Rates of convergence in the strong invariance principle under projective criteria

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Abstract

We give rates of convergence in the strong invariance principle for stationary sequences satisfying some projective criteria. The conditions are expressed in terms of conditional expectations of partial sums of the initial sequence. Our results apply to a large variety of examples. We present some applications to a reversible Markov chain, to symmetric random walks on the circle, and to functions of dependent sequences.

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1 Introduction and notations

The almost sure invariance principle is a powerful tool in both probability and statistics. It says that the partial sums of random variables can be approximated by those of independent Gaussian random variables, and that the approximation error between the trajectories of the two processes is negligible in a certain sense. In this paper, we are interested in studying rates in the almost sure invariance principle for dependent sequences. When $(X_i)_{i \geq 1}$ is a sequence of independent and identically distributed (iid) centered real-valued random variables with a finite second moment, it is known from Strassen (1964) that a sequence $(Z_i)_{i \geq 1}$ of iid centered Gaussian variables with variance $\sigma^2 = \mathbf{E}(X_0^2)$ may be constructed is such a way that

$$\sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Z_i) \right| = o(b_n) \text{ almost surely, as } n \to \infty, \tag{1.1}$$

where $b_n = (n \log \log n)^{1/2}$. To get smaller (b_n) additional information on the moments of X_1 is necessary. In the iid setting, Komlós, Major and Tusnády (1976) and Major (1976) obtained (1.1) with $b_n = n^{1/p}$ as soon as $\mathbf{E}(|X_1|^p) < \infty$ for $p > 2$. There has

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been a great amount of works to extend these results to dependent sequences, under various conditions: see for instance Heyde (1975), Philipp and Stout (1975), Berkes and Philipp (1979), Dabrowski (1982), Bradley (1983), Utev (1984), Eberlein (1986), Shao and Lu (1987), Sakhanenko (1988), Shao (1993), Rio (1995), and more recently, Wu (2007), Zhao and Woodroofe (2008), Liu and Lin (2009), Gouëzel (2010), Merlevède and Rio (2012). Having explicit rates in the strong invariance principle (1.1) may be useful to derive results in asymptotic statistics. We refer to the monograph by Csörgő and Horváth (1997) which illustrates the importance of strong approximation principles for change-point and trend analysis, and also to the paper by Horváth and Steinebach (2000), showing that limit results for the so-called CUSUM and MOSUM-type test procedures, which are used to detect mean and variance changes, can be proved with the help of strong invariance principles. For instance, Aue, Berkes and Horváth (2006) use a strong approximation principle with an explicit rate for the sums of squares of augmented GARCH sequences to study the limiting behavior of statistical tests, which are used to decide whether the volatility of the underlying variables is stable over time or if it changes in the observation period. Let us also mention the recent paper by Wu and Zhao (2007) who consider statistical inference of trends in mean non-stationary models. Starting from a strong approximation principle with an explicit rate for the partial sums of stationary processes, they propose a statistical test concerning the existence of structural breaks in trends, and they construct simultaneous confidence bands with asymptotically correct nominal coverage probabilities. In their paper, they point out that an explicit rate in the strong approximation principle is crucial to control certain errors terms (see their Remark 2). In this paper, we obtain rates of convergence of order $b_n = n^{1/p} L(n)$ in (1.1) ($L(n)$ is a slowly varying function) when $p \in]2,4]$, for stationary sequence satisfying some projective conditions. To describe our results more precisely, we need to introduce some notations. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbf{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let X_0 be a square integrable, zero mean and $\mathcal{F}_0$-measurable random variable, and define the stationary sequence $(X_i)_{i \in \mathbf{Z}}$ by $X_i = X_0 \circ T^i$. Define then the partial sum $S_n = X_1 + X_2 + \cdots + X_n$. Finally, let H_i be the space of $\mathcal{F}_i$-measurable and square integrable random variables, and denote by $H_i \ominus H_{i-1}$ the orthogonal of H_{i-1} in H_i. Let P_i be the projection operator from $\mathbf{L}^2$ to $H_i \ominus H_{i-1}$, that is

$$P_i(f) = \mathbf{E}(f|\mathcal{F}_i) - \mathbf{E}(f|\mathcal{F}_{i-1}) \quad \text{for any } f \text{ in } \mathbf{L}^2.$$

We shall denote sometimes by $\mathbf{E}_i$ the conditional expectation with respect to $\mathcal{F}_i$. The following notations will be also frequently used: For any two positive sequences $a_n \ll b_n$ means that for a certain numerical constant C not depending on n, we have $a_n \leq C b_n$ for all n; $[x]$ denotes the largest integer smaller or equal to x. Our starting point is the same as in Shao and Lu (1987) and Wu (2007): to obtain a rate of order $b_n = n^{1/p} L(n)$ in (1.1), we shall always assume that

$$\text{the series} \quad d_0 = \sum_{i \geq 0} P_0(X_i) \quad \text{converges in } \mathbf{L}^p. \tag{1.2}$$

We then define the approximating martingale M_n as in Gordin (1969) and Heyde (1974):

$$M_n = \sum_{i=1}^{n} d_0 \circ T^i. \tag{1.3}$$

Now to prove (1.1), it remains to find appropriate conditions under which (1.1) is true for M_n instead of S_n, and $|M_n - S_n| = o(b_n)$ almost surely. To prove that M_n satisfies

(1.1), we shall apply the Skorohod embedding theorem (see Proposition 5.1 in the appendix). Let us now describe the main differences between our paper and that by Shao and Lu (1987) or Wu (2007). It is quite easy to see that one of the assumptions of Shao and Lu is that the sequence $\mathbf{E}(S_n|\mathcal{F}_0)$ converges in $\mathbf{L}^p$, which is equivalent to a coboundary decomposition: $X_0 = d_0 + Z - Z \circ T$, for some random variable Z in $\mathbf{L}^p$. Clearly this coboundary decomposition implies (1.2). However, for $p = 2$, this condition is known to be too restrictive for the almost sure invariance principle: see the recent paper by Zhao and Woodroofe (2008). We shall not require this coboundary decomposition in our results. In his 2007's paper, Wu does not assume the existence of a coboundary decomposition, but a polynomial decay of

$$\sum_{i \geq n} \|P_0(X_i)\|_p.$$

He also assumes that the quantity $\|\mathbf{E}(d_n^2|\mathcal{F}_0) - \mathbf{E}(d_0^2)\|_{p/2}$ converges to zero fast enough as n tend to infinity. He then gives a large class of functions of iid sequences to which his results apply. In our first result (Theorem 2.2), we give slightly weaker conditions than those required in Theorem 4 in Wu (2007), and we provide some examples to which our result applies whereas Wu's conditions are not satisfied. However the condition on d_n can be very difficult to check if the sequence (X_i) has not an explicit expression as a function of an iid sequence. In Theorem 2.4 and its corollaries, we give conditions expressed in terms of conditional expectations of the random variables X_i and $X_i X_j$ with respect to the past σ-algebra $\mathcal{F}_0$, to obtain rates in the almost sure invariance principle. The proofs of these results are postponed to the section 4. As we shall see, our results apply to a large variety of examples, including mixing processes of different kinds. However, with this direct approximating martingale method, there seems to be no hope to get the rate $n^{1/p}$ instead of $n^{1/p}L(n)$ with only a moment of order p, whereas this can be done in some situations *via* other approaches (for ϕ-mixing sequences in the sense of Ibragimov (1962) and $2 < p < 5$, it can be deduced from a paper by Utev (1984)). In Section 3, we have chosen to restrict our attention to four different classes of examples. We first apply Theorem 2.2 to a function of an absolutely regular Markov chain (see Section 3.1). We obtain a rate of order $n^{1/p}$ in (1.1) under the same conditions implying a Rosenthal type inequality of order p (see Rio (2009) and also Merlevède and Peligrad (2012)). Note that we get these rates of convergence in the case where the β-mixing coefficients of the chain are not summable. For the three other classes of examples, we apply Theorem 2.4. In Section 3.2 we show that our projective conditions apply to the well known example of the symmetric random walk on the circle. We obtain rates of convergence in the strong invariance principle for a function f of the stationary Markov chain with transition $Kf(x) = \frac{1}{2}(f(x + a) + f(x - a))$, when a is irrational and badly approximable by rationals (see definitions (3.7) and (3.8)), and the Fourier coefficients $\hat{f}$ of f satisfy $\hat{f}(k) = O(k^{-b})$ for some $b > 1$. In particular, we obtain the rate $n^{1/4}L(n)$ in (1.1) when f is three times differentiable (see Remark 3.5). Up to our knowledge, this is the first strong approximation result for this chain. In Section 3.3, we give an application of Theorem 2.4 to the case of τ-dependent sequences in the sense of Dedecker and Prieur (2005). The nice coupling properties of τ-dependent sequences enable to get results for sums of Hölder functions of the random variables. We apply our results to a functional auto-regressive process whose auto-regression function is not strictly contracting, so that the τ-dependence coefficients decrease with an arithmetical rate. In Section 3.4, we give an application of Theorem 2.4 to the case of α-dependent sequences in the sense of Dedecker and Prieur (2005). This class contains the class of α-mixing sequences in the sense of Rosenblatt, and as a consequence, we improve on the result given by Shao and Lu (1987) in the α-mixing case. We also give an example

of a non α-mixing sequence to which our result apply, by considering the Markov chain associated to an intermittent map of the interval.

2 Main results.

In this section, we give rates of convergence in the strong invariance principle for stationary sequences satisfying projective criteria. We shall use the notations of Section 1 (recall in particular that $d_0 = \sum_{i \geq 0} P_0(X_i)$, where the series converges in $\mathbf{L}^p$). We start this section by recalling Theorem 4 in Wu (2007).

Theorem 2.1 (Wu (2007)). *Let* $2 < p \leq 4$. *Assume that*

$$\sum_{\ell \geq n} \|P_0(X_\ell)\|_p = O(n^{-(1/2-1/p)}) \ \text{ and } \ \|\mathbf{E}(d_n^2|\mathcal{F}_0) - \mathbf{E}(d_n^2)\|_{p/2} = O(n^{-(1-2/p)}) \,. \quad (2.1)$$

Then, enlarging Ω *if necessary, there exists a sequence* $(Z_i)_{i \geq 1}$ *of iid gaussian random variables with zero mean and variance* $\sigma^2 = \mathbf{E}(d_0^2)$ *such that*

$$\sup_{1 \leq k \leq n} \left| S_k - \sum_{i=1}^k Z_i \right| = o(n^{1/p}(\log n)^{3/2}) \text{ almost surely, as } n \to \infty.$$

Let us mention that in the statement of Theorem 4 in Wu (2007), the bound in the first part of condition (2.1) appears with the power $-(1-2/p)$ (instead of $-(1/2-1/p)$ but his proof reveals that it is a missprint in the statement. Let us continue now with some comments concerning the method used to prove this result. The second part of condition (2.1) comes from an application of the Shorohod-Strassen embedding theorem (see Proposition 5.1 given in Appendix for more details) to the martingale M_n. Hence the conclusion of Theorem 2.1 holds provided that

$$R_n = S_n - M_n = o(n^{1/p}(\log n)^{3/2}) \text{ almost surely} \,, \quad (2.2)$$

which is true if

$$\sum_{n \geq 2} \frac{\|R_n\|_p^p}{n^2(\log n)^{p/2}} < \infty \,. \quad (2.3)$$

(see Proposition 1 in Wu (2007) combined with an application of Hölder's inequality). Next, Wu proved the following upper bound (see his Theorem 1):

$$\|R_n\|_p^2 \ll \sum_{k=1}^n \left(\sum_{\ell \geq k} \|P_0(X_\ell)\|_p \right)^2 \,, \quad (2.4)$$

which leads to the first part of condition (2.1). In Proposition 4.1 of Section 4, we give another condition under which (2.2) is satisfied. As a consequence we obtain the following result:

Theorem 2.2. *Let* $2 < p \leq 4$ *and* $t > 2/p$. *Assume that* X_0 *belongs to* $\mathbf{L}^p$ *and that (1.2) is satisfied. Assume in addition that*

$$\sum_{n \geq 2} \frac{\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p^p}{n^2(\log n)^{(t-1)p/2}} < \infty \ \text{ and } \ \sum_{n \geq 2} \frac{\|\mathbf{E}(M_n^2|\mathcal{F}_0) - \mathbf{E}(M_n^2)\|_{p/2}^{p/2}}{n^2(\log n)^{(t-1)p/2}} < \infty \,. \quad (2.5)$$

Then $n^{-1}\mathbf{E}(S_n^2)$ *converges to some nonnegative number* σ^2 *and, enlarging* Ω *if necessary, there exists a sequence* $(Z_i)_{i \geq 1}$ *of iid gaussian random variables with zero mean and variance* σ^2 *such that*

$$\sup_{1 \leq k \leq n} \left| S_k - \sum_{i=1}^k Z_i \right| = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \text{ almost surely, as } n \to \infty.$$

Since $\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p = \|\mathbf{E}(R_n|\mathcal{F}_0)\|_p \leq \|R_n\|_p$, by taking into account (2.4), it follows that the first part of (2.5) holds under the first part of (2.1). Therefore Theorem 2.2 contains Theorem 2.1. Notice also that the first part of (2.5) can be satisfied whereas the first part of (2.1) fails to hold. Indeed, let us consider the following linear process $(X_k)_{k\in\mathbf{Z}}$ defined by $X_k = \sum_{j\geq 0} a_j \varepsilon_{k-j}$ where $(\varepsilon_k)_{k\in\mathbf{Z}}$ is a strictly stationary sequence of martingale differences in $\mathbf{L}^p$ and $(a_k)_{k\in\mathbf{Z}}$ is a sequence of reals defined by:

$$a_0 = 1 + u_0 \text{ and } a_k = \frac{1}{k^{\alpha+1}} + (-1)^k u_k \text{ for all } k \geq 1,$$

where $\alpha > 0$ and $(u_k)_{k\in\mathbf{Z}}$ is a sequence of reals in ℓ^2 but not in ℓ^1. Taking $\mathcal{F}_0 = \sigma(\varepsilon_k, k \leq 0)$, it follows that $P_0(X_i) = a_i \varepsilon_0$, showing that (1.2) is satisfied but the first part of (2.1) is not. In addition, from Burkholder's inequality,

$$\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p^2 \ll \|\varepsilon_0\|_p^2 \sum_{j\geq 0} \Big(\sum_{k=j+1}^{n+j} a_k \Big)^2 \ll n^{1-2\alpha} + \sum_{k\geq 0} u_k^2 \,.$$

Hence the first part of (2.5) is satisfied as soon as $\alpha \geq 1/2 - 1/p$. In this situation, notice that the second part of (2.5) is satisfied as soon as it is with $\sum_{k=1}^n \varepsilon_k$ instead of M_n. This last condition for $\sum_{k=1}^n \varepsilon_k$ can be then verified in different situations. We refer, for instance, to Section 4.3 in Merlevède and Peligrad (2012) where this condition has been verified in case when the sequence of martingale differences, $(\varepsilon_k)_{k\in\mathbf{Z}}$, has an ARCH($\infty$) structure, or also to the example given in Section 4.1 in Dedecker *et al.* (2009) where $(\varepsilon_k)_{k\in\mathbf{Z}}$ is, in addition, a certain function of a homogeneous Markov chain as described in Davydov (1973). Theorem 2.1 as well as Theorem 2.2 gives explicit approximation

rates that are optimal up to multiplicative logarithmic factors. As we just mentioned before, the conditions involved in these results are well adapted to linear processes even generated by martingale differences sequences, and we would like to refer to Section 3 in Wu (2007) where it is shown that they are also satisfied for a large variety of functions of iid sequences. In Section 3.1, we shall also give an application of Theorem 2.2 to the case where $(X_n)_{n\geq 0}$ is a function of a stationary Markov chain for which the knowledge of the transition probability allows us to verify both parts of condition (2.5). However, a condition expressed in terms of $\|\mathbf{E}(S_n^2|\mathcal{F}_0) - \mathbf{E}(S_n^2)\|_{p/2}$ rather than the second part of condition (2.5) would give a nice counterpart to Theorem 2.2. It would be much easier to check, and would allow to consider general classes of weakly dependent processes that are not explicit functions of iid sequences. The forthcoming Theorem 2.4 and its corollaries are in this direction. To replace M_n by S_n in the second part of condition (2.5), a first step is to give a precise decomposition of $R_n = S_n - M_n$.

Proposition 2.3. *Let $p \geq 1$ and assume that (1.2) holds. Then, for any positive integers n and N,*

1. $R_n = \mathbf{E}(S_n|\mathcal{F}_0) - \mathbf{E}(S_{n+N} - S_n|\mathcal{F}_n) + \mathbf{E}(S_{n+N} - S_n|\mathcal{F}_0) - \sum_{k=1}^n \sum_{j\geq n+N+1} P_k(X_j)$.

2. $\|R_n\|_p^{p'} \ll \|\mathbf{E}(S_n|\mathcal{F}_0)\|_p^{p'} + \|\mathbf{E}(S_N|\mathcal{F}_0)\|_p^{p'} + \sum_{k=1}^n \big\| \sum_{j\geq k+N} P_0(X_j) \big\|_p^{p'}$ *where* $p' = \min(2, p)$.

Let us now consider the following reinforcement of the first part of (2.5): there exists a sequence $(u_n)_{n\geq 1}$ of positive reals such that $u_n \geq n$ and

$$\sum_{n\geq 2} \frac{\max_{k\in\{n,u_n\}} \|\mathbf{E}(S_k|\mathcal{F}_0)\|_p^p}{n^2(\log n)^{(t-1)p/2}} < \infty \text{ and } \sum_{n\geq 2} \frac{\big(\sum_{k=1}^n \big\| \sum_{j\geq k+u_n} P_0(X_j) \big\|_p^2 \big)^{p/2}}{n^2(\log n)^{(t-1)p/2}} < \infty \,,$$

$$(2.6)$$

and

$$\sum_{n\geq 2} \frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}} \Big(\sum_{k=1}^{n} \Big\| \sum_{j\geq k+n} P_0(X_j)\Big\|_2^2\Big)^{p/4} < \infty. \tag{2.7}$$

With the help of Proposition 2.3, we then obtain the following counterpart to Theorem 2.2:

Theorem 2.4. *Let $2 < p \leq 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$ and that (1.2) is satisfied. Assume in addition that the conditions (2.6) and (2.7) hold and that*

$$\sum_{n\geq 2} \frac{1}{n^2(\log n)^{(t-1)p/2}} \big\|\mathbf{E}(S_n^2|\mathcal{F}_0) - \mathbf{E}(S_n^2)\big\|_{p/2}^{p/2} < \infty. \tag{2.8}$$

Then the conclusion of Theorem 2.2 holds.

In view of applications to mixingale-like sequences, we give the following results:

Proposition 2.5. *Let $2 < p \leq 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$. If*

$$\sum_{n\geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} \|\mathbf{E}(X_n|\mathcal{F}_0)\|_p^p < \infty \text{ and } \sum_{n\geq 2} \frac{n^{3p/4}}{n^2(\log n)^{\frac{(t-1)p}{2}}} \|\mathbf{E}(X_n|\mathcal{F}_0)\|_2^{p/2} < \infty, \tag{2.9}$$

then (1.2) holds, and the conditions (2.6) and (2.7) are satisfied with $u_n = [n^{p/2}]$. In addition $n^{-1}\mathbf{E}(S_n^2)$ converges to $\sum_{k\in\mathbf{Z}} \mathrm{Cov}(X_0, X_k)$ as n tends to infinity.

Corollary 2.6. *Let $2 < p \leq 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$ and that there exists $\gamma \in]0,1]$ such that*

$$\sum_{n>0} \frac{n^{(\frac{p}{2}-1)(\frac{1}{\gamma}+1)}}{n^{1/2}(\log n)^{(t-1)p/2}} \|\mathbf{E}(X_n|\mathcal{F}_0)\|_p^{p/2} < \infty, \tag{2.10}$$

and

$$\sum_{n>0} \frac{n^{(\gamma+1)p/2}}{n^2(\log n)^{(t-1)p/2}} \sup_{i\geq j\geq n} \|\mathbf{E}(X_iX_j|\mathcal{F}_0) - \mathbf{E}(X_iX_j)\|_{p/2}^{p/2} < \infty. \tag{2.11}$$

Then the conclusion of Theorem 2.2 holds with $\sigma^2 = \sum_{k\in\mathbf{Z}} \mathrm{Cov}(X_0, X_k)$.

The next result has a different range of applicability than Corollary 2.6.

Corollary 2.7. *Let $2 < p \leq 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$ and that (2.9) holds. Assume in addition that*

$$\sum_{n>0} \frac{n^p}{n^2(\log n)^{(t-1)p/2}} \|X_0\mathbf{E}(X_n|\mathcal{F}_0)\|_{p/2}^{p/2} < \infty, \tag{2.12}$$

and

$$\sum_{n>0} \frac{n^p}{n^2(\log n)^{(t-1)p/2}} \sup_{i\geq j\geq n} \|\mathbf{E}(X_iX_j|\mathcal{F}_0) - \mathbf{E}(X_iX_j)\|_{p/2}^{p/2} < \infty. \tag{2.13}$$

Then the conclusion of Theorem 2.2 holds with $\sigma^2 = \sum_{k\in\mathbf{Z}} \mathrm{Cov}(X_0, X_k)$.

3 Applications

3.1 Application to an example of irreducible Markov chain.

In this section we apply Theorem 2.2 to a Markov chain which is a symmetrized version of the Harris recurrent Markov chain defined in Doukhan, Massart and Rio (1994) and that has been considered recently in Rio (2009). Let $E = [-1, 1]$ and let υ be a symmetric atomless law on E. The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\upsilon(A),$$

where δ_x denotes the Dirac measure at point x. Assume that $\theta = \int_E |x|^{-1}\upsilon(dx) < \infty$. Then there is an unique invariant measure

$$\pi(dx) = \theta^{-1}|x|^{-1}\upsilon(dx), \tag{3.1}$$

and the stationary Markov chain $(\zeta_i)_i$ is reversible and positively recurrent. Let f be a measurable function on E and $X_i = f(\zeta_i)$. We denote by $S_n(f)$ the partial sum S_n. Assuming that the measure υ satisfies

$$\upsilon([0, t]) \leq ct^{a+1} \text{ for some } a > (p - 2)/2 \text{ and some } c > 0, \tag{3.2}$$

and that f is an odd function satisfying $|f(x)| \leq C|x|^{1/2}$ for any x in E with C is a positive constant, Rio (2009) (for $p \in]2, 3]$) and Merlevède and Peligrad (2012) (for any $p > 2$) have shown that $\|\max_{1 \leq k \leq n} |S_k(f)|\|_p$ satisfies a Rosenthal-type inequality. When $p \in]2, 4[$, applying Theorem 2.4, we shall prove that under the same assumptions, $S_n(f)$ satisfies the strong approximation (1.1) with rate $b_n = n^{1/p}$.

Corollary 3.1. *Let f be such that $f(-x) = -f(x)$ for any $x \in E$. Assume that there exist $C > 0$ and $\gamma \geq 1/2$ such that $|f(x)| \leq C|x|^{1/2}$ for any x in E. Let $p \in]2, 4]$ be a real number and assume that (3.2) holds true. Then $S_n(f)$ satisfies the strong approximation (1.1) with $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$ and rate $b_n = n^{1/p}$ if $p \in]2, 4[$ and $b_n = n^{1/4}(\log n)^{3/4+\varepsilon}$ for any $\varepsilon > 0$ if $p = 4$.*

Remark 3.2. *Let $\beta_\zeta(n) = 2^{-1}\int_E \|Q^n(x, \cdot) - \pi(\cdot)\|\pi(dx)$ where $\|\mu(\cdot)\|$ denotes the total variation of the signed measure μ. According to Lemma 2 in Doukhan et al. (1994), the absolute regularity coefficients $\beta_\zeta(n)$ of the sequence $(\zeta_i)_i$ are exactly of order n^{-a}. Therefore, for $p \in]2, 4[$, as soon as γ is big enough, $S_n(f)$ satisfies the almost sure invariance principle with the rate $n^{1/p}$ even if the absolute regularity coefficients of the Markov chain $(\zeta_i)_i$ do not satisfy $\sum_{n \geq 1} \beta_\zeta(n) < \infty$ which corresponds to the ergodicity of degree two (see Nummelin (1984)). Notice also that an application of Theorem 2.1 in Merlevède and Rio (2012) would require $a > p - 1$ if $p \in]2, 3]$ to get the rate $n^{1/p}$ (up to some logarithmic terms) in the almost sure invariance principle for $S_n(f)$.*

3.2 Symmetric random walk on the circle

Let K be the Markov kernel defined by

$$Kf(x) = \frac{1}{2}(f(x + a) + f(x - a))$$

on the torus $\mathbf{R}/\mathbf{Z}$, with a irrational in $[0, 1]$. The Lebesgue-Haar measure m is the unique probability which is invariant by K. Let $(\xi_i)_{i \in \mathbf{Z}}$ be the stationary Markov chain with transition kernel K and invariant distribution m. For $f \in \mathbf{L}^2(m)$, let

$$X_k = f(\xi_k) - m(f). \tag{3.3}$$

This example has been considered by Derriennic and Lin (2001) who showed (see their section 2) that the central limit theorem holds for $n^{-1/2} \sum_{k=1}^{n} X_k$ as soon as the series of covariances

$$\sigma^2(f) = m((f - m(f))^2) + 2 \sum_{n>0} m(fK^n(f - m(f))) \tag{3.4}$$

is convergent, and that the limiting distribution is $\mathcal{N}(0, \sigma^2(f))$. In fact the convergence of the series in (3.4) is equivalent to

$$\sum_{k \in \mathbf{Z}^*} \frac{|\hat{f}(k)|^2}{d(ka, \mathbf{Z})^2} < \infty \,, \tag{3.5}$$

where $\hat{f}(k)$ are the Fourier coefficients of f and $d(ka, \mathbf{Z}) = \min_{i \in \mathbf{Z}} |ka - i|$. Hence, for any irrational number a, the criterion (3.5) gives a class of function f satisfying the central limit theorem, which depends on the sequence $((d(ka, \mathbf{Z}))_{k \in \mathbf{Z}^*}$. Note that a function f such that

$$\liminf_{k \to \infty} k|\hat{f}(k)| > 0 \,, \tag{3.6}$$

does not satisfies (3.5) for any irrational number a. Indeed, it is well known from the theory of continued fraction that if p_n/q_n is the n-th convergent of a, then $|p_n - q_n a| < q_n^{-1}$, so that $d(ka, \mathbf{Z}) < k^{-1}$ for an infinite number of positive integers k. Hence, if (3.6) holds, then $|\hat{f}(k)|/d(ka, \mathbf{Z})$ does not even tend to zero as k tends to infinity. Our aim in this section is to give conditions on f and on the properties of the irrational number a ensuring rates of convergence in the almost sure sure invariance principle. Let us then introduce the following definitions: a is said to be *badly approximable* by rationals if

$$d(ka, \mathbf{Z}) \geq c(a)|k|^{-1} \quad \text{for some positive constant } c(a). \tag{3.7}$$

An irrational number is badly approximable iff the terms a_n of its continued fraction are bounded. In particular, the quadratic irrationals are badly approximable. However, note that the set of numbers in $[0, 1]$ satisfying (3.7) has Lebesgue measure 0. A much bigger set is the following: a is said to be *badly approximable in the weak sense* by rationals if for any positive ε,

$$\text{the inequality } d(ka, \mathbf{Z}) < |k|^{-1-\varepsilon} \text{ has only finitely many solutions for } k \in \mathbf{Z}. \tag{3.8}$$

From Roth's theorem the algebraic numbers are badly approximable in the weak sense (cf. Schmidt (1980)). Note also that the set of numbers in $[0, 1]$ satisfying (3.8), has Lebesgue measure 1. Let us note that in Section 5.3 of Dedecker and Rio (2008), it is proved that the condition (3.5) (and hence the central limit theorem for $n^{-1/2} \sum_{k=1}^{n} X_k$) holds for any number a satisfying (3.8) as soon as

$$\sup_{k \neq 0} |k|^{1+\varepsilon}|\hat{f}(k)| < \infty \quad \text{for some positive } \varepsilon. \tag{3.9}$$

Note that, in view of (3.6), one cannot take $\varepsilon = 0$ in the condition (3.9).

Corollary 3.3. *Let X_k be defined by (3.3). Suppose that a satisfies (3.7). Let $p \in]2, 4]$ and assume that for some positive ε,*

$$\sup_{k \neq 0} |k|^s \big(\log(1 + |k|) \big)^{1+\varepsilon}|\hat{f}(k)| < \infty \quad \text{where } s = \frac{\sqrt{1 + 4p(p-2)}}{p} - \frac{3}{p} + 2. \tag{3.10}$$

Then $S_n(f)$ satisfies the strong approximation (1.1) with $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$ and rate $b_n = n^{1/p} \log n$.

When the condition on a is weaker, we obtain:

Corollary 3.4. *Let X_k be defined by (3.3). Suppose that a satisfies (3.8). Let $p \in]2, 4]$ and assume that for s defined in (3.10) and some positive ε,*

$$\sup_{k \neq 0} |k|^{s+\varepsilon} |\hat{f}(k)| < \infty \, .$$

Then $S_n(f)$ satisfies the strong approximation (1.1) with $\sigma^2 = \sum_{k \in \mathbf{Z}} \operatorname{Cov}(X_0, X_k)$ and rate $b_n = n^{1/p}$ if $p \in]2, 4[$ and $b_n = n^{1/4} (\log n)^{3/4+\delta}$ for any $\delta > 0$ if $p = 4$.

Remark 3.5. *Applying Corollary 3.4 with p close enough to 2, we derive that if the function f satisfies (3.9) then, enlarging Ω if necessary, there exists a sequence $(Z_i)_{i \geq 1}$ of iid gaussian random variables with zero mean and variance σ^2 such that, for some $\eta > 0$,*

$$\sum_{i=1}^{n} (X_i - Z_i) = o(n^{1/2-\eta}) \text{ almost surely, as } n \to \infty, \tag{3.11}$$

which could be also deduced from Theorem 1 in Eberlein (1986). As a consequence of (3.11), the weak invariance principle as well as the almost sure invariance principle hold true under (3.9). Note also that if f is three times differentiable then $\sum_{i=1}^{n} X_i$ satisfies the strong approximation (1.1) with rate $b_n = n^{1/4} (\log n)^{3/4+\delta}$ for any $\delta > 0$.

3.3 Application to a class of weak dependent sequences

In this section we give rates of convergence in the almost sure invariance principle for a stationary sequence $(X_i)_{i \in \mathbf{Z}}$ satisfying some weak dependence conditions specified below.

Definition 3.6. *Let $\Lambda_1(\mathbf{R})$ be the set of the functions f from $\mathbf{R}$ to $\mathbf{R}$ such that $|f(x) - f(y)| \leq |x - y|$. For any σ-algebra $\mathcal{F}$ of $\mathcal{A}$ and any real-valued integrable random variable X, we consider the coefficient $\theta(\mathcal{F}, X)$ defined by*

$$\theta(\mathcal{F}, X) = \sup_{f \in \Lambda_1(\mathbf{R})} \|\mathbf{E}(f(X)|\mathcal{F}) - \mathbf{E}(f(X))\|_1 \, . \tag{3.12}$$

We now define the coefficients $\gamma(n)$, $\theta_2(n)$ and $\lambda_2(n)$ of the sequence $(X_i)_{i \in \mathbf{Z}}$.

Definition 3.7. *For any positive integer k, define*

$$\theta_2(n) = \sup_{i \geq j \geq n} \max\{\theta(\mathcal{F}_0, X_i + X_j), \theta(\mathcal{F}_0, X_i - X_j)\} \quad \text{and} \quad \gamma(n) = \|\mathbf{E}(X_n|\mathcal{F}_0)\|_1 \, . \tag{3.13}$$

Let now

$$\lambda_2(n) = \max(\theta_2(n), \gamma(n)) \, . \tag{3.14}$$

Definition 3.8. *For any integrable random variable X, define the "upper tail" quantile function Q_X by $Q_X(u) = \inf \{t \geq 0 : \mathbf{P}(|X| > t) \leq u\}$. Note that, on the set $[0, \mathbf{P}(|X| > 0)]$, the function $H_X : x \to \int_0^x Q_X(u) du$ is an absolutely continuous and increasing function with values in $[0, \mathbf{E}|X|]$. Denote by G_X the inverse of H_X.*

Corollary 3.9. *Let $2 < p \leq 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$. Let $Q = Q_{X_0}$ and $G = G_{X_0}$. Assume in addition that*

$$\sum_{n \geq 2} \frac{n^{p-1}}{n^{2/p} (\log n)^{\frac{(t-1)p}{2}}} \int_0^{\lambda_2(n)} Q^{p-1} \circ G(u) du < \infty \, . \tag{3.15}$$

Then the conclusion of Theorem 2.2 holds with $\sigma^2 = \sum_{k \in \mathbf{Z}} \operatorname{Cov}(X_0, X_k)$.

Denote by $\mathcal{F}_0 = \sigma(X_i, i \leq 0)$ and by $\mathcal{G}_n = \sigma(X_i, i \geq n)$. Notice that if α denote the usual strong mixing coefficient of Rosenblatt (1956) of the stationary sequence $(X_i)_{i \in \mathbf{Z}}$ defined by

$$\alpha(n) = \alpha(\mathcal{F}_0, \mathcal{G}_n) \text{ for } n \geq 0 \,,$$

where $\alpha(\mathcal{F}, \mathcal{G}) = \sup_{A \in \mathcal{F}, B \in \mathcal{G}} |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|$, then according to Lemma 1 in Dedecker and Doukhan (2003), condition (3.15) is implied by

$$\sum_{n \geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} \int_0^{\alpha(n)} Q^p(u)du < \infty \,.$$

As a consequence, it follows that, if for $r > p$,

$$\sup_{x > 0} x^r \mathbf{P}(|X_0| > x) < \infty \quad \text{and} \quad \sum_{n \geq 2} \frac{n^{p-1}}{n^{2/p}}(\alpha(n))^{(r-p)/r} < \infty \,, \tag{3.16}$$

then (1.1) holds with $b_n = n^{1/p}(\log n)$. Therefore, Corollary 3.16 improves on Shao and Lu's result (1987), which requires: $\sum_{n=1}^{\infty}(\alpha(n))^{(r-p)/(rp)} < \infty$. Notice however that (3.16) is stronger than $\sum_{n=1}^{\infty} n^{p-2}(\alpha(n))^{(r-p)/r} < \infty$ which is the condition obtained by Merlevède and Rio (2012) to get the rate $n^{1/p}$ (up to logarithmic terms) in (1.1), but only for $p \in]2, 3]$. Therefore, compared to their Theorem 2.1, Corollary 3.9 allows better rates than $n^{1/3}$.

3.3.1 Application to τ-dependent sequences

As we shall see Corollary 3.9 is well adapted to obtain rates of convergence in the almost sure invariance principle for functions of τ-dependent sequences. Before stating the result, some definitions are needed.

Definition 3.10. *Let $\Lambda_1(\mathbf{R})$ be the set of the functions f from $\mathbf{R}$ to $\mathbf{R}$ such that $|f(x) - f(y)| \leq |x - y|$. Let $\Lambda_1(\mathbf{R}^2)$ be the set of functions f from $\mathbf{R}^2$ to $\mathbf{R}$ such that*

$$|f(x_1, y_1) - f(x_2, y_2)| \leq \frac{1}{2}|x_1 - y_1| + \frac{1}{2}|x_2 - y_2| \,.$$

Let Y_0 be a $\mathcal{F}_0$-measurable random variable in $\mathbf{L}^1$, and $Y_i = Y_0 \circ T^i$. Define the dependence coefficients $\tau_{1,\mathbf{Y}}$ and $\tau_{2,\mathbf{Y}}$ of the sequence $(Y_i)_{i \in \mathbf{Z}}$ by

$$\begin{aligned}
\tau_{1,\mathbf{Y}}(k) &= \left\| \sup_{f \in \Lambda_1(\mathbf{R})} \left| \mathbf{E}(f(Y_k)|\mathcal{F}_0) - \mathbf{E}(f(Y_k)) \right| \right\|_1 , \\
\tau_{2,\mathbf{Y}}(k) &= \max \left\{ \tau_{1,\mathbf{Y}}(k), \sup_{i > j \geq k} \left\| \sup_{f \in \Lambda_1(\mathbf{R}^2)} \left| \mathbf{E}(f(Y_i, Y_j)|\mathcal{F}_0) - \mathbf{E}(f(Y_i, Y_j)) \right| \right\|_1 \right\}.
\end{aligned}$$

Many examples of τ-dependent sequences are given in Dedecker and Prieur (2005). We now define the classes of functions which are adapted to this kind of dependence.

Definition 3.11. *Let c be any concave function from $\mathbf{R}^+$ to $\mathbf{R}^+$, with $c(0) = 0$. Let $\mathcal{L}_c$ be the set of functions f from $\mathbf{R}$ to $\mathbf{R}$ such that*

$$|f(x) - f(y)| \leq Kc(|x - y|), \quad \text{for some positive } K.$$

An application of Corollary 3.9 gives:

Corollary 3.12. *Let $f \in \mathcal{L}_c$, and let $X_k = f(Y_k) - \mathbf{E}(f(Y_k))$. Let $2 < p \le 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbf{L}^p$. Let $Q = Q_{X_0}$ and $G = G_{X_0}$. Assume in addition that*

$$\sum_{n \ge 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} \int_0^{c(\tau_{2,\mathbf{Y}}(n))} Q^{p-1} \circ G(u)du < \infty \,. \tag{3.17}$$

Then the conclusion of Theorem 2.2 holds with $\sigma^2 = \sum_{k \in \mathbf{Z}} \mathrm{Cov}(X_0, X_k)$.

Notice that if for $r > p$ and $t > 2/p$,

$$\sup_{x>0} x^r \mathbf{P}(|X_0| > x) < \infty \quad \text{and} \quad \sum_{n \ge 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} (c(\tau_{2,\mathbf{Y}}(k)))^{(r-p)/(r-1)} < \infty \,,$$

the condition (3.17) is satisfied.

Proof of Corollary 3.12. Let us first prove that the condition (3.17) implies the condition (3.15). Let $X_k = f(Y_k) - \mathbf{E}(f(Y_k))$. Applying the coupling result given in Dedecker and Prieur (2005, Section 7.1) (see also Proposition 4 in Rüschendorf (1985)), we infer that there exists $\bar{Y}_n$ distributed as Y_n and independent of $\mathcal{F}_0$ such that

$$\mathbf{E}(|Y_n - \bar{Y}_n|) = \tau_{1,\mathbf{Y}}(n) \le \tau_{2,\mathbf{Y}}(n) \,.$$

In the same way, for $n \le i < j$ there exists (Y_i^*, Y_j^*) distributed as (Y_i, Y_j) and independent of $\mathcal{F}_0$ such that

$$\frac{1}{2}\mathbf{E}(|Y_i - Y_i^*| + |Y_j - Y_j^*|) = \left\| \sup_{h \in \Lambda_1(\mathbf{R}^2)} \left| \mathbf{E}(h(Y_i, Y_j)|\mathcal{F}_0) - \mathbf{E}(h(Y_i, Y_j)) \right| \right\|_1 \le \tau_{2,\mathbf{Y}}(i) \le \tau_{2,\mathbf{Y}}(n) \,.$$

Clearly
$$\gamma(n) = \|\mathbf{E}(f(Y_n)|\mathcal{F}_0) - \mathbf{E}(f(Y_n))\|_1 \le \|f(Y_n) - f(\bar{Y}_n)\|_1 \,.$$
Consequently, if $f \in \mathcal{L}_c$, one has

$$\gamma(n) \le K\mathbf{E}(c(|Y_n - \bar{Y}_n|)) \le Kc(\|Y_n - \bar{Y}_n\|_1) = Kc(\tau_{1,\mathbf{Y}}(n)) \,.$$

In the same way, if g is in $\Lambda_1(\mathbf{R})$,

$$\|\mathbf{E}(g(X_i + X_j)|\mathcal{F}_0) - \mathbf{E}(g(X_i + X_j))\|_1 \le \mathbf{E}(|f(Y_i) - f(Y_i^*)| + |f(Y_j) - f(Y_j^*)|) \,.$$

Hence, if $f \in \mathcal{L}_c$,

$$\|\mathbf{E}(g(X_i + X_j)|\mathcal{F}_0) - \mathbf{E}(g(X_i + X_j))\|_1 \le 2Kc(\tau_{2,\mathbf{Y}}(n)) \,.$$

Note that the same inequalities hold with $X_i - X_j$ instead of $X_i + X_j$. As a consequence, we obtain that if $f \in \mathcal{L}_c$, then $\lambda_2(n) \le 2Kc(\tau_{2,\mathbf{Y}}(n))$. Hence, Corollary 3.12 follows from Corollary 3.9. $\qquad\square$

Example: Autoregressive Lipschitz model. Let us give an example of an iterative Lipschitz model, which may fail to be irreducible and to which Corollary 3.12 applies. For δ in $[0, 1[$ and C in $]0, 1]$, let $\mathcal{L}(C, \delta)$ be the class of 1-Lipschitz functions h which satisfy

$$h(0) = 0 \quad \text{and} \quad |h'(t)| \le 1 - C(1 + |t|)^{-\delta} \quad \text{almost everywhere.}$$

Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of independent identically distributed real-valued random variables. For $S \ge 1$, let $ARL(C, \delta, S)$ be the class of Markov chains on $\mathbf{R}$ defined by

$$Y_n = h(Y_{n-1}) + \varepsilon_n \quad \text{with} \quad h \in \mathcal{L}(C, \delta) \quad \text{and} \quad \mathbf{E}(|\varepsilon_0|^S) < \infty \,. \tag{3.18}$$

For this model, there exists an unique invariant probability μ such that $\mu(|x|^{S-\delta}) < \infty$ (see Proposition 2 of Dedecker and Rio (2000)). In addition starting from the inequality (7.7) in Dedecker and Prieur (2005) and arguing as in Dedecker and Rio (2000), one can prove that $\tau_{2,\mathbf{Y}}(n) = O(n^{(\delta+1-S)/\delta})$ if $S > 1 + \delta$. Therefore an application of Corollary 3.12 leads directly to the following result:

Corollary 3.13. *Assume that $(Y_i)_{i\in\mathbf{Z}}$ belongs to $ARL(C, \delta, S)$. Let f be some Hölder function of order $\gamma \in]0,1]$, that is $|f(x) - f(y)| \leq K|x - y|^\gamma$ for some $K > 0$. Let $X_i = f(Y_i) - \mathbf{E}(f(Y_i))$ and $S_n(f) = \sum_{k=1}^n X_k$. If for some $p \in]2,4]$,*

$$S > 1 + \delta \quad and \quad \frac{(S - 1 - \delta)(S - \delta - \gamma p)}{S - \gamma - \delta} > \frac{\delta}{\gamma}\left(p - \frac{2}{p}\right), \tag{3.19}$$

then $S_n(f)$ satisfies the strong approximation (1.1) with $\sigma^2 = \sum_{k\in\mathbf{Z}} \mathrm{Cov}(X_0, X_k)$ and rate $b_n = n^{1/p}$ if $p \in]2,4[$ and $b_n = n^{1/4}(\log n)^{3/4+\varepsilon}$ for any $\varepsilon > 0$ if $p = 4$.

Notice that the condition $S > p + \delta\big(1 + \gamma^{-1}(p - 2/p)\big)$ implies the condition (3.19) (both conditions are identical if $\gamma = 1$, that is when g is Lipschitz). An element of $ARL(C, \delta, \eta)$ may fail to be irreducible and then strongly mixing in the general case. However, if the common distribution of the ε_i's has an absolutely continuous component which is bounded away from 0 in a neighborhood of the origin, then the chain is irreducible and fits in the example of Tuominen and Tweedie (1994), Section 5.2. In this case, the rate of ergodicity can be derived from Theorem 2.1 in Tuominen and Tweedie (1994).

3.4 Application to α-dependent sequences

In this section we want to consider a weaker coefficient than the Rosenblatt strong mixing coefficient defined by (3.3), and which may computed for instance for many Markov chains associated to dynamical systems that fail to be strongly mixing. We start with the definition of the α-dependent coefficients.

Definition 3.14. *For any integrable random variable X, let us write $X^{(0)} = X - \mathbf{E}(X)$. For any random variable $Y = (Y_1, \cdots, Y_k)$ with values in $\mathbf{R}^k$ and any σ-algebra $\mathcal{F}$, let*

$$\alpha(\mathcal{F}, Y) = \sup_{(x_1,\ldots,x_k)\in\mathbf{R}^k} \left\| \mathbf{E}\Big(\prod_{j=1}^k (\mathbf{I}_{Y_j \leq x_j})^{(0)} \Big| \mathcal{F}\Big)^{(0)} \right\|_1.$$

For the sequence $\mathbf{Y} = (Y_i)_{i\in\mathbf{Z}}$, let

$$\alpha_{k,\mathbf{Y}}(0) = 1 \text{ and } \alpha_{k,\mathbf{Y}}(n) = \max_{1 \leq l \leq k} \sup_{n \leq i_1 \leq \ldots \leq i_l} \alpha(\mathcal{F}_0, (Y_{i_1}, \ldots, Y_{i_l})) \text{ for } n > 0. \tag{3.20}$$

Let $\Lambda_1(\mathbf{R})$ be the set of the functions f from $\mathbf{R}$ to $\mathbf{R}$ such that $|f(x) - f(y)| \leq |x - y|$.

Notice that $\alpha_{k,\mathbf{Y}}(n) \leq \alpha(n)$ for any positive n, where $\alpha(n)$ is the strong mixing coefficient of Rosenblatt of $\mathbf{Y}$ as defined by (3.3). For examples of Markov chains satisfying $\lim_{n\to\infty} \alpha_{k,\mathbf{Y}}(n) = 0$ and which are not strongly mixing in the sense of Rosenblatt, see the section 3.4.1. We now define the classes of functions which are adapted to this kind of weak dependence.

Definition 3.15. *Let μ be the probability distribution of a random variable X. If Q is an integrable quantile function (see definition 3.8), let $\mathrm{Mon}(Q, \mu)$ be the set of functions g which are monotonic on some open interval of $\mathbf{R}$ and null elsewhere and such that $Q_{|g(X)|} \leq Q$. Let $\mathcal{F}(Q, \mu)$ be the closure in $\mathbf{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and f_ℓ belongs to $\mathrm{Mon}(Q, \mu)$.*

For functions of α-dependent sequences, the following result holds:

Corollary 3.16. *Let $2 < p \leq 4$ and $t > 2/p$. Let Y_0 be a $\mathcal{F}_0$-measurable random variable, $Y_i = Y_0 \circ T^i$ and let P_{Y_0} denote the distribution of Y_0. Let $X_i = f(Y_i) - \mathbf{E}(f(Y_i))$ where f belongs to $\mathcal{F}(Q, P_{Y_0})$ with Q^p integrable. Let $\alpha_{2,\mathbf{Y}}(n)$ be defined as in (3.20). Assume that*

$$\sum_{n \geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} \int_0^{\alpha_{2,\mathbf{Y}}(n)} Q^p(u)du < \infty \,, \tag{3.21}$$

Then the conclusion of Theorem 2.2 holds with $\sigma^2 = \sum_{k \in \mathbf{Z}} \mathrm{Cov}(X_0, X_k)$.

When p is close to 2, the condition (3.21) is close to $\sum_{k \geq 1} \int_0^{\alpha_{2,\mathbf{Y}}(k)} Q^2(u)du < \infty$ which is the best known condition (and optimal in a sense) for the strong invariance principle of α-dependent sequences (see Theorem 1.13 of Dedecker, Gouëzel and Merlevède (2010)). However when $p \in]2,3]$, Theorem 2.1 in Merlevède and Rio (2012) provides a sharper condition than (3.21). As a counterpart, our Corollary 3.16 allows rates of convergence of order $n^{1/p}$ (up to logarithmic terms) with $p \in]3,4]$ in the almost sure invariance principle that are not reached in Merlevède and Rio's paper.

3.4.1 Application to functions of Markov chains associated to intermittent maps

For γ in $]0, 1[$, we consider the intermittent map T_γ from $[0, 1]$ to $[0, 1]$, which is a modification of the Pomeau-Manneville map (1980):

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \,. \end{cases}$$

We denote by ν_γ the unique T_γ-invariant probability measure on $[0, 1]$ which is absolutely continuous with respect to the Lebesgue measure. We denote by K_γ the Perron-Frobenius operator of T_γ with respect to ν_γ. Recall that for any bounded measurable functions f and g,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(K_\gamma(f)g) \,.$$

Let $(Y_i)_{i \geq 0}$ be a stationary Markov chain with invariant measure ν_γ and transition Kernel K_γ. Applying Corollary 3.16, we shall see that for f belonging to a certain class of functions defined below, $\sum_{k=1}^n (f(Y_i) - \nu_\gamma(f))$ satisfies the strong approximation principle (1.1) with rate $b_n = n^{1/p}(\log n)$.

Definition 3.17. *A function H from $\mathbb{R}_+$ to $[0, 1]$ is a tail function if it is non-increasing, right continuous, converges to zero at infinity, and $x \to xH(x)$ is integrable. If μ is a probability measure on $\mathbb{R}$ and H is a tail function, let $\mathrm{Mon}^*(H, \mu)$ denote the set of functions $f : \mathbf{R} \to \mathbf{R}$ which are monotonic on some open interval and null elsewhere and such that $\mu(|f| > t) \leq H(t)$. Let $\mathcal{F}^*(H, \mu)$ be the closure in $\mathbf{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and $f_\ell \in \mathrm{Mon}^*(H, \mu)$.*

Corollary 3.18. *Let $(Y_i)_{i \geq 1}$ be a stationary Markov chain with transition kernel K_γ and invariant measure ν_γ. Let $p \in]2, 4]$ and let H be a tail function with*

$$\int_0^\infty x^{p-1}(H(x))^{\frac{1-\gamma\delta}{1-\gamma}} dx < \infty \text{ where } \delta = p + 1 - 2/p \,. \tag{3.22}$$

Then, for any $f \in \mathcal{F}^*(H, \nu_\gamma)$, the series

$$\sigma^2 = \nu_\gamma((f - \nu_\gamma(f))^2) + 2 \sum_{k > 0} \nu_\gamma((f - \nu_\gamma(f))f \circ T_\gamma^k)$$

converges absolutely, and $\sum_{k=1}^{n}(f(Y_i) - \nu_\gamma(f))$ satisfies the strong invariance principle (1.1) with $b_n = n^{1/p}(\log n)$.

To prove this corollary, it suffices to see that (3.22) implies (3.21) with $t = 1$. In this purpose, we use Proposition 1.17 in Dedecker, Gouëzel and Merlevède (2010) stating that there exist two positive constant B, C such that, for any $n > 0$, $Bn^{(\gamma-1)/\gamma} \leq \alpha_{2,\mathbf{Y}}(n) \leq Cn^{(\gamma-1)/\gamma}$, together with their computations page 817. Note that Corollary 3.18 can be extended to functions of Markov chains associated to generalized Pomeau-Manneville maps (or GPM maps) of parameter $\gamma \in (0, 1)$ as defined in Dedecker, Gouëzel and Merlevède (2010). Notice also that when f is a bounded variation function, Corollary 3.18 applies as soon as $\gamma \leq \delta^{-1}$. Therefore when $\gamma < 3/10$, we obtain better rates than the one obtained by Merlevède and Rio (2012, Corollary 3.1). In particular, if $\gamma \leq 2/9$, we obtain the rate $b_n = n^{1/4}\log n$ in (1.1).

4 Proofs

Recall that $d_0 = \sum_{j\geq0} P_0(X_j)$, so it is an element of $H_0 \ominus H_{-1}$ and by (1.2), it is in $\mathbf{L}^p$. Recall also that $M_n = \sum_{i=1}^{n} d_0 \circ T^i$ and let $R_n = S_n - M_n$.

4.1 Proof of Theorem 2.2

We start the proof by stating the following proposition concerning the almost sure convergence of R_n (its proof will be given later).

Proposition 4.1. *Let $p > 1$. Assume that X_0 belongs to $\mathbf{L}^p$ and that (1.2) is satisfied. Let $(\psi(n))_{n\geq1}$ be a positive and nondecreasing sequence such that there exists a positive constant C satisfying $\psi(2n) \leq C\psi(n)$ for all $n \geq 1$. Assume that*

$$\sum_{n\geq2} \frac{\|R_n\|_q^q}{n(\psi(n))^q} < \infty \text{ for some } q \in [1, p] \text{ and } \sum_{n\geq2} \frac{1}{(\psi(n))^p}\left(\sum_{k=1}^{n} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_p}{k^{1+1/p}}\right)^p < \infty,$$
(4.1)

then $R_n = o(\psi(n))$ almost surely.

With the help of the above proposition, we prove now that under the first part of (2.5),
$$R_n = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \text{ almost surely.}$$
(4.2)

Taking $\psi(n) = n^{1/p}(\log n)^{(t+1)/2}$, we observe that $(\psi(n))_{n\geq1}$ satisfies the assumptions of Proposition (4.1). With the above selection of $\psi(n)$, we infer that a suitable application of Hölder's inequality implies that the second part of (4.1) holds provided that the first part of condition (2.5) is satisfied. We prove now that the first part of condition (2.5) implies the first part of (4.1) with $\psi(n) = n^{1/p}(\log n)^{(t+1)/2}$ and $q = 2$. Note that the first part of (2.5) implies that

$$\sum_{k>0} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_2}{k^{3/2}} < \infty.$$
(4.3)

Applying Proposition 1 in Merlevède *et al.* (2012), we derive that:

$$\|R_n\|_2 \ll n^{1/2} \sum_{k\geq n} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_2}{k^{3/2}}$$

(notice that under (1.2), the approximating martingale considered in the paper by Merlevède *et al.* is almost surely equals to $\sum_{k=1}^{n} d_0 \circ T^i$ where $d_0 = \sum_{j\geq0} P_0(X_j)$). Next,

using Hölder's inequality, we derive that for any $\gamma \in]0, 1 - 2/p[$

$$\sum_{n \geq 2} \frac{1}{n^{1+2/p}(\log n)^{t+1}} \|R_n\|_2^2 \ll \sum_{n \geq 2} \frac{1}{n^{2/p}(\log n)^{t+1}} \left(\sum_{k \geq n} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_2}{k^{3/2}} \right)^2$$

$$\ll \sum_{n \geq 2} \frac{1}{n^{\gamma+2/p}(\log n)^{t+1}} \sum_{k \geq n} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_2^2}{k^{2-\gamma}} \ll \sum_{k \geq 2} \frac{\|\mathbf{E}(S_k|\mathcal{F}_0)\|_2^2}{n^{1+2/p}(\log n)^{t+1}} \,,$$

which is finite under the first part of condition (2.5) (to see this it suffices to apply Hölder's inequality). Therefore, due to the almost sure convergence (4.2), to complete the proof of Theorem 2.2, it suffices to notice that under the second part of condition (2.5), $(M_n)_{n \geq 1}$ satisfies the condition (5.1) of Proposition 5.1 given in Appendix with $\psi(n) = n^{2/p}(\log n)^t$. Hence, enlarging Ω is necessary, there exists a sequence $(Z_i)_{i \geq 1}$ of iid centered Gaussian variables with variance $\mathbf{E}(d_0^2)$ such that (1.1) holds with $b_n = n^{1/p}(\log n)^{(t+1)/2}$. In addition, note that (4.3) is a sufficient condition for $n^{-1}\mathbf{E}(S_n^2)$ to converge (see for instance Theorem 1 in Peligrad and Utev (2005)). To end the proof

of the theorem, it remains to prove Proposition 4.1. With this aim, we first notice that due to the properties of monotonicity of the sequence $(\psi(n))_{n \geq 1}$, the almost sure convergence (4.2) will follow if we can prove that for any $\lambda > 0$,

$$\sum_{r \geq 1} \mathbf{P}\left(\max_{1 \leq i \leq 2^r} |S_i - M_i| \geq \lambda\psi(2^r) \right) < \infty \,. \tag{4.4}$$

Let $q \in [1, p]$. Applying inequality (8) of Proposition 5 of Merlevède and Peligrad (2012) with $\varphi(u) = u^q$ and $x = \lambda\psi(2^r)$, and using stationarity, we derive that for any integer $r \geq 1$,

$$\mathbf{P}\left(\max_{1 \leq i \leq 2^r} |S_i - M_i| \geq \lambda\psi(2^r) \right) \ll \frac{\|R_{2^r}\|_q^q}{\lambda^q(\psi(2^r))^q} + \frac{2^r}{\lambda^p(\psi(2^r))^p} \left(\sum_{l=0}^{r-1} 2^{-l/p} \|\mathbf{E}(S_{2^l}|\mathcal{F}_0)\|_p \right)^p \,.$$

Notice now that by stationarity, we have that, for all $i, j \geq 0$, $\|R_{i+j}\|_q \leq \|R_i\|_q + \|R_j\|_q$ and $\|\mathbf{E}(S_{i+j}|\mathcal{F}_0)\|_p \leq \|\mathbf{E}(S_i|\mathcal{F}_0)\|_p + \|\mathbf{E}(S_j|\mathcal{F}_0)\|_p$. Therefore applying Lemma 5.3 of the appendix respectively with $V_n = (\psi(n))^{-q}\|R_n\|_q^q$ and $V_n = \|\mathbf{E}(S_n|\mathcal{F}_0)\|_p$, we infer that (4.4) will hold true if (4.1) is satisfied. $\square$

4.2 Proof of Proposition 2.3

Notice first that the following decomposition is valid: for any positive integer n,

$$R_n = \sum_{k=1}^n \left(X_k - \sum_{j=1}^n P_j(X_k) \right) - \sum_{k=1}^n \sum_{j \geq n+1} P_k(X_j) = \mathbf{E}(S_n|\mathcal{F}_0) - \sum_{k=1}^n \sum_{j \geq n+1} P_k(X_j) \,. \tag{4.5}$$

Let N be a positive integer and write

$$\sum_{k=1}^n \sum_{j \geq n+1} P_k(X_j) = \sum_{j=n+1}^{n+N} \sum_{k=1}^n P_k(X_j) + \sum_{k=1}^n \sum_{j \geq n+N+1} P_k(X_j)$$

$$= \mathbf{E}(S_{n+N} - S_n|\mathcal{F}_n) - \mathbf{E}(S_{n+N} - S_n|\mathcal{F}_0) + \sum_{k=1}^n \sum_{j \geq n+N+1} P_k(X_j) \,. \tag{4.6}$$

Starting from (4.5) and considering (4.6), item 1 follows. To prove item 2, we start from item 1 and use stationarity, to derive that for any positive integers n and N,

$$\|R_n\|_p \leq \|\mathbf{E}(S_n|\mathcal{F}_0)\|_p + 2\|\mathbf{E}(S_N|\mathcal{F}_0)\|_p + \left\| \sum_{k=1}^n \sum_{j \geq n+N+1} P_k(X_j) \right\|_p \,. \tag{4.7}$$

Applying then Burkholder's inequality and using the stationarity, we obtain that for any positive integer n, there exists a positive constant c_p such that

$$\Big\| \sum_{k=1}^{n} \sum_{j \geq n+N+1} P_k(X_j) \Big\|_p^{p'} \leq c_p \sum_{k=1}^{n} \Big\| \sum_{j \geq n+N+1} P_k(X_j) \Big\|_p^{p'} = c_p \sum_{k=1}^{n} \Big\| \sum_{j \geq N+k} P_0(X_j) \Big\|_p^{p'}, \quad (4.8)$$

where $p' = \min(2, p)$. Starting from (4.7) and using (4.8), item 2 follows. $\square$

4.3 Proof of Theorem 2.4

The proof will follow from Theorem 2.2 if we can show that under (2.6), (2.7) and (2.8), the second part of condition (2.5) is satisfied. With this aim, let $M_n = S_n - R_n$ and write that

$$\big\| \mathbf{E}(M_n^2 | \mathcal{F}_0) - \mathbf{E}(M_n^2) \big\|_{p/2} \leq \big\| \mathbf{E}(S_n^2 | \mathcal{F}_0) - \mathbf{E}(S_n^2) \big\|_{p/2} + 2 \big\| \mathbf{E}_0(S_n R_n) - \mathbf{E}(S_n R_n) \big\|_{p/2} + 2 \| R_n \|_p^2.$$

Let $\beta_n = n^2 (\log n)^{(t-1)p/2}$. Since (2.8) holds true, the second part of condition (2.5) will be satisfied if

$$\sum_{n \geq 1} \frac{1}{\beta_n} \| R_n \|_p^p < \infty \text{ and } \sum_{n \geq 1} \frac{1}{\beta_n} \| \mathbf{E}_0(S_n R_n) - \mathbf{E}(S_n R_n) \|_{p/2}^{p/2} < \infty. \quad (4.9)$$

By using item 2 of Proposition 2.3 with $N = u_n$, the first part of (4.9) clearly holds under (2.6). To prove the second part of (4.9), we first notice that

$$\| \mathbf{E}(S_n \mathbf{E}(S_n | \mathcal{F}_0) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_n | \mathcal{F}_0)) \|_{p/2} \leq 2 \| \mathbf{E}(S_n | \mathcal{F}_0) \|_p^2.$$

Hence the first part of (2.6) implies that

$$\sum_{n \geq 1} \frac{1}{\beta_n} \| \mathbf{E}^2(S_n | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_n | \mathcal{F}_0)) \|_{p/2}^{p/2} < \infty. \quad (4.10)$$

In addition, $\| \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_0) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_0)) \|_{p/2} \leq 2 \| \mathbf{E}(S_n | \mathcal{F}_0) \|_p^2$. Therefore, we also have that

$$\sum_{n \geq 1} \frac{1}{\beta_n} \| \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_0) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_0)) \|_{p/2}^{p/2} < \infty. \quad (4.11)$$

Now since S_n is $\mathcal{F}_n$-measurable, we get that

$$\| \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n)) \|_{p/2}$$
$$= \| \mathbf{E}(S_n(S_{2n} - S_n) | \mathcal{F}_0) - \mathbf{E}(S_n(S_{2n} - S_n)) \|_{p/2}.$$

Next, using the identity $2ab = (a+b)^2 - a^2 - b^2$ and the stationarity, we obtain that

$$2 \| \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n)) \|_{p/2}$$
$$\leq \| \mathbf{E}(S_{2n}^2 | \mathcal{F}_0) - \mathbf{E}(S_{2n}^2) \|_{p/2} + 2 \| \mathbf{E}(S_n^2 | \mathcal{F}_0) - \mathbf{E}(S_n^2) \|_{p/2},$$

which combined with (2.8) implies that

$$\sum_{n \geq 1} \frac{1}{\beta_n} \| \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n) | \mathcal{F}_0) - \mathbf{E}(S_n \mathbf{E}(S_{2n} - S_n | \mathcal{F}_n)) \|_{p/2}^{p/2} < \infty. \quad (4.12)$$

Therefore by combining (4.10), (4.11) and (4.12), we derive that the second part of (4.9) will be satisfied provided that

$$\sum_{n \geq 1} \frac{1}{\beta_n^{2/p} n^{1+2/p}} \| \mathbf{E}(S_n \widetilde{R}_n | \mathcal{F}_0) - \mathbf{E}(S_n \widetilde{R}_n) \|_{p/2} < \infty, \quad (4.13)$$

where $\widetilde{R}_n = R_n - \mathbf{E}(S_n|\mathcal{F}_0) - \mathbf{E}(S_{2n} - S_n|\mathcal{F}_n) + \mathbf{E}(S_{2n} - S_n|\mathcal{F}_0)$. To prove (4.13), we first write that

$$\|\mathbf{E}_0(S_n\widetilde{R}_n) - \mathbf{E}(S_n\widetilde{R}_n)\|_{p/2} \le 2\|\mathbf{E}_0(S_n\widetilde{R}_n)\|_{p/2} \le 2\|\mathbf{E}_0^{1/2}(S_n^2)\mathbf{E}_0^{1/2}(\widetilde{R}_n^2)\|_{p/2}$$
$$\le 2\|(\mathbf{E}_0(S_n^2) - \mathbf{E}(S_n^2))^{1/2}\mathbf{E}_0^{1/2}(\widetilde{R}_n^2)\|_{p/2} + 2(\mathbf{E}(S_n^2))^{1/2}\|\mathbf{E}_0^{1/2}(\widetilde{R}_n^2)\|_{p/2}$$
$$\le \|\mathbf{E}_0(S_n^2) - \mathbf{E}(S_n^2)\|_{p/2} + \|\widetilde{R}_n\|_p^2 + 2(\mathbf{E}(S_n^2))^{1/2}\|\mathbf{E}_0^{1/2}(\widetilde{R}_n^2)\|_{p/2}. \quad (4.14)$$

In addition,

$$\|\widetilde{R}_n\|_p \le \|R_n\|_p + 3\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p \;\; \text{and} \;\; (\mathbf{E}(S_n^2))^{1/2}\|\mathbf{E}_0^{1/2}(\widetilde{R}_n^2)\|_{p/2} \ll n^{1/2}\|\widetilde{R}_n\|_2, \quad (4.15)$$

where for the last inequality we have used the fact that the function $x \mapsto |x|^{p/4}$ is concave and that $\mathbf{E}(S_n^2) \ll n$. Next, taking into account item 1 of Proposition 2.3 with $N = n$, the following inequality holds:

$$\|\widetilde{R}_n\|_2^2 = \sum_{k=1}^n \Big\| \sum_{j \ge k+n} P_0(X_j) \Big\|_2^2, \quad (4.16)$$

Therefore, starting from (4.14) and using (4.15) and (4.16), we then infer that (4.13) holds by taking into account (2.7), (2.8) and the first part of (4.9). This ends the proof of the second part of (4.9) and then of the theorem. $\square$

4.4 Proof of Proposition 2.5

The first part of (2.9) implies that $\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p = o(n^{2/p^2-1}(\log n)^{(t-1)/2})$, so that $\sum_{n>0} n^{-1/p}\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p < \infty$, since $p > 2$. This implies that X_0 is regular in the sense that $\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0$ almost surely. Applying Lemma 5.2 with $q = 1$, it follows that $\sum_{n>0} \|P_0(X_n)\|_p < \infty$ and the condition (1.2) is satisfied. Now, since X_0 is regular, the condition $\sum_{n>0} \|P_0(X_n)\|_2 < \infty$ implies that the series $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$ converges absolutely, and consequently $n^{-1}\mathbf{E}(S_n^2)$ converges to σ^2. Let $\beta_n = n^2(\log n)^{(t-1)p/2}$. Let us prove that (2.6) holds with $u_n = [n^{p/2}]$ as soon as the first part of (2.9) is satisfied. Since $\max_{k \in \{n, u_n\}} \|\mathbf{E}(S_k|\mathcal{F}_0)\|_p \le \sum_{k=1}^{u_n} \|\mathbf{E}(X_k|\mathcal{F}_0)\|_p$, Hölder's inequality implies that for any $\gamma \in]0, 2/p[$,

$$\sum_{n \ge 2} \frac{1}{\beta_n} \max_{k \in \{n, u_n\}} \|\mathbf{E}(S_k|\mathcal{F}_0)\|_p^p \ll \sum_{n \ge 2} \frac{n^{\gamma p/2}}{\beta_n} \sum_{k=1}^{[n^{p/2}]} k^{p-1-\gamma} \|\mathbf{E}(X_k|\mathcal{F}_0)\|_p^p.$$

Therefore changing the order of summation, we infer that the first part of (2.6) holds with $u_n = [n^{p/2}]$ as soon as the first part of (2.9) does. Let us prove now that the second part of (2.6) is satisfied with $u_n = [n^{p/2}]$. Using Lemma 5.2 given in Appendix, we first get that

$$\sum_{n \ge 2} \frac{1}{\beta_n} \Big(\sum_{k=1}^n \Big\| \sum_{j \ge k+u_n} P_0(X_j) \Big\|_p^2 \Big)^{p/2} \ll \sum_{n \ge 2} \frac{n^{p/2}}{\beta_n} \Big(\sum_{k \ge [n^{p/2}/2]} \frac{\|\mathbf{E}(X_k|\mathcal{F}_0)\|_p}{k^{1/p}} \Big)^p.$$

Next Hölder's inequality implies that for any $\gamma \in]1-1/p, 1-2/p^2[$,

$$\sum_{n \ge 2} \frac{1}{\beta_n} \Big(\sum_{k=1}^n \Big\| \sum_{j \ge k+u_n} P_0(X_j) \Big\|_p^2 \Big)^{p/2} \ll \sum_{n \ge 2} \frac{n^{p(p-\gamma p)/2}}{\beta_n} \sum_{k \ge [n^{p/2}]} \frac{k^{\gamma p}\|\mathbf{E}(X_k|\mathcal{F}_0)\|_p^p}{k}.$$

Changing the order of summation, we then derive that the second part of (2.6) holds with $u_n = [n^{p/2}]$ as soon as the first part of (2.9) does. It remains to prove that the

second part of (2.9) entails that (2.7) holds. Using again Lemma 5.2 followed by an application of Hölder's inequality, we first obtain that for any $\gamma \in]1-2/p, 2-4/p[$,

$$\sum_{n \geq 2} \frac{n^{p/4}}{\beta_n} \Big(\sum_{k=1}^{n} \Big\| \sum_{j \geq k+n} P_0(X_j) \Big\|_2^2 \Big)^{p/4} \ll \sum_{n \geq 2} \frac{n^{p/2}}{\beta_n} \Big(\sum_{k \geq [n/2]} \frac{\|\mathbf{E}(X_k|\mathcal{F}_0)\|_2}{k^{1/2}} \Big)^{p/2}$$

$$\ll \sum_{n \geq 2} \frac{n^{p(2-\gamma-2/p)/2}}{\beta_n} \sum_{k \geq n} k^{\gamma p/2} \frac{\|\mathbf{E}(X_k|\mathcal{F}_0)\|_2^{p/2}}{k^{p/4}}.$$

Changing the order of summation, it follows that (2.7) holds provided that the second part of (2.9) does. This ends the proof of the proposition. $\square$

4.5 Proof of Corollary 2.6.

Notice first that, since $\gamma \in]0,1]$ and $p > 2$, the condition (2.10) implies the second part of (2.9). Now, since $(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p)_{n \geq 1}$ is a nonincreasing sequence, condition (2.10) implies that

$$\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p^p = o\big((\log n)^{(t-1)p} n^{3-2p}\big), \tag{4.17}$$

which in turn entails that the first part of (2.9) is satisfied. By Proposition 2.5, it follows that (1.2), (2.6) and (2.7) are satisfied. It remains to prove that condition (2.8) holds as soon as (2.10) and (2.11) hold. With this aim, setting

$$\gamma(\mathcal{F}_0, m, k) := \|\mathbf{E}(X_m X_{k+m}|\mathcal{F}_0) - \mathbf{E}(X_m X_{k+m})\|_{p/2},$$

we first write that for any $\gamma \in [0,1]$,

$$\|\mathbf{E}(S_n^2|\mathcal{F}_0) - \mathbf{E}(S_n^2)\|_{p/2} \leq 2 \sum_{m=1}^{n} \sum_{k=0}^{n-m} \gamma(\mathcal{F}_0, m, k)$$

$$\leq 2 \sum_{m=1}^{n} \sum_{k=0}^{[m^\gamma]} \gamma(\mathcal{F}_0, m, k) + 2 \sum_{m=1}^{n} \sum_{k=[m^\gamma]+1}^{n-m} \gamma(\mathcal{F}_0, m, k). \tag{4.18}$$

We shall bound up $\gamma(\mathcal{F}_0, m, k)$ in two ways. First we consider the bound: for any $k \geq 0$,

$$\gamma(\mathcal{F}_0, m, k) \leq \sup_{i \geq j \geq m} \|\mathbf{E}(X_i X_j|\mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_{p/2} = \widetilde{\gamma}(m).$$

Therefore, by Hölder's inequality,

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{m=1}^{n} \sum_{k=0}^{[m^\gamma]} \gamma(\mathcal{F}_0, m, k) \Big)^{p/2} \ll \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{m=1}^{n} m^\gamma \widetilde{\gamma}(m) \Big)^{p/2}$$

$$\ll \sum_{n \geq 2} \frac{n^{(1-\alpha)(p-2)/2}(\log n)^{(1-t)p/2}}{n^2} \sum_{m=1}^{n} m^{\gamma p/2 + \alpha(p-2)/2} (\widetilde{\gamma}(m))^{p/2},$$

for any $\alpha \in]1-2/(p-2), 1[$. Changing the order of summation, it follows that under (2.11),

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{m=1}^{n} \sum_{k=0}^{[m^\gamma]} \gamma(\mathcal{F}_0, m, k) \Big)^{p/2} \leq \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} n^{(1+\gamma)p/2}}{n^2} (\widetilde{\gamma}(n))^{p/2} < \infty. \tag{4.19}$$

We write now that

$$\gamma(\mathcal{F}_0, m, k) \leq 2\|\mathbf{E}(X_m X_{k+m}|\mathcal{F}_0)\|_{p/2}$$
$$\leq 2\|\mathbf{E}\big((X_m - \mathbf{E}_0(X_m))(X_{k+m} - \mathbf{E}_0(X_{k+m}))|\mathcal{F}_0\big)\|_{p/2} + 2\|\mathbf{E}_0(X_m)\mathbf{E}_0(X_{k+m})\|_{p/2}. \tag{4.20}$$

We notice first that Hölder's inequality entails that

$$\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}}{n^2}\Big(\sum_{m=1}^{n}\sum_{k=0}^{n-m}\|\mathbf{E}_0(X_m)\mathbf{E}_0(X_{k+m})\|_{p/2}\Big)^{p/2}$$

$$\ll\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}}{n^2}\Big(\sum_{k=1}^{n}\|\mathbf{E}_0(X_k)\|_p\Big)^p$$

$$\ll\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}\,n^{(1-\alpha)(p-1)}}{n^2}\sum_{k=1}^{n}k^{\alpha(p-1)}\|\mathbf{E}_0(X_k)\|_p^p\,,$$

for any $\alpha\in]1-1/(p-1),1[$. Changing the order of summation and using (4.17), it follows that

$$\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}}{n^2}\Big(\sum_{m=1}^{n}\sum_{k=0}^{n-m}\|\mathbf{E}_0(X_m)\mathbf{E}_0(X_{k+m})\|_{p/2}\Big)^{p/2}$$

$$\ll\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}\,n^p}{n^2}\|\mathbf{E}_0(X_n)\|_p^p<\infty\,. \qquad (4.21)$$

Starting from (4.18) and using (4.19), (4.20) and (4.21), condition (2.8) will be satisfied if we can prove that

$$\sum_{n\geq 2}\frac{(\log n)^{(1-t)p/2}}{n^2}\Big(\sum_{m=1}^{n}\sum_{k=[m^\gamma]+1}^{n}\gamma^*(m,k)\Big)^{p/2}<\infty\,. \qquad (4.22)$$

where $\gamma^*(m,k):=\|\mathbf{E}\big((X_m-\mathbf{E}_0(X_m))(X_{k+m}-\mathbf{E}_0(X_{k+m}))|\mathcal{F}_0\big)\|_{p/2}$. With this aim, since $X_m-\mathbf{E}_0(X_m)=\sum_{\ell=1}^{m}P_\ell(X_m)$, we first observe that

$$\gamma^*(m,k)\leq\Big\|\mathbf{E}\big(\sum_{\ell=1}^{m}P_\ell(X_m)P_\ell(X_{k+m})|\mathcal{F}_0\big)\Big\|_{p/2}\leq\sum_{\ell=1}^{m}\|P_\ell(X_m)\|_p\|P_\ell(X_{k+m})\|_p\,.$$

Therefore by stationarity,

$$\gamma^*(m,k)\leq\sum_{\ell=0}^{m-1}\|P_0(X_\ell)\|_p\|P_0(X_{\ell+k})\|_p\,. \qquad (4.23)$$

Now we write that

$$\sum_{m=1}^{n}\sum_{k=[m^\gamma]+1}^{n}\gamma^*(m,k)\ll\sum_{k=1}^{[n^\gamma]}\sum_{m=1}^{[k^{1/\gamma}]+1}\gamma^*(m,k)+\sum_{k=[n^\gamma]+1}^{n}\sum_{m=1}^{n}\gamma^*(m,k)\,. \qquad (4.24)$$

Using (4.23) and Hölder's inequality twice, we first derive that

$$\Big(\sum_{k=1}^{[n^\gamma]}\sum_{m=1}^{[k^{1/\gamma}]+1}\gamma^*(m,k)\Big)^{p/2}\ll\Big(\sum_{\ell=0}^{n-1}\|P_0(X_\ell)\|_p\sum_{k=1}^{[n^\gamma]}k^{1/\gamma}\,\|P_0(X_{\ell+k})\|_p\Big)^{p/2}$$

$$\ll\Big(\sum_{i=0}^{n-1}\|P_0(X_i)\|_p\Big)^{\frac{p-2}{2}}\sum_{\ell=0}^{n-1}\|P_0(X_\ell)\|_p\Big(\sum_{k=1}^{[n^\gamma]}k^{1/\gamma}\,\|P_0(X_{\ell+k})\|_p\Big)^{p/2}$$

$$\ll n^{\gamma(1-\alpha)(p-2)/2}\Big(\sum_{i=0}^{n-1}\|P_0(X_i)\|_p\Big)^{\frac{p-2}{2}}\sum_{\ell=0}^{n-1}\|P_0(X_\ell)\|_p\sum_{k=1}^{[n^\gamma]}k^{p/(2\gamma)}k^{\alpha(p-2)/2}\,\|P_0(X_{\ell+k})\|_p^{p/2}\,.$$

for any $\alpha \in]1 - 2/(p-2), 1[$. We then infer that

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{k=1}^{[n^\gamma]} \sum_{m=1}^{[k^{1/\gamma}]+1} \gamma^*(m,k) \Big)^{p/2}$$

$$\ll \Big(\sum_{\ell \geq 0} \|P_0(X_\ell)\|_p \Big)^{p/2} \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} \, n^{p/(2\gamma)+p/2}}{n^{1+1/\gamma}} \, \|P_0(X_n)\|_p^{p/2} \, .$$

Notice now that by Lemma 5.2, $\sum_{\ell \geq 0} \|P_0(X_\ell)\|_p < \infty$ as soon as $\sum_{k>0} k^{-1/p} \|\mathbf{E}_0(X_\ell)\|_p < \infty$, which clearly holds under (4.17) since $p > 2$. On an other hand, using Lemma 5.2 again, we get that

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} \, n^{p/(2\gamma)+p/2}}{n^{1+1/\gamma}} \, \|P_0(X_n)\|_p^{p/2}$$

$$\ll \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} \, n^{p/(2\gamma)+p/2}}{n^{2+1/\gamma}} \sum_{k \geq n} \|P_0(X_k)\|_p^{p/2}$$

$$\ll \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} \, n^{p/(2\gamma)+p/2}}{n^{2+1/\gamma}} \sum_{k \geq [n/2]} k^{-1/2} \|\mathbf{E}_0(X_k)\|_p^{p/2} \, . \qquad (4.25)$$

It follows that under (2.10),

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{k=1}^{[n^\gamma]} \sum_{m=1}^{[k^{1/\gamma}]+1} \gamma^*(m,k) \Big)^{p/2} < \infty \, .$$

Therefore from (4.24), (4.22) will follow if we can prove that

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{k=[n^\gamma]+1}^{n} \sum_{m=1}^{n} \gamma^*(m,k) \Big)^{p/2} < \infty \, . \qquad (4.26)$$

Using (4.23) and Hölder's inequality twice,

$$\Big(\sum_{k=[n^\gamma]+1}^{n} \sum_{m=1}^{n} \gamma^*(m,k) \Big)^{p/2} \ll n^{p/2} \Big(\sum_{\ell=0}^{n-1} \|P_0(X_\ell)\|_p \sum_{k=[n^\gamma]+1}^{n} \|P_0(X_{\ell+k})\|_p \Big)^{p/2}$$

$$\ll n^{p/2} \Big(\sum_{i=0}^{n-1} \|P_0(X_i)\|_p \Big)^{(p-2)/2} \sum_{\ell=0}^{n-1} \|P_0(X_\ell)\|_p \Big(\sum_{k=[n^\gamma]+1}^{n} \|P_0(X_{\ell+k})\|_p \Big)^{p/2}$$

$$\ll n^{p/2} n^{\gamma(1-\alpha)(p-2)/2} \Big(\sum_{i=0}^{n-1} \|P_0(X_i)\|_p \Big)^{\frac{p-2}{2}} \sum_{\ell=0}^{n-1} \|P_0(X_\ell)\|_p \sum_{k=[n^\gamma]+1}^{n} k^{\frac{\alpha(p-2)}{2}} \|P_0(X_{\ell+k})\|_p^{p/2} \, ,$$

for any $\alpha \in]1, 1+1/\gamma[$. It follows that

$$\sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2}}{n^2} \Big(\sum_{k=[n^\gamma]+1}^{n} \sum_{m=1}^{n} \gamma^*(m,k) \Big)^{p/2}$$

$$\ll \Big(\sum_{\ell \geq 0} \|P_0(X_\ell)\|_p \Big)^{p/2} \sum_{n \geq 2} \frac{(\log n)^{(1-t)p/2} \, n^{p/(2\gamma)+p/2}}{n^{1+1/\gamma}} \, \|P_0(X_n)\|_p^{p/2} \, ,$$

which is finite under (2.10) (see (4.25)). This ends the proof of Corollary 2.6. $\square$

4.6 Proof of Corollary 2.7.

Starting from (4.18) with $\gamma = 1$ combined with the bound

$$\gamma(\mathcal{F}_0, m, k) \leq \left(\sup_{i \geq j \geq m} \|\mathbf{E}(X_i X_j | \mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_{p/2} \right) \wedge \left(2 \|X_0 \mathbf{E}(X_k | \mathcal{F}_0)\|_{p/2} \right),$$

where $\gamma(\mathcal{F}_0, m, k)$ is defined in (4.18), we infer that (2.8) is satisfied provided that (2.12) and (2.13) are. The corollary then follows from an application of Theorem 2.4 together with Proposition 2.5. $\square$

4.7 Proof of Corollary 3.1.

Let us first check that (1.2) is satisfied. By definition of the transition probability, and since f is an odd function and υ is symmetric,

$$\mathbf{E}_0(X_n) = \mathbf{E}(f(\zeta_n)|\zeta_0) = (1 - |\zeta_0|)^n f(\zeta_0) \text{ a.s.} \tag{4.27}$$

Taking into account that $|f(x)| \leq C|x|^{1/2}$ and the assumption on υ it follows that

$$\|\mathbf{E}_0(X_n)\|_p^p \ll \int_0^1 (1-x)^{np} x^{p/2} x^{a-1} dx,$$

and using the properties of the Beta functions, $\|\mathbf{E}_0(X_n)\|_p^p = O(n^{-a-p/2})$. Hence, if $a > (p-2)/2$, $\sum_{n>0} n^{-1/p} \|\mathbf{E}_0(X_n)\|_p < \infty$ and by Lemma 5.2 of the appendix, $\sum_{n>0} \|P_0(X_n)\|_p < \infty$, proving that (1.2) is satisfied. We also get that

$$\|\mathbf{E}_0(S_n(f))\|_p \ll \max(\log n, n^{1/2 - a/p}). \tag{4.28}$$

We prove now that for $M_n = \sum_{k=1}^n d_i$ where $d_i = \sum_{n \geq i} P_i(X_n)$,

$$\|\mathbf{E}_0(M_n^2) - \mathbf{E}(M_n^2)\|_{p/2} \ll \max(\log n, n^{1-2a/p}). \tag{4.29}$$

Starting from (4.27), we infer that

$$d_1 = \frac{f(\zeta_1)}{|\zeta_1|} - \frac{f(\zeta_0)}{|\zeta_0|} + f(\zeta_0) \text{ a.s.},$$

yielding to

$$M_n = \frac{f(\zeta_n)}{|\zeta_n|}(1 - |\zeta_n|) - \frac{f(\zeta_0)}{|\zeta_0|}(1 - |\zeta_0|) + S_n(f) \text{ a.s.}.$$

Hence (4.29) will be proven if we can show that

$$\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \ll \max(\log n, n^{1-2a/p}), \tag{4.30}$$

$$\left\| \mathbf{E}_0 \left(\left(f(\zeta_n)(|\zeta_n|^{-1} - 1) - f(\zeta_0)(|\zeta_0|^{-1} - 1) \right)^2 \right) \right\|_{p/2} \ll \max(\log n, n^{1-2a/p}), \tag{4.31}$$

and

$$\left\| \mathbf{E}_0 \left(S_n(f) \left(f(\zeta_n)(|\zeta_n|^{-1} - 1) - f(\zeta_0)(|\zeta_0|^{-1} - 1) \right) \right) \right\|_{p/2} \ll \max(\log n, n^{1-2a/p}). \tag{4.32}$$

The bound (4.30) has been proved in Rio (2009) (see the computations leading to his bound (4.15)). We turn now to the proof of (4.31). According to the definition of the transition probability, we first notice that for any positive integer n, and any function g on E,

$$\mathbf{E}_0\big(g(\zeta_n)\big) = \mathbf{E}_0\Big((1 - |\zeta_{n-1}|)g(\zeta_{n-1})\Big) + \mathbf{E}_0(|\zeta_{n-1}|) \int_E g(y)\upsilon(dy) \text{ a.s}, \tag{4.33}$$

and since f is an odd function and υ is symmetric,

$$\mathbf{E}_0\big(f(\zeta_n)(|\zeta_n|^{-1}-1)\big) = (1-|\zeta_0|)^{n+1}\frac{f(\zeta_0)}{|\zeta_0|} \text{ a.s.} \tag{4.34}$$

Therefore, using (4.33) with $g(x) = y^{-2}(1-|y|)^2 f^2(y)$ and (4.34), we get that for any positive integer n,

$$\left\|\mathbf{E}_0\Big(\big(f(\zeta_n)(|\zeta_n|^{-1}-1) - f(\zeta_0)(|\zeta_0|^{-1}-1)\big)^2\Big)\right\|_{p/2} \ll \|\zeta_{n-1}\|_{p/2}\int_E (1-|y|)^2\frac{f^2(y)}{y^2}d\upsilon(y)$$

$$+\left\|\mathbf{E}_0\Big(\frac{f^2(\zeta_{n-1})}{\zeta_{n-1}^2}(1-|\zeta_{n-1}|)^3 - 2\frac{f^2(\zeta_0)}{\zeta_0^2}(1-|\zeta_0|)^{n+2} + \frac{f^2(\zeta_0)}{\zeta_0^2}(1-|\zeta_0|)^2\Big)\right\|_{p/2}.$$

With the help of (4.33), we then derive after $n-1$ steps that

$$\left\|\mathbf{E}_0\Big(\big(f(\zeta_n)(|\zeta_n|^{-1}-1) - f(\zeta_0)(|\zeta_0|^{-1}-1)\big)^2\Big)\right\|_{p/2} \ll \|\zeta_0\|_{p/2}\sum_{k=0}^{n-1}\int_E (1-|y|)^{2+k}\frac{f^2(y)}{y^2}d\upsilon(y)$$

$$+\left\|\frac{f^2(\zeta_0)}{\zeta_0^2}(1-|\zeta_0|)^2\big(1-(1-|\zeta_0|)^n\big)\right\|_{p/2}. \tag{4.35}$$

Taking into account that $|f(x)| \le C|x|^{1/2}$, the assumption on υ and using the properties of the Beta functions, we get that for any positive integer n

$$\sum_{k=0}^{n-1}\int_E (1-|y|)^{2+k}\frac{f^2(y)}{y^2}d\upsilon(y) \ll \max(\log n, n^{1-a}). \tag{4.36}$$

Since $|f(x)| \le C|x|^{1/2}$, by taking into account (3.1), it follows that for any positive integer n,

$$\left\|\frac{f^2(\zeta_0)}{\zeta_0^2}(1-|\zeta_0|)^2\big(1-(1-|\zeta_0|)^n\big)\right\|_{p/2}^{p/2} \ll \int_0^1 x^{-p/2}(1-x)^p\big(1-(1-x)^n\big)^{p/2}x^{a-1}dx.$$

Using the fact that $1-(1-x)^n \le \min(1, nx)$, we then infer that for any positive integer n,

$$\left\|\frac{f^2(\zeta_0)}{\zeta_0^2}(1-|\zeta_0|)^2\big(1-(1-|\zeta_0|)^n\big)\right\|_{p/2}^{p/2} \ll \max(\log n, n^{p/2-a}). \tag{4.37}$$

The inequality (4.35) combined with the bounds (4.36) and (4.37) gives (4.31). It remains to show that (4.32) holds true. Taking into account the definition of the transition probability, the fact that f is an odd function and υ is symmetric, it follows that

$$\mathbf{E}_0\big(S_n(f)f(\zeta_n)(|\zeta_n|^{-1}-1)\big) = \sum_{k=1}^{n}\mathbf{E}_0\Big(\frac{f^2(\zeta_k)}{|\zeta_k|}(1-|\zeta_k|)^{n-k+1}\Big). \tag{4.38}$$

Next, by using (4.33) k times, we infer that for any positive integer k

$$\left\|\mathbf{E}_0\Big(\frac{f^2(\zeta_k)}{|\zeta_k|}(1-|\zeta_k|)^{n-k+1}\Big)\right\|_{p/2} \le \|\zeta_0\|_{p/2}\sum_{\ell=0}^{k-1}\int_E (1-|y|)^{n-\ell}\frac{f^2(y)}{|y|}d\upsilon(y)$$

$$+\left\|f^2(\zeta_0)|\zeta_0|^{-1}(1-|\zeta_0|)^{n+1}\right\|_{p/2}. \tag{4.39}$$

On the other hand, since f is an odd function and υ is symmetric, by the definition of the transition probability, it follows that

$$\mathbf{E}_0\big(S_n(f)f(\zeta_0)(|\zeta_0|^{-1}-1)\big) = \frac{f^2(\zeta_0)}{|\zeta_0|}\sum_{k=1}^{n}(1-|\zeta_0|)^{k+1}. \tag{4.40}$$

Therefore, taking into account (4.38), (4.39) and (4.40) together with (3.1) and the fact $|f(x)| \leq C|x|^{1/2}$, we infer that

$$\left\| \mathbf{E}_0 \Big(S_n(f) \big(f(\zeta_n)(|\zeta_n|^{-1} - 1) - f(\zeta_0)(|\zeta_0|^{-1} - 1) \big) \Big) \right\|_{p/2} \ll \sum_{k=1}^{n} \sum_{\ell=k}^{n} \int_0^1 (1-x)^\ell x^a dx$$

$$+ n \Big(\int_0^1 (1-x)^{np/2} x^{a-1} dx \Big)^{2/p} + \sum_{k=1}^{n} \Big(\int_0^1 (1-x)^{kp/2} x^{a-1} dx \Big)^{2/p} .$$

The bound (4.32) then follows from the properties of the Beta functions. This ends to proof of (4.29). It remains to use Theorem 2.2 combined with the bounds (4.28) and (4.29) and the assumption that $a > (p-2)/2$, to end the proof of the corollary. Indeed when $p = 4$, this directly implies that $S_n(f)$ satisfies the strong approximation (1.1) with $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$ and rate $b_n = n^{1/4}(\log n)^{3/4+\varepsilon}$ for any $\varepsilon > 0$. Now when $p \in]2, 4[$, since $a > (p-2)/2$, the arguments also hold for some $p' > p$ leading to the rate $b_n = n^{1/p}$ in (1.1). $\square$

4.8 Proof of Corollaries 3.3 and 3.4.

To prove Corollary 3.3, it suffices to prove that the sequence $X_i = f(\xi_i) - m(f)$ satisfies the conditions (2.10) and (2.11) of Corollary 2.6 with $t = 1$. Note that, according to the proofs of Lemmas 5.2 and 5.3 in Dedecker and Rio (2008) and to their inequality (5.18),

$$\|\mathbf{E}(X_n|\xi_0)\|_\infty \leq C_1(f) \sum_{k \in \mathbf{Z}^*} |k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)} |\cos(2\pi ka)|^n ,$$

and

$$\sup_{i \geq j \geq n} \|\mathbf{E}(X_i X_j | \mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_\infty \leq C_2(f) \sum_{k \in \mathbf{Z}^*} |k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)} |\cos(2\pi ka)|^n .$$

Hence to verify the conditions (2.10) and (2.11), it suffices to take γ such that $2/p - \gamma = 1/p - \gamma^{-1}(1 - 2/p)$ that is

$$\gamma = \frac{1 + \sqrt{1 + 4p(p-2)}}{2p} , \tag{4.41}$$

and to show that

$$\sum_{n \geq 1} n^{\gamma - 2/p} \sum_{k \in \mathbf{Z}^*} |k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)} |\cos(2\pi ka)|^n < \infty . \tag{4.42}$$

Note first that by the properties of the Gamma function there exists a positive constant K such that, for any irrational number a

$$\sum_{n \geq 1} n^{\gamma - 2/p} |\cos(2\pi ka)|^n \leq \frac{K}{(1 - |\cos(2\pi ka)|)^{\gamma - 2/p + 1}} .$$

Since $(1 - |\cos(\pi u)|) \geq \pi(d(u, \mathbf{Z}))^2$, we derive that

$$\sum_{n \geq 1} n^{\gamma - 2/p} \sum_{k \in \mathbf{Z}^*} |k|^{-s-\varepsilon} |\cos(2\pi ka)|^n \leq \frac{K}{\pi^{\gamma - 2/p + 1}} \sum_{k \in \mathbf{Z}^*} \frac{|k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)}}{(d(2ka, \mathbf{Z}))^{2\gamma - 4/p + 2}} .$$

Note that, if a is badly approximable by rationals, then so is $2a$. Therefore if a satisfies (3.7), proceeding as in the proof of Lemma 5.1 in Dedecker and Rio (2008), we get that

$$\sum_{k=2^N}^{2^{N+1}-1} \frac{1}{(d(2ka, \mathbf{Z}))^{2\gamma - 4/p + 2}} \ll 2^{(2\gamma - 4/p + 2)N} .$$

Therefore

$$\sum_{n \geq 1} n^{\gamma - 2/p} \sum_{k \in \mathbf{Z}^*} |k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)} |\cos(2\pi ka)|^n$$

$$\ll \sum_{N \geq 0} 2^{(2\gamma - 4/p + 2)N} \max_{2^N \leq k \leq 2^{N+1}} |k|^{-s} \big(\log(1 + |k|) \big)^{-(1+\varepsilon)} < \infty ,$$

proving, by the choices of s and γ, that the condition (4.42) is satisfied. This ends the proof of Corollary 3.3. Corollary 3.4 follows the same lines and the main point is to prove that for γ defined by (4.41) and a satisfying (3.8),

$$\sum_{n \geq 1} n^{\gamma - 2/p} \sum_{k \in \mathbf{Z}^*} |k|^{-s-\varepsilon} |\cos(2\pi ka)|^n < \infty .$$

When $p = 4$ this will give the rate $b_n = n^{1/4}(\log n)^{3/4 + \delta}$, for a $\delta > 0$, in the strong invariance principle (1.1). For $p \in]2, 4[$, the arguments also hold for some $p' > p$ leading to the rate $b_n = n^{1/p}$ in (1.1). $\square$

4.9 Proof of Corollary 3.9.

To prove the result, we apply Corollary 2.7, so that we need to check the conditions (2.9), (2.12) and (2.13). Notice that since $p > 2$, (2.12) and the first part of (2.9) are both satisfied if

$$\sum_{n \geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{\frac{(t-1)p}{2}}} \max_{0 \leq k \leq n} \|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)\|_{p/2}^{p/2} < \infty . \tag{4.43}$$

We start by proving this condition. Using Proposition 1 in Dedecker and Doukhan (2003), we get that for any integers k and n,

$$\|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)\|_{p/2} = \mathbf{E}\big(Z_0|\mathbf{E}_0(X_k)|X_n\big) \leq \int_0^{G(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1)} Q_{|Z_0\mathbf{E}_0(X_k)|}(u) Q_{|X_n|}(u) du ,$$

where $Z_0 = \|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)\|_{p/2}^{1-p/2}|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)|^{p/2-1}\mathrm{sign}(\mathbf{E}_0(X_n))$. Next using successively Lemma 2.1 and Inequality (4.6) in Rio (2000), we obtain that

$$\|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)\|_{p/2} \leq \int_0^{G(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1)} Q_{|Z_0|}(u) Q^2(u) du .$$

Now, applying Hölder's inequality and noting that $\|Z_0\|_{p/(p-2)} = 1$, we obtain that

$$\max_{0 \leq k \leq n} \|\mathbf{E}_0(X_k)\mathbf{E}_0(X_n)\|_{p/2}^{p/2} \leq \int_0^{G(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1)} Q^p(u) du ,$$

so that (4.43) is satisfied under (3.15), since $\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1 \leq \lambda_2(n)$. Let us prove now that the second part of (2.9) is satisfied under (3.15). Using the same arguments as before, we infer that

$$\|\mathbf{E}_0(X_n)\|_2 \leq \Big(\int_0^{G(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1)} Q^2(u) du \Big)^{1/2}$$

$$\leq \Big(\int_0^1 Q^p(u) du \Big)^{1/p - 1/2} \Big(\int_0^{G(\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1)} Q^p(u) du \Big)^{1/2} .$$

For the second inequality we have used Hölder's inequality together with the fact that, since Q is nonincreasing, for any $x \in [0, 1]$,

$$\int_0^1 Q^p(u) du \leq \frac{1}{x} \int_0^x Q^p(u) du .$$

Therefore the second part of (2.9) will be satisfied if

$$\sum_{n \geq 2} \frac{n^{3p/4}}{n^2 (\log n)^{\frac{(t-1)p}{2}}} \Big(\int_0^{\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1} Q^{p-1} \circ G(u) du \Big)^{p/4} < \infty \,,$$

which clearly holds if (3.15) does, since $\|\mathbf{E}(X_n|\mathcal{F}_0)\|_1 \leq \lambda_2(n)$, $(\int_0^{\lambda_2(n)} Q^{p-1} \circ G(u)du)_n$ is nonincreasing and $p > 2$. It remains to prove that (2.13) holds if (3.15) does. Let $B_{i,j} = X_i X_j - \mathbf{E}(X_i X_j)$ and $B_0 = |\mathbf{E}_0(B_{i,j})|^{p/2-1} \|\mathbf{E}_0(B_{i,j})\|_{p/2}^{1-p/2} \text{sign}(\mathbf{E}_0(B_{i,j}))$. Applying again Proposition 1 in Dedecker and Doukhan (2003), we derive that

$$\|\mathbf{E}(X_i X_j|\mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_{p/2} = \mathbf{E}(B_0 B_{i,j}) \leq \int_0^{G_{B_{i,j}}(\|\mathbf{E}_0(B_{i,j})\|_1)} Q_{B_0}(u) Q_{B_{i,j}}(u) du \,.$$

Since $\|B_0\|_{p/(p-2)} = 1$, Hölder's inequality gives

$$\|\mathbf{E}(X_i X_j|\mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_{p/2} \leq \Big(\int_0^{G_{B_{i,j}}(\|\mathbf{E}_0(B_{i,j})\|_1)} Q_{B_{i,j}}^{p/2}(u) du \Big)^{2/p} \,. \tag{4.44}$$

Now, since $Q_{B_{i,j}}(u) \leq Q_{|X_i X_j|}(u) + \mathbf{E}(|X_i X_j|)$, we get that

$$\int_0^x Q_{B_{i,j}}(u) du \leq 2 \int_0^x Q_{|X_i X_j|}(u) du \leq 2 \int_0^x Q^2(u) du \,,$$

where for the last inequality we have used Lemma 2.1 in Rio (2000). It follows that $G_2(u/2) \leq G_{B_{i,j}}(u)$ where G_2 is the inverse of $x \mapsto H_2(x) = \int_0^x Q^2(u) du$. In particular, $G_{B_{i,j}}(u) \geq G_2(u/c)$ for any $c \geq 2$. Since Q is non increasing, it follows that for any $c \geq 2$,

$$\int_0^{G_{B_{i,j}}(\|\mathbf{E}_0(B_{i,j})\|_1)} Q_{B_{i,j}}^{p/2}(u) du = \int_0^{\|\mathbf{E}_0(B_{i,j})\|_1} Q_{B_{i,j}}^{p/2-1} \circ G_{B_{i,j}}(u) du$$

$$\leq \int_0^{\|\mathbf{E}_0(B_{i,j})\|_1} Q_{B_{i,j}}^{p/2-1} \circ G_2(u/c) du \,,$$

so that

$$\int_0^{G_{B_{i,j}}(\|\mathbf{E}_0(B_{i,j})\|_1)} Q_{B_{i,j}}^{p/2}(u) du$$

$$\leq c \int_0^{\|\mathbf{E}_0(B_{i,j})\|_1/c} Q_{B_{i,j}}^{p/2-1} \circ G_2(u) du = c \int_0^{G_2(\|\mathbf{E}_0(B_{i,j})\|_1/c)} Q_{B_{i,j}}^{p/2-1}(v) Q^2(v) dv. \tag{4.45}$$

Notice now that $Q_{B_{i,j}}^{p/2}(u) \leq 2^{p/2} Q_{|X_i X_j|}^{p/2}(u) + 2^{p/2} \mathbf{E}(|X_i X_j|^{p/2})$, which implies that

$$\int_0^x Q_{B_{i,j}}^{p/2}(u) du \leq 2^{1+p/2} \int_0^x Q_{|X_i X_j|}^{p/2}(u) du \leq 2^{1+p/2} \int_0^x Q^p(u) du \,, \tag{4.46}$$

where for the last inequality we have used Lemma 2.1 in Rio (2000). Therefore starting from (4.44) and using (4.45) together with Hölder's inequality and (4.46), we get that

$$\|\mathbf{E}(X_i X_j|\mathcal{F}_0) - \mathbf{E}(X_i X_j)\|_{p/2}^{p/2} \leq c^{p/2} 2^{(p^2-4)/(2p)} \int_0^{G_2(\|\mathbf{E}_0(B_{i,j})\|_1/c)} Q^p(v) dv \,. \tag{4.47}$$

We show now that for any $i \geq n$ and $j \geq n$, there exists $c \geq 2$ such that

$$G_2(\|\mathbf{E}_0(B_{i,j})\|_1/c) \leq G(\theta_2(n)) \,. \tag{4.48}$$

With this aim, let $M = Q \circ G(\theta_2(n))$ and, for $i \in \mathbf{Z}$, define the variables

$$X_i' = X_i \mathbf{I}_{|X_i| \leq M} \quad \text{and} \quad X_i'' = X_i \mathbf{I}_{|X_i| > M}.$$

With this notation and using the stationarity, we have that

$$
\begin{aligned}
\|\mathbf{E}_0(B_{i,j})\|_1 &\leq \|\mathbf{E}_0(X_i'X_j') - \mathbf{E}(X_i'X_j')\|_1 + 4M\mathbf{E}|X_0''| + 2\mathbf{E}|X_0''|^2 \\
&\leq \|\mathbf{E}_0(X_i'X_j') - \mathbf{E}(X_i'X_j')\|_1 + 6\mathbf{E}(X_0^2 \mathbf{I}_{|X_0|>M}).
\end{aligned} \tag{4.49}
$$

Notice now that for any reals x and y:

$$
\begin{aligned}
&\left| |x\mathbf{I}_{|x| \leq M} + y\mathbf{I}_{|y| \leq M}|^2 - |x+y|^2 \mathbf{I}_{|x+y| \leq 2M} \right| \\
&\leq |y|^2 \mathbf{I}_{|y| \leq M} \mathbf{I}_{|x| > M} + |x|^2 \mathbf{I}_{|x| \leq M} \mathbf{I}_{|y| > M} + |x+y|^2 \mathbf{I}_{|x+y| \leq 2M} \mathbf{I}_{|x| > M} \\
&\quad + |x+y|^2 \mathbf{I}_{|x+y| \leq 2M} \mathbf{I}_{|y| > M} + |x+y|^2 \mathbf{I}_{|x+y| \leq 2M} \mathbf{I}_{|x| > M} \mathbf{I}_{|y| > M} \\
&\leq 5M^2 \mathbf{I}_{|x| > M} + 9M^2 \mathbf{I}_{|y| > M}.
\end{aligned} \tag{4.50}
$$

In addition setting for any real $u \geq 0$ and any $T > 0$, $g_T(u) = u^2 \wedge T^2$, we have

$$\left| |x+y|^2 \mathbf{I}_{|x+y| \leq 2M} - g_{2M}(|x+y|) \right| \leq 4M^2 \mathbf{I}_{|x+y| > 2M}. \tag{4.51}$$

Using (4.50) and (4.51), the fact that $4xy = |x+y|^2 - |x-y|^2$, and the stationarity of $(X_i)_{i \in \mathbf{Z}}$, it follows that

$$
\sup_{j \geq k \geq q} \|4X_j'X_k' - g_{2M}(|X_j + X_k|) + g_{2M}(|X_j - X_k|)\|_1
$$
$$
\leq 44M\mathbf{E}(|X_0|\mathbf{I}_{|X_0|>M}) \leq 44\mathbf{E}(X_0^2 \mathbf{I}_{|X_0|>M}). \tag{4.52}
$$

In addition, since g_{2M} is $2M$-Lipschitz, it follows that

$$\sup_{j \geq k \geq n} \|\mathbf{E}(g_{2M}(|X_j + X_k|)|\mathcal{F}_0) - \mathbf{E}(g_{2M}(|X_j + X_k|))\|_1 \leq 2M\theta_2(n), \tag{4.53}$$

and the same holds true with $|X_j + X_k|$ in place of $|X_j - X_k|$. Starting from (4.49) and using (4.52), (4.53), together with the fact that $\mathbf{E}(X_0^2 \mathbf{I}_{|X_0|>M}) \leq \int_0^{G(\theta_2(n))} Q^2(u)du$, we obtain that for any $i \geq n$ and $j \geq n$,

$$\|\mathbf{E}_0(B_{i,j})\|_1 \leq Q \circ G(\theta_2(n))\theta_2(n) + 28 \int_0^{G(\theta_2(n))} Q^2(u)du \leq 29 \int_0^{\theta_2(n)} Q \circ G(u)du$$

so that

$$\|\mathbf{E}_0(B_{i,j})\|_1 \leq 29 \int_0^{G(\theta_2(n))} Q^2(u)du = 29H_2(G(\theta_2(n))),$$

which proves (4.48) with $c = 29$. Starting from (4.47), using (4.48) and the fact that $\theta_2(n) \leq \lambda_2(n)$, it follows that (2.13) will be satisfied if

$$\sum_{n \geq 2} \frac{n^{p-2}}{(\log n)^{\frac{(t-1)p}{2}}} \int_0^{\lambda_2(n)} Q^{p-1} \circ G(u)du < \infty,$$

which holds if (3.15) does, since $(\int_0^{\lambda_2(n)} Q^{p-1} \circ G(u)du)_n$ is nonincreasing and $p > 2$. This ends the proof of Corollary 3.9. $\square$

4.10 Proof of Corollary 3.16.

It comes from an application of Corollary 2.7. We omit the details since to prove that (2.9), (2.12) and (2.13) are satisfied if (3.21) is, we follow the lines of the beginning of the proof of Corollary 3.9 and, to take care of the covariances terms, we use the arguments developped in the proof of Proposition 5.3 in Merlevède and Rio (2012). $\square$

5 Appendix

The next proposition gives simple criteria to obtain rates of convergence in the almost sure invariance principle for a strictly stationary sequence of martingale differences. It is based on Theorem 2.1 in Shao (1993) that gives, with the help of the Skorohod-Strassen theorem, sufficient conditions for partial sums of non necessarily stationary sequences of martingale differences to satisfy the almost sure invariance principle with rates.

Proposition 5.1. *Let $d_i = d_0 \circ T^i$ where d_0 is an element of $H_0 \ominus H_{-1}$. Assume furthermore that d_0 is in $\mathbf{L}^p$ with $p \in]2, 4]$. Let $M_n = \sum_{i=1}^n d_i$. Let ψ be a nondecreasing positive function. Assume also that there exists a positive constant C such that $\psi(2n) \leq C\psi(n)$ for all $n \geq 1$ and $\sum_{n>0} \psi^{-p/2}(n) < \infty$. Define $v_n = n^{-1}\psi^{p/2}(n)\sum_{k \geq n} \psi^{-p/2}(k)$. Assume that*

$$\sum_{n \geq 1} v_n^{p/2} \frac{\left\|\mathbf{E}(M_n^2|\mathcal{F}_0) - \mathbf{E}(M_n^2)\right\|_{p/2}^{p/2}}{n\psi^{p/2}(n)} < \infty. \tag{5.1}$$

Enlarging Ω if necessary, there exists a sequence $(Z_i)_{i \geq 1}$ of iid gaussian random variables with zero mean and variance $\mathbf{E}(d_0^2)$ such that

$$\sup_{1 \leq k \leq n} \left| M_k - \sum_{i=1}^k Z_i \right| = o\left(\psi^{1/2}(n)\left| \log \frac{n}{\psi(n)} + \log\log(\psi(n)) \right|^{1/2}\right) \text{ a.s. as } n \to \infty.$$

Proof of Proposition 5.1. From Theorem 2.1 in Shao (1993), it suffices to verify that

$$\sum_{i=1}^n (\mathbf{E}(d_i^2|\mathcal{F}_{i-1}) - \mathbf{E}(d_i^2)) = o(\psi(n)) \text{ almost surely}. \tag{5.2}$$

Since ψ is nondecreasing and $\psi(2n) \leq C\psi(n)$ for any $n \geq 1$ and some fixed $C > 0$, the almost sure convergence (5.2) will follow if we can prove that, for every $\varepsilon > 0$,

$$\sum_{r \geq 1} \mathbf{P}\left(\max_{1 \leq k \leq 2^{r+1}} \left| \sum_{i=1}^k \mathbf{E}(d_i^2|\mathcal{F}_{j-1}) - \mathbf{E}(d_i^2)) \right| \geq \varepsilon\psi(2^r)\right) < \infty.$$

Therefore (5.2) and then the proposition will be proved if we can show that

$$\sum_{r \geq 1} \frac{1}{\psi^{p/2}(2^r)} \mathbf{E}\left(\max_{1 \leq k \leq 2^r} \left| \sum_{i=1}^k \mathbf{E}(d_i^2|\mathcal{F}_{i-1}) - \mathbf{E}(d_i^2)) \right|^{p/2}\right) < \infty.$$

Applying Theorem 3 in Wu and Zhao (2008) (since $1 < p/2 \leq 2$) and using the martingale property, we get that

$$\mathbf{E}\left(\max_{1 \leq k \leq 2^r} \left| \sum_{i=1}^k (\mathbf{E}(d_i^2|\mathcal{F}_{i-1}) - \mathbf{E}(d_i^2)) \right|^{p/2}\right)$$
$$\ll 2^r\|d_1^2\|_{p/2}^{p/2} + 2^r\left(\sum_{k=0}^{r-1} \frac{\|\mathbf{E}(M_{2^k}^2|\mathcal{F}_0) - \mathbf{E}(M_{2^k}^2)\|_{p/2}}{2^{2k/p}}\right)^{p/2}.$$

Since ψ is nondecreasing and since $\sum_n \psi^{-p}(n) < \infty$, it follows that $\sum_r 2^r\psi^{-p}(2^r) < \infty$. Hence the result will be proved if we can show that

$$\sum_{r \geq 1} \frac{1}{\psi^{p/2}(2^r)}\left(\sum_{k=0}^{r-1} 2^{2(r-k)/p}\|\mathbf{E}(M_{2^k}^2|\mathcal{F}_0) - \mathbf{E}(M_{2^k}^2)\|_{p/2}\right)^{p/2} < \infty.$$

But the latter is implied by (5.1), which may be proved as in the proof of (3) $\Rightarrow$ (2) of Proposition 2.2 in Cuny (2011) (see Appendix A there). $\qquad\square$

The following lemma is useful to compare conditions involving the projection operator P_0 with a mixingale-type condition.

Lemma 5.2. *Let $p \geq 2$. For any real $1 \leq q \leq p$ and any positive integer n,*

$$\sum_{k \geq 2n} \|P_0(X_k)\|_p^q \ll \sum_{k \geq n} \frac{\|\mathbf{E}(X_k|\mathcal{F}_0)\|_p^q}{k^{q/p}} \,.$$

Proof of Lemma 5.2. If the upper bound is infinite, the inequality is clear. Let us consider now the case where the upper bound is finite. In that case, since $\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p$ is nonincreasing, we infer that $n^{1-q/p}\|\mathbf{E}(X_n|\mathcal{F}_0)\|_p^q$ converges to 0 as n tends to infinity, so that $\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0$ almost surely. Now, for any sequence of nonnegative numbers $(u_k)_{k \in \mathbf{N}}$ and any real $\alpha > 1$, the following inequality holds: for any positive integer n, there exists a positive constant C_α depending only on α such that

$$\sum_{k \geq 2n} u_k \leq C_\alpha \sum_{k \geq n+1} \frac{1}{k^{1/\alpha}} \Big(\sum_{\ell \geq k} u_\ell^\alpha \Big)^{1/\alpha} \tag{5.3}$$

(to prove this inequality, it suffices to slightly adapt the proof of Lemma 6.1 in Dedecker *et al.* (2011) and to use the fact that the sequence $\big(\sum_{\ell \geq k} u_\ell^\alpha \big)_{k \geq 1}$ is nonincreasing). We proceed now as in the proof of Remark 3.3 in Dedecker *et al.* (2011). We first consider the case $p > q$. Applying inequality (5.3) with $u_k = \|P_0(X_k)\|_p^q$ and $\alpha = p/q$, we get that

$$\sum_{k \geq 2n} \|P_0(X_k)\|_p^q \ll \sum_{k \geq n+1} \frac{1}{k^{q/p}} \Big(\sum_{\ell \geq k} \|P_0(X_\ell)\|_p^p \Big)^{q/p} \tag{5.4}$$

Next, using the stationarity and applying the Rosenthal's inequality given in Theorem 2.12 of Hall and Heyde (1980), we then derive that for any $p \in [2, \infty[$, there exists a constant c_p depending only on p such that

$$\sum_{\ell \geq k} \|P_0(X_\ell)\|_p^p = \sum_{\ell \geq k} \|P_{-\ell}(X_0)\|_p^p \leq c_p \Big\| \sum_{\ell \geq k} P_{-\ell}(X_0) \Big\|_p^p = c_p \|\mathbf{E}(X_k|\mathcal{F}_0)\|_p^p, \tag{5.5}$$

the last equality being true because $\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0$ almost surely. Therefore when $p > q$, the lemma follows by taking into account (5.5) in (5.4). Now when $p = q$, inequality (5.5) together with the fact that $\|\mathbf{E}(X_k|\mathcal{F}_0)\|_p$ is nonincreasing implies the result. $\square$

The next lemma is useful to deal with sequences having a subadditive property (see Lemma 38 in Merlevède and Peligrad (2012) for a proof).

Lemma 5.3. *Let $(V_i)_{i \geq 0}$ be a sequence of non negative numbers such that $V_0 = 0$ and for all $i, j \geq 0$,*

$$V_{i+j} \leq C(V_i + V_j) \,,$$

where $C \geq 1$ is a constant not depending on i and j. Then For any integer $r \geq 1$, any integer n satisfying $2^{r-1} \leq n < 2^r$ and any real $q \geq 0$

$$\sum_{i=0}^{r-1} \frac{1}{2^{iq}} V_{2^i} \leq C 2^{q+2} (2^{q+1} - 1)^{-1} \sum_{k=1}^{n} \frac{V_k}{k^{1+q}} \,.$$

References

[1] Aue, A., Berkes, I. and Horváth, L.: Strong approximation for the sums of squares of augmented GARCH sequences. *Bernoulli* **12**, (2006), 583-608. MR-2248229

[2] Berkes, I and Philipp, W.: Approximation theorems for independent and weakly dependent random vectors. *Ann. Probab. 7*, (1979), 29-54. MR-0515811

[3] Bradley, R.C.: Approximation theorems for strongly mixing random variables. *Michigan Math. J.* **30**, (1983), 69-81. MR-0694930

[4] Cuny, C.: Almost everywhere convergence of generalized ergodic transforms for invertible power-bounded operators in L^p. *Coll. Math.* **124**, (2011), 61-77. MR-2820555

[5] Csörgő, M. and Horváth, L.: Limit Theorems in Change-Point Analysis. With a foreword by David Kendall. Wiley Series in Probability and Statistics. *John Wiley & Sons, Ltd., Chichester*, 1997. xvi+414 pp. MR-2743035

[6] Dabrowski, A. R.: A note on a theorem of Berkes and Philipp for dependent sequences. *Statist. Probab. Lett.* **1**, (1982), 53-55. MR-0687971

[7] Davydov, Y. A.: Mixing conditions for Markov chains. *Theor. Probab. Appl.* **18**,(1973), 312-328. MR-0321183

[8] Dedecker, J. and Doukhan, P.: A new covariance inequality and applications. *Stochastic Process. Appl.* **106**, (2003), 63-80. MR-1983043

[9] Dedecker, J., Gouëzel, S. and Merlevède, F.: Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.* **46**, (2010), 796-821. MR-2682267

[10] Dedecker, J., Merlevède, F. and Peligrad, M.: Invariance principles for linear processes with application to isotonic regression. *Bernoulli.* 17, (2011), 88-113. MR-2797983

[11] Dedecker, J., Merlevède, F. and Rio, E.: Rates of convergence for minimal distances in the central limit theorem under projective criteria *Electron. J. Probab.*, **14**, (2009), 978-1011. MR-2506123

[12] Dedecker, J. and Prieur, C.: New dependence coefficients. Examples and applications to statistics. *Probab. Theory Related Fields* **132**, (2005), 203-236. MR-2199291

[13] Dedecker, J. and Rio, E.: On the functional central limit theorem for stationary processes. *Ann. Inst. Henri Poincaré, Prob. Stat.* **36**, (2000), 1-34. MR-1743095

[14] Dedecker, J. and Rio, E.: On mean central limit theorems for stationary sequences. *Ann. Inst. Henri Poincaré, Prob. Stat.* **44**, (2008), 693-726. MR-2446294

[15] Derriennic, Y. and Lin, M.: The central limit theorem for Markov chains with normal transition operators, started at a point. *Probab. Theory Relat. Fields.* **119**, (2001), 508-528. MR-1826405

[16] Doukhan, P., Massart, P. and Rio, E.: The functional central limit theorem for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.* **30**, (1994), 63-82. MR-1262892

[17] Eberlein, E.: On strong invariance principles under dependence assumptions. *Ann. Probab.* **14**, (1986), 260-270. MR-0815969

[18] Gordin, M. I.: The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR.* **188**, (1969), 739-741. MR-0251785

[19] Gouëzel, S. Almost sure invariance principle for dynamical systems by spectral methods, *Ann. Probab.* **38**, (2010), 1639-1671. MR-2663640

[20] Hall, P. and Heyde, C. C.: Martingale Limit Theory and its Applications. *Academic Press, New York-London*, 1980. xii+308 pp. MR-0624435

[21] Heyde, C. C.: On the central limit theorem for stationary processes. *Z. Wahrscheinlichkietstheorie und Verw. Gebiete.* **30**, (1974), 315-320 MR-0372955

[22] Heyde, C.C.: On the central limit theorem and iterated logarithm law for stationary processes. *Bull. Austral. Math. Soc.* **12**, (1975), 1-8. MR-0372954

[23] Horváth, L. and Steinebach, J.: Testing for changes in the mean or variance of a stochastic process under weak invariance. Prague Workshop on Perspectives in Modern Statistical Inference: Parametrics, Semi-parametrics, Non-parametrics. *J. Statist. Plann. Inference* **91**, (2000), 365-376. MR-1814790

[24] Ibragimov, I.A.: Some limit theorems for stationary processes. *Theory Probab. Appl.* **7**, (1962), 349-382. MR-0148125

[25] Komlós, J., Major, P. and Tusnády, G.: An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34**, (1976), 33-58. MR-0402883

[26] Liu, W. and Lin, Z.: Strong approximation for a class of stationary processes. *Stochastic Process. Appl.* **119**, (2009), 249-280. MR-2485027

[27] Major, P.: The approximation of partial sums of independent RV's. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **35**, (1976), 213-220. MR-0415743

[28] Merlevède, F. and Peligrad, M.: Rosenthal inequalities for martingales and stationary sequences and examples. *to appear in Ann. Probab. (on line)* (2012). arXiv:1103.3242

[29] Merlevède, F., Peligrad, C. and Peligrad, M.: Almost Sure Invariance Principles via Martingale Approximation. *Stochastic Process. Appl.* **122**, (2012), 170-190.

[30] Merlevède, F. and Rio, E.: Strong approximation of partial sums under dependence conditions with application to dynamical systems. *Stochastic Process. Appl.* **122**, (2012), 386-417.

[31] Peligrad, M. and Utev, S.: A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33**, (2005), 798-815. MR-2123210

[32] Nummelin, E. (1984). General irreducible Markov chains and nonnegative operators. Cambridge Tracts in Mathematics, **83**, *Cambridge University Press, Cambridge*, 1984. xi+156 pp. MR-0776608

[33] Philipp, W. and Stout, W.F.: Almost sure invariance principle for partial sums of weakly dependent random variables. *Mem. of the Amer. Math. Soc.* **161**, (1975), Providence, RI: Amer. Math. Soc. MR-0433597

[34] Rio, E.: The functional law of the iterated logarithm for stationary strongly mixing sequences. *Ann. Probab.* **23**, (1995), 1188-1203. MR-1349167

[35] Rio, E.: Théorie asymptotique des processus aléatoires faiblement dépendants. Mathématiques et Applications **31**, *Springer-Verlag, Berlin*, 2000. x+169 pp. MR-2117923

[36] Rio, E.: Moment inequalities for sums of dependent random variables under projective conditions. *J. Theoret. Probab.* **22**, (2009), 146-163. MR-2472010

[37] Rosenblatt, M.: A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U. S. A.* **42**, (1956), 43-47. MR-0074711

[38] Rüschendorf, L. The Wasserstein Distance and Approximation Theorems. *Z. Wahrsch. Verw. Gebiete 70*, (1985), 117-129. MR-0795791

[39] Sakhanenko, A. I.: Simple method of obtaining estimates in the invariance principle. Probability theory and mathematical statistics (Kyoto, 1986), *Lecture Notes in Math.* **1299**, (1988), 430-443, Springer, Berlin. MR-0936018

[40] Schmidt, W.M.: Diophantine approximation. Lecture Notes in Mathematics, **785**. *Springer, Berlin*, 1980. x+299 pp. MR-0568710

[41] Shao, Q.M.: Almost sure invariance principles for mixing sequences of random variables. *Stochastic Process. Appl.* **48**, (1993), 319-334. MR-1244549

[42] Shao, Q.M. and Lu, C.R. Strong approximations for partial sums of weakly dependent random variables. *Sci. Sinica Ser. A* **30**, (1987), 575-587. MR-1000718

[43] Strassen, V.: An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3**, (1964), 211-226. MR-0175194

[44] Tuominen, P. and R. L. Tweedie, R. L.: Subgeometric rates of convergence of f-ergodic Markov chains, *Adv. Appl. Prob.* **26**, (1994), 775-798. MR-1285459

[45] Utev, S. A.: Inequalities for sums of weakly dependent random variables and estimates of the rate of convergence in an invariance principle, *Trudy Inst. Mat.* **3**, (1984), 50-77. MR-0749758

[46] Wu, W. B.: Strong invariance principles for dependent random variables. *Ann. Probab.* **35**, (2007), 2294-2320. MR-2353389

[47] Wu, W. B. and Zhao, Z.: Inference of trends in time series. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **69**, (2007), 391-410. MR-2323759

[48] Wu, W. B. and Zhao, Z.: Moderate deviations for stationary processes. *Statist. Sinica* **18**, (2008), 769-782. MR-2411619

[49] Zhao, O and Woodroofe, M.: Law of the iterated logarithm for stationary processes. *Ann. Probab.* **36**, (2008), 127-142. MR-2370600