q} \le 1\}} p_{r_{0,1}}(x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)} - x_{\pi_0(q-1)}) \\
&\quad \int_{\{\sum_{q=1}^{m_1} r_{1,q} \le 1\}} p_{(1-\sum_{q=1}^{m_0} r_{0,q}) + r_{1,1}}(x_{\pi_1(1)} - x_{\pi_0(m_0)}) \\
&\qquad \prod_{q=2}^{m_1} p_{r_{1,q}}(x_{\pi_1(q)} - x_{\pi_1(q-1)}) \\
&\quad \left(\int p_{(1-\sum_{q=1}^{m_1} r_{1,q})}(y' - x_{\pi_1(m_1)}) g(y')\, dy' \right) \prod_{q=1}^{m_1} dr_{1,q} \prod_{q=1}^{m_0} dr_{0,q}.
\end{aligned}
$$

Iterating this procedure, and recalling (H.6) we see that

$$
\begin{aligned}
&E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \left((L_1^{x_{j,k,i}} \circ \theta_j)\ (L_1^{x_{j,k,i}} \circ \theta_k) \right) \right) \qquad\qquad \text{(H.17)}\\
&= \sum_{\pi_0,\ldots,\pi_K} \prod_{l=0}^{K} \int_{\{\sum_{q=1}^{m_l} r_{l,q} \le 1\}} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q}) + r_{l,1}}(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}) \\
&\qquad \prod_{q=2}^{m_l} p_{r_{l,q}}(x_{\pi_l(q)} - x_{\pi_l(q-1)}) \prod_{q=1}^{m_l} dr_{l,q},
\end{aligned}
$$

where $\pi_{-1}(m_{-1}) := 0$ and $1 - \sum_{q=1}^{m_{-1}} r_{-1,q} := 0$. In (H.17) the sum runs over all $\pi_0, \ldots, \pi_K$ such that each π_l is a bijection from $[1, m_l]$ to

$$
\bar{I}_l = \bigcup_{j=0}^{l-1} \{(j,l,i),\ 1 \le i \le m_{j,l}\} \bigcup_{k=l+1}^{K} \{(l,k,i),\ 1 \le i \le m_{l,k}\}. \tag{H.18}
$$

As in the observations about $\bar{I}_0$ and $\bar{I}_1$, we see that $\bar{I}_l$ is the set of subscripts of the terms $x_\cdot$ terms appearing in the local time factors in H_l. Since there are $2m$ local time factors we have that $\sum_{l=0}^{K} m_l = 2m$.

We now use (H.17) in (H.4) and continue to develop an expression for the left-hand side of (H.4). Let $\mathcal{B}$ to denote the set of $K+1$ tuples $\pi = (\pi_0, \ldots, \pi_K)$ of bijections described in (H.18). Clearly

$$|\mathcal{B}| = \prod_{l=0}^{K} m_l! \leq (2m)!. \tag{H.19}$$

Also, similarly to the way we obtain the first equality in (H.6), we see that

$$\prod_{\substack{j,k=0\\ j<k}}^{K} \prod_{i=1}^{m_{j,k}} \Delta^{h,j}_{x_{j,k,i}} \Delta^{h,k}_{x_{j,k,i}} = \prod_{l=0}^{K} \prod_{q=1}^{m_l} \Delta^{h,l}_{x_{\pi_l(q)}}. \tag{H.20}$$

Consequently

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K} (J_{j,k,h})^{m_{j,k}} \right) = \sum_{\pi_0,\ldots,\pi_K} \int \widetilde{\mathcal{T}}_h(x;\,\pi) \prod_{j,k,i} dx_{j,k,i} \tag{H.21}$$

where we take the product over $\{0 \leq j < k \leq K,\, 1 \leq i \leq m_{j,k}\}$, $\pi \in \mathcal{B}$ and

$$\begin{aligned}
&\widetilde{\mathcal{T}}_h(x;\,\pi) \\
&= \prod_{l=0}^{K} \prod_{q=1}^{m_l} \Delta^{h}_{x_{\pi_l(q)}} \int_{\{\sum_{q=1}^{m_l} r_{l,q} \leq 1\}} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}\big(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}\big) \\
&\qquad\qquad \prod_{q=2}^{m_l} p_{r_{l,q}}\big(x_{\pi_l(q)} - x_{\pi_l(q-1)}\big) \prod_{q=1}^{m_l} dr_{l,q}.
\end{aligned} \tag{H.22}$$

We continue to rewrite the right-hand side of (H.21).

In (H.22), each difference operators, say Δ^h_u is applied to the product of two terms, say $p_\cdot(u-a)p_\cdot(u-b)$, using the product rule for difference operators we see that

$$\begin{aligned}
&\Delta^h_u\{p_\cdot(u-a)p_\cdot(u-b)\} \\
&\quad = \Delta^h_u\, p_\cdot(u-a)p_\cdot(u+h-b) + p_\cdot(u-a)\Delta^h_u\, p_\cdot(u-b)
\end{aligned} \tag{H.23}$$

Consider an example of how the term $\Delta^h_a\Delta^h_u p_\cdot(u-a)$ may appear. It could be by the application

$$\Delta^h_a \left(\Delta^h_u\, p_\cdot(u-a)p_\cdot(v-a)\right), \tag{H.24}$$

in which we take account of the two terms to which Δ^h_a is applied. Using the product rule in (H.23) we see that (H.24)

$$= \left(\Delta^h_a\Delta^h_u\, p_\cdot(u-a)\right) p_\cdot(v-(a+h)) + \Delta^h_u\, p_\cdot(u-a)\Delta^h_a p_\cdot(v-a). \tag{H.25}$$

Consider one more example

$$\begin{aligned}
&\Delta^h_a \left(\Delta^h_u\, p_\cdot(u-a)\Delta^h_v\, p_\cdot(v-a)\right) \\
&\quad = \left(\Delta^h_a\Delta^h_u\, p_\cdot(u-a)\right) \Delta^h_v\, p_\cdot(v-(a+h)) \\
&\qquad\qquad + \Delta^h_u\, p_\cdot(u-a)\Delta^h_a\Delta^h_v p_\cdot(v-a).
\end{aligned} \tag{H.26}$$

Note that in both examples the arguments of probability densities with two difference operators applied to it does not contain an h. This is true in general because the difference formula, (H.23), does not add an h to the argument of a term to which a difference operator is applied. Otherwise we may have a $\pm h$ added to the arguments of probability densities to which one difference operator is applied, as in (H.26), or to the arguments of probability densities to which no difference operator is applied, as in (H.25).

Based on the argument of the preceding paragraph we write (H.22) in the form

$$E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}(J_{j,k,h})^{m_{j,k}}\right)=\sum_{a}\sum_{\pi_0,\ldots,\pi_K}\int \mathcal{T}'_h(x;\,\pi,a)\prod_{j,k,i}dx_{j,k,i}, \tag{H.27}$$

where

$$\mathcal{T}'_h(x;\,\pi,a)=\prod_{l=0}^{K}\int_{\mathcal{R}_l}\left(\left(\Delta^h_{x_{\pi_l(1)}}\right)^{a_1(l,1)}\left(\Delta^h_{x_{\pi_{l-1}(m_{l-1})}}\right)^{a_2(l,1)}\right. \tag{H.28}$$
$$\left. p^\sharp_{(1-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})\right)$$
$$\prod_{q=2}^{m_l}\left(\left(\Delta^h_{x_{\pi_l(q)}}\right)^{a_1(l,q)}\left(\Delta^h_{x_{\pi_l(q-1)}}\right)^{a_2(l,q)}p^\sharp_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\right)\prod_{q=1}^{m_l}dr_{l,q},$$

and where $\mathcal{R}_l=\{\sum_{q=1}^{m_l}r_{l,q}\le 1\}$. In (H.27) the first sum is taken over all

$$a=(a_1,a_2)\,:\,\{(l,q),\,0\le l\le K,\,1\le q\le m_l\}\mapsto\{0,1\}\times\{0,1\} \tag{H.29}$$

with the restriction that for each triple j,k,i, there are exactly two factors of the form $\Delta^h_{x_{j,k,i}}$, each of which is applied to one of the terms $p^\sharp_{r\cdot}(\cdot)$ that contains $x_{j,k,i}$ in its argument. This condition can be stated more formally by saying that for each l and $q=1,\ldots,m_l-1$, if $\pi_l(q)=(j,k,i)$, then $\{a_1(l,q),a_2(l,q+1)\}=\{0,1\}$ and if $q=m_l$ then $\{a_1(l,m_l),a_2(l+1,1)\}=\{0,1\}$. (Note that when we write $\{a_1(l,q),a_2(l,q+1)\}=\{0,1\}$ we mean as two sets, so, according to what a is, we may have $a_1(l,q)=1$ and $a_2(l,q+1)=0$ or $a_1(l,q)=0$ and $a_2(l,q+1)=1$ and similarly for $\{a_1(l,m_l),a_2(l+1,1)\}$.) Also, in (H.28) we define $(\Delta^h_{x_i})^0=1$ and $(\Delta^h_0)=1$.

In (H.28), $p^\sharp_{r\cdot}(z)$ can take any of the three values $p_{r\cdot}(z)$, $p_{r\cdot}(z+h)$, or $p_{r\cdot}(z-h)$. (We must consider all three possibilities.) Finally, it is important to emphasize that in (H.28) each of the difference operators is applied to only one of the terms $p^\sharp_{r\cdot}(\cdot)$.

To avoid confusion caused by the ambiguity of $p^\sharp_{\cdot}$, we first analyze

$$\sum_{a}\sum_{\pi_0,\ldots,\pi_K}\int \mathcal{T}_h(x;\,\pi,a)\prod_{j,k,i}dx_{j,k,i}, \tag{H.30}$$

where

$$\mathcal{T}_h(x;\,\pi,a)=\prod_{l=0}^{K}\int_{\mathcal{R}_l}\left(\left(\Delta^h_{x_{\pi_l(1)}}\right)^{a_1(l,1)}\left(\Delta^h_{x_{\pi_{l-1}(m_{l-1})}}\right)^{a_2(l,1)}\right. \tag{H.31}$$
$$\left. p_{(1-\sum_{q=1}^{m_{l-1}}r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})\right)$$
$$\prod_{q=2}^{m_l}\left(\left(\Delta^h_{x_{\pi_l(q)}}\right)^{a_1(l,q)}\left(\Delta^h_{x_{\pi_l(q-1)}}\right)^{a_2(l,q)}p_{r_{l,q}}(x_{\pi_l(q)}-x_{\pi_l(q-1)})\right)\prod_{q=1}^{m_l}dr_{l,q}.$$

The difference between $\mathcal{T}_h(x;\,\pi,a)$ and $\mathcal{T}'_h(x;\,\pi,a)$ is that in the former we replace $p^\sharp$ by p. (I.e. we set $h=0$ in the arguments of the $p^\sharp$ terms in (H.28).) At the conclusion of this proof we show that both (H.30) than (H.27) have the same asymptotic limit as h goes to zero.

We first obtain (H.2). Let $m=2n$, since $m_{j,k}=2n_{j,k}$, $m_l=2n_l$ for some integer n_l. (Recall (H.9)). To begin we consider the case in which $a=e$, where

$$e(l,2q)=(1,1)\quad\text{and}\quad e(l,2q-1)=(0,0)\qquad\forall q. \tag{H.32}$$

When $a=e$ we have

$$\mathcal{T}_h(x;\,\pi,e)=\prod_{l=0}^{K}\int_{\mathcal{R}_l} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)}-x_{\pi_{l-1}(m_{l-1})})$$
$$\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\pi_l(2q-1)}-x_{\pi_l(2q-2)}) \tag{H.33}$$
$$\prod_{q=1}^{n_l}\Delta^h\Delta^{-h}\,p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)})\prod_{q=1}^{m_l} dr_{l,q}.$$

Here we use the following notation: $\Delta^h p(u-v)=p(u-v+h)-p(u-v)$, i.e., when Δ^h has no subscript, the difference operator is applied to the whole argument of the function. In this notation,

$$\Delta^h_u\Delta^h_v p(u-v)=\Delta^h\Delta^{-h}p(u-v). \tag{H.34}$$

H.1 $\mathbf{a=e}$, with all cycles of order two

Consider the multigraph G_π with vertices $\{(j,k,i),\,0\le j<k\le K,\,1\le i\le m_{j,k}\}$. Assign an edge between the vertices $\pi_l(2q-1)$ and $\pi_l(2q)$ for each $0\le l\le K$ and $1\le q\le n_l$. Each vertex is connected to two edges. To see this suppose that $\pi_l(2q)=\{(j,k,i)\}$, with $j=l$ and $k=l'\neq l$, then there is a unique q' such that $\pi_{l'}(2q')$ or $\pi_{l'}(2q'-1)$ is equal to $\{(j,k,i)\}$. Therefore all the vertices lie in some cycle. Assume that there are S cycles. We denote them by C_s, $s=1,\dots,S$. Clearly, it is possible to have cycles of order two, in which case two vertices are connected by two edges.

It is important to note that the graph G_π does not assign edges between $\pi_l(2q)$ and $\pi_l(2q+1)$, although these vertices may be connected by the edge assigned between $\pi_{l'}(2q'-1)$ and $\pi_{l'}(2q')$ for some l' and q'.

We estimate (H.31) by breaking the calculation into two cases. In this section we consider the case when $a=e$ and all the cycles of G_π are of order two. In Section H.2 we consider the cases when $a=e$ and not all the cycles of G_π are of order two, and when $a\neq e$.

Let $\mathcal{P}=\{(\gamma_{2v-1},\gamma_{2v})\,,\,1\le v\le n\}$ be a pairing of the m vertices

$$\{(j,k,i),\,0\le j<k\le K,\,1\le i\le m_{j,k}\}$$

of G_π, that satisfies the following special property: whenever (j,k,i) and (j',k',i') are paired together, $j=j'$ and $k=k'$. Equivalently,

$$\mathcal{P}=\bigcup_{\substack{j,k=0\\ j<k}}^{K}\mathcal{P}_{j,k} \tag{H.35}$$

where each $\mathcal{P}_{j,k}$ is a pairing of the $m_{j,k}$ vertices

$$\{(j,k,i),\,1\le i\le m_{j,k}\}.$$

We refer to such a pairing $\mathcal{P}$ as a special pairing and denote the set of special pairings by $\mathcal{S}$.

Given a special pairing $\mathcal{P}\in\mathcal{S}$, let π be such that for each $0\le l\le K$ and $1\le q\le n_l$,

$$\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2v-1},\gamma_{2v}\} \tag{H.36}$$

for some, necessarily unique, $1 \le v \le n_l$. In this case we say that π is compatible with the pairing $\mathcal{P}$ and write this as $\pi \sim \mathcal{P}$. (Recall that when we write $\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}$, we mean as two sets, so, according to what π_l is, we may have $\pi_l(2q-1) = \gamma_{2v-1}$ and $\pi_l(2q) = \gamma_{2v}$ or $\pi_l(2q-1) = \gamma_{2v}$ and $\pi_l(2q) = \gamma_{2v-1}$.) Clearly

$$|\mathcal{S}| \le \frac{(2n)!}{2^n n!} \tag{H.37}$$

the number of pairings of $m = 2n$ objects.

Let $\pi \in \mathcal{B}$ be such that G_π consists of cycles of order two. It is easy to see that $\pi \sim \mathcal{P}$ for some $\mathcal{P} \in \mathcal{S}$. To see this note that if $\{(j,k,i),(j',k',i')\}$ form a cycle of order two, there must exist l and l' with $l \ne l'$ and q and q' such that both $\{(j,k,i),(j',k',i')\} = \{\pi_l(2q-1), \pi_l(2q)\}$ and $\{(j,k,i),(j',k',i')\} = \{\pi_{l'}(2q'-1), \pi_{l'}(2q')\}$. This implies that $j = j'$, $k = k'$ and $\{j,k\} = \{l,l'\}$. Furthermore, by (H.36) we have

$$\{\pi_l(2q-1), \pi_l(2q)\} = \{\pi_{l'}(2q'-1), \pi_{l'}(2q')\} = \{\gamma_{2v-1}, \gamma_{2v}\} \tag{H.38}$$

When $\pi \sim \mathcal{P}$ and all cycles are of order two we can write

$$\begin{aligned} &\prod_{l=0}^{K} \prod_{q=1}^{n_l} \Delta^h \Delta^{-h} p_{r_{l,2q}}(x_{\pi_l(2q)} - x_{\pi_l(2q-1)}) \qquad \text{(H.39)} \\ &\quad = \prod_{v=1}^{n} \Delta^h \Delta^{-h} p_{r_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \Delta^h \Delta^{-h} p_{r'_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}), \end{aligned}$$

where $r_{2\nu}$ and $r'_{2\nu}$ are the rearranged indices $r_{l,2q}$ and $r_{l',2q'}$. We also use the fact that $\sum_{l=0}^{K} n_l = 2n$.

For use in (H.45) below we note that

$$\begin{aligned} &\int_0^1 \int_0^1 |\Delta^h \Delta^{-h} p_{r_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| \, |\Delta^h \Delta^{-h} p_{r'_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| \, dr_{2\nu} \, dr'_{2\nu} \\ &= \left(\int_0^1 |\Delta^h \Delta^{-h} p_r(x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| \, dr \right)^2 = w^2(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}), \qquad \text{(H.40)} \end{aligned}$$

(see (G.7).)

We want to estimate the integrals in (H.30). However, it is difficult to integrate $\mathcal{T}_h(x;\pi,e)$ directly, because the variables,

$$\begin{aligned} \{x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})},\, x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)},\, x_{\pi_l(2q)} - x_{\pi_l(2q-1)}; \\ l \in [0,K],\, q \in [1,n_l]\}, \end{aligned}$$

are not independent. We begin the estimation by showing that over much of the domain of integration, the integral is negligible, asymptotically, as $h \to 0$. To begin, we write

$$1 = \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \le \sqrt{h}\}} + 1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \ge \sqrt{h}\}} \right) \tag{H.41}$$

and expand it as a sum of 2^n terms and use it to write

$$\begin{aligned} &\int \mathcal{T}_h(x;\pi,e) \prod_{j,k,i} dx_{j,k,i} \qquad \text{(H.42)} \\ &\quad = \int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \le \sqrt{h}\}} \right) \mathcal{T}_h(x;\pi,e) \prod_{j,k,i} dx_{j,k,i} + E_{1,h}. \end{aligned}$$

We now show that

$$E_{1,h} = O\left(h^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^n\right). \tag{H.43}$$

Note that every term in $E_{1,h}$ can be written in the form

$$B_h(\pi, e, D) := \int \prod_{v=1}^{n} 1_{D_v} \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{H.44}$$

where each D_v is either $\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \leq \sqrt{h}\}$ or $\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \geq \sqrt{h}\}$, and at least one of the D_v is of the second type.

Consider (H.44) and the representation of $\mathcal{T}_h(x;\, \pi, e)$ in (H.33). We take absolute values in the integrand in (H.33) and take all the integrals with $r.$ between 0 and 1 and use (H.40) followed by (G.2) to get

$$\begin{aligned} |B_h(\pi, e, D)| \quad \leq \quad & \int \prod_{v=1}^{n} 1_{D_v} w^2(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \prod_{l=0}^{K} u(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}) \\ & \prod_{q=2}^{n_l} u(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}) \prod_{j,k,i} dx_{j,k,i}. \end{aligned} \tag{H.45}$$

We now take

$$\{x_{\gamma_{2v}} - x_{\gamma_{2v-1}},\, v = 1, \ldots, n\} \tag{H.46}$$

and an additional n variables from the $2n$ arguments of the u terms,

$$\cup_{l=0}^{K}\{x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}, x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)},\, q = 2, \ldots, n_l\} \tag{H.47}$$

so that the chosen $2n$ variables generate the space spanned by the 2n variables $\{x_{j,k,i}\}$. There are n variables in (H.47) that are not used. We bound the functions u of these variables by their sup norm, which by (G.2) is finite. Then we make a change of variables and get that

$$\begin{aligned} |B_h(\pi, e, D)| \quad & \leq \quad \int \prod_{v=1}^{n} 1_{D_v} w^2(y_v) \prod_{v=n+1}^{2n} u(y_v) \prod_{v=1}^{2n} dy_v \\ & \leq \quad C \int \prod_{v=1}^{n} 1_{D_v} w^2(y_v) \prod_{v=1}^{n} dy_v, \\ & = \quad O\left(h^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^n\right). \end{aligned} \tag{H.48}$$

Here we use (G.2) to see that the integrals of the u terms is finite. Then we use (G.9) and (G.10) to obtain (H.43). (Note that it is because at least one of the D_v is of the second type that we can use (G.10).)

We now study

$$\int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \leq \sqrt{h}\}}\right) \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i}. \tag{H.49}$$

Recall that for each $0 \leq l \leq K$ and $1 \leq q \leq n_l$, $\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}$ for some $1 \leq v \leq n$. We identify these relationships by setting $v = \sigma_l(q)$ when $\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}$, and sometimes write this last term as $\{\gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)}\}$.

For $q \geq 2$ we write

$$
\begin{aligned}
&p_{r_{l,2q-1}}(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}) \qquad \text{(H.50)}\\
&\quad = p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}) + \Delta^{h_{l,q}} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}),
\end{aligned}
$$

where $h_{l,q} = (x_{\pi_l(2q-1)} - x_{\gamma_{2\sigma_l(q)-1}}) + (x_{\gamma_{2\sigma_l(q-1)-1}} - x_{\pi_l(2q-2)})$. When $q = 1$ we can make a similar decomposition

$$
\begin{aligned}
&p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}) \qquad \text{(H.51)}\\
&\quad = p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\\
&\qquad + \Delta^{h_{l,1}} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}),
\end{aligned}
$$

where $h_{l,1} = (x_{\pi_l(1)} - x_{\gamma_{2\sigma_l(1)-1}}) + (x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}) - x_{\pi_{l-1}(m_{l-1})})$. Note that because of the presence of the term $\prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \leq \sqrt{h}\}}\right)$ in the integral in (H.49) we need only be concerned with values of $|h_{l,q}| \leq 2\sqrt{h}$, for $0 \leq l \leq K$ and $1 \leq q \leq n_l$.

For $q = 1, \ldots, n_l$, $l = 0 \ldots, K$, we substitute (H.50) and (H.51) into the term $\mathcal{T}_h(x;\, \pi, e)$ in (H.49), (see also (H.33)), and expand the products so that we can write (H.49) as a sum of $2^{\sum_{l=0}^{K} n_l}$ terms, which we write as

$$
\begin{aligned}
&\int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \leq \sqrt{h}\}}\right) \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \qquad \text{(H.52)}\\
&\quad = \int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}} - x_{\gamma_{2v-1}}| \leq \sqrt{h}\}}\right) \mathcal{T}_{h,1}(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} + E_{2,h},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}_{h,1}(x;\, \pi, e) = \prod_{l=0}^{K} \int_{\mathcal{R}_l} & p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\\
& \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}) \qquad \text{(H.53)}\\
& \prod_{q=1}^{n_l} \Delta^h \Delta^{-h}\, p_{r_{l,2q}}(x_{\pi_l(2q)} - x_{\pi_l(2q-1)}) \prod_{q=1}^{m_l} dr_{l,q}.
\end{aligned}
$$

Using (H.39) we can rewrite this as

$$
\begin{aligned}
&\mathcal{T}_{h,1}(x;\, \pi, e) \qquad \text{(H.54)}\\
&\quad = \int_{\mathcal{R}_0 \times \cdots \times \mathcal{R}_K} \left(\prod_{l=0}^{K} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}) \right.\\
&\qquad\qquad \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}) \right)\\
&\quad \left(\prod_{v=1}^{n} \Delta^h \Delta^{-h}\, p_{r_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \Delta^h \Delta^{-h}\, p_{r'_{2\nu}}(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \right)\\
&\qquad\qquad \prod_{l=0}^{K} \prod_{q=1}^{m_l} dr_{l,q},
\end{aligned}
$$

where $r_{2\nu}$ and $r'_{2\nu}$ are the rearranged indices $r_{l,2q}$ and $r_{l',2q'}$. Since the variables $x_{\gamma_{2v}}$, $v=1,\dots,n$, occur only in the last line of (H.54), we make the change of variables $x_{\gamma_{2v}} - x_{\gamma_{2v-1}} \to x_{\gamma_{2v}}$ and $x_{\gamma_{2v-1}} \to x_{\gamma_{2v-1}}$ and get that

$$\begin{aligned}
&\int \mathcal{T}_{h,1}(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \qquad\qquad (H.55)\\
&= \int\int_{\mathcal{R}_0\times\cdots\times\mathcal{R}_K} \left(\prod_{l=0}^{K} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right.\\
&\qquad\qquad \left.\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}})\right)\\
&\qquad \left(\prod_{v=1}^{n}\Delta^h\Delta^{-h}\,p_{r_{2\nu}}(x_{\gamma_{2v}})\Delta^h\Delta^{-h}\,p_{r'_{2\nu}}(x_{\gamma_{2v}})\right)\prod_{l=0}^{K}\prod_{q=1}^{m_l} dr_{l,q}\prod_{j,k,i} dx_{j,k,i}.
\end{aligned}$$

Since the variables $x_{\gamma_{2v}}$, $v=1,\dots,n$ occur only in the last line of (H.55) and the variables $x_{\gamma_{2v-1}}$, $v=1,\dots,n$ occur only in the second and third lines of (H.55), we can write (H.55) as

$$\begin{aligned}
&\int \mathcal{T}_{h,1}(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \qquad\qquad (H.56)\\
&= \int_{\mathcal{R}_0\times\cdots\times\mathcal{R}_K}\int \left(\prod_{l=0}^{K} p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right.\\
&\qquad\qquad \left.\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}})\right)\prod_{v=1}^{n} dx_{\gamma_{2v-1}}\\
&\qquad \left(\prod_{v=1}^{n}\int\Delta^h\Delta^{-h}\,p_{r_{2\nu}}(x_{\gamma_{2v}})\Delta^h\Delta^{-h}\,p_{r'_{2\nu}}(x_{\gamma_{2v}})\,dx_{\gamma_{2v}}\right)\prod_{l=0}^{K}\prod_{q=1}^{m_l} dr_{l,q}.
\end{aligned}$$

Note that we also use Fubini's Theorem which is justified since the absolute value of the integrand is integrable, (as we point out in the argument preceding (H.45)). (In the rest of this section use Fubini's Theorem frequently for integrals like (H.56) without repeating the explanation about why it is justified.)

Analogous to (H.42) we note that

$$\begin{aligned}
&\int \mathcal{T}_{h,1}(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \qquad\qquad (H.57)\\
&= \int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\le\sqrt{h}\}}\right)\mathcal{T}_{h,1}(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} + \widetilde{E}_{1,h},
\end{aligned}$$

where $\widetilde{E}_{1,h} = O\left(h^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^n\right)$. The proof of (H.57) is the same as the proof of (H.43).

We now show that

$$E_{2,h} = O\left(\left(\frac{1}{h\psi(1/h)}\right)^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^n\right). \qquad (H.58)$$

To see this note that the terms in $E_{2,h}$ are of the form

$$\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\le\sqrt{h}\}}\right) \qquad (H.59)$$

$$\prod_{l=0}^{K}\int_{\mathcal{R}_l}\widetilde{p}_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})$$
$$\prod_{q=2}^{n_l}\widetilde{p}_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})$$
$$\prod_{q=1}^{n_l}\Delta^h\Delta^{-h}\, p_{r_{l,2q}}(x_{\pi_l(2q)}-x_{\pi_l(2q-1)})\prod_{q=1}^{m_l}dr_{l,q}\prod_{j,k,i}dx_{j,k,i},$$

where $\widetilde{p}_{r_{l,2q-1}}$ is either $p_{r_{l,2q-1}}$ or $\Delta^{h_{l,q}}p_{r_{l,2q-1}}$. Furthermore, at least one of the terms $\widetilde{p}_{r_{l,2q-1}}$ is of the form a $\Delta^{h_{l,q}}p_{r_{l,2q-1}}$.

As in the transition from (H.44) to (H.45) we bound the absolute value of (H.59) by

$$\int\prod_{v=1}^{n}\left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}|\le\sqrt{h}\}}\right)w^2(x_{\gamma_{2v}}-x_{\gamma_{2v-1}}) \tag{H.60}$$
$$\prod_{l=0}^{K}\widetilde{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\prod_{q=2}^{n_l}\widetilde{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\prod_{j,k,i}dx_{j,k,i},$$

where each $\widetilde{u}$ is either of the form u or v, in Lemma G.1, and where, obviously, the h in (G.4) is $h_{l,q}$. Furthermore, we have J terms of the type v, for some $J\ge 1$. It follows from (G.4), the regular variation of ψ and the fact that $|h_{l,q}|\le 2\sqrt{h}$, that

$$v(\,\cdot\,)\le C\left(\frac{1}{h\psi(1/h)}\right)^{1/2}\frac{1}{1+x^2} \tag{H.61}$$

Using this and (G.2) we can bound the integral in (H.60) by

$$C\left(\frac{1}{h\psi(1/h)}\right)^{J/2}\int\prod_{v=1}^{n}w^2(x_{\gamma_{2v}}-x_{\gamma_{2v-1}}) \tag{H.62}$$
$$\prod_{l=0}^{K}\bar{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\prod_{q=2}^{n_l}\bar{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\prod_{j,k,i}dx_{j,k,i}$$

where all the terms $\bar{u}(y)=(1+y^2)^{-1}$.

Since the variables $x_{\gamma_{2\nu}}$, $\nu=1,\ldots,n$, occur only in the w terms in (H.62) and the variables $x_{\gamma_{2v-1}}$, $v=1,\ldots,n$ occur only in the $\overline{u}$ terms in (H.62) , (refer to the change of variables arguments in (H.55) and (H.56)), we can write (H.62) as

$$C\left(\frac{1}{h\psi(1/h)}\right)^{J/2}\int\left(\prod_{l=0}^{K}\overline{u}(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\right. \tag{H.63}$$
$$\left.\prod_{q=2}^{n_l}\overline{u}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})\right)\prod_{v=1}^{n}dx_{\gamma_{2v-1}}\prod_{v=1}^{n}w^2(x_{\gamma_{2v}})\prod_{v=1}^{n}dx_{\gamma_{2v}}.$$

As we have been doing we extract a linearly independent set of variables from the arguments of the $\overline{u}$ terms. The other $\overline{u}$ terms we bound by one. Then we make a change of variables and integrate the remaining $\overline{u}$ terms and the w^2 terms using (G.2) and (G.9). Since $J\ge 1$, we get (H.58).

Since ψ is regularly varying with index $\beta>1$ we see that there exists an $\epsilon>0$ such that

$$E_{1,h}+E_{2,h}+\widetilde{E}_{2,h}=O\left(h^{(2\beta-1)n+\epsilon}\right). \tag{H.64}$$

Therefore, it follows from (H.42), (H.52) and (H.57) that

$$\int \mathcal{T}_h(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \tag{H.65}$$
$$= \int \mathcal{T}_{h,1}(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} + O\left(h^{(2\beta-1)n+\epsilon}\right).$$

Let $\widetilde{\mathcal{R}}_l(s)=\{\sum_{q=1}^{n_l} r_{l,2q-1}\le 1-s\}$ and $\widetilde{\sigma}_l(q):=\gamma_{2\sigma_l(q)-1}$. We define

$$F(\widetilde{\sigma},s_0,\ldots,s_K) \tag{H.66}$$
$$=\int\left(\int_{\widetilde{\mathcal{R}}_0(s_0)\times\cdots\times\widetilde{\mathcal{R}}_K(s_K)}\prod_{l=0}^{K} p_{(1-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1})+r_{l,1}}\right.$$
$$\left.(x_{\widetilde{\sigma}_l(1)}-x_{\widetilde{\sigma}_{l-1}(n_{l-1})})\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\widetilde{\sigma}_l(q)}-x_{\widetilde{\sigma}_l(q-1)})\prod_{q=1}^{n_l} dr_{l,2q-1}\right)dx,$$

where $(1-\sum_{q=1}^{n_{-1}} r_{-1,2q-1}-s_{-1}):=0$ and $\widetilde{\sigma}_{-1}(n_{-1}):=0$. Here the generic term dx indicates integration with respect to all the variables $x_\cdot$ that appear in the integrand.

Since $\widetilde{\sigma}_l(q)=\gamma_{2\sigma_l(q)-1}$ we can also write (H.66) as

$$F(\widetilde{\sigma},s_0,\ldots,s_K) \tag{H.67}$$
$$=\int\left(\int_{\widetilde{\mathcal{R}}_0(s_0)\times\cdots\times\widetilde{\mathcal{R}}_K(s_K)}\prod_{l=0}^{K} p_{(1-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1})+r_{l,1}}\right.$$
$$(x_{\gamma_{2\sigma_l(1)-1}}-x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}}-x_{\gamma_{2\sigma_l(q-1)-1}})$$
$$\left.\prod_{q=1}^{n_l} dr_{l,2q-1}\right)dx,$$

$x_{\gamma_{2\sigma_{-1}(n_{-1})-1}}:=0$.

Consider the mappings $\widetilde{\sigma}_l$ that are used in (H.66). Recall that $\sigma_l(q)$ is defined by the relationship $\{\pi_l(2q-1),\pi_l(2q)\}=\{\gamma_{2\sigma_l(q)-1},\gamma_{2\sigma_l(q)}\}$. Therefore, since $\widetilde{\sigma}_l(q)=\gamma_{2\sigma_l(q)-1}$ we can have that either $\widetilde{\sigma}_l(q)=\pi_l(2q-1)$ or $\widetilde{\sigma}_l(q)=\pi_l(2q)$. However, since the terms $\widetilde{\sigma}_l(q)$ are subscripts of the terms x, all of which are integrated, it is more convenient to define $\widetilde{\sigma}_l$ differently.

Recall that $\mathcal{P}$, (see (H.35)), is a union of pairings $\mathcal{P}_{j,k}$ of the $m_{j,k}$ vertices

$$\{(j,k,i),\,1\le i\le m_{j,k}\}.$$

Each $\mathcal{P}_{j,k}$ consists of $n_{j,k}$ pairs, that can ordered arbitrarily. If $\{\gamma_{2\sigma_l(q)-1},$ $\gamma_{2\sigma_l(q)}\}$ is the i-th pair in $\mathcal{P}_{j,k}$, we set $\widetilde{\sigma}_l(q)=(j,k,i)$. (Necessarily, l will be either j or k, as we point out in the paragraph containing (H.38)). Thus, each $\widetilde{\sigma}_l$ is a bijection from $[1,n_l]$ to

$$\bigcup_{k=l+1}^{K}\{(l,k,i),\,1\le i\le n_{l,k}\}\bigcup_{j=0}^{l-1}\{(j,l,i),\,1\le i\le n_{j,l}\}. \tag{H.68}$$

Let $\widetilde{\mathcal{B}}$ denote the set of $K+1$ tuples, $\widetilde{\sigma}=(\widetilde{\sigma}_0,\ldots,\widetilde{\sigma}_K)$ of such bijections. Note that with this definition of $\widetilde{\sigma}_l(q)$ (H.66) remains unchanged since we have simply renamed the variables of integration.

By interchanging the elements in any of the $2n$ pairs $\{\pi_l(2q-1),\pi_l(2q)\}$ we obtain a new $\pi' \sim \mathcal{P}$. In fact we obtain 2^{2n} different permutations π, in this way, all of which are compatible with $\mathcal{P}$, and all of which give the same $\widetilde{\sigma}$ in (H.66). Furthermore, by permuting the pairs $\{\pi_l(2q-1),\pi_l(2q)\}$, $1\le q\le n_l$, for each l, we get all the possible permutation $\widetilde{\pi}\sim\mathcal{P}$, and these give all possible mappings $\widetilde{\sigma}\in\widetilde{\mathcal{B}}$. Note that $|\widetilde{\mathcal{B}}| = \prod_{l=0}^K n_l! \le (2n)!$.

Consider (H.67). Since $x_{\gamma_{2\sigma_{-1}(n_{-1})-1}}=0$, $x_{\gamma_{2\sigma_0-1}}$ appears alone as the argument of one of the density functions. Therefore we can extract a linearly independent set from the arguments of the densities that spans the space spanned by all the arguments of the densities. We use (G.1) to bound the density functions with arguments that are not in the spanning set by $C\psi^{-1}(1/s)$. We then integrate them with respect to the time variables. Since the time variables are bounded, all this contributes only some constant. With what is left we can make a change of variables and use (G.1) again to see that

$$F(\widetilde{\sigma},s_0,\ldots,s_K)\le C, \tag{H.69}$$

for some constant depending only on m.

Let $\widehat{\mathcal{R}}_l=\{\sum_{q=1}^{n_l} r_{l,2q}\le 1\}$. We break up the integration over $\mathcal{R}_l$ into integration over $\widetilde{\mathcal{R}}_l(s)$ and $\widehat{\mathcal{R}}_l$ in (H.56) and (H.67). If one carefully examines the time indices in (H.31) and (H.66) and uses Fubini's Theorem, one sees that

$$\begin{aligned}
&\int \mathcal{T}_h(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \qquad\qquad (\text{H.70})\\
&\quad=\int_{\widehat{\mathcal{R}}_0\times\cdots\times\widehat{\mathcal{R}}_K} F(\widetilde{\sigma},\sum_{q=1}^{n_0} r_{0,2q},\ldots,\sum_{q=1}^{n_K} r_{K,2q})\\
&\qquad\qquad \prod_{i=1}^n\left(\int\left(\Delta^h\Delta^{-h}\,p_{r_i}(x)\right)\left(\Delta^h\Delta^{-h}\,p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'.
\end{aligned}$$

The variables $\{r_i,r_i'\,|\,i=1,\ldots,n\}$ are simply a relabeling of the variables $\{r_{l,2q}\,|\,0\le l\le K,\,1\le q\le n_l\}$. (The exact form of this relabeling does not matter in what follows.) Here, as always, we set $p_r(x)=0$, if $r\le 0$.

By Parseval's Theorem,

$$\begin{aligned}
&\int\left(\Delta^h\Delta^{-h}\,p_r(x)\right)\left(\Delta^h\Delta^{-h}\,p_{r'}(x)\right)\,dx \qquad\qquad (\text{H.71})\\
&\quad=\frac{1}{2\pi}\int|2-e^{iph}-e^{-iph}|^2e^{-r\psi(p)}e^{-r'\psi(p)}\,dp\ge 0.
\end{aligned}$$

Using this, (H.69) and Fubini's Theorem, we see that

$$\begin{aligned}
&\int_{\left(\widehat{\mathcal{R}}_0\times\cdots\times\widehat{\mathcal{R}}_K\right)\cap([0,\sqrt{h}]^{2n})^c} F(\widetilde{\sigma},\sum_{q=1}^{n_0} r_{0,2q},\ldots,\sum_{q=1}^{n_K} r_{K,2q}) \qquad\qquad (\text{H.72})\\
&\qquad \prod_{i=1}^n\left(\int\left(\Delta^h\Delta^{-h}\,p_{r_i}(x)\right)\left(\Delta^h\Delta^{-h}\,p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'\\
&\quad\le C\int_{([0,\sqrt{h}]^{2n})^c}\prod_{i=1}^n\left(\int\left(\Delta^h\Delta^{-h}\,p_{r_i}(x)\right)\left(\Delta^h\Delta^{-h}\,p_{r_i'}(x)\right)\,dx\right)\prod_{i=1}^n dr_i\,dr_i'\\
&\quad\le C\left(\int\left(\int\left(\Delta^h\Delta^{-h}\,p_r(x)\right)dr\right)^2\,dx\right)^{n-1}
\end{aligned}$$

$$\int \left\{ \int_0^\infty \int_{\sqrt{h}}^\infty \left(\Delta^h \Delta^{-h}\, p_{r_i}(x)\right) \left(\Delta^h \Delta^{-h}\, p_{r_i'}(x)\right)\, dr_i\, dr_i' \right\} dx$$
$$= C c_{\psi,h,1}^{n-1} \int \left\{ \int_0^\infty \int_{\sqrt{h}}^\infty \left(\Delta^h \Delta^{-h}\, p_{r_i}(x)\right) \left(\Delta^h \Delta^{-h}\, p_{r_i'}(x)\right)\, dr_i\, dr_i' \right\} dx,$$

by (4.12). The integral in the final line of (H.72)

$$\leq c_{\psi,h,1} - \int \left(\int_0^{\sqrt{h}} \Delta^h \Delta^{-h}\ p_s(x)\, ds \right)^2 dx. \tag{H.73}$$

Therefore, it follows from Lemma G.2 that the first integral in (H.72) is $O(h^{(2\beta-1)n+\epsilon})$, for some $\epsilon > 0$.

Since $\left(\widehat{\mathcal{R}}_0 \times \cdots \times \widehat{\mathcal{R}}_K\right) \supseteq [0,\sqrt{h}]^{2n}$ for $2n\sqrt{h} \leq 1$, it follows from (H.70) and the preceding sentence, that

$$\int \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{H.74}$$
$$= \int_{[0,\sqrt{h}]^{2n}} F(\widetilde{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}\,, \ldots, \sum_{q=1}^{n_K} r_{K,2q}) \prod_{i=1}^n \left(\int \left(\Delta^h \Delta^{-h}\, p_{r_i}(x)\right)\right.$$
$$\left.\left(\Delta^h \Delta^{-h}\, p_{r_i'}(x)\right)\, dx \right) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q} + O(h^{(2\beta-1)n+\epsilon}).$$

We use the next lemma which is proved in Subsection H.3.

Lemma H.2 *For any fixed m and $s_0, \ldots, s_K \leq m\sqrt{h}$, there exists an $\epsilon > 0$ such that for all $h > 0$, sufficiently small,*

$$|F(\widetilde{\sigma}, s_0, \ldots, s_K) - F(\widetilde{\sigma}, 0, \ldots, 0)| \leq C h^\epsilon. \tag{H.75}$$

Proof of Lemma H.1 continued It follows from (H.74) and Lemmas H.2 and G.2, that

$$\int \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{H.76}$$
$$= F(\widetilde{\sigma}, 0, \ldots, 0) \int_{[0,\sqrt{h}]^{2n}} \prod_{i=1}^n \left(\int \left(\Delta^h \Delta^{-h}\, p_{r_i}(x)\right)\right.$$
$$\left.\left(\Delta^h \Delta^{-h}\, p_{r_i'}(x)\right)\, dx \right) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q} + O(h^{(2\beta-1)n+\epsilon})$$
$$= \left(c_{\psi,h,1}\right)^n F(\widetilde{\sigma}, 0, \ldots, 0) + O(h^{(2\beta-1)n+\epsilon}). \tag{H.77}$$

We now use the notation introduced in the paragraph containing (H.68), and the fact that there are 2^{2n} permutations that are compatible with $\mathcal{P}$, to see that

$$\sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_h(x;\, \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{H.78}$$
$$= \left(4 c_{\psi,h,1}\right)^n \sum_{\widetilde{\sigma} \in \widetilde{\mathcal{B}}} F(\widetilde{\sigma}, 0, \ldots, 0) + O(h^{(2\beta-1)n+\epsilon}).$$

Since $|\widetilde{\mathcal{B}}| \leq (2n)!$, we see that the error term only depends on m, (recall that $m = 2n$). Consider (H.78) and the definition of $F(\widetilde{\sigma}, 0, \ldots, 0)$ in (H.66) and use (H.17), with $m_{j,k}$

replaced by $n_{j,k}$, to see that

$$\sum_{\pi\sim\mathcal{P}}\int \mathcal{T}_h(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \tag{H.79}$$

$$= (4c_{\psi,h,1})^n\, E\left(\prod_{\substack{j,k=0\\ j<k}}^{K}(\alpha_{j,k})^{n_{j,k}}\right) + O(h^{(2\beta-1)n+\epsilon}).$$

Recall the definition of $\mathcal{S}$, to set of special pairings, given in the first paragraph of this subsection. Since there are $\frac{(2n_{j,k})!}{2^{n_{j,k}}n_{j,k}!}$ pairings of the $2n_{j,k}$ elements $\{1,\dots,m_{j,k}\}$, (recall that $m_{j,k}=2n_{j,k}$), we see that when we sum over all the special pairings we get

$$\sum_{\mathcal{P}\in\mathcal{S}}\sum_{\pi\sim\mathcal{P}}\int \mathcal{T}_h(x;\,\pi,e)\prod_{j,k,i} dx_{j,k,i} \tag{H.80}$$

$$= \prod_{\substack{j,k=0\\ j<k}}^{K}\frac{(2n_{j,k})!}{2^{n_{j,k}}n_{j,k}!}\,(4c_{\psi,h,1})^{n_{j,k}}\, E\left\{\prod_{\substack{j,k=0\\ j<k}}^{K}(\alpha_{j,k})^{n_{j,k}}\right\} + O\left(h^{(2\beta-1)n+\epsilon}\right).$$

It follows from (H.37) that the error term, still, only depends on m.

The right-hand side of (H.80) is precisely the desired expression in (H.2). Therefore, to complete the proof of Lemma H.1, we show that for all the other possible values of a, the integral in (H.27) can be absorbed in the error term.

H.2 $\mathrm{a}=\mathrm{e}$ but not all cycles are of order two or $\mathrm{a}\neq\mathrm{e}$

Lemma H.3 *Suppose that $a=e$ but not all cycles are of order two or $a\neq e$. Then*

$$\int \mathcal{T}_h(x;\,\pi,a)\prod_{j,k,i} dx_{j,k,i} = O\left(\frac{h^{\epsilon}}{h\psi^2(1/h)}\right)^n, \tag{H.81}$$

for some $\epsilon>0$.

In the rest of this section we ignore all factors of $\log 1/h$.

Proof Consider the basic formula (H.31). Since we only need an upper bound, we take absolute values in the integrand and extend the time integral to $[0,1]$, as we have done several times above. We take the time integral and get an upper bound for (H.31) involving the terms u, v and w. Since $a\neq e$, the number of w terms is less than $2n$.

We obtain (H.81) by dividing the u, v and w terms in $\mathcal{T}_h(x;\,\pi,a)$ into sets. Clearly, if a set contains k terms of the form w and k' terms of the form v, there are $2k+k'$ difference operators $\Delta^h_{\cdot}$ associated with this set. There are no difference operators associated with sets of u terms.

Consider a set of two w terms that lies in a cycle of order two. There are four difference operators $\Delta^h_{\cdot}$ associated with this set. We show this set contributes a bound to (H.81) that is

$$O\left(\frac{1}{h\psi^2(1/h)}\right). \tag{H.82}$$

(By contributes a bound we mean that this is what we get after we make an appropriate change of variables and integrate out the w terms in this set.) Thus we may say that each difference operator in a cycle of order two contributes a bound of

$$O\left(\left(\frac{1}{h\psi^2(1/h))}\right)^{1/4}\right). \tag{H.83}$$

We show that any set that has $k>0$ associated difference operators except for a set of two w terms that forms a cycle of order two contributes a bound that is

$$O\left(\left(\frac{1}{h\psi^2(1/h))}\right)^{k/4}\right)h^{\epsilon}, \tag{H.84}$$

for some $\epsilon>0$.

There are $4n$ difference operators $\Delta^h_{\cdot}$, in $\mathcal{T}_h(x;\pi,a)$. Consequently unless the graph associated with $\mathcal{T}_h(x;\pi,a)$ consists solely of cycles of order two, we obtain (H.81).

As we construct the sets of u, v and w terms, we also choose a collection $\mathcal{I}\cup\mathcal{I}'$ of m terms with arguments that are linearly independent. To bound the contribution of each set we bound all the terms not in $\mathcal{I}\cup\mathcal{I}'$ by their supremum, and, after changing variables, integrate the terms in $\mathcal{I}\cup\mathcal{I}'$. Using (G.4), (G.5), (G.7) and (G.8) we verify the bounds given in the preceding paragraph. (Actually, there is an exceptions to this rule which we also deal with.)

This is how we divide the u, v and w terms into sets. For each π and a we define a multigraph $G_{\pi,a}$ with vertices $\{(j,k,i),\ 0\le j<k\le K,\ 1\le i\le m_{j,k}\}$, and an edge between the vertices $\pi_l(q-1)$ and $\pi_l(q)$ whenever $a(l,q)=(1,1)$. This graph divides the w terms into cycles and chains. Assume that there are S cycles. We denote them by $C_s=\{\phi_{s,1},\ldots,\phi_{s,l(s)}\}$, written in cyclic order, where the cycle length $l(s)=|C_s|\ge 1$ and $1\le s\le S$. For each $1\le s\le S$ we take the set of $l(s)$ terms

$$\mathcal{G}_s^{\text{cycle}}=\{w(x_{\phi_{s,2}}-x_{\phi_{s,1}}),\cdots,w(x_{\phi_{s,l(s)}}-x_{\phi_{s,l(s)-1}}),w(x_{\phi_{s,1}}-x_{\phi_{s,l(s)}})\}. \tag{H.85}$$

Let

$$y_{\phi_{s,i}}=x_{\phi_{s,i}}-x_{\phi_{s,i-1}},\qquad i=2,\ldots,l(s). \tag{H.86}$$

It is easy to see that $\{y_{\phi_{s,i}}\,|i=2,\ldots,l(s)\}$, are linearly independent. We put the corresponding w terms, $w(x_{\phi_{s,2}}-x_{\phi_{s,1}}),\cdots,w(x_{\phi_{s,l(s)}}-x_{\phi_{s,l(s)-1}})$ into $\mathcal{I}$. (On the other hand, since

$$\sum_{i=2}^{l(s)}y_{\phi_{s,i}}=-(x_{\phi_{s,1}}-x_{\phi_{s,l(s)}}), \tag{H.87}$$

we see that we can only extract $l(s)-1$ linearly independent variables from the $l(s)$ arguments of w for a given s.) A cycle of length 1 consists of a single point $\phi_{s,1}=\phi_{l(s),1}$ in the graph, so in this case

$$\mathcal{G}_s^{\text{cycle}}=\{w(0)\}. \tag{H.88}$$

We explain below how this can occur. Obviously, $w(0)$ is not put into $\mathcal{I}$.

Next, suppose there are S' chains. We denote them by $C'_s=\{\phi'_{s,1},\ldots,\phi'_{s,l'(s)}\}$, written in order, where $l'(s)=|C'_s|\ge 2$ and $1\le s\le S'$. Note that there are $l'(s)-1$, w terms corresponding to C'_s. Then for each $1\le s\le S'$ we form the set of $l'(s)+1$ terms

$$\begin{aligned}\mathcal{G}_s^{\text{chain}}=\{&v(x_{\phi'_{s,1}}-x_{a(s)}),w(x_{\phi'_{s,2}}-x_{\phi'_{s,1}}),\cdots,\\&\cdots,w(x_{\phi'_{s,l(s)}}-x_{\phi'_{s,l(s)-1}}),v(x_{b(s)}-x_{\phi'_{s,l(s)}})\}\end{aligned} \tag{H.89}$$

where $v(x_{\phi'_{s,1}}-x_{a(s)})$ is the unique v term associated with $\Delta_h^{x_{\phi'_{s,1}}}$, and similarly, $v(x_{b(s)}-x_{\phi'_{s,l(s)}})$ is the unique v term associated with $\Delta_h^{x_{\phi'_{s,l(s)}}}$. (This deserves further clarification. There may be other v terms containing the variable $x_{\phi'_{s,1}}$. But there is only one v term of the form

$$\int_0^1\left|\Delta_h^{x_{\phi'_{s,1}}}p_s(x_{\phi'_{s,1}}-u)\right|ds \tag{H.90}$$

where u is some other $x_{\cdot}$ variable which we denote by $x_{a(s)}$. This is because one operator $\Delta_h^{x_{\phi'_{s,1}}}$ is associated with $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}})$ and there are precisely two operators $\Delta_h^{x_{\phi'_{s,1}}}$ in (H.81)).

It is easy to see that variables $y_{\phi'_{s,i}} = x_{\phi'_{s,i}} - x_{\phi'_{s,i-1}}$, $i = 2, \ldots, l(s)$, are linearly independent. We put the w terms, $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}}), \cdots, w(x_{\phi'_{s,l(s)}} - x_{\phi'_{s,l(s)-1}})$ into $\mathcal{I}$. We leave the v terms in $\mathcal{G}_s^{\text{chain}}$ out of $\mathcal{I}$.

At this stage we emphasize that the terms we have put in $\mathcal{I}$ from all cycles and chains have linearly independent arguments. If fact, the set of x's appearing in the different chains and the cycles are disjoint. This is obvious for the cycles and the interior of the chains since there are exactly two difference operators Δ_h^x for each x. It also must be true for the endpoints of the chains, since if this is not the case they could be made into larger chains or cycles.

For the same reason, if a v term involving $\Delta_{x'}^h$ is not in any of the sets of chains, then x' will not appear in the arguments of the terms that are put in $\mathcal{I}$ from all the cycles and chains.

Suppose, after considering the w terms and the v terms associated with the chains of w terms, that there are p pairs of v terms left, each pair corresponding to difference operators $\Delta_{z_j}^h$, $j = 1, \ldots, p$. (p may be 0). Let

$$\mathcal{Z} := \{z_1, \ldots, z_p\} \tag{H.91}$$

A typical v term is of the form

$$v^{(j)}(z_j - u_{j'}) := v(z_j - u_{j'}) = \int_0^1 |\Delta_{z_j}^h p_t(z_j - u_{j'})| \, dt. \tag{H.92}$$

where $u_{j'}$ is some $x_{\cdot}$ term. We use the superscript (j) is to keep track of the fact that this v term is associated with the difference operator $\Delta_{z_j}^h$. We distinguish between the variables z_j and $u_{j'}$ by referring to z_j as a marked variable. Note that if $u_{j'}$ is also in $\mathcal{Z}$, say $u_{j'} = z_k$, then $u_{j'}$ is also a marked variable but in a different v term. (In this case, in $v^{(k)}(z_k - u_{k'})$, where $u_{k'}$ is some other $x_{\cdot}$ variable.)

Thus $\mathcal{Z}$ is the collection of marked variables. Consider the corresponding terms

$$v^{(j)}(z_j - u_j) \quad \text{and} \quad v^{(j)}(z_j - v_j), \qquad j = 1, \ldots, p \tag{H.93}$$

where u_j and v_j represent whatever terms $x_{\cdot}$ and $x'_{\cdot}$ are coupled with the two variables z_j.

There may be some j for which u_j and v_j in (H.93) are both in $\mathcal{Z}$. Choose such a j. Suppose $u_j = v_j = z_k$. We set

$$\begin{aligned} \mathcal{G}_j^{\mathcal{Z},1} = \{ & v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_k), \\ & v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k)\} \end{aligned} \tag{H.94}$$

and put $v^{(j)}(z_j - z_k)$ into $\mathcal{I}$. Here u_k and v_k are whatever two variables appear with the two marked variables z_k.

On the other hand, suppose u_j and v_j are both in $\mathcal{Z}$ but $u_j = z_k$ and $v_j = z_l$ with $k \neq l$. We set

$$\begin{aligned} \mathcal{G}_j^{\mathcal{Z},2} = \{ & v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_l), \\ & v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k), v^{(l)}(z_l - u_l), v^{(l)}(z_l - v_l)\} \end{aligned} \tag{H.95}$$

and put both $v^{(j)}(z_j - z_k)$ and $v^{(j)}(z_j - z_l)$ into $\mathcal{I}$.

We then turn to the elements in $\mathcal{Z}$ which have not yet appeared in the arguments of the terms that have been put into $\mathcal{I}$. If there is another j' for which $u_{j'}$ and $v_{j'}$ are both in $\mathcal{Z}$, choose such a j' and proceed as above. If there are no longer any such elements in $\mathcal{Z}$, choose some remaining element, say, z_i. Set

$$\mathcal{G}_i^{\mathcal{Z},3} = \{v^{(i)}(z_i - u_i), v^{(i)}(z_i - v_i)\} \tag{H.96}$$

and if $u_i \notin \mathcal{Z}$, place $v^{(i)}(z_i - u_i)$ into $\mathcal{I}$. If $u_i \in \mathcal{Z}$, so that $v_i \notin \mathcal{Z}$, place $v^{(i)}(z_i - v_i)$ into $\mathcal{I}$.

We then continue until we have exhausted $\mathcal{Z}$. We form a final set $\mathcal{G}^u$ which contains all the u terms, so that all u, v and w terms have been divided into sets.

It is possible that there are no cycles of length one. We show how we get (H.81) in this case.

We have constructed $\mathcal{I}$ so that all its members have linearly independent arguments. However, $\mathcal{I}$ may contain less than m terms. We simply add to $\mathcal{I}$ a set $\mathcal{I}'$ of enough of the remaining u and v terms so that $\mathcal{I} \cup \mathcal{I}'$ has m terms, whose arguments span R^{2n}, the space spanned by the original $x_\cdot$ terms. (It follows from (H.87) that no further w terms can be added to $\mathcal{I}'$). We bound the v terms in $\mathcal{I}'$ as follows:

$$|v(x' - x'')| \leq \frac{C}{h\psi(1/h)(1 + (x' - x'')^2)}. \tag{H.97}$$

We then make a change of variables setting the arguments of the terms in $\mathcal{I} \cup \mathcal{I}'$ equal to $y_1, \ldots, y_m$ and bound the v terms not in $\mathcal{I} \cup \mathcal{I}'$ by $C(h\psi(1/h))^{-1}$ and the u terms not in $\mathcal{I} \cup \mathcal{I}'$ by C. Finally we integrate. We have m one dimensional integrals which we bound by (G.5) for the v terms in $\mathcal{I}$, by $C(h\psi(1/h))^{-1}$ for the v terms in $\mathcal{I}'$, and by (G.8) for w terms in $\mathcal{I}$. The integrals of the u terms in $\mathcal{I}$ we bound by a constant; (see (G.2)).

Clearly $\mathcal{G}^u$ gives a bounded contribution. We now show that (H.84) holds for all other sets of v and w terms, with the exception of sets of w terms in cycles of length 2.

Consider first $\mathcal{G}_s^{\text{cycle}}$ for a cycle of lengths $l(s)$. We integrate the $l(s) - 1$, w terms which were put in $\mathcal{I}$ and bound the remaining w term by $C(h\psi(1/h))^{-1}$ to obtain the bound

$$C\left(\frac{1}{\psi(1/h)}\right)^{l(s)-1} \frac{1}{h\psi(1/h)} = C\left(\frac{1}{\psi(1/h)}\right)^{l(s)-2} \frac{1}{h\psi^2(1/h)}. \tag{H.98}$$

Since

$$\frac{1}{\psi(1/h)} = h^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{1/2} \tag{H.99}$$

(H.98) is bounded by

$$C\Big\{h^{(l(s)-2)/2}\Big\}\left(\frac{1}{h\psi^2(1/h)}\right)^{l(s)/2}. \tag{H.100}$$

Since a cycle of length $l(s)$ involves $2l(s)$ difference operators Δ_h, and $l(s)/2 = 2l(s)/4$, we are in the situation of (H.84), unless all cycles are of order two. (This shows, incidentally, that when $a = e$, (H.81) holds unless all cycles are of order two.)

Consider next $\mathcal{G}_s^{\text{chain}}$. Recall that there are $l'(s) - 1$, w terms in a chain, where $l'(s) \geq 2$. We have put all $l'(s) - 1$ terms w in $\mathcal{I}$, and we can bound their integrals by

$$C\left(\frac{1}{\psi(1/h)}\right)^{l'(s)-1}. \tag{H.101}$$

In addition there are two v terms in $\mathcal{G}_s^{\text{chain}}$. The ones not in $\mathcal{I}'$ can be bounded by $C(h\psi(1/h))^{-1}$ and the ones in $\mathcal{I}'$ are bounded by (H.97), which after integration also contributes $C(h\psi(1/h))^{-1}$. Thus we obtain the following bound for for $\mathcal{G}_s^{\text{chain}}$:

$$\begin{aligned} &C\left(\frac{1}{\psi(1/h)}\right)^{l'(s)-1}\left(\frac{1}{h\psi(1/h)}\right)^2 \\ &\quad = C\left(\frac{1}{\psi(1/h)}\right)^{l'(s)-3}\left(\frac{1}{h\psi^2(1/h)}\right)^2 \\ &\quad \leq Ch^{(l'(s)-3)/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{l'(s)/2}. \end{aligned} \tag{H.102}$$

Note that each chain of length $l'(s)$ together with the two v terms associated with the end points involves $2l'(s)$ difference operators Δ_h. Clearly if $l'(s) \geq 3$ we are in the situation of (H.84). This holds even for chains of length $l'(s) = 2$ since

$$h^{(2-3)/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{1/2} = \frac{1}{h\psi(1/h)}. \tag{H.103}$$

Note that the v terms that were not initially in $\mathcal{I}$ contribute a bound of $C(h\psi(1/h))^{-1}$, whether or not they are placed in $\mathcal{I}'$. We continue to use this fact below without commenting on it further.

We next consider $\mathcal{G}_j^{\mathcal{Z},1}$. We integrate the one v term in $\mathcal{I}$ and any that are in $\mathcal{I}'$ and bound the remaining ones. This gives a bound of

$$\frac{1}{h^2\psi^3(1/h)} = \frac{1}{h\psi(1/h)}\left(\frac{1}{h\psi^2(1/h)}\right). \tag{H.104}$$

Since $\mathcal{G}_j^{\mathcal{Z},1}$ involves four $\Delta^h_\cdot$ operators we are in the situation of (H.84).

For $\mathcal{G}_j^{\mathcal{Z},2}$ we integrate two v terms in $\mathcal{I}$ and any that are in $\mathcal{I}'$ and bound the remaining ones. This gives a bound of

$$\left(\frac{1}{h\psi^2(1/h)}\right)^2 = \left(\frac{1}{h\psi^2(1/h)}\right)^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{3/2}; \tag{H.105}$$

Since $\mathcal{G}_j^{\mathcal{Z},2}$ involves six $\Delta^h_\cdot$ operators we are in the situation of (H.84).

Finally, for $\mathcal{G}_j^{\mathcal{Z},3}$ we integrate the one v term in $\mathcal{I}$ and the other if it is in $\mathcal{I}'$. Otherwise we bound it. This gives a bound of

$$\frac{1}{\psi(1/h)} = h^{1/2}\left(\frac{1}{h\psi^2(1/h)}\right)^{1/2}; \tag{H.106}$$

Since $\mathcal{G}_j^{\mathcal{Z},3}$ involves two $\Delta^h_\cdot$ operators we are in the situation of (H.84).

This shows that if a and the partition π does not generate exclusively w terms in cycles of order two and are such that there are no cycles of length one, then (H.81) holds.

We now remove the restriction that a and π does not give rise to cycles of length one. The only way this anomaly can occur is in terms of the type

$$\Delta^h\Delta^{-h}p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}}) \tag{H.107}$$

when $\gamma_{2\sigma_l(1)-1} = \gamma_{2\sigma_{l-1}(n_{l-1})-1}$. Note that in this case

$$\int_0^t \Delta^h\Delta^{-h}p_s(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}})\,ds \quad = \quad w(0). \tag{H.108}$$

This is what we call a cycle of length one. In this case we have

$$\Delta^h\Delta^{-h}p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(0)=-2\Delta^h p_{(1-\sum_{q=1}^{m_{l-1}} r_{l-1,q})+r_{l,1}}(0). \tag{H.109}$$

We now show how to deal with (H.109). We return to the basic formulas (H.30) and (H.31). We obtain an upper bound for (H.31) by taking the absolute value of the integrand. However, we do not, initially extend the region of integration with respect to time. Instead we proceed as follows: Let l' be the largest value of l for which (H.109) occurs. We extend the integral with respect to $r_{l,q}$ for all $l>l'$, and also for $l=l'$ and $q>1$, and bound these integrals with terms of the form u, v and w. We then consider the integral of the term in (H.109) with respect to $r_{l',1}$.

Clearly

$$\int_0^1 |\Delta^h p_{(1-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q})+r_{l',1}}(0)|\,dr_{l',1}\le\int_{1-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}}^{2-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}}|\Delta^h p_s(0)|\,ds \tag{H.110}$$

If $\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}\le 1/2$ this last integral

$$\le\int_{1/2}^2|\Delta^h p_s(0)|\,ds\le Ch^2 \tag{H.111}$$

by (G.14). Since we have only used two $\Delta^h_{\cdot}$ operators we are in the situation of (H.84).

If $\sum_{q=1}^{m_{l'-1}} r_{l'-1,q}\ge 1/2$ then for some q' we have $r_{l'-1,q'}\ge 1/2m$. Note that the variable $r_{l'-1,q'}$ appears in (H.107) and in only one other term. If $q'>1$, then using the fact that $r_{l'-1,q'}\ge 1/2m$, we use one of the bounds in Lemma G.5, to bound a term which in the non-exceptional case would be u, v or w, or their integrals with respect to x, by $\overline{u}_{1/2m}$, $\overline{v}_{1/2m}$ or $\overline{w}_{1/2m}$, or their integrals with respect to x. One sees from Lemma G.1 that we don't loose anything in comparison with the non-exceptional case. The case $r_{l'-1,1}\ge 1/2m$ and $\gamma_{2\sigma_{l'-1}(1)-1}\neq\gamma_{2\sigma_{l'-2}(n_{l'-2})-1}$ is handled the same way.

On the other hand if $r_{l'-1,1}\ge 1/2m$ and $\gamma_{2\sigma_{l'-1}(1)-1}=\gamma_{2\sigma_{l'-2}(n_{l'-2})-1}$, we use Lemma G.4 to get the same bound of Ch^2.

After completing the procedure described in the previous two paragraphs we integrate in (H.31) with respect to $r_{l'-1,q'}$ and $r_{l',1}$, since these variables now appear only in the term in (H.107). What we are left with is bounded by

$$\int_{1/2m}^{1-\sum_{q\neq q'} r_{l'-1,q}}\int_0^1|\Delta^h p_{(1-\sum_{q=1}^{m_{l'-1}} r_{l'-1,q})+r_{l',1}}(0)|\,dr_{l',1}\,dr_{l'-1,q'} \tag{H.112}$$

Let $\alpha=1-\sum_{q\neq q'} r_{l'-1,q}$. We make the change of variables $r=r_{l',1}$ and $s=-r_{l'-1,q'}+\alpha$ to get that (H.112)

$$\begin{aligned}&\le\int_0^1\int_0^1|\Delta^h p_{r+s}(0)|\,dr\,dsw \qquad\qquad (H.113)\\&\le\int_0^2 r|\Delta^h p_r(0)|\,dr\le C\int_0^2 r\left(\int\sin^2(ph)e^{-r\psi(p)}\,dp\right)dr\\&\le Ch^\beta\int_0^2 r\left(\int p^\beta e^{-r\psi(p)}\,dp\right)dr\\&\le Ch^\beta\int\frac{p^\beta}{1+\psi^2(p)}\,dp=O(h^\beta).\end{aligned}$$

Since

$$h^\beta=h^{\beta+1/2}\psi(1/h)\left(\frac{1}{h\psi^2(1/h)}\right)^{1/2} \tag{H.114}$$

we are once again in the situation of (H.84).

We then apply a similar procedure for each l in decreasing order, skipping those for which (H.109) occurs, if they were already bounded by the procedure described in the paragraph preceding the one containing (H.112). Thus we see that cycles of length one are in the situation of (H.84). We proceed to deal with remaining terms as we did when we assumed that there were no cycles of length one and see that (H.81) holds. This completes the proof of Lemma H.3. □

It follows from (H.80) and Lemma H.3 that when m is even

$$\sum_{a}\sum_{\pi_0,\ldots,\pi_K}\int \mathcal{T}_h(x;\,\pi,a)\prod_{j,k,i} dx_{j,k,i} \tag{H.115}$$

$$= \prod_{\substack{j,k=0\\ j<k}}^{K} \frac{(2n_{j,k})!}{2^{n_{j,k}} n_{j,k}!} \left(4c_{\psi,h,1}\right)^{n_{j,k}} E\left\{\prod_{\substack{j,k=0\\ j<k}}^{K} \left(\alpha_{j,k}\right)^{n_{j,k}}\right\} + O\left(h^{(2\beta-1)n+\epsilon}\right).$$

We now show that we get the same estimates when $\mathcal{T}_h(x;\,\pi,a)$ is replaced by $\mathcal{T}'_h(x;\,\pi,a)$; (see (H.28) and (H.30)).

We point out, in the paragraph containing (H.26) that terms of the form $\Delta^h\Delta^{-h}p^\sharp_\cdot$ in (H.28) are always of the form $\Delta^h\Delta^{-h}p_\cdot$. Therefore, in showing that (H.28) and (H.30) have the same asymptotic behavior as $h\to 0$ we need only consider how the proof of (H.115) must be modified when the arguments of the density functions with one or no difference operators applied is effected by adding $\pm h$.

It is easy to see that the presence of these terms has no effect on the integrals that are $O\left(h^{(2\beta-1)n+\epsilon}\right)$ as $h\to 0$. This is because in evaluating these expressions we either integrate over all of R^1 or else use bounds that hold on all of R^1. Since terms with one difference operator only occur in these estimations, we no longer need to be concerned with them.

Consider the terms with no difference operators applied to them, now denoted by $p^\sharp$. So, for example, instead of $F(\widetilde{\sigma},0,\ldots,0)$ on the right-hand side of (H.76), we now have

$$\int\left(\int_{\widetilde{\mathcal{R}}_0(0)\times\cdots\times\widetilde{\mathcal{R}}_K(0)}\prod_{l=0}^{K} p^\sharp_{(1-\sum_{q=1}^{n_{l-1}} r_{l-1,2q-1}-s_{l-1})+r_{l,1}} \right. \tag{H.116}$$

$$\left. (x_{\widetilde{\sigma}_l(1)}-x_{\widetilde{\sigma}_{l-1}(n_{l-1})})\prod_{q=2}^{n_l} p^\sharp_{r_{l,2q-1}}(x_{\widetilde{\sigma}_l(q)}-x_{\widetilde{\sigma}_l(q-1)})\prod_{q=1}^{n_l} dr_{l,2q-1}\right) dx.$$

Suppose that $p^\sharp_r(y_{\sigma(i)}-y_{\sigma(i-1)})=p_r(y_{\sigma(i)}-y_{\sigma(i-1)}\pm h)$. We write this term as

$$p^\sharp_r(y_{\sigma(i)}-y_{\sigma(i-1)})=p_r(y_{\sigma(i)}-y_{\sigma(i-1)})+\Delta^{\pm h}p_r(y_{\sigma(i)}-y_{\sigma(i-1)}). \tag{H.117}$$

Substituting all such terms into (H.116) and expanding we get (H.115) and many other terms with at least one $p_r(y_{\sigma(i)}-y_{\sigma(i-1)})$ replaced by $\Delta^{\pm h}p_r(y_{\sigma(i)}-y_{\sigma(i-1)})$. In this case simply take these terms, extend their integrals to $[0,1]$ and bound them as in (G.4). Then follow the procedure in the paragraph containing (H.69) to deal with the remaining terms and the functions $1/(1+(y_{\sigma(i)}-y_{\sigma(i-1)})^2)$. In this the integral in (H.116) is bounded by $C(1/(h\psi(1/h)))^j$, where j is the number of terms that have the difference operator applied. Thus we see that replacing $\mathcal{T}_h(x;\,\pi,a)$ by $\mathcal{T}'_h(x;\,\pi,a)$ does not change (H.115) when m is even.

When m is odd we can not construct a graph with all cycles of order 2. Therefore, we are not in the situation covered by Section H.1. Moreover, in Section H.2 we never use the fact that m is even. We actually obtain (H.81) with n replaced by $m/2$, which is what we assert in (H.3). This also holds when when $p.$ is replaced by $p^\sharp_.$ for the reasons given in the preceding two paragraphs. □

H.3 Proof of Lemma H.2

For any $A \subseteq [0,3]^n$ we set

$$F_A = \int \left\{ \int_A \prod_{l=0}^{K} p_{r_{l,1}}(x_{\widetilde{\sigma_l(1)}} - x_{\widetilde{\sigma_{l-1}(n_{l-1})}}) \right. \\ \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\widetilde{\sigma_l(q)}} - x_{\widetilde{\sigma_l(q-1)}}) \prod_{l=0}^{K}\prod_{q=1}^{n_l} dr_{l,2q-1} \right\} \prod_{q=1}^{n_l} dx_{\widetilde{\sigma_l(q)}}. \tag{H.118}$$

Then by Hölder's inequality, for any $1/a + 1/b = 1$

$$\begin{aligned} &\left\{ \int_A \prod_{l=0}^{K} p_{r_{l,1}}(x_{\widetilde{\sigma_l(1)}} - x_{\widetilde{\sigma_{l-1}(n_{l-1})}}) \right. \\ &\qquad \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\widetilde{\sigma_l(q)}} - x_{\widetilde{\sigma_l(q-1)}}) \prod_{l=0}^{K}\prod_{q=1}^{n_l} dr_{l,2q-1} \right\} \qquad (\text{H.119}) \\ &\le |A|^{1/a} \left\{ \int_{[0,3]^n} \prod_{l=0}^{K} p^b_{r_{l,1}}(x_{\widetilde{\sigma_l(1)}} - x_{\widetilde{\sigma_{l-1}(n_{l-1})}}) \right. \\ &\qquad \left. \prod_{q=2}^{n_l} p^b_{r_{l,2q-1}}(x_{\widetilde{\sigma_l(q)}} - x_{\widetilde{\sigma_l(q-1)}}) \prod_{l=0}^{K}\prod_{q=1}^{n_l} dr_{l,2q-1} \right\}^{1/b}, \end{aligned}$$

where $|A|$ denotes the volume of A in R^n_+.

Since $\beta > 1$ we can choose a $1 < b < \beta$ such that

$$\int_0^3 \left(\psi^{-1}(1/s)\right)^b \, ds \le C. \tag{H.120}$$

Therefore it follows from (G.1) that

$$\int_0^3 p^b_r(x)\, dr \le C\frac{1}{1+x^2}. \tag{H.121}$$

Thus there exists a finite constant C, depending only on n and b, that is independent of A, such that

$$F_A \le C|A|^{1/a}. \tag{H.122}$$

It follows from (H.66), paying special attention to the time variable of $p.$ in the second line, that

$$F(\sigma, s_0, \ldots, s_K) = F_{A_{s_0,\ldots,s_K}} \tag{H.123}$$

where

$$\begin{aligned} A_{s_0,\ldots,s_K} \quad = \quad & \left\{ r \in R^n_+ \,\middle|\, \sum_{\lambda=0}^{l-1}(1 - \sum_{q=1}^{n_\lambda} r_{\lambda,2q-1} - s_\lambda) \le \sum_{q=1}^{n_l} r_{l,2q-1} \right. \qquad (\text{H.124}) \\ & \left. \le \sum_{\lambda=0}^{l-1}(1 - \sum_{q=1}^{n_\lambda} r_{\lambda,2q-1} - s_\lambda) + (1 - s_l);\; l = 0,1,\ldots,K \right\}. \end{aligned}$$

In particular

$$\begin{aligned} A_{0,\ldots,0} \quad = \quad & \left\{ r \in [0,3]^n \,\middle|\, \sum_{\lambda=0}^{l-1}(1-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1}) \le \sum_{q=1}^{n_l} r_{l,2q-1} \right. \\ & \left. \le \sum_{\lambda=0}^{l-1}(1-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})+1);\ l=0,1,\ldots,K \right\}. \end{aligned} \tag{H.125}$$

Let $\phi_l(r) = \sum_{\lambda=0}^{l}(1-\sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})$. We have

$$\begin{aligned} & A_{s_0,\ldots,s_K} \,\Delta\, A_{0,\ldots,0} \\ & \subseteq \bigcup_{l=1}^{K} \left\{ r \in [0,3]^n \,\middle|\, \phi_{l-1}(r) - \sum_{\lambda=0}^{l-1} s_\lambda \le \sum_{q=1}^{n_l} r_{l,2q-1} \le \phi_{l-1}(r) \right\} \\ & \quad \bigcup_{l=0}^{K} \left\{ r \in [0,3]^n \,\middle|\, \phi_{l-1}(r)+1 - \sum_{\lambda=0}^{l} s_\lambda \le \sum_{q=1}^{n_l} r_{l,2q-1} \le \phi_{l-1}(r)+1 \right\}. \end{aligned} \tag{H.126}$$

(Note that the first union are the points in $A_{s_0,\ldots,s_K}$ that are not in $A_{0,\ldots,0}$ and the second union are the points in $A_{0,\ldots,0}$ that are not in $A_{s_0,\ldots,s_K}$.)

Since for fixed $a \ge b \ge 0$

$$\left| \left\{ r \in [0,3]^{n_l} \,\middle|\, a-b \le \sum_{q=1}^{n_l} r_{l,2q-1} \le a \right\} \right| \le C b^{n_l} \tag{H.127}$$

we have that

$$\begin{aligned} |A_{s_0,\ldots,s_K} \,\Delta\, A_{0,\ldots,0}| \quad & \le \quad CK \left(\sum_{\lambda=0}^{K} s_\lambda \right)^{2n} \\ & \le \quad CK^{m+1} (\max_{0\le\lambda\le K} s_\lambda)^m, \end{aligned} \tag{H.128}$$

when $\max_{0\le\lambda\le K} s_\lambda$ is sufficiently small. Let K' be the cardinality of $\{l \,|\, n_l > 0\}$. It is easy to see that we have actually proved (H.128) with K replaced by K'. Since $K' \le m$, the last line in (H.128) can be written in terms of $\{s_\lambda\}$ and m. Lemma H.2 follows from (H.122) and (H.128). □

I Proof of Lemmas 4.1–4.3

Proof of Lemma 4.1 Using the multinomial theorem we have

$$E\left(\left(\widetilde{J}_{l,h}\right)^m\right) = \sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)} \right) E\left(\prod_{\substack{j,k=0\\ j<k}}^{l-1} (J_{j,k,l,h})^{m_{j,k}} \right), \tag{I.1}$$

where

$$\mathcal{M} = \left\{ \widetilde{m} = \{m_{j,k}, 0 \le j < k \le l-1\} \,\middle|\, \sum_{\substack{j,k=0\\ j<k}}^{l-1} m_{j,k} = m \right\}. \tag{I.2}$$

We now use Lemma H.1 to compute the expectation on the right-hand side of (I.1). Even though Lemma H.1 is proved for time intervals of length 1, (see H.1), it is straight forward to check that it holds for any fixed time interval, if the term $\alpha_{j,k}$, in $(H.1)$, is altered to reflect the new length. Therefore, for some $\epsilon > 0$

$$\begin{aligned} & E\left(\left(\widetilde{J}_{l,h}\right)^m\right) \\ &= \sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) \prod_{\substack{j,k=0\\ j<k}}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)} \left(4c_{\psi,h,1}\right)^{n_{j,k}} E\left\{\prod_{\substack{j,k=0\\ j<k}}^{l-1} \left(\alpha_{j,k,l}\right)^{n_{j,k}}\right\} \\ &\qquad + O\left(l^m h^{(2\beta-1)n+\epsilon}\right). \end{aligned} \tag{I.3}$$

when $m_{j,k} = 2n_{j,k}$ for all j and k, and is $O\left(l^m h^{(2\beta-1)n+\epsilon}\right)$ if any of the $m_{j,k}$ are odd. Here we use the fact that

$$\sum_{\widetilde{m}\in\mathcal{M}} \left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) = l^m \tag{I.4}$$

to compute the error term. (Lemma H.1 is for a fixed partition of m. Here we include the factor l^m, to account for the number of possible partitions. Note that l is a function of h.)

When $m_{j,k} = 2n_{j,k}$ for all j and k,

$$\left(\frac{m!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(m_{j,k}!)}\right) \prod_{\substack{j,k=0\\ j<k}}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)} = \frac{(2n)!}{2^n n!} \frac{n!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1}(n_{j,k}!)}. \tag{I.5}$$

Using this in (I.3) we get

$$\begin{aligned} & E\left(\left(\widetilde{J}_{l,h}\right)^m\right) \\ &= \frac{(2n)!}{2^n n!} \left(4c_{\psi,h,1}\right)^n \sum_{\mathcal{N}} \left(\frac{n!}{\prod_{\substack{j,k=0\\ j<k}}^{l-1} n_{j,k}!}\right) E\left\{\prod_{\substack{j,k=0\\ j<k}}^{l-1} \left(\alpha_{j,k,l}\right)^{n_{j,k}}\right\} \\ &\qquad + O\left(l^m h^{(2\beta-1)n+\epsilon}\right), \end{aligned} \tag{I.6}$$

where $\mathcal{N}$ is defined similarly as $\mathcal{M}$. Using the multinomial theorem as in (I.1) we see that the sum in (I.6) is equal to $E\left\{\left(\widetilde{\alpha}_l\right)^n\right\}$, which completes the proof of (4.11). $\square$

Proof of Lemma 4.2 By the Kac Moment Formula

$$\begin{aligned} E\left\{(\alpha_t)^n\right\} &= E\left(\left(\int (L_t^x)^2\, dx\right)^n\right) \\ &= 2^n \sum_{\pi} \int \int_{\{\sum_{i=1}^{2n} r_i \le t\}} \prod_{i=1}^{2n} p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^{2n} dr_i \prod_{i=1}^{n} dx_i, \end{aligned} \tag{I.7}$$

where the sum runs over all maps $\pi : [1, 2n] \mapsto [1, n]$ with $|\pi^{-1}(i)| = 2$ for each i. The factor 2^n comes from the fact that we can interchange each $x_{\pi(i)}$ and $x_{\pi(i-1)}$, $i = 1, \ldots, 2n$.

It is not difficult to see that we can find a subset $J = \{i_1, \ldots, i_n\} \subseteq [1, 2n]$, such that each of $x_1, \ldots, x_n$ can be written as a linear combination of $y_j := x_{\pi(i_j)} - x_{\pi(i_j-1)}$,

$j = 1, \ldots, n$. For $i \in J^c$ we use the bound $p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \leq p_{r_i}(0)$, then change variables and integrate out the y_j, to see that

$$\begin{aligned} &\int \left(\prod_{i=1}^{2n} \int_0^t p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)})\, dr_i \right) \prod_{i=1}^{n} dx_i \qquad \text{(I.8)} \\ &\leq C \left(\int_0^t p_r(0)\, dr \right)^n \int \left(\prod_{i \in J} \int_0^t p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)})\, dr_i \right) \prod_{i=1}^{n} dx_i \\ &\leq C \left(\int_0^t p_r(0)\, dr \right)^n \left(\prod_{i \in J} \int \int_0^t p_{r_i}(y_i)\, dr_i\, dy_i \right) \\ &= Ct^n \left(\int_0^t p_r(0)\, dr \right)^n \leq C \left(t^2 \psi^{-1}(1/t) \right)^n , \end{aligned}$$

for all t sufficiently small, where we use (G.3) and (J.7) for the last line.

It follows from (I.8) that

$$\|\alpha_{1/l}\|_n \leq C \frac{\psi^{-1}(l)}{l^2} \tag{I.9}$$

for all l sufficiently large. Consequently, for l sufficiently large,

$$\begin{aligned} \Big| \|2\widetilde{\alpha}_l\|_n - \|\alpha_1\|_n \Big| \quad &\leq \quad \|2\widetilde{\alpha}_l - \alpha_1\|_n = \| \sum_{j=0}^{l-1} \alpha_{j,j,1/l} \|_n \qquad \text{(I.10)} \\ &\leq \quad l \| \, \alpha_{0,0,1/l} \|_n = l \| \, \alpha_{1/l} \|_n \\ &\leq \quad C \frac{\psi^{-1}(l)}{l}. \end{aligned}$$

This gives (4.13). □

The next three lemmas give estimates for the mean and variance of $\int (L_1^{x+h} - L_1^x)^2\, dx$. They are proved in Section K.

Let

$$c_{\psi,h,0} := \int_0^\infty \left(p_s(0) - p_s(h) \right)\, ds. \tag{I.11}$$

Lemma I.1 *Under the hypotheses of Theorem 1.2,*

$$\lim_{h \to 0} h\psi(1/h) c_{\psi,h,0} = c_{\beta,0}. \tag{I.12}$$

Lemma I.2 *Under the hypotheses of Theorem 1.2; for small h and $t(h) = 1/(\log 1/h)$,*

$$E \left(\int (L_t^{x+h} - L_t^x)^2\, dx \right) = 4 c_{\psi,h,0} t + O\left(g(h,t) \right) \tag{I.13}$$

as $h \to 0$, where

$$g(h,t) = \begin{cases} h^2 t^2 \left(\psi^{-1}(1/t) \right)^3 & 3/2 < \beta \leq 2 \\ h^2 L(1/h) & \beta = 3/2 \\ (h \psi^2(1/h))^{-1} & 1 < \beta < 3/2 \end{cases} \tag{I.14}$$

and $L(\,\cdot\,)$ is some function that is slowly varying at infinity. Also

$$\begin{aligned} &\mathit{Var} \left(\int (L_t^{x+h} - L_t^x)^2\, dx \right) \qquad \text{(I.15)} \\ &\leq C \left(\frac{t g(h,t)}{h \psi(1/h)} + \frac{t^2 \psi^{-1}(1/t)}{h \psi^2(1/h)} + \frac{Ct}{h^{3/2} \psi^{5/2}(1/h)} + \frac{Ct \log 1/h}{h^2 \psi^3(1/h)} \right). \end{aligned}$$

The proof of this lemma shows that we can take any function $t := t(h)$ such that $\psi^{-1}(1/t) << 1/h$ and $\lim_{h\to 0} t(h) = 0$.

Lemma I.3 *Under the hypotheses of Theorem 1.2,*

$$E\left(\int (L_1^{x+h} - L_1^x)^2\,dx\right) = 4c_{\psi,h,0} + O\left(\overline{g}(h)\right) \tag{I.16}$$

as $h \to 0$, where

$$\overline{g}(h) = \begin{cases} h^2 & 3/2 < \beta \le 2 \\ h^2 L(1/h) & \beta = 3/2 \\ (h\psi^2(1/h))^{-1} & 1 < \beta < 3/2 \end{cases} \tag{I.17}$$

and $L(\,\cdot\,)$ is slowly varying at infinity.

Proof of Lemma 4.3 We use (4.4) with $l = [\log 1/h]$. Since $J_{j,j,l,h}$, $0 \le j \le l-1$, are independent and identically distributed, $E(J_{j,j,l,h}) = E(J_{0,0,l,h})$, for all $j = 0,\ldots,l-1$ and

$$\text{Var}\left(\sqrt{h\psi^2(1/h)}\sum_{j=0}^{l-1}(J_{j,j,l,h} - E(J_{j,j,l,h}))\right) = lh\psi^2(1/h)\text{Var}\,(J_{0,0,l,h}) \tag{I.18}$$

Consequently, to obtain (4.14) it suffices to show that

$$\lim_{h\to 0}\sqrt{h\psi^2(1/h)}\left(lE(J_{0,0,l,h}) - E\int (L_1^{x+h} - L_1^x)^2\,dx\right) = 0 \tag{I.19}$$

and

$$\lim_{h\to 0} lh\psi^2(1/h)\text{Var}\,(J_{0,0,l,h}) = 0. \tag{I.20}$$

Using (I.13) and (I.16) on the expectations in (I.19), and recalling that $l = 1/t$, we see that

$$lE(J_{0,0,l,h}) - E\int (L_1^{x+h} - L_1^x)^2\,dx = O(g(h,t)/t) + O(\overline{g}(h)) \tag{I.21}$$

It is easy to verify that (I.19) holds.

Showing that (I.20) holds is a little more subtle so we provide some details. We first consider the last three terms in (I.15) and multiply them by $lh\psi^2(1/h) = h\psi^2(1/h)/t$ as in (I.18). The first of these is

$$\frac{h\psi^2(1/h)}{t}\frac{t^2\psi^{-1}(1/t)}{h\psi^2(1/h)} = t\psi^{-1}(1/t). \tag{I.22}$$

This last function is regularly varying as $t \to 0$ with index $1 - 1/\beta$ which is positive by hypothesis.

The next term is

$$\frac{h\psi^2(1/h)}{t}\frac{t}{h^{3/2}\psi^{5/2}(1/h)} = \frac{1}{h^{1/2}\psi^{1/2}(1/h)}. \tag{I.23}$$

Here $(h^{1/2}\psi^{1/2})^{-1}$ is regularly varying as $h \to 0$ with index $(\beta - 1)/2$ which is positive.

The third of the last three terms is

$$\frac{h\psi^2(1/h)}{t}\frac{t\log 1/h}{h^2\psi^3(1/h)}=\frac{\log 1/h}{h\psi^(1/h)}. \tag{I.24}$$

Here $(\log 1/h)(h\psi(1/h))^{-1}$ is regularly varying as $h\to 0$ with index $(\beta-1)$ which is positive. Thus (I.20) holds for these three terms.

We now consider

$$\frac{h\psi^2(1/h)}{t}\frac{tg(h,t)}{h\psi(1/h)}=g(h,t)\psi(1/h). \tag{I.25}$$

We use (I.14) to see that when $\beta>3/2$ this is equal to

$$t^2(\psi^{-1}(1/t))^3h^2\psi(1/h). \tag{I.26}$$

Here we note that $t^2(\psi^{-1}(1/t))^3$ is regularly varying at zero with index $2-(3/\beta)$ which is positive since $\beta>3/2$. In addition by (G.12), $\lim_{h\to 0}h^2\psi(1/h)<\infty$.

When $\beta=3/2$, (I.25) is equal to

$$h^2L(1/h)\psi(1/h). \tag{I.27}$$

This function is regularly varying at zero with index $2-(3/2)$.

When $\beta<3/2$, (I.25) is equal to $(h\psi(1/h))^{-1}$, which is regularly varying at zero with index $\beta-1$. Thus we have verified (I.20). This completes the proof. □

J Proofs of Lemmas G.1–G.5 and Lemma 4.4

Since the Lévy processes, X, that we are concerned with satisfy

$$\int\frac{1}{1+\psi(p)}\,dp<\infty \tag{J.1}$$

it follows from the Riemann Lebesgue Lemma that they have transition probability density functions, which we designate as $p_s(\,\cdot\,)$. Taking the inverse Fourier transform of the characteristic function X_s, and using the symmetry of ψ, we see that

$$\begin{aligned}p_s(x)&=\frac{1}{2\pi}\int e^{ipx}\,e^{-s\psi(p)}\,dp\\&=\frac{1}{\pi}\int_0^\infty\cos px\,e^{-s\psi(p)}\,dp.\end{aligned} \tag{J.2}$$

We begin with a technical lemma.

Lemma J.1 *Let X be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at infinity with index $1<\beta\le 2$ and satisfies (1.16). Then for any $r\ge 0$ and $t>0$*

$$\int_0^t s^re^{-s\psi(p)}\,ds\le C\left(t\wedge\frac{1}{\psi(p)}\right)^{r+1}\le\frac{2Ct^{r+1}}{1+(t\psi(p))^{r+1}}; \tag{J.3}$$

$$\int_0^\infty\psi^r(p)\left(\int_0^1 s^re^{-s\psi(p)}\,ds\right)dp\le C; \tag{J.4}$$

$$\int|\sin(hp)|\psi^r(p)\left(\int_0^1 s^re^{-s\psi(p)}\,ds\right)dp\le\frac{C}{h\psi(1/h)}, \tag{J.5}$$

and

$$\int_0^1(p_s(0)-p_s(h))\,ds\le C\frac{1}{h\psi(1/h)} \tag{J.6}$$

as $h \to 0$*.*

In addition for all $t \le 1$ *and all* $y \in R^1$

$$\int_0^t p_s(y)\, ds \le Ct\psi^{-1}(1/t). \tag{J.7}$$

Proof The first part of the bound in the first inequality in (J.3) comes from taking $e^{-s\psi(p)} \le 1$; the second from letting $t = \infty$. The second inequality in (J.3) is trivial.

Note that for any $y > 0$

$$y^r \int_0^1 s^r e^{-sy}\, ds = \frac{1}{y}\int_0^y s^r e^{-s}\, ds. \tag{J.8}$$

Consequently

$$\begin{aligned} y^r \int_0^1 s^r e^{-sy}\, ds &\le \left(\sup_{x\ge 0} x^r e^{-x}\right) \wedge \left(\frac{1}{y}\int_0^\infty s^r e^{-s}\, ds\right) \\ &\le C\left(1 \wedge \frac{1}{y}\right) \le 2C\frac{1}{1+y}. \end{aligned} \tag{J.9}$$

Using this it is easy to see that

$$\begin{aligned} \int \psi^r(p) \int_0^1 s^r e^{-s\psi(p)}\, ds\, dp &\le C \int \left(1 \wedge \frac{1}{\psi(p)}\right) dp \\ &\le C\int_0^1 1\, dp + C\int_1^\infty \frac{1}{\psi(p)}\, dp \end{aligned} \tag{J.10}$$

which gives (J.4).

Similarly we obtain (J.5),

$$\begin{aligned} &\int_0^\infty |\sin(hp)|\psi^r(p) \int_0^1 s^r e^{-s\psi(p)}\, ds\, dp \\ &\le C\int_0^\infty |\sin(hp)| \left(1 \wedge \frac{1}{\psi(p)}\right) dp \\ &\le C\int_0^\infty \frac{hp \wedge 1}{1+\psi(p)}\, dp \\ &\le C\left(h\int_0^{1/h} \frac{p}{1+\psi(p)}\, dp + \int_{1/h}^\infty \frac{1}{1+\psi(p)}\, dp\right) \le \frac{C}{h\psi(1/h)}. \end{aligned} \tag{J.11}$$

(In (J.11) we use the regular variation of ψ at infinity. We continue to do so throughout the rest of this paper without further comment.)

For (J.6) we first note that by (J.2)

$$\begin{aligned} p_s(0) - p_s(h) &= \frac{1}{\pi}\int_0^\infty (1-\cos ph)\, e^{-s\psi(p)}\, dp \\ &= \frac{2}{\pi}\int_0^\infty \sin^2 ph/2\, e^{-s\psi(p)}\, dp. \end{aligned} \tag{J.12}$$

Therefore by Fubini's Theorem and (J.3),

$$\begin{aligned} &\int_0^1 (p_s(0) - p_s(h))\, ds \\ &= \frac{2}{\pi}\int_0^\infty \sin^2 ph/2 \int_0^1 e^{-s\psi(p)}\, ds\, dp \end{aligned} \tag{J.13}$$

$$
\begin{aligned}
&\le C\int_0^\infty \left(1\wedge \frac{p^2h^2}{2}\right)\left(1\wedge\frac{1}{\psi(p)}\right)\,dp\\
&\le Ch^2\int_0^{1/h}\frac{p^2}{\psi(p)}\,dp + C\int_{1/h}^\infty \frac{1}{\psi(p)}\,dp \le C\frac{1}{h\psi(1/h)}.
\end{aligned}
$$

For (J.7) we use (J.3) to see that

$$
\begin{aligned}
\int_0^t p_s(y)\,ds &\le \frac{1}{2\pi}\int_0^t\int e^{-s\psi(p)}\,dp\,ds\\
&\le C\int_0^\infty \left(t\wedge\frac{1}{\psi(p)}\right)dp \qquad\qquad (J.14)\\
&\le C\left(t\psi^{-1}(1/t)+\int_{\psi^{-1}(1/t)}^\infty \frac{1}{\psi(p)}\,dp\right)\\
&\le Ct\psi^{-1}(1/t).
\end{aligned}
$$

□

Proof of Lemma G.1 We first note that

$$
p_s(x)\le C\left(\psi^{-1}(1/s)\vee 1\right). \qquad (J.15)
$$

Refer to (J.2). It is obvious that for $s\ge 1$, $p_s(x)\le C$, for all x. In addition,

$$
p_s(x) \le \frac{1}{\pi}\psi^{-1}(1/s)+\frac{1}{\pi}\int_{\psi^{-1}(1/s)}^\infty e^{-s\psi(p)}\,dp. \qquad (J.16)
$$

Also, for all s sufficiently small, the last integral is equal to

$$
\int_1^\infty e^{-u}\,d\psi^{-1}(u/s) < \int_1^\infty \psi^{-1}(u/s)\,e^{-u}\,du \qquad (J.17)
$$

by integration by parts, where we drop a negative term. The final integral in (J.17)

$$
\begin{aligned}
&\le \psi^{-1}(1/s)\int_1^\infty \frac{\psi^{-1}(u/s)}{\psi^{-1}(1/s)}\,e^{-u}\,du \qquad\qquad (J.18)\\
&\le \psi^{-1}(1/s)K\int_1^\infty u^{1/\beta+\delta}\,e^{-u}\,du \le C\psi^{-1}(1/s),
\end{aligned}
$$

for all $\delta>0$; where the constant K depends on δ. (See e.g. [3, Theorem 1.5.6].) Thus we get (J.15).

By integration by parts

$$
\begin{aligned}
p_s(x) &= \frac{1}{\pi x}\int_0^\infty e^{-s\psi(p)}\,d(\sin px) \qquad\qquad (J.19)\\
&= -\frac{1}{\pi x}\int_0^\infty \sin px\left(\frac{d}{dp}e^{-s\psi(p)}\right)dp\\
&= -\frac{1}{\pi x^2}\int_0^\infty \cos px\left(\frac{d^2}{dp^2}e^{-s\psi(p)}\right)dp.
\end{aligned}
$$

Furthermore

$$
\frac{d^2}{dp^2}e^{-s\psi(p)} = \left(s^2(\psi'(p))^2 - s\psi''(p)\right)e^{-s\psi(p)}. \qquad (J.20)
$$

Therefore, by (1.17)

$$\left|\int_0^1 \cos px \left(\frac{d^2}{dp^2}e^{-s\psi(p)}\right)\, dp\right| \le C\left(\int_0^1 \left((\psi'(p))^2+|\psi''(p)|\right)\right)\, dp \le C. \tag{J.21}$$

(We use (1.17) repeatedly in the rest of the paper without comment.) In addition, by (1.16), for all s sufficiently small

$$\begin{aligned}&\left|\int_1^\infty \cos px \left(\frac{d^2}{dp^2}e^{-s\psi(p)}\right)\, dp\right| \\ &\qquad\le C\int_1^\infty \frac{1}{p^2}\left(\psi^2(p)s^2e^{-s\psi(p)}+s\psi(p)e^{-s\psi(p)}\right)\, dp \\ &\qquad\le C\int_1^\infty \frac{1}{p^2}\, dp \le C,\end{aligned} \tag{J.22}$$

since $\sup_{x\ge 0} x^r e^{-x}\le C$. Using (J.15) and (J.19)–(J.22) we get (G.1).

The inequality in (G.2) follows immediately from (G.1).

The equality in (G.3) is trivial since $\int p_s(x)\, dx=1$.

Note that

$$\begin{aligned}\Delta^h p_s(x) &= p_s(x+h)-p_s(x) \\ &= \frac{1}{\pi}\int_0^\infty (\cos p(x+h)-\cos px)\; e^{-s\psi(p)}\, dp \\ &= -\frac{2}{\pi}\int_0^\infty \cos(px)\sin^2(hp/2)e^{-s\psi(p)} \\ &\qquad -\frac{1}{\pi}\int_0^\infty \sin(px)\sin(hp)\, e^{-s\psi(p)}\, dp\end{aligned} \tag{J.23}$$

and

$$\begin{aligned}\Delta^h\Delta^{-h}p_s(x) &= 2p_s(x)-p_s(x+h)-p_s(x-h) \\ &= \frac{4}{\pi}\int_0^\infty \cos(px)\sin^2(hp/2)\, e^{-s\psi(p)}\, dp.\end{aligned} \tag{J.24}$$

Thus

$$\Delta^h p_s(x) = -\frac{1}{2}\Delta^h\Delta^{-h}p_s(x)-\frac{1}{\pi}\int_0^\infty \sin(px)\sin(hp)\, e^{-s\psi(p)}\, dp. \tag{J.25}$$

We now note that

$$\sup_x \int_0^1 |\Delta^h p_s(x)|\, ds \le \frac{C}{h\psi(1/h)} \tag{J.26}$$

and

$$\sup_x \int_0^1 |\Delta^h\Delta^{-h} p_s(x)|\, ds \le \frac{C}{h\psi(1/h)}. \tag{J.27}$$

To obtain (J.27) we use (J.24) to see that

$$\sup_x \int_0^1 |\Delta^h\Delta^{-h}p_s(x)|\, ds \le \frac{4}{\pi}\int_0^1\int_0^\infty \sin^2(hp/2)\, e^{-s\psi(p)}\, dp\, ds. \tag{J.28}$$

Using the calculation in (J.13) we get (J.27).

To obtain (J.26) we note that by (J.3), similarly to (J.13)

$$\begin{aligned}
&\sup_x \int_0^1 \left| \int_0^\infty \sin(px)\sin(hp)\, e^{-s\psi(p)}\, dp \right| ds && \text{(J.29)}\\
&\quad \le C \int_0^\infty \left(1 \wedge \frac{ph}{2}\right)\left(1 \wedge \frac{1}{\psi(p)}\right) dp \\
&\quad \le C \left(h \int_0^{1/h} \frac{p}{1+\psi(p)}\, dp + \int_{1/h}^\infty \frac{1}{\psi(p)}\, dp \right) \le C \frac{1}{h\psi(1/h)}.
\end{aligned}$$

Thus (J.26) follows from (J.25), (J.27) and (J.29).

We now show that

$$\Delta^h \Delta^{-h} p_s(x) = \frac{8}{\pi} \frac{K}{x^2} \tag{J.30}$$

where

$$K = K(s,x,h) := \int_0^\infty \sin^2(px/2) \left(\sin^2(hp/2)\, e^{-s\psi(p)} \right)'' dp. \tag{J.31}$$

To get this we integrate by parts in (J.24),

$$\begin{aligned}
&\int_0^\infty \cos px \sin^2(hp/2)\, e^{-s\psi(p)}\, dp && \text{(J.32)}\\
&\quad = \frac{1}{x} \int_0^\infty \sin^2(hp/2)\, e^{-s\psi(p)}\, d(\sin px) \\
&\quad = -\frac{1}{x} \int_0^\infty \sin px \left(\sin^2(hp/2)\, e^{-s\psi(p)} \right)' dp \\
&\quad = -\frac{1}{x} \int_0^\infty \left(\sin^2(hp/2)\, e^{-s\psi(p)} \right)' d\left(\int_0^p \sin rx\, dr \right) \\
&\quad = -\frac{1}{x^2} \int_0^\infty \left(\sin^2(hp/2)\, e^{-s\psi(p)} \right)' d\,(1-\cos px) \\
&\quad = \frac{2}{x^2} \int_0^\infty \sin^2(px/2) \left(\sin^2(hp/2)\, e^{-s\psi(p)} \right)'' dp.
\end{aligned}$$

Let $g(p) = e^{-s\psi(p)}$ and note that

$$\left(2\sin^2(hp/2)\, e^{-s\psi(p)} \right)' = g(p) h \sin hp + 2g'(p)\sin^2(hp/2) \tag{J.33}$$

and

$$\begin{aligned}
&\left(2\sin^2(hp/2)\, e^{-s\psi(p)} \right)'' && \text{(J.34)}\\
&\quad = g(p)h^2 \cos hp + 2g'(p) h \sin hp + 2g''(p)\sin^2(hp/2).
\end{aligned}$$

Substituting (J.34) in (J.32) we write $K = I + II + III$. Using (J.3) we see that

$$\begin{aligned}
\int_0^1 |I|\, ds &= h^2 \int_0^1 \left| \int_0^\infty \cos hp \sin^2(px/2) e^{-s\psi(p)}\, dp \right| ds && \text{(J.35)}\\
&\le h^2 \int_0^\infty \left(\int_0^1 e^{-s\psi(p)}\, ds \right) dp \\
&\le Ch^2 \int_0^1 \frac{1}{1+\psi(p)}\, dp = O\left(h^2\right).
\end{aligned}$$

Then using (1.16), (1.17) and (J.4) with $r=1$ we get

$$\begin{aligned}\int_0^1 |II|\,ds &= 2h\int_0^1 \Big|\int_0^\infty \sin hp \sin^2(px/2) g'(p)\,dp\Big|\,ds \\ &\le 2h\int_0^\infty |\sin(hp)\,\psi'(p)| \left(\int_0^1 se^{-s\psi(p)}\,ds\right)\,dp \\ &\le Ch^2\int_0^\infty |p\,\psi'(p)| \left(\int_0^1 se^{-s\psi(p)}\,ds\right)\,dp \\ &\le Ch^2\left(C_1+\int_1^\infty \psi(p)\left(\int_0^1 se^{-s\psi(p)}\,ds\right)\,dp\right) = O\left(h^2\right).\end{aligned} \tag{J.36}$$

Similarly, and also using (J.4) with $r=1$ we get

$$\begin{aligned}\int_0^1 |III|\,ds &= 2\int_0^1 \Big|\int_0^\infty \sin^2(hp/2)\sin^2(px/2) g''(p)\,dp\Big|\,ds \\ &\le Ch^2\int_0^\infty p^2\left(\int_0^1 \left(s|\psi''(p)|+s^2|\psi'(p)|^2\right)e^{-s\psi(p)}\,ds\right)\,dp \\ &\le Ch^2\left\{\int_0^1 p^2\left(|\psi''(p)|+|\psi'(p)|^2\right)\,dp\right. \\ &\qquad \left.+\int_1^\infty\left(\int_0^1 \left(s\psi(p)+s^2\psi^2(p)\right)e^{-s\psi(p)}\,ds\right)\,dp\right\} \\ &= O\left(h^2\right).\end{aligned} \tag{J.37}$$

Combining (J.35)–(J.37) with (J.30) we get the third bound in (G.7). The first bound in (G.7) follows from (J.27).

To get the second bound in (G.7) we use (J.24) and the third integral in (J.32) to see that

$$\Delta^h\Delta^{-h}p_s(x) = -\frac{4}{\pi}\frac{L}{x} \tag{J.38}$$

where

$$L = L(s,x,h) := \int_0^\infty \sin px\left(\sin^2(hp/2)\,e^{-s\psi(p)}\right)'\,dp. \tag{J.39}$$

Using (J.33), (1.16), (1.17) and (J.5) with $r=0$ and 1, we see that

$$\begin{aligned}&\int_0^1 |L|\,ds \\ &\le C\int_0^1\left(h\int_0^\infty|\sin hp|g(p)\,dp+\int_0^\infty \sin^2(hp/2)|g'(p)|\,dp\right)\,ds \\ &\le Ch\int_0^\infty|\sin hp|\int_0^1 e^{-s\psi(p)}\,ds\,dp \\ &\qquad +Ch\left(C_1+\int_1^\infty|\sin(hp/2)||p\psi'(p)|\int_0^1 se^{-s\psi(p)}\,ds\,dp\right) \\ &\le O\left(\frac{1}{\psi(1/h)}\right)+Ch\int_0^\infty|\sin(hp/2)|\psi(p)\int_0^1 se^{-s\psi(p)}\,ds\,dp \\ &\le O\left(\frac{1}{\psi(1/h)}\right).\end{aligned} \tag{J.40}$$

Thus we get the second bound on the right–hand side of (G.7). This completes the proof of (G.7).

To prove (G.4) we first note that by (J.25) it is less than $w(x)/2$ plus

$$C\int_0^1 \Big| \int_0^\infty \sin(px)\sin(hp)\, e^{-s\psi(p)}\, dp\Big|\, ds \tag{J.41}$$

Integrating by parts twice we obtain

$$\begin{aligned}
&\int_0^\infty \sin(px)\sin(hp)\, e^{-s\psi(p)}\, dp \\
&\quad = -\frac{1}{x}\int_0^\infty \sin(hp)\, e^{-s\psi(p)}\, d(\cos px) \\
&\quad = \frac{1}{x}\int_0^\infty \cos px\left(\sin(hp)\, e^{-s\psi(p)})\right)'\, dp \\
&\quad = \frac{1}{x^2}\int_0^\infty \left(\sin(hp)\, e^{-s\psi(p)}\right)'\, d\left(\sin px\right) \\
&\quad = -\frac{1}{x^2}\int_0^\infty \sin px\left(\sin(hp)\, e^{-s\psi(p)}\right)''\, dp.
\end{aligned} \tag{J.42}$$

Note that

$$\left(\sin(hp)\, e^{-s\psi(p)}\right)' = (h\cos hp - \sin hp(s\,\psi'(p))\, e^{-s\psi(p)} \tag{J.43}$$

Thus the left hand side of (J.42) is bounded by $\dfrac{\bar J}{x}$ where

$$\bar J = \bar J(s,x,h) := \int_0^1 \Big| \int_0^\infty \cos px\left(\sin(hp)\, e^{-s\psi(p)}\right)'\, dp\Big|\, ds. \tag{J.44}$$

We write

$$\bar J \le \bar J_1 + \bar J_2 \tag{J.45}$$

where

$$\begin{aligned}
|\bar J_1| &\le h\int_0^1\int_0^\infty |\cos px\cos(hp)|\, e^{-s\psi(p)}\, dp\, ds \\
&\le Ch\int_0^\infty \frac{1}{1+\psi(p)}\, dp \le C'h
\end{aligned} \tag{J.46}$$

and using (1.16), (1.17) and (J.4)

$$\begin{aligned}
|\bar J_2| &\le \int_0^1\int_0^\infty |\cos px\sin(hp)||\psi'(p)|\, se^{-s\psi(p)}\, dp\, ds \\
&\le h\int_0^1\int_0^\infty p|\psi'(p)|\, se^{-s\psi(p)}\, dp\, ds \\
&\le h\int_0^1 |\psi'(p)| \int_0^1 se^{-s\psi(p)}\, ds\, dp \\
&\quad + Ch\int_0^1\int_1^\infty \psi(p)\, se^{-s\psi(p)}\, dp\, ds \le C'h.
\end{aligned} \tag{J.47}$$

Therefore

$$\frac{\bar J}{|x|} \le C\frac{h}{|x|}. \tag{J.48}$$

In addition (J.41) is $\dfrac{G}{x^2}$ where

$$G = G(x,h) := \int_0^1 \Big| \int_0^\infty \sin px\left(\sin(hp)\, e^{-s\psi(p)}\right)''\, dp\Big|\, ds. \tag{J.49}$$

Since

$$\left(\sin(hp)\, e^{-s\psi(p)}\right)'' \tag{J.50}$$
$$= \left(-h^2 \sin hp + 2hs\cos hp\,\psi'(p) - \sin hp(s\,\psi''(p) - s^2(\psi'(p))^2\right) e^{-s\psi(p)},$$

we can write

$$G \le G_1 + G_2 + G_3. \tag{J.51}$$

Using (J.50) and (J.4) we get

$$\begin{aligned} |G_1| &= h^2 \int_0^1 \Big| \int_0^\infty \sin px \left(\sin(hp)\, e^{-s\psi(p)}\right)\, dp\Big|\, ds \\ &\le Ch^2 \int_0^\infty \int_0^1 e^{-s\psi(p)}\, ds\, dp \le Ch^2. \end{aligned} \tag{J.52}$$

Using (1.16) , (1.17) and (J.4) we see that

$$\begin{aligned} |G_2| &= 2h \int_0^1 \Big| \int_0^\infty \sin px \cos hp \left(\psi'(p)\, se^{-s\psi(p)}\right)\, dp\Big|\, ds \\ &\le 2h \int_0^1 \int_0^\infty |\psi'(p)|\, se^{-s\psi(p)}\, dp\, ds \\ &\le 2h \left(C_1 + \int_1^\infty p|\psi'(p)| \left(\int_0^1 se^{-s\psi(p)}\, ds\right)\, dp\right) \\ &\le Ch \left(C_1 + \int_1^\infty \psi(p) \left(\int_0^1 se^{-s\psi(p)}\, ds\right)\, dp\right) \le Ch. \end{aligned} \tag{J.53}$$

Similarly

$$|G_3| \tag{J.54}$$
$$\begin{aligned} &= \int_0^1 \Big| \int_0^\infty \sin px\ \sin hp \left(s\psi''(p) - s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, dp\Big|\, ds \\ &\le h \int_0^\infty p \left(\int_0^1 \left(s|\psi''(p)| + s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, ds\right)\, dp \\ &\le Ch \left\{C_1 + \int_1^\infty p^2 \left(\int_0^1 \left(s|\psi''(p)| + s^2(\psi'(p))^2\right) e^{-s\psi(p)}\, ds\right)\, dp\right\} \\ &\le Ch \left\{C_1 + \int_1^\infty \left(\int_0^1 \left(s\psi(p) + s^2(\psi(p))^2\right) e^{-s\psi(p)}\, ds\right)\, dp\right\} \\ &\le Ch. \end{aligned}$$

Thus we see that for all $|x| > 0$

$$G \le Ch, \tag{J.55}$$

for some $C < \infty$ independent of $|x|$. Combining (J.26), (J.48) and (J.55) and taking into account the value of $w(x)$, we get (G.4).

For (G.5) we use (G.4) to see that

$$\int \left(\int_0^1 |\Delta^h\, p_s(x)|\, ds\right)\, dx \tag{J.56}$$
$$\le C \left(\int_0^a \frac{1}{h\psi(1/h)}\, dx + h \int_a^1 \frac{1}{x}\, dx + h \int_1^\infty \frac{1}{x^2}\, dx\right),$$

Set $a = a(h) = h^2\psi(1/h)$. For Lévy processes excluding Brownian Motion, $\lim_{h\to 0} h^2\psi(1/h) = 0$; (see [11, Lemma 4.2.2]), and we can estimate (J.56) to obtain (G.5). For Brownian Motion take $a = 1$ in (J.56) to obtain (G.5).

Similarly, to obtain (G.6) we use (G.4) to get

$$\begin{aligned}\int \left(\int_0^1 |\Delta^h\, p_s(x)|\, ds \right)^p dx &\leq C\left(\int_0^a \frac{1}{h^p\psi^p(1/h)}\, dx + h^p \int_a^\infty \frac{1}{x^p}\, dx \right) \\ &\leq C\left(\frac{a}{h^p\psi^p(1/h)} + \frac{h^p}{a^{p-1}} \right). \end{aligned} \tag{J.57}$$

For (G.9) we use (G.7) to see that

$$\begin{aligned} &\int \left(\int_0^1 |\Delta^h\Delta^{-h}\, p_s(x)|\, ds \right)^2 dx \qquad (J.58) \\ &\quad = \int_0^h \left(\int_0^1 |\Delta^h\Delta^{-h}\, p_s(x)|\, ds \right)^2 dx \\ &\qquad + \int_h^\infty \left(\int_0^1 |\Delta^h\Delta^{-h}\, p_s(x)|\, ds \right)^2 dx \\ &\quad \leq \frac{C}{h\psi^2(1/h)} + \frac{C}{\psi^2(1/h)} \int_h^\infty \frac{1}{x^2}\, dx = O\left(\frac{1}{h\psi^2(1/h)} \right). \end{aligned}$$

The inequality in (G.10) follows similarly,

$$\int_u^\infty \left(\int_0^1 |\Delta^h\Delta^{-h}\, p_s(x)|\, ds \right)^2 dx \leq \frac{C}{\psi^2(1/h)} \int_u^\infty \frac{1}{x^2}\, dx = \frac{C}{u\psi^2(1/h)}. \tag{J.59}$$

To obtain (G.8) we use (G.7) to see that

$$\begin{aligned} &\int \int_0^1 \left| \Delta^h\Delta^{-h}\, p_s(x) \right| ds\, dx \qquad (J.60) \\ &= \int_0^h \int_0^1 \left| \Delta^h\Delta^{-h}\, p_s(x) \right| ds\, dx + \int_h^1 \int_0^1 \left| \Delta^h\Delta^{-h}\, p_s(x) \right| ds\, dx \\ &\qquad + \int_1^\infty \int_0^1 \left| \Delta^h\Delta^{-h}\, p_s(x) \right| ds\, dx \\ &\leq \frac{C}{h\psi(1/h)} \int_0^h 1\, dx + \frac{C}{\psi(1/h)} \int_h^1 \frac{1}{|x|}\, dx + Ch^2 \int_1^\infty \frac{1}{|x|^2}\, dx \\ &\leq \frac{C}{\psi(1/h)} + \frac{C\log 1/h}{\psi(1/h)} + Ch^2 \end{aligned}$$

□

Proof of Lemma 4.4 Using $2 - e^{iph} - e^{-iph} = 4\sin^2(hp/2)$ we see that

$$\begin{aligned} \int_0^\infty \Delta^h\Delta^{-h} p_t(x)\, dt &= \frac{1}{2\pi} \int_0^\infty \int e^{-ipx}(2 - e^{iph} - e^{-iph}) e^{-t\psi(p)}\, dp\, dt \\ &= \frac{4}{2\pi} \int e^{-ipx} \sin^2(hp/2) \int_0^\infty e^{-t\psi(p)}\, dt\, dp \\ &= \frac{4}{2\pi} \int e^{-ipx} \frac{\sin^2(hp/2)}{\psi(p)}\, dp. \end{aligned} \tag{J.61}$$

It follows from Parseval's Theorem that

$$c_{\psi,h,1} = \int \left(\int_0^\infty \Delta^h\Delta^{-h} p_t(x)\, dt \right)^2 dx = \frac{8}{\pi} \int \frac{\sin^4(hp/2)}{\psi^2(p)}\, dp. \tag{J.62}$$

Using this we write

$$h\psi^2(1/h)c_{\psi,h,1} = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin^2(p/2)}{\psi(p/h)/\psi(1/h)}\right)^2 dp \tag{J.63}$$

For a fixed $0 < a < 1$,

$$\begin{aligned}\int_0^a \left(\frac{\sin^2 p/2}{\psi(p/h)/\psi(1/h)}\right)^2 dp &= h\psi^2(1/h)\int_0^{a/h}\left(\frac{\sin^2(ph/2)}{\psi(p)}\right)^2 dp \\ &\le \frac{h^5\psi^2(1/h)}{4}\int_0^{a/h}\frac{p^4}{\psi^2(p)}\,dp. \end{aligned}\tag{J.64}$$

For any $\epsilon > 0$ we can find an $h_0 > 0$, such that for all $0 < h \le h_0$, the last line above

$$\le \frac{(1+\epsilon)h^5\psi^2(1/h)}{4(5-2\beta)}\frac{(a/h)^5}{\psi^2(a/h)} \le \frac{a^{5-2\beta}}{2(5-2\beta)}. \tag{J.65}$$

Note that for any $\epsilon' > 0$ and $p \ge a > 0$, we can find an $h_0' > 0$, such that for all $0 < h \le h_0' \le h_0$,

$$\frac{\psi^2(1/h)}{\psi^2(p/h)} \le C\max\left(\frac{1}{p^{2\beta-\epsilon}},\frac{1}{p^{2\beta+\epsilon}}\right). \tag{J.66}$$

(See [3, Theorem 1.5.6].) Therefore, it follows from the Dominated Convergence Theorem that

$$\lim_{h\to 0}\int_a^\infty \left(\frac{\sin^2 p/2}{\psi(p/h)/\psi(1/h)}\right)^2 dp = \int_a^\infty \frac{\sin^4 p/2}{p^{2\beta}}\,dp. \tag{J.67}$$

Since (J.64), (J.65) and (J.67) hold for all $a > 0$ sufficiently small, we get (4.15). □

Proof of Lemma G.2 We now consider (G.11). Just as we obtained (J.61) and (J.62) we see that

$$\begin{aligned}&\int_{[0,\sqrt{h}]^2}\int \left(\Delta^h\Delta^{-h}\,p_r(x)\right)\left(\Delta^h\Delta^{-h}\,p_{r'}(x)\right)\,dx\,dr\,dr' \\ &\qquad = \frac{8}{\pi}\int \frac{\sin^4(ph/2)}{\psi^2(p)}\left(1-e^{-\sqrt{h}\psi(p)}\right)^2\,dp. \end{aligned}\tag{J.68}$$

We show below that

$$h\psi^2(1/h)\int \frac{\sin^4(ph/2)}{\psi^2(p)}e^{-\sqrt{h}\psi(p)}\,dp = O(h^{1/2}), \tag{J.69}$$

which proves (G.11).

To obtain (J.69) we note that

$$\begin{aligned}&h\psi^2(1/h)\int \frac{\sin^4(ph/2)}{\psi^2(p)}e^{-\sqrt{h}\psi(p)}\,dp \qquad\qquad (J.70)\\ &= h\psi^2(1/h)\int_{0\le|p|\le 1}\frac{\sin^4(ph/2)}{\psi^2(p)}e^{-\sqrt{h}\psi(p)}\,dp \\ &\quad + h\psi^2(1/h)\int_{1\le|p|\le 1/h}\frac{\sin^4(ph/2)}{\psi^2(p)}e^{-\sqrt{h}\psi(p)}\,dp \\ &\quad + h\psi^2(1/h)\int_{|p|\ge 1/h}\frac{\sin^4(ph/2)}{\psi^2(p)}e^{-\sqrt{h}\psi(p)}\,dp \\ &\le Ch^5\psi^2(1/h)\int_{0\le|p|\le 1}\frac{p^4}{\psi^2(p)}\,dp \end{aligned}$$

$$+Ch^5\psi^2(1/h)\int_{1\leq|p|\leq 1/h}\frac{p^4}{\psi^2(p)}\frac{1}{\sqrt{h}\psi(p)}\,dp$$
$$+Ch\psi^2(1/h)e^{-\sqrt{h}\psi(1/h)}\int_{|p|\geq 1/h}\frac{1}{\psi^2(p)}\,dp,$$

where, in the next to last line of (J.70), we use the fact that for $s\geq 0$, $e^{-s}\leq(\sup_{s\geq 0}se^{-s})/s$. It is obvious that the first and last integral in the last inequality in (J.70) is $O(\sqrt{h})$. As for the second integral, if $1<\beta<5/3$

$$h^5\psi^2(1/h)\int_{1\leq|p|\leq 1/h}\frac{p^4}{\psi^2(p)}\frac{1}{\sqrt{h}\psi(p)}\,dp\leq C\frac{1}{h^{1/2}\psi(1/h)}=O\left(\sqrt{h}\right); \tag{J.71}$$

if $5/3<\beta\leq 2$

$$h^5\psi^2(1/h)\int_{1\leq|p|\leq 1/h}\frac{p^4}{\psi^2(p)}\frac{1}{\sqrt{h}\psi(p)}\,dp\leq C\frac{h^5\psi^2(1/h)}{h^{1/2}}=O\left(\sqrt{h}\right), \tag{J.72}$$

where we use Remark G.3 when $\beta=2$. When $\beta=5/3$

$$h^5\psi^2(1/h)\int_{1\leq|p|\leq 1/h}\frac{p^4}{\psi^2(p)}\frac{1}{\sqrt{h}\psi(p)}\,dp\leq C\frac{L(1/h)}{h^{1/2}\psi(1/h)}\leq O(h) \tag{J.73}$$

for some function $L(\,\cdot\,)$ that is slowly varying at infinity. This gives us (J.69). □

Lemma J.2 *For $r\geq 0$*

$$\sup_{\delta\leq s\leq 1}s^re^{-s\psi(p)}\leq C\left(1\wedge\frac{1}{\psi^r(p)}\right)\leq\frac{2C}{1+\psi^r(p)}, \tag{J.74}$$

and for $k>0$

$$\sup_{\delta\leq s\leq 1}s^re^{-s\psi(p)}\leq\sup_{\delta\leq s\leq 1}\frac{s^{r+k}}{\delta^k}e^{-s\psi(p)}\leq\frac{1}{\delta^k}\frac{2C}{1+\psi^{r+k}(p)}. \tag{J.75}$$

Proof The first inequality in (J.74) follows from the facts that $y^re^{-y}\leq C$ and, of course, $\sup_{\delta\leq s\leq 1}s^re^{-s\psi(p)}\leq 1$. The second inequality in (J.74) is elementary. The inequality in (J.75) follows from (J.74). □

Proof of Lemma G.4 The inequality in (G.13) follows immediately from (G.1).

By (J.75) with $r=0$ and $k=3$

$$\begin{aligned}\sup_{\delta\leq s\leq 1}|\Delta^hp_s(0)| &= \sup_{\delta\leq s\leq 1}\frac{1}{\pi}\int_0^\infty\sin^2(ph/2)\,e^{-s\psi(p)}\,dp \\ &\leq \sup_{\delta\leq s\leq 1}\frac{h^2}{2\pi}\int_0^\infty p^2\,e^{-s\psi(p)}\,dp \\ &\leq C\frac{h^2}{\delta^3}\int_0^\infty\frac{p^2}{1+\psi^3(p)}\,dp\leq\frac{C}{\delta^3}h^2,\end{aligned} \tag{J.76}$$

Note that $\Delta^hp_r(0)=p_r(h)-p_r(0)<0$ and $\Delta^h\Delta^{-h}p_r(0)=2(p_r(0)-p_r(h))$. Thus (G.15) follows immediately from (G.14). □

Proof of Lemma G.5 The inequality in (G.17) follows immediately from (G.1).

To obtain (G.19) consider the material in the proof of Lemma G.1 from (J.30) to the statement that $K = I + II + III$. Now, instead of integrating I, II and III we take their supremum as $\delta \le s \le 1$. We have

$$\begin{aligned}\sup_{\delta\le s\le 1}|I| &\le h^2 \sup_{\delta\le s\le 1}\Big|\int_0^\infty \cos hp\sin^2(px/2)e^{-s\psi(p)}\,dp\Big| \\ &\le h^2 \sup_{\delta\le s\le 1}\int_0^\infty e^{-s\psi(p)}\,dp \\ &\le h^2\frac{C}{\delta}\int_0^\infty \frac{1}{1+\psi(p)}\,dp \le \frac{C}{\delta}h^2,\end{aligned} \tag{J.77}$$

where we use (J.75) with $r=0$ and $k=1$.

$$\begin{aligned}\sup_{\delta\le s\le 1}|II| &= 2\sup_{\delta\le s\le 1} h\Big|\int_0^\infty \sin hp\sin^2(px/2)g'(p)\,dp\Big| \\ &\le 2h\sup_{\delta\le s\le 1}\int_0^\infty |\sin(hp)\,\psi'(p)|\,e^{-s\psi(p)}\,dp \\ &\le Ch^2\sup_{\delta\le s\le 1}\int_0^\infty |p\,\psi'(p)|\,se^{-s\psi(p)}\,dp \\ &\le Ch^2\sup_{\delta\le s\le 1}\left(C_1+\int_1^\infty \psi(p)\,se^{-s\psi(p)}\,dp\right) \\ &\le \frac{C}{\delta}h^2\left(C_1'+\int_1^\infty \frac{\psi(p)}{1+\psi^2(p)}\,dp\right),\end{aligned} \tag{J.78}$$

where we use (J.75) with $r=1$ and $k=1$. Similarly, but with $r,k=0,1$ and $r,k=2,1$

$$\begin{aligned}\sup_{\delta\le s\le 1}|III| &\le \sup_{\delta\le s\le 1}\Big|\int_0^\infty \sin^2(hp/2)\sin^2(px/2)g''(p)\,dp\Big| \\ &\le Ch^2\sup_{\delta\le s\le 1}\int_0^\infty p^2\left(|s\psi''(p)|+s^2|\psi'(p)|^2\right)e^{-s\psi(p)}\,dp \\ &\le Ch^2\left\{\int_0^1 p^2\left(|\psi''(p)|+|\psi'(p)|^2\right)\,dp\right. \\ &\quad\left.+\sup_{\delta\le s\le 1}\int_1^\infty \left(s\psi(p)+s^2\psi^2(p)\right)e^{-s\psi(p)}\,dp\right\} \\ &\le Ch^2\left\{C_1+\sup_{\delta\le s\le 1}\int_1^\infty \left(s\psi(p)+s^2\psi^2(p)\right)e^{-s\psi(p)}\,dp\right\} \\ &\le \frac{C}{\delta}h^2\left\{C_1+\int_1^\infty \frac{\psi(p)}{1+\psi^2(p)}\,dp+\int_1^\infty \frac{\psi^2(p)}{1+\psi^3(p)}\,dp\right\}\end{aligned} \tag{J.79}$$

Combining (J.77)–(J.79) with (J.30) we get the second bound in (G.19).

The first bound on the right–hand side of (G.19) follows from (G.15) since,

$$\left|\sup_{\delta\le r\le 1}\Delta^h\Delta^{-h}p_r(x)\right| \le \sup_{\delta\le r\le 1}\Delta^h\Delta^{-h}p_r(0), \tag{J.80}$$

(see (J.24).)

To get the second bound on the right–hand side of (G.18) consider the material in the paragraph containing (J.41). For our purposes here we need to obtain

$$\sup_{\delta\le s\le 1}\Big|\int_0^\infty \sin(px)\sin(hp)\,e^{-r\psi(p)}\,dp\Big| \tag{J.81}$$

Integrating by parts twice as in (J.42) we see that (J.81) is bounded by

$$\sup_{\delta\le s\le 1}\left|\frac{1}{x^2}\int_0^\infty \sin px\left(\sin(hp)\,e^{-s\psi(p)}\right)''\,dp\right|. \tag{J.82}$$

Thus we have to take $\sup_{\delta\le s\le 1}$ of the terms in (J.52)–(J.54), but without the integral on s. It is easy to see that we get the same bounds as in (J.52)–(J.54) but with the factor $1/\delta$ as in (J.77)–(J.79),

By (J.23), (G.18) is bounded by (G.15) plus

$$\begin{aligned}
&Ch\int p\,e^{-\delta\psi(p)}\,dp \qquad\qquad\qquad\qquad (J.83)\\
&\quad\le Ch\left(\int_0^{\psi^{-1}(1/\delta)} p\,dp+\frac{1}{\delta^2}\int_{\psi^{-1}(1/\delta)}^\infty \frac{p}{\psi^2(p)}\,dp\right)\\
&\quad\le Ch(\psi^{-1}(1/\delta))^2\le C\frac{h}{\delta^2}.
\end{aligned}$$

(For the second integral in the middle line of (J.83) see the comment following (J.70).) This gives the first bound on the right–hand side of (G.18).

The inequalities in (G.20)–(G.22) follow easily from (G.17)–(G.19). □

K Proofs of Lemmas I.1–I.3

Proof of Lemma I.1 By (J.2)

$$\begin{aligned}
h\psi(1/h)c_{\psi,h,0} &= \frac{h\psi(1/h)}{\pi}\int_0^\infty \frac{1-\cos(ph)}{\psi(p)}\,dp \qquad (K.1)\\
&= \frac{2h\psi(1/h)}{\pi}\int_0^\infty \frac{\sin^2(ph/2)}{\psi(p)}\,dp\\
&= \frac{2}{\pi}\int_0^\infty \frac{\sin^2(p/2)}{\psi(p/h)/\psi(1/h)}\,dp.
\end{aligned}$$

Compare this to (J.63). Following the proof of (4.15), from (J.64) to (J.67), with obvious modifications, we get (I.12).

Proof of Lemma I.2 By the Kac Moment Formula,

$$\begin{aligned}
&E\left(\int (L_t^{x+h}-L_t^x)^2\,dx\right) \qquad\qquad\qquad (K.2)\\
&\quad= 2\int\int_{\{\sum_{i=1}^2 r_i\le t\}} \Delta^h p_{r_1}(x)\Delta^h p_{r_2}(0)\,dr_1\,dr_2\,dx\\
&\quad\quad +2\int\int_{\{\sum_{i=1}^2 r_i\le t\}} p_{r_1}(x)\Delta^h\Delta^{-h} p_{r_2}(0)\,dr_1\,dr_2\,dx\\
&\quad= 2\int_{\{\sum_{i=1}^2 r_i\le 1\}} \Delta^h\Delta^{-h} p_{r_2}(0)\,dr_1\,dr_2\\
&\quad= 4\int_0^t (t-r)\left(p_r(0)-p_r(h)\right)\,dr\\
&\quad= \frac{8}{\pi}\int_0^\infty \sin^2(hp/2)\int_0^t (t-r)e^{-r\psi(p)}\,dr\,dp.
\end{aligned}$$

Here we use the facts that when we integrate the second and third integrals with respect to x we get zero in the second integral and one in the third.

Note that

$$\int_0^t (t-r)e^{-r\psi(p)}\,dr = \frac{t}{\psi(p)} - \frac{1-e^{-t\psi(p)}}{\psi^2(p)}. \tag{K.3}$$

By (K.1)

$$\frac{8t}{\pi}\int_0^\infty \frac{\sin^2(ph/2)}{\psi(p)}\,dp = 4c_{\psi,h,0}t. \tag{K.4}$$

This gives the dominant term in (I.13). The absolute value of the remainder is

$$\frac{8}{\pi}\int_0^\infty \frac{\sin^2(ph/2)}{\psi^2(p)}\left(1-e^{-t\psi(p)}\right)\,dp \le \frac{8}{\pi}\int_0^\infty \frac{\sin^2(ph/2)}{\psi^2(p)}\left(1\wedge t\psi(p)\right)\,dp. \tag{K.5}$$

We break this last integral into three parts and see that it is bounded by

$$C\left(h^2 t\int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)}\,dp + h^2\int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)}\,dp + \int_{1/h}^\infty \frac{1}{\psi^2(p)}\,dp\right) \tag{K.6}$$

We have

$$h^2 t\int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)}\,dp \le Ch^2t^2\left(\psi^{-1}(1/t)\right)^3, \tag{K.7}$$

(Since $\lim_{p\to 0}\psi(p)/p^2 > 0$ this integral is finite; (see [11, Lemma 4.2.2]).

In addition

$$\int_{1/h}^\infty \frac{1}{\psi^2(p)}\,dp \le C\frac{1}{h\psi^2(1/h)} \tag{K.8}$$

If $\beta > 3/2$

$$h^2\int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)}\,dp \le Ch^2t^2\left(\psi^{-1}(1/t)\right)^3. \tag{K.9}$$

If $\beta = 3/2$

$$h^2\int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)}\,dp \le Ch^2 L(1/h) \tag{K.10}$$

for some function $L(\,\cdot\,)$ that is slowly varying at infinity. If $\beta < 3/2$

$$h^2\int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)}\,dp \le C\frac{1}{h\psi^2(1/h)}. \tag{K.11}$$

Using (K.5)–(K.11) we get (I.14).

Let

$$Z = \int (L_t^{x+1} - L_t^x)^2\,dx. \tag{K.12}$$

We get an upper bound for the variance of Z by finding an upper bound for EZ^2 and using (I.13) to estimate $(EZ)^2$. We proceed as in the beginning of the proof of Lemma H.1, however there are enough differences that it is better to repeat some of the arguments.

By the Kac Moment Formula

$$\begin{aligned} &E\left(\prod_{i=1}^2 \left(\Delta_{x_i}^h L_t^{x_i}\right)\left(\Delta_{y_i}^h L_t^{y_i}\right)\right) \qquad \text{(K.13)}\\ &\quad = \prod_{i=1}^2 \left(\Delta_{x_i}^h \Delta_{y_i}^h\right) \sum_\sigma \int_{\{\sum_{i=1}^4 r_i \le t\}} \prod_{i=1}^4 p_{r_i}(\sigma(i)-\sigma(i-1)) \prod_{i=1}^4 dr_i \end{aligned}$$

where the sum runs over all bijections $\sigma : [1,4] \mapsto \{x_i, y_i, 1 \leq i \leq 2\}$ and we take $\sigma(0) = 0$. We rewrite (K.13) so that each $\Delta^h_\cdot$ applies to a single $p_\cdot$ factor and then set $y_i = x_i$ and then integrate with respect to $x_1, \ldots, x_m$ to get

$$
\begin{aligned}
&E\left(\left(\int (L_t^{x+h} - L_t^x)^2\,dx\right)^2\right) \qquad\qquad \text{(K.14)}\\
&= 4\sum_{\pi,a} \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} \prod_{i=1}^4 \left(\Delta^h_{x_{\pi(i)}}\right)^{a_1(i)} \left(\Delta^h_{x_{\pi(i-1)}}\right)^{a_2(i)} \\
&\qquad p^\sharp_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^4 dr_i \prod_{i=1}^2 dx_i,
\end{aligned}
$$

as in (H.28). As we did following (H.28) we continue the analysis with $p^\sharp$ replaced by p.

In (K.14) the sum runs over all maps $\pi : [1,4] \mapsto [1,2]$ with $|\pi^{-1}(i)| = 2$ for each i and over all $a = (a_1, a_2) : [1, \ldots, 4] \mapsto \{0,1\} \times \{0,1\}$ with the property that for each i there are exactly two factors of the form $\Delta^h_{x_i}$. The factor 4 comes from the fact that we can interchange each y_i and x_i, $i = 1, 2$. As usual we take $\pi(0) = 0$

Note that in (K.14) it is possible to have 'bound states', that is values of i for which $\pi(i) = \pi(i-1)$. We first consider the terms in (K.14) with two bound states. There are two possible maps. They are $(\pi(1), \pi(2), \pi(3), \pi(4)) = (1,1,2,2)$ and $(\pi(1), \pi(2), \pi(3), \pi(4)) = (2,2,1,$
$1)$. The terms in (K.14) for the map $(\pi(1), \pi(2), \pi(3), \pi(4)) = (1,1,2,$
$2)$ are of the form

$$\prod_{i=1}^4 \left(\Delta^h_{x_{\pi(i)}}\right)^{a_1(i)} \left(\Delta^h_{x_{\pi(i-1)}}\right)^{a_2(i)} p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}), \tag{K.15}$$

where the density terms have the form

$$p_{r_1}(x_1) p_{r_2}(y_1 - x_1) p_{r_3}(x_2 - y_1) p_{r_4}(y_2 - x_2), \tag{K.16}$$

and where $y_i - x_i = 0$. The value of the integrals of the terms in (K.15) depend upon how the difference operators are distributed. In many cases the integrals are equal to zero. For example suppose we have

$$\Delta^h_{x_1} p_{r_1}(x_1) \Delta^h_{x_1} p_{r_2}(0) \Delta^h_{x_2} p_{r_3}(x_2 - x_1) \Delta^h_{x_2} p_{r_4}(0), \tag{K.17}$$

which we obtain by setting $y_1 = x_1$. (Note that $\Delta^h_{x_1} p_{r_2}(0)$ should be interpreted as $\Delta^h_{x_1} p_{r_2}(x_1 - y_1)$ or $\Delta^h_{x_1} p_{r_2}(y_1 - x_1)$). Written out this term is

$$
\begin{aligned}
&(p_{r_1}(x_1 + h) - p_{r_1}(x_1))\, \Delta^h_{x_1} p_{r_2}(0) \\
&\qquad (p_{r_3}(x_2 - x_1 + h) - p_{r_3}(x_2 - x_1))\, \Delta^h_{x_2} p_{r_4}(0)
\end{aligned} \tag{K.18}
$$

By a change of variables one sees that the integral of this term with respect to x_1 and x_2 is zero.

The only non-zero integrals in (K.15) comes from

$$p_{r_1}(x_1) \Delta^h \Delta^{-h} p_{r_2}(0) p_{r_3}(x_2 - x_1) \Delta^h \Delta^{-h} p_{r_4}(0). \tag{K.19}$$

(Similar to the above $\Delta^h \Delta^{-h} p_{r_2}(0)$ is $\Delta^h_{x_1} \Delta^{-h}_{y_1} p_{r_2}(x_1 - y_1)$ where $y_1 = x_1$.) The integral of this term with respect to x_1 and x_2 is

$$\Delta^h \Delta^{-h} p_{r_2}(0) \Delta^h \Delta^{-h} p_{r_4}(0). \tag{K.20}$$

We get the same contribution when $(\pi(1),\pi(2),\pi(3),\pi(4)) = (2,2,1,1)$. Consequently, the contribution to (K.14) of maps with two bound states is

$$\begin{aligned}
&8\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta_x^h\Delta_x^{-h}p_{r_2}(0)\,\Delta_x^h\Delta_x^{-h}p_{r_4}(0)\prod_{i=1}^4 dr_i \qquad \text{(K.21)}\\
&\quad = 32\int_{\{\sum_{i=1}^4 r_i\le t\}} (p_{r_2}(0)-p_{r_2}(h))\,(p_{r_4}(0)-p_{r_4}(h))\prod_{i=1}^4 dr_i\\
&\quad = 16\int_{\{u+v\le t\}} (t-u-v)^2\,(p_u(0)-p_u(h))\,((p_v(0)-p_v(h))\,du\,dv.\\
&\quad \le 16t^2\left(\int_0^\infty (p_u(0)-p_u(h))\,du\right)^2 = (4c_{\psi,h,0}t)^2,
\end{aligned}$$

see (I.11).

We next consider the contribution from terms with exactly one bound state. These come from maps of the form $(\pi(1),\pi(2),\pi(3),\pi(4)) = (1,2,2,1)$ or $(\pi(1),\pi(2),\pi(3),\pi(4)) = (2,1,1,2)$. These terms give non-zero contributions of the form

$$\begin{aligned}
Q_2 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^h p_{r_2}(y-x)\,\Delta_y^h\Delta_y^{-h}p_{r_3}(0)\,\Delta_x^h p_{r_4}(x-y)\\
&\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(K.22)}\\
&= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^{-h}p_{r_2}(y)\,\Delta^h\Delta^{-h}p_{r_3}(0)\,\Delta_y^{-h}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy;\\
Q_3 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^h\Delta_y^h p_{r_2}(y-x)\,p_{r_3}(0)\,\Delta_x^h\Delta_y^h p_{r_4}(x-y)\\
&\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(K.23)}\\
&= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^h\Delta^{-h}p_{r_2}(y)\,p_{r_3}(0)\,\Delta^h\Delta^{-h}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy;
\end{aligned}$$

and

$$\begin{aligned}
Q_4 &:= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} p_{r_1}(x)\Delta_x^h\Delta_y^h p_{r_2}(y-x)\,\Delta_y^h p_{r_3}(0)\,\Delta_x^h p_{r_4}(x-y)\\
&\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad \text{(K.24)}\\
&= \int\int_{\{\sum_{i=1}^4 r_i\le t\}} \Delta^h\Delta^{-h}p_{r_2}(y)\,\Delta^h p_{r_3}(0)\,\Delta^{-h}p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy.
\end{aligned}$$

For further explanation consider Q_2. This arrangement comes from the sequence (x_1,y_2,x_2,y_1). The expression it is equal to comes by making the change of variables, $y-x\to y$ and then integrating with respect to x.

Integrating and using (G.7) we see that

$$\begin{aligned}
Q_2 &\le t\left(\int_0^1 |\Delta^h\Delta^{-h}p_s(0)|\,ds\right)\int\left(\int_0^t \Delta^{-h}p_r(y)\,dr\right)^2 dy \qquad \text{(K.25)}\\
&\le \frac{Ct}{h\psi(1/h)}\int\left(\int_0^t \Delta^{-h}p_r(y)\,dr\right)^2 dy.
\end{aligned}$$

Here we use the fact that $\int \Delta^{-h}p_{r_2}(y)\Delta^{-h}p_{r_4}(y)\,dy \geq 0$ to extend the region of integration with respect to r_2 and r_4. By Parseval's Theorem and (J.3)

$$\begin{aligned}&\int \left(\int_0^t \Delta^{-h}p_r(y)\ dr\right)^2 dy \qquad\qquad (K.26)\\ &= \frac{1}{2\pi}\int |1-e^{iph}|^2 \left(\int_0^t e^{-r\psi(p)}\ dr\right)^2 dp\\ &\leq \frac{8}{\pi}\int \sin^2(hp/2)\left(t\wedge \frac{1}{\psi(p)}\right)^2 dp.\end{aligned}$$

Similar to the transition between (K.5) and (K.6) the last integral is bounded by

$$C\left(h^2t^2\int_0^{\psi^{-1}(1/t)} p^2\,dp + h^2\int_{\psi^{-1}(1/t)}^{1/h}\frac{p^2}{\psi^2(p)}\,dp + \int_{1/h}^{\infty}\frac{1}{\psi^2(p)}\,dp\right). \tag{K.27}$$

Note that

$$h^2t^2\int_0^{\psi^{-1}(1/t)} p^2\,dp \leq Ch^2t^2\left(\psi^{-1}(1/t)\right)^3. \tag{K.28}$$

This bound is the right hand side of (K.7). Bounds for the other integrals are given in (K.8)–(K.11). Since the bounds in (K.7)–(K.11) give (I.14), we see that

$$Q_2 \leq \frac{Ctg(h,t)}{h\psi(1/h)}. \tag{K.29}$$

To obtain a bound for Q_3 we use (G.9) and (J.7) to see that it is bounded in absolute value by

$$t\left(\int_0^t p_s(0)\ ds\right)\int\left(\int_0^1 |\Delta^h\Delta^{-h}p_r(y)|\ dr\right)^2 dy \leq C\frac{t^2\psi^{-1}(1/t)}{h\psi^2(1/h)}. \tag{K.30}$$

Integrating Q_4 and using the Cauchy-Schwarz Inequality we see that it is bounded in absolute value by

$$t\left|\int_0^1 \Delta^h p_r(0)\,dr\right|\left(\int\left|\int_0^1 \Delta^h\Delta^{-h}p_r(y)\,dr\right|^2 dy\int\left|\int_0^1 \Delta^{-h}p_r(y)\,dr\right|^2 dy\right)^{1/2}. \tag{K.31}$$

By (G.4), (G.9) and (G.6) we get

$$Q_4 \leq \frac{Ct}{h^{3/2}\psi^{5/2}(1/h)}. \tag{K.32}$$

Finally, we consider the contribution from terms in (K.14) with no bound states. These have to be from π of the form $(\pi(1),\pi(2),\pi(3),\pi(4)) = (1,2,1,2)$ or of the form $(\pi(1),\pi(2),\pi(3),\pi(4)) = (2,1,2,1)$. They give contributions of the form

$$\begin{aligned}Q_5 &:= \int\int_{\{\sum_{i=1}^4 r_i\leq t\}} p_{r_1}(x)\Delta^h_x p_{r_2}(y-x)\,\Delta^h_y\Delta^h_x p_{r_3}(x-y)\,\Delta^h_y p_{r_4}(y-x)\\ &\qquad\qquad \prod_{i=1}^4 dr_i\,dx\,dy \qquad\qquad (K.33)\\ &= \int\int_{\{\sum_{i=1}^4 r_i\leq t\}} \Delta^{-h}p_{r_2}(y)\,\Delta^h\Delta^{-h}p_{r_3}(y)\,\Delta^h p_{r_4}(y)\prod_{i=1}^4 dr_i\,dy\end{aligned}$$

and

$$
\begin{aligned}
Q_6 := \int\int_{\{\sum_{i=1}^4 r_i \le t\}} & p_{r_1}(x)\Delta_x^h\Delta_y^h p_{r_2}(y-x)\, p_{r_3}(x-y)\, \Delta_x^h\Delta_y^h p_{r_4}(x-y) \\
& \prod_{i=1}^4 dr_i\, dx\, dy \qquad \text{(K.34)}\\
= \int\int_{\{\sum_{i=1}^4 r_i \le t\}} & \Delta^h\Delta^{-h} p_{r_2}(y)\, p_{r_3}(y)\, \Delta^h\Delta^{-h} p_{r_4}(y) \prod_{i=1}^4 dr_i\, dy.
\end{aligned}
$$

Clearly

$$
\begin{aligned}
Q_5 \le t \int & \left(\int_0^1 |\Delta^{-h} p_r(y)|\, dr \right) \qquad \text{(K.35)}\\
& \left(\int_0^1 |\Delta^{h} p_r(y)|\, dr \right) \left(\int_0^1 |\Delta^h\Delta^{-h} p_r(y)|\, dr \right) dy.
\end{aligned}
$$

Using (G.4), and (G.8) we see that

$$
Q_5 \le \frac{Ct \log 1/h}{h^2\psi^3(1/h)}. \qquad \text{(K.36)}
$$

The term Q_6 is bounded the same way we bounded Q_3 and has the same bound.

It follows from (I.13), Lemma I.1 and (K.21) that

$$
\text{Var Z} \le C \left(\sum_{j=2}^6 |Q_j| + \left(\frac{tg(h,t)}{h\psi(1/h)} \right) \right) \qquad \text{(K.37)}
$$

as $h \to 0$, since $g(h,t) < t/(h\psi(1/h)$. (We need a large constant because expressions for Q_j, $j = 2,\dots,6$ occur may ways, according to combinatorics of the distribution of the difference operators.)

We leave it to the reader to verify that replacing p by $p^\sharp$ only adds error terms that do not change (I.16) and (I.17). □

Proof of Lemma I.3 Use (K.2)–(K.6) with $\psi^{-1}(1/t)$ replaced by 1. In place of (K.7) we have

$$
h^2 \int_0^1 \frac{p^2}{\psi(p)}\, dp \le Ch^2. \qquad \text{(K.38)}
$$

(Since $\lim_{p\to 0} \psi(p)/p^2 > 0$ this integral is finite; (see [11, Lemma 4.2.2]).

In place of (K.9) we have, if $\beta > 3/2$

$$
h^2 \int_1^{1/h} \frac{p^2}{\psi^2(p)}\, dp \le Ch^2. \qquad \text{(K.39)}
$$

The statements in (K.10) and (K.11) remain the same when $\psi^{-1}(1/t)$ replaced by 1. With these changes the proof of (I.14) gives (I.16). □