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Small-time Behaviour of Lévy Processes

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Abstract. In this paper a necesary and sufficient condition is established for the probability that a Lévy process is positive at time t to tend to 1 as t tends to 0. This condition is expressed in terms of the characteristics of the process, and is also shown to be equivalent to two probabilistic statements about the behaviour of the process for small t.

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1 Introduction

The quantity $\rho(t) = P(X_t > 0)$ where $X = (X_t, t \geq 0)$ is a Lévy process is of fundamental importance in fluctuation theory. For example, combining results in [1] and [2] shows that, *both as* $t \to \infty$ *and as* $t \downarrow 0$,

$$\begin{aligned} \rho(t) &\to \rho \in [0,1] \\ &\Longleftrightarrow \frac{1}{t}\int_0^t \rho(s)ds \to \rho \\ &\Longleftrightarrow \frac{1}{t}\int_0^t \mathbf{1}_{\{X_s>0\}}ds \xrightarrow{d} A_\rho, \end{aligned} \tag{1}$$

where A_ρ denotes a random variable with an arc-sine law of parameter ρ if $0 < \rho < 1$, and a random variable degenerate at ρ if $\rho = 0, 1$. It would therefore be useful to find a necessary and sufficient condition for (1) to hold, ideally expressed in terms of the *characteristics* of X, that is its Lévy measure Π, its Brownian coefficient σ^2, and γ, the coefficient of the linear term in the Lévy-Itô decomposition (2) below.

This problem is obviously difficult, and has so far only been solved for large t in the special case $\rho = 0, 1$. This result is in Theorem 3.3 of [5], and in an extended form in Theorem 1.3 in [4]. In both cases the results are deduced from the corresponding results for random walks due to Kesten and Maller in [7] and [8]. Here we consider the corresponding question for small t, where apparently the large t results will have no relevance, but in fact it turns out that there is a striking formal similarity, both in the statement and proof.

Our Lévy process will be written as

$$X_t = \gamma t + \sigma B_t + Y_t^{(1)} + Y_t^{(2)}, \tag{2}$$

where B is a standard BM, $Y^{(1)}$ is a pure jump martingale formed from the jumps whose absolute values are less than or equal to 1, $Y^{(2)}$ is a compound Poisson process formed from the jumps whose absolute values exceed 1, and $B, Y^{(1)}$, and $Y^{(2)}$ are independent.

We denote the Lévy measure of X by Π, and introduce the tail functions

$$N(x) = \Pi\{(x,\infty)\},\ M(x) = \Pi\{(-\infty,-x)\},\ x > 0, \tag{3}$$

and the tail sum and difference

$$T(x) = N(x) + M(x),\ D(x) = N(x) - M(x),\quad x > 0. \tag{4}$$

The rôles of truncated first and second moments are played by

$$A(x)=\gamma+D(1)+\int_1^x D(y)dy,\quad U(x)=\sigma^2+2\int_0^x yT(y)dy,\ x>0. \tag{5}$$

We will denote the jump process of X by $\Delta=(\Delta_s, s\geq 0)$, and put

$$\Delta_t^{(1)}=\sup_{s\leq t}\Delta_s^- \tag{6}$$

for the magnitude of the largest negative jump which occurs by time t.

For the case $t\downarrow 0$, a sufficient condition for (1) with $\rho=1$ was given in Theorem 2.3 of [5]; the following, together with Lemma 5 shows that the condition given there is also necessary:

Theorem 1 *Suppose that the Lévy process X has $\sigma=0, \Pi(\mathbb{R})=\infty$, and $M(0+)>0$; then the following are equivalent.*

$$\rho_t=P(X_t>0)\to 1 \text{ as } t\downarrow 0; \tag{7}$$

$$\frac{X_t}{\Delta_t^{(1)}}\xrightarrow{P}\infty \text{ as } t\downarrow 0; \tag{8}$$

for some deterministic d which decreases to 0 and is regularly varying of index 1 at 0, $\dfrac{X_t}{d(t)}\xrightarrow{P}\infty$ *as* $t\downarrow 0$; (9)

and

$$\frac{A(x)}{\sqrt{U(x)M(x)}}\to+\infty \text{ as } x\downarrow 0. \tag{10}$$

Remark 2 *None of the above assumptions are really restrictive. First, if $\sigma\neq 0$, it was shown in [5] that $P(X_t>0)\to 1/2$. It was also shown there that when $M(0+)=0$, i.e. X is spectrally positive, and $N(0+)=\infty$, then (7) occurs iff X is a subordinator iff $A(x)\geq 0$ for all small x, and it is easy to see that these are equivalent to (9) in this case. (Of course (8) is not relevant here, as $\Delta_t^{(1)}\equiv 0$.) Finally the case when $\Pi(\mathbb{R})<\infty$ is of no real interest; then X is a compound Poisson process plus linear drift, and the behaviour of ρ_t as $t\downarrow 0$ is determined by whether the drift is positive or not.*

Remark 3 *Comparing the above results with the large-time results we see that each of (7)-(10) has a formally similar counterpart at ∞. (Actually the counterpart of (8) was omitted in [4], but it is easy to establish.) One difference is that (7) as $t \to \infty$ implies that $X_t \stackrel{P}{\to} \infty$, and of course this can't happen as $t \downarrow 0$. At first sight the appearance of A in both (10) and its counterpart at ∞ is surprising. However this can be understood by realising that A acts both as a generalised mean at ∞ and a generalised drift at 0. To be precise, $t^{-1}X_t \stackrel{P}{\to} c$, as $t \to \infty$ or as $t \downarrow 0$ is equivalent to $xT(x) \to 0$ and $A(x) \to c$ as $x \to \infty$ or as $x \downarrow 0$; in the first case if X has finite mean μ then $c = \mu$, and in the second if X has bounded variation and drift δ then $c = \delta$. (See Theorems 2.1 and 3.1 of [5].)*

Remark 4 *The structure of the following proof also shows a strong similarity to the proof of the random walk results in [7] and [8]. There are of course differences in detail, and some simplifications due to the advantages of working in continuous time and the ability to decompose X into independent components in various ways. There are also some extra difficulties; for example we need to establish results related to the Cental Limit Theorem which are standard for random walks but apparently not previously written down for Lévy processs at zero. Also the case where Π has atoms presents technical difficulties which are absent in the random walk situation; compare the argument on page 1499 of [7] to the upcoming Lemma 9.*

2 Preliminary Results

We start by showing that (10) can be replaced by the simpler

$$\frac{A(x)}{xM(x)} \to \infty \text{ as } x \downarrow 0. \tag{11}$$

Lemma 5 *(i) If (10) holds then (11) holds.*

(ii) If (11) holds then

$$\limsup_{x\downarrow 0} \frac{U(x)}{xA(x)} \leq 2, \tag{12}$$

and consequently (10) holds.

Proof For i) just note that

$$U(x) = \int_0^x 2yT(y)dy \geq T(x)\int_0^x 2ydy = x^2T(x), \tag{13}$$

so that

$$\liminf_{x\downarrow 0} \frac{A(x)}{xM(x)} \geq \liminf_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)M(x)}}\sqrt{\frac{U(x)}{x^2M(x)}} \geq \liminf_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)M(x)}}.$$

For (ii) we first show that

$$\lim_{x\downarrow 0} \frac{U_-(x)}{xM(x)} = 0, \tag{14}$$

where we write

$$U_-(x) = \int_0^x 2yM(y)dy, \; U_+(x) = \int_0^x 2yN(y)dy, \tag{15}$$

so that $U(x) = U_-(x) + U_+(x)$. Given $\varepsilon > 0$ take $x_0 > 0$ such that $\varepsilon A(x) \geq xM(x)$ for $x \in (0, x_0]$, and hence

$$U_-(x) = \int_0^x 2yM(y)dy \leq 2\varepsilon \int_0^x A(y)dy.$$

Note also that for $0 < x < 1$ we can write

$$\begin{aligned} A(x) &= \tilde{\gamma} + A_-(x) - A_+(x), \text{ where } \tilde{\gamma} = \gamma + D(1), \\ A_-(x) &= \int_x^1 M(y)dy, \text{ and } A_+(x) = \int_x^1 N(y)dy. \end{aligned}$$

Then

$$\begin{aligned} U_-(x) &= \int_0^x 2yM(y)dy = -2\int_0^x ydA_-(y) \\ &= 2\int_0^x A_-(y)dy - 2xA_-(x), \end{aligned}$$

and similarly

$$U_+(x) = 2\int_0^x A_+(y)dy - 2xA_+(x).$$

Thus for $0 < x < 1 \wedge x_0$

$$\begin{aligned} U_-(x) &\leq 2\varepsilon \int_0^x \{\tilde{\gamma} + A_-(y) - A_+(y)\} dy \\ &= \varepsilon\{2\tilde{\gamma}x + 2xA_-(x) + U_-(x) - 2xA_+(x) - U_+(x)\} \\ &\leq \varepsilon\{2xA(x) + U_-(x)\}. \end{aligned}$$

Since ε is arbitrary, (14) follows. Also for $0 < x < 1 \wedge x_0$

$$\begin{aligned} U_+(x) - U_-(x) &= 2\left(\int_0^x A_+(y)dy - xA_+(x) - \int_0^x A_-(y)dy + xA_-(x)\right) \\ &= 2\left(xA(x) - \int_0^x A(y)dy\right) \leq 2xA(x). \end{aligned}$$

So

$$U(x) \leq 2xA(x) + 2U_-(x) = 2xA(x) + o\{xA(x)\} \text{ as } x \downarrow 0,$$

and (12) follows. Since

$$\frac{A(x)}{\sqrt{U(x)M(x)}} = \sqrt{\frac{A(x)}{xM(x)}} \cdot \sqrt{\frac{xA(x)}{U(x)}},$$

(11) is immediate. ■

The main part of the proof consists of showing that (11) holds whenever $\rho_t \to 1$. We first dispose of one situation where the argument is straightforward.

Lemma 6 *Let X be any Lévy process satisfying the assumptions of Theorem 1, having $\rho_t \to 1$ as $t \downarrow 0$, and additionally having $M(0+) < \infty$. Then (11) holds.*

Proof In this case we can write

$$X_t = X_t^{(0)} + X_t^{(1)}, \; t \geq 0,$$

where

$$X_t^{(1)} = \sum_{s<t} \Delta_s \mathbf{1}_{\{\Delta_s < 0\}}$$

is a compound Poisson process which is independent of the spectrally positive process $X^{(0)}$. Clearly

$$P\{X_t^{(1)} = 0\} = e^{-tM(0)} \to 1 \text{ as } t \downarrow 0,$$

so we have

$$P\{X_t^{(0)} > 0\} \to 1 \text{ as } t \downarrow 0.$$

But, as previously mentioned, it was shown in [5] that this happens iff $X^{(0)}$ is a subordinator, i.e. it has bounded variation, so that $\int_0^1 x\Pi^{(0)}(dx) = \int_0^1 x\Pi(dx) < \infty$, $xN(x) \to 0$ as $x \downarrow 0$, and we can write

$$X_t^{(0)} = \sum_{s<t} \Delta_s \mathbf{1}_{\{\Delta_s > 0\}} + \delta^{(0)} t,$$

where the drift $\delta^{(0)}$ is non-negative. Comparing this to the representation (2) of X we see that $\delta^{(0)} = \gamma - \int_0^1 x\Pi(dx) + \int_0^1 x\Pi^*(dx)$, where Π^* denotes the Lévy measure of $-X$. If $\delta^{(0)} > 0$ the alternative expression

$$A(x) = \gamma - \int_x^1 y\Pi(dy) + \int_x^1 y\Pi^*(dy) + xD(x), \tag{16}$$

which results from (5) by integration by parts shows that $A(x) \to \delta^{(0)}$ as $x \downarrow 0$. Thus $A(x)/\{xM(x)\} \sim \delta^{(0)}/\{xM(0+)\} \to \infty$. If $\delta^{(0)} = 0$ the same conclusion follows because

$$\frac{A(x)}{x} \sim \frac{\int_0^x D(y)dy}{x} \to \infty,$$

since $D(0+) = N(0+) - M(0+) = \infty$. ■

The next result allows us to make some additional assumptions about X in the remaining case.

Lemma 7 *Let $X^{\#}$ be any Lévy process with no Brownian component which has $M^{\#}(0+) = \infty$ and $\rho_t^{\#} = P(X_t^{\#} > 0) \to 1$ as $t \downarrow 0$. Then there is a Lévy process X with no Brownian component such that $\rho_t = P(X_t > 0) \to 1$ as $t \downarrow 0$ whose Lévy measure can be chosen so that*

(i) $N(1) = M(1) = 0$, $M(0+) = \infty$;

(ii) each of N and M is continuous and strictly decreasing on $(0, c]$ for some $c > 0$.

Moreover, (iii) (11) holds for $X^{\#}$ if and only if it holds for X.

Proof Note that if $X^{(1)}$ has the same characteristics as $X^{\#}$ except that $\Pi^{\#}$ is replaced by

$$\Pi^{(1)}(dx) = \{\Pi^{\#}(dx) + \lambda(dx)\}\mathbf{1}_{\{-1<x<1\}},$$

where λ denotes Lebesgue measure, we have $X^{(1)}+Y^{(1)} = X^{\#}+Y^{(2)}$, where $Y^{(1)}$ is a compound Poisson process independent of $X^{(1)}$ and $Y^{(2)}$ is a compound Poisson process independent of $X^{\#}$. Since $P(Y_t^{(i)} = 0) \to 1$ as $t \downarrow 0$ for $i = 1, 2$ it is immediate that $P(X_t^{(1)} > 0) \to 1$ as $t \downarrow 0$. By construction $N^{(1)}(1) = M^{(1)}(1) = 0$, for $0 < x < 1$ both $N^{(1)}(x) = N^{\#}(x)+1-x-N^{\#}(1)$ and $M^{(1)}(x) = M^{\#}(x)+1-x-M^{\#}(1)$ are strictly decreasing, and (16) shows that $A^{(1)}(x) = A^{\#}(x)-xD^{\#}(1)$. Thus $M^{(1)}(x) \sim M^{\#}(x)$ as $x \downarrow 0$ and we see that (11) holds for $X^{(1)}$ if and only if it holds for $X^{\#}$. This establishes (i), and allows us to assume in the remainder of the proof that $N^{\#}(1) = M^{\#}(1) = 0$, and both $N^{\#}$ and $M^{\#}$ are strictly decreasing on $(0, 1]$. For (ii) it remains only to show that we can take N and M to be continuous.

So suppose that $\Pi^{\#}$ has atoms of size a_n and b_n located at $x_1 > x_2 > \cdots > 0$ and $-y_1 < -y_2 < \cdots < 0$ respectively, for $n = 1, 2, \cdots$; clearly if $\Pi^{\#}$ has only finitely many atoms there is nothing to prove, and the case when the restriction of $\Pi^{\#}$ to $(0, 1]$ or $[-1, 0)$ has only finitely many atoms can be dealt with in a similar way to what follows. Note that from $\int_{-1}^{1} x^2\Pi^{\#}(dx) < \infty$ we have

$$\sum_1^\infty a_n x_n^2 + \sum_1^\infty b_n y_n^2 < \infty. \tag{17}$$

Now let $\Pi^{(c)}$ denote the continuous part of $\Pi^{\#}$, so that

$$\Pi^{\#} = \Pi^{(c)} + \sum_1^\infty a_n\delta(x_n) + \sum_1^\infty b_n\delta(-y_n),$$

where $\delta(x)$ denotes a unit mass at x. With $U[a, b]$ denoting a uniform probability distribution on $[a, b]$ we introduce the measure

$$\Pi = \Pi^{(c)} + \sum_1^\infty a_n U[x_n, x_n + \alpha_n] + \sum_1^\infty b_n U[-y_n, -y_n + \beta_n].$$

We choose $\alpha_n > 0, \beta_n > 0$ to satisfy the following conditions; for $n = 1, 2, \cdots,$

$$x_n + \alpha_n < x_{n+1}, \; - y_n + \beta_n < -y_{n+1}; \tag{18}$$

and

$$\alpha_n \le x_n^3, \; \beta_n \le y_n^3. \tag{19}$$

Note that (17) and (19) imply that

$$c := \frac{1}{2}\sum_1^\infty a_n \alpha_n < \infty,\ c^* := \frac{1}{2}\sum_1^\infty b_n \beta_n < \infty,$$

and hence

$$\lim_{\varepsilon\downarrow 0}\left(\int_\varepsilon^1 x\Pi(dx) - \int_\varepsilon^1 x\Pi^{\#}(dx)\right) = \lim_{n\to\infty}\left(\sum_1^n a_k(x_k + \frac{\alpha_k}{2}) - \sum_1^n a_k x_k\right) = c, \tag{20}$$

and

$$\lim_{\varepsilon\downarrow 0}\left(\int_{-1}^{-\varepsilon} x\Pi(dx) - \int_{-1}^{-\varepsilon} x\Pi^{\#}(dx)\right) = \lim_{n\to\infty}\left(-\sum_1^n b_k(y_k - \frac{\beta_k}{2}) + \sum_1^n b_k y_k\right) = c^*. \tag{21}$$

Now let X be a Lévy process with Lévy measure Π, no Brownian component, and having $\gamma = \gamma^{\#} + c + c^*$. Since $T(1) = T^{\#}(1) = 0$ and we have got Π by 'moving some of the mass of $\Pi^{\#}$ to the right', we have, for each fixed $t > 0$,

$$\begin{aligned} X_t &= \gamma t + \lim_{\varepsilon\downarrow 0}\left(\sum_{0<s<t}\Delta_s \mathbf{1}_{\{|\Delta_s|>\varepsilon\}} - t\int_{\varepsilon<|x|<1} x\Pi(dx)\right) \\ &\stackrel{P}{\geq} \gamma^{\#}t + (c+c^*)t + \lim_{\varepsilon\downarrow 0}\left(\sum_{0<s<t}\Delta_s^{\#}\mathbf{1}_{\{|\Delta_s^{\#}|>\varepsilon\}} - t\int_{\varepsilon<|x|<1} x\Pi^{\#}(dx)\right) \\ &\quad -t\lim_{\varepsilon\downarrow 0}\left(\int_{\varepsilon<|x|<1} x\Pi(dx) - \int_{\varepsilon<|x|<1} x\Pi^{\#}(dx)\right) = X_t^{\#}. \end{aligned}$$

Thus $P(X_t > 0) \to 1$, and to conclude we only need to show that (11) holds for $X^{\#}$ if and only if it holds for X.

Again using $D(1) = D^{\#}(1) = 0$ we have, for $x \in (0,1)$,

$$\begin{aligned} A(x) &= \gamma - \int_x^1 N(y)dy + \int_x^1 M(y)dy \\ &= \gamma^{\#} + c + c^* - \int_x^1 N^{\#}(y)dy + \int_x^1 M^{\#}(y)dy \\ &\quad + \int_x^1 \{N^{\#}(y) - N(y)\}dy + \int_x^1 \{M(y) - M^{\#}(y)\}dy \end{aligned}$$

$$= A^{\#}(x) + c + c^* + \int_x^1 \{N^{\#}(y) - N(y)\}dy + \int_x^1 \{M(y) - M^{\#}(y)\}dy$$
$$= A^{\#}(x) + \int_0^x \{N(y) - N^{\#}(y)\}dy + \int_0^x \{M(y) - M^{\#}(y)\}dy, \qquad (22)$$

where we have used (20) to see that

$$\int_0^1 \{N(y) - N^{\#}(y)\}dy = \lim_{\varepsilon \downarrow 0} \left(\int_\varepsilon^1 x\Pi(dx) - \int_\varepsilon^1 x\Pi^{\#}(dx) \right) = c,$$

and similarly for c^*. Since (19) gives

$$0 \leq \int_0^x \{N(y) - N^{\#}(y)\}dy \leq \frac{1}{2} \sum_{n: x_n < x} a_n \alpha_n$$
$$\leq \frac{1}{2} \sum_{n: x_n < x} a_n x_n^3 \leq \frac{x}{2} \sum_{n: x_n < x} a_n x_n^2 = o(x),$$

and the same argument applies to the second integral in (22), we see that

$$\frac{A^{\#}(x)}{x} = \frac{A(x)}{x} + o(1) \text{ as } x \downarrow 0. \qquad (23)$$

Next note that if $-x \notin \bigcup_1^\infty (-y_n, -y_n + \beta_n]$, then $M(x) = M^{\#}(x)$. On the other hand, if $-x = -y_n + \theta\beta_n$, with $0 < \theta \leq 1$, then

$$M^{\#}(x) = M(x) + (1 - \theta)b_n \sim M(x) \text{ as } x \downarrow 0,$$

so (iii) follows. ■

The next piece of information we need is reminiscent of the Berry-Esseen Theorem;

Lemma 8 *Let μ be any Lévy measure, and write μ_t for the restriction of μ to the interval $[-b_t, b_t]$, where $b_t \downarrow 0$ as $t \downarrow 0$. Suppose that for each $t > 0$ Z_t has an infinitely divisible distribution determined by*

$$E(e^{i\theta Z_t}) = \exp -t\{\int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x)\mu_t(dx)\} := \psi_t(\theta). \qquad (24)$$

Put

$$t\sigma_t^2 = t \int_{\mathbb{R}} x^2 \mu_t(dx) = EZ_t^2,$$

and write Φ for the standard Normal distribution function. Then for any $\varepsilon > 0$ there is a positive constant W_ε such that for all x

$$|P\{Z_t \leq x\sqrt{t}\sigma_t\} - \Phi(x)| \leq \varepsilon \tag{25}$$

for all t satisfying

$$\frac{\sqrt{t}\sigma_t}{b_t} \geq W_\varepsilon. \tag{26}$$

Proof Note first that $EZ_t = 0$ and $EZ_t^3 = t\int_{\mathbb{R}} x^3 \mu_t(dx) := t\zeta_t$, and write

$$\nu_t = \frac{EZ_t^3}{6(EZ_t^2)^{\frac{3}{2}}} = \frac{\zeta_t}{6\sqrt{t}\sigma_t^3}.$$

We will apply the inequality (3.13), p 512 of [6], with t fixed,

$$F(x) = P\{Z_t \leq x\sqrt{t}\sigma_t\}, \text{ and } G(x) = \Phi(x) - \nu_t(x^2-1)\phi(x),$$

where ϕ is the standard Normal density function. From it we deduce that for any $T > 0$ the LHS of (25) is bounded above by

$$|\nu_t(x^2-1)|\phi(x) + \frac{1}{\pi}\int_{-T}^{T} \frac{|\psi_t(\frac{\theta}{\sqrt{t}\sigma_t}) - e^{\frac{-\theta^2}{2}}(1+\nu_t(i\theta)^3)|}{|\theta|} d\theta + \frac{24m_t}{\pi T}. \tag{27}$$

Here

$$m_t = \sup_x |G'(x)| = \sup_x \phi(x)\{1 + |\nu_t(2x^2 - 3x + 1)|\} \leq M,$$

where M is an absolute constant, for all t satisfying $|\nu_t| \leq 1$. But if (26) holds we have

$$|\nu_t| \leq \frac{b_t\sigma_t^2}{6\sqrt{t}\sigma_t^3} = \frac{b_t}{6\sqrt{t}\sigma_t} \leq \frac{1}{6W_\varepsilon}, \tag{28}$$

so this will hold provided $6W_\varepsilon \geq 1$. Now fix $T = \frac{72M}{\pi\varepsilon}$, so that the third term in (27) is no greater than $\varepsilon/3$. The same argument shows that the first term in (27) is also no greater than $\varepsilon/3$ provided (26) holds and $3W_\varepsilon \geq 1/\varepsilon$. Finally, to deal with the middle term we write $\tilde{\theta} = \frac{\theta}{\sqrt{t}\sigma_t}$, and note that

$$\begin{aligned}
\psi_t(\tilde{\theta}) &= e^{\frac{-\theta^2}{2}} \exp t\{\int_{|x|\leq b_t} (e^{i\tilde{\theta}x} - (1 + i\tilde{\theta}x - \frac{1}{2}\tilde{\theta}^2x^2)\mu(dx)\} \\
&= e^{\frac{-\theta^2}{2}} \exp t\{\int_{|x|\leq b_t} \frac{(i\tilde{\theta}x)^3}{6}\mu(dx)\} \cdot \exp t\{\int_{|x|\leq b_t} r(i\tilde{\theta}x)\mu(dx)\} \\
&= e^{\frac{-\theta^2}{2}} \exp \nu_t(i\theta)^3 \cdot \exp t\{\int_{|x|\leq b_t} r(i\tilde{\theta}x)\mu(dx)\},
\end{aligned} \tag{29}$$

where for some positive constant c_ε

$$|r(z)| = |e^z - (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3)| \le \varepsilon |z|^3 \tag{30}$$

whenever $|z| \le c_\varepsilon$. Since for $|\theta| \le T$ and $|x| \le b_t$ we have

$$|\tilde{\theta} x| \le \frac{T b_t}{\sqrt{t}\sigma_t} \le \frac{T}{W_\varepsilon}$$

we see that when (26) holds and $W_\varepsilon \ge (T/c_\varepsilon) \vee 1$ we can apply (30) to deduce that for $|\theta| \le T$

$$\begin{aligned} \left| t \left(\int_{|x|\le b_t} r(i\tilde{\theta}x)\mu(dx) \} \right) \right| &\le \varepsilon t |\tilde{\theta}|^3 \int_{|x|\le b_t} |x|^3 \mu(dx) \\ &\le \varepsilon t |\tilde{\theta}|^3 b_t \sigma_t^2 \le \frac{\varepsilon |\theta|^3 b_t}{\sqrt{t}\sigma_t} \le \varepsilon |\theta|^3. \end{aligned}$$

It follows from this and (28) that, increasing the value of W_ε if necessary, we can make

$$\frac{1}{\pi} \int_{-T}^{T} \frac{|\psi_t(\frac{\theta}{\sqrt{t}\sigma_t}) - e^{\frac{-\theta^2}{2} + \nu_t (i\theta)^3} (-1 + \exp t\{\int_{|x|\le b_t} r(i\tilde{\theta}x)\mu(dx)\})|}{|\theta|} d\theta \le \frac{\varepsilon}{6},$$

and clearly we can also arrange that

$$\frac{1}{\pi} \int_{-T}^{T} \frac{|e^{\frac{-\theta^2}{2}} \{ e^{\nu_t (i\theta)^3} - 1 - \nu_t (i\theta)^3 \}|}{|\theta|} d\theta \le \frac{\varepsilon}{6},$$

whenever (26) holds. Putting these bounds into (27) finishes the proof. ■

Next we record a variant of the Lévy-Khintchine decomposition (2) which is important for us:

Lemma 9 *If X is any Lévy process with no Brownian component and $b, b^* \in (0,1)$ and $t > 0$ are fixed we can write*

$$X_t = \tilde{\gamma}(b, b^*)t + Y_t^{(1,+)} + Y_t^{(1,-)} + Y_t^{(2,+)} + Y_t^{(2,-)}, \tag{31}$$

where

$$\begin{aligned} Y_t^{(1,+)} &= \lim_{\varepsilon \downarrow 0} \left\{ \sum_{s \le t} \Delta_s \mathbf{1}_{\{\varepsilon < \Delta_s < b\}} - t \int_\varepsilon^b x \Pi(bx) \right\}, \ Y_t^{(2,+)} = \sum_{s\le t} \Delta_s \mathbf{1}_{\{\Delta_s \ge b\}}, \\ Y_t^{(1,-)} &= \lim_{\varepsilon \downarrow 0} \left\{ \sum_{s \le t} \Delta_s \mathbf{1}_{\{-b^* < \Delta_s < -\varepsilon\}} - t \int_{-b^*}^{-\varepsilon} x \Pi(dx) \right\}, \ Y_t^{(2,-)} = \sum_{s\le t} \Delta_s \mathbf{1}_{\{\Delta_s \le -b^*\}}, \end{aligned}$$

are independent, and

$$\tilde{\gamma}(b, b^*) = \gamma - \int_b^1 x\Pi(dx) + \int_{b^*}^1 x\Pi^*(dx).$$

Proof This is proved in the same way as (2), except we compensate over the interval $(-b^*, b)$ rather than $(-1, 1)$. ■

Finally we are in a position to establish the main technical estimate we need in the proof of Theorem 1;

Proposition 10 *Suppose that X is a Lévy process with no Brownian component whose Lévy measure satisfies $N(0+) = M(0+) = \infty$, and suppose $d(t)$ and $d^*(t)$ satisfy*

$$N(d(t)) = M(d^*(t)) = \frac{1}{t} \tag{32}$$

for all small enough $t > 0$. Then there is a finite constant K such that, for any $\lambda > 0, \rho > 0, L \geq 0$ there exists $C = C(X, \lambda, \rho, L) > 0$ with

$$P\{X_t \leq t\tilde{\gamma}(d(\lambda t), d^*(\rho\lambda t)) + Kd(\lambda t) - Ld^*(\rho\lambda t)\} \geq C \tag{33}$$

for all small enough t.

Proof We start by noting that if we use decomposition (31) for each fixed t with b and b^* replaced by $d(\lambda t)$ and $d^*(\rho\lambda t)$, (32) gives

$$P(Y_t^{(2,+)} = 0) = e^{-tN(d(\lambda t))} = e^{-\frac{1}{\lambda}}. \tag{34}$$

Also $Y_t^{(1,+)}$ has mean zero and since $d(t) \downarrow 0$, because $N(0+) = \infty$, we can apply Lemma 8 to $Y_t^{(1,+)}$ with $b_t = d(\lambda t)$ and $\sigma_t^2 = \int_0^{d(\lambda t)} x^2\Pi(dx)$. Choosing $x = 0$ and writing W for the W_ε of Lemma 8 with $\varepsilon = 1/4$ we conclude from (25) that $P\{Y_t^{(1,+)} \leq 0\} \geq \frac{1}{4}$ whenever $t\sigma_t^2 \geq \{Wd(\lambda t)\}^2$. On the other hand, if $t\sigma_t^2 \leq \{Wd(\lambda t)\}^2$ it follows from Chebychev's inequality that

$$P\{|Y_t^{(1,+)}| > Kd(\lambda t)\} \leq \frac{t\sigma_t^2}{\{Kd(\lambda t)\}^2} \leq \frac{W^2}{K^2}.$$

Thus in all cases we can fix K large enough that

$$P\{Y_t^{(1,+)} \leq Kd(\lambda t)\} \geq \frac{1}{4}. \tag{35}$$

An exactly similar argument shows that we can fix a finite K^* with

$$P\{Y_t^{(1,-)} \geq -K^* d^*(\rho\lambda t)\} \geq \frac{1}{4}. \tag{36}$$

Finally we note that if Z is a random variable with a Poisson$(1/\rho\lambda)$ distribution

$$P\{Y_t^{(2,-)} \leq -(K^* + L)d^*(\rho\lambda t)\} \geq P\{Z \geq (K^* + L)\} > 0. \tag{37}$$

Combining (34)-(37) gives the required conclusion. ■

3 Proofs

Proof of Theorem 1.1 Since we have demonstrated in Lemma 5 the equivalence of (10) and (11), and Theorem 2.3 of [5] shows that (11) implies (7) under our assumptions, we will first show that (7) implies (11), and later their equivalence to (9) and (8)

So assume (7), and also that $M(0+) = \infty$, since Lemma 6 deals with the contrary case. Then X satisfies the assumptions we made about $X^{\#}$ in Lemma 7, so that result allows us to save extra notation by assuming that the conclusions (i) and (ii) of that lemma apply to X. For the moment assume also that $N(0+) = \infty$, so that we can define d and d^* as the unique solutions of (32) on $(0, t_0]$ for some fixed $t_0 > 0$. Our first aim is to show that

$$\liminf_{x \downarrow 0} \frac{\tilde{\gamma}(x, x)}{xN(x)} \geq 0. \tag{38}$$

To see this we use Proposition 10 with $L = 0$, which since $P(X_t \leq 0) \to 0$ implies that for all sufficiently small t

$$t\tilde{\gamma}(d(\lambda t), d^*(\lambda\rho t)) + Kd(\lambda t) \geq 0. \tag{39}$$

Writing $\nu(x) = \int_x^1 y\Pi(dy)$ and $\nu^*(x) = \int_x^1 y\Pi^*(dy)$ we have

$$\tilde{\gamma}(x_1, x_2) = \gamma - \nu(x_1) + \nu^*(x_2),$$

and clearly $\tilde{\gamma}(x_1, x_2)$ is a decreasing function of x_2 for fixed x_1. Thus if $d(\lambda t) \leq d^*(\lambda\rho t)$ then from (39)

$$t\tilde{\gamma}(d(\lambda t), d(\lambda t)) + Kd(\lambda t) \geq t\tilde{\gamma}(d(\lambda t), d^*(\lambda\rho t)) + Kd(\lambda t) \geq 0.$$

However if $d^*(\lambda\rho t) < d(\lambda t)$ then

$$\begin{aligned}\nu^*(d^*(\lambda\rho t)) - \nu^*(d(\lambda t)) &= \int_{d^*(\lambda\rho t)}^{d(\lambda t)} y\Pi^*(dy) \\ &\leq d(\lambda t)M(d^*(\lambda\rho t)) = \frac{d(\lambda t)}{\lambda\rho t}.\end{aligned}$$

Consequently in both cases

$$\tilde{\gamma}(d(\lambda t), d(\lambda t)) \geq -(K\lambda + \frac{1}{\rho})\frac{d(\lambda t)}{\lambda t} = -(K\lambda + \frac{1}{\rho})d(\lambda t)N(d(\lambda t)),$$

and since Lemma 7 allows us to assume the continuity of d, we have

$$\liminf_{x\downarrow 0} \frac{\tilde{\gamma}(x,x)}{xN(x)} \geq -(K\lambda + \frac{1}{\rho}).$$

But in this we may choose λ arbitrarily small and ρ arbitrarily large, so (38) follows.

Assume next that (11) fails, and recall that

$$A(x) = \tilde{\gamma}(x,x) + xD(x) = \tilde{\gamma}(x,x) + xN(x) - xM(x).$$

Then for some sequence $x_k \downarrow 0$ and some $D < \infty$

$$\tilde{\gamma}(x,x) + xN(x) \leq DxM(x) \text{ when } x = x_1, x_2 \cdots. \tag{40}$$

Let

$$t_k = \frac{K}{2DM(x_k)}, \text{ or equivalently } x_k = d^*(\frac{2Dt_k}{K}),$$

then from (38) we have $\tilde{\gamma}(x_k, x_k) \geq -\frac{1}{2}x_kN(x_k)$ for all large enough k, and hence, using (40),

$$2Dx_kM(x_k) \geq x_kN(x_k).$$

Thus

$$N(x_k) \leq 2DM(x_k) = \frac{K}{t_k},$$

and hence

$$d(K^{-1}t_k) \leq x_k. \tag{41}$$

We now invoke Proposition 10 again, this time choosing $t = t_k, \lambda = K^{-1}$, and $\rho = 2D$, to get

$$P\{X_t \leq t\tilde{\gamma}(d(K^{-1}t), d^*(2DK^{-1}t)) + Kd(K^{-1}t) - Ld^*(2DK^{-1}t)\} \geq C > 0, \tag{42}$$

whenever $t = t_k$ and k is large enough. However, in view of (41) the term on the right of the inequality is bounded above by

$$\begin{aligned} t_k\{\gamma + \nu^*(x_k) - \nu(x_k)\} + Kx_k - Lx_k &= t_k\tilde{\gamma}(x_k, x_k) + (K-L)x_k \\ &\leq Dt_kx_kM(x_k) + (K-L)x_k \\ &= (\frac{3}{2}K - L)x_k. \end{aligned}$$

If now we choose $L = 2K$ we see that (42) contradicts (7); this contradiction implies that (11) is in fact correct.

We reached this conclusion making the additional assumption that $N(0+) = \infty$, but it is easy to see that it also holds if $N(0+) < \infty$. In this case by an argument we have used previously there is no loss of generality in taking $N(0+) = 0$, so that X is spectrally negative. We can then repeat the proof of Proposition 10 with $b(t) \equiv 0$, the conclusion being that for any $L \geq 0$ there exists $C = C(X, L) > 0$ with

$$P(X_t \leq \{t[\gamma + \nu^*(d^*(t))] - Ld^*(t)\}) \geq C \tag{43}$$

for all sufficiently small t. When (7) holds this clearly implies that $\gamma + \nu^*(x) \geq 0$ for all sufficiently small x. If (11) were false, we would have

$$\gamma + \nu^*(x) \leq DxM(x)$$

along some sequence $x_k \downarrow 0$. Choosing $t_k = 1/M(x_k)$, or equivalently $x_k = d^*(t_k)$, (43) becomes

$$P\{X_{t_k} \leq \{t_k[\gamma + \nu^*(x_k)] - Lx_k\} \geq C.$$

Since

$$\gamma + \nu^*(x_k) - \frac{Lx_k}{t_k} = \gamma + \nu^*(x_k) - Lx_kM(x_k) \leq (D-L)x_kM(x_k),$$

we again get a contradiction by choosing L sufficiently large. This completes the proof of the equivalence of (7) and (10).

As (9) obviously implies (7), our next aim is to show the reverse implication, or in view of Lemma 5, that (11) implies (9).

Since $M(0+) > 0$ a first consequence of (11) is that there is a $x_0 > 0$ with $A(y) > 0$ for $0 < y \le x_0$, and a second is that $y^{-1}A(y) \to \infty$ as $y \downarrow 0$. For $\delta \ge 1$ define a function $b_\delta(x)$ by

$$b_\delta(x) = \inf\{0 < y \le x_0 : \frac{A(y)}{y} \ge \frac{\delta}{x}\}. \tag{44}$$

Then $b_\delta(x) \downarrow 0$ as $x \downarrow 0$, and since A is continuous, there is a $x_1 > 0$ such that

$$xA(b_\delta(x)) = \delta b_\delta(x) \text{ for } 0 < x \le x_1. \tag{45}$$

Our first aim is to show that there is a slowly varying function $f_\delta(x)$ which increases as $x \downarrow 0$ and satisfies

$$\frac{A(b_\delta(x))}{\delta} = \frac{b_\delta(x)}{x} := \gamma_\delta(x) \le f_\delta(x) \text{ for } 0 < x \le x_1. \tag{46}$$

First, using (5) we see that for $x \le x_1/2$,

$$\begin{aligned}\frac{\gamma_\delta(2x)}{\gamma_\delta(x)} &= \frac{A(b_\delta(2x))}{A(b_\delta(x))} = 1 + \frac{\int_{b_\delta(x)}^{b_\delta(2x)} D(y)dy}{A(b_\delta(x))} \\ &\ge 1 - \frac{\int_{b_\delta(x)}^{b_\delta(2x)} M(y)dy}{A(b_\delta(x))} \ge 1 - \frac{\{b_\delta(2x) - b_\delta(x)\}M(b_\delta(x))}{A(b_\delta(x))} \\ &= 1 - \left(\frac{b_\delta(2x) - b_\delta(x)}{b_\delta(x)}\right)\varepsilon_\delta(x) \end{aligned} \tag{47}$$

say, where

$$\varepsilon_\delta(x) = \frac{b_\delta(x)M(b_\delta(x))}{A(b_\delta(x))} = \frac{xM(b_\delta(x))}{\delta}. \tag{48}$$

¿From (11) we have $\varepsilon_\delta(x) \to 0$ as $x \downarrow 0$ for each fixed δ, and it is also the case that

$$\frac{\gamma_\delta(2x)}{\gamma_\delta(x)} \ge 1 - \varepsilon_\delta(x) \text{ for } 0 < 2x \le x_1. \tag{49}$$

To see this observe that if $b_\delta(2x) - b_\delta(x) \le b_\delta(x)$ then this is immediate from (47), whereas if $b_\delta(2x) - b_\delta(x) > b_\delta(x)$ then

$$\frac{\gamma_\delta(2x)}{\gamma_\delta(x)} = \frac{b_\delta(2x)/2x}{b_\delta(x)/x} = \frac{1}{2} \cdot \frac{b_\delta(2x)}{b_\delta(x)} > 1.$$

Next, given $0 < x \le x_1/2$ choose $k = k(x)$ such that

$$2^{-(k+1)} < x \le 2^{-k}$$

and k_0 such that $\varepsilon_\delta(2^{-j}) \le 1/2$ when $j \ge k_0$. Applying (49) we get for $k(x) \ge k_0$

$$\begin{aligned}
\gamma_\delta(x) &= \frac{b_\delta(x)}{x} \le \frac{b_\delta(2^{-k})}{2^{-(k+1)}} = 2\gamma_\delta(2^{-k}) \\
&= 2\frac{\gamma_\delta(2^{-k})}{\gamma_\delta(2^{-(k-1)})} \cdot \frac{\gamma_\delta(2^{-(k-1)})}{\gamma_\delta(2^{-(k-2)})} \cdot \cdots \cdot \frac{\gamma_\delta(2^{-k_0})}{\gamma_\delta(2^{-(k_0-1)})}\gamma_\delta(2^{-(k_0-1)}) \\
&\le C_\delta \prod_{k_0}^{k} (1 - \varepsilon_\delta(2^{-j}))^{-1},
\end{aligned}$$

where $C_\delta = 2\gamma_\delta(2^{-(k_0-1)})$. If ln denotes the base 2 logarithm, we have $k(x) \ge \ln 1/x$, so (46) holds with

$$f_\delta(x) = C_\delta \prod_{k_0 \le j \le \ln 1/x} (1 - \varepsilon_\delta(2^{-j}))^{-1}.$$

Clearly $f_\delta(x)$ increases as $x \downarrow 0$, and if $j = j(x)$ is the unique integer with

$$\ln 1/x < j \le \ln 2/x = 1 + \ln 1/x,$$

we have

$$\frac{f_\delta(x)}{f_\delta(x/2)} = (1 - \varepsilon_\delta(2^{-j})) \to 1,$$

and we conclude that $f_\delta(x)$ is slowly varying as $x \downarrow 0$. (See [3], Prop 1.10.1, p. 54.)

If we now put $L_\delta(x) = (f_\delta(x))^2$ when $f_\delta(0+) = \infty$, and $L_\delta(x) = \log 1/x$ when $f_\delta(0+) < \infty$, we have that $L_\delta(x)$ is increasing and slowly varying as $x \downarrow 0$, and

$$\frac{b_\delta(x)}{xL_\delta(x)} \to 0 \text{ as } x \downarrow 0. \tag{50}$$

Furthermore, since $b_\delta(x) \le b_1(x)$ for $\delta \ge 1$, we automatically have

$$\frac{b_\delta(x)}{xL_1(x)} \to 0 \text{ as } x \downarrow 0 \text{ for } \delta \ge 1. \tag{51}$$

We are now in a position to prove (9). We use Lemma 9 with $b = b^* = b_\delta(t)$. Replacing $M\{b_\delta(t)\}$ by 0, observing that $Y_t^{(2,+)} \geq b_\delta(t) Z_t$, where Z is a Poisson process with rate $N\{b_\delta(t)\}$, and combining $Y_t^{(1,+)}$ and $Y_t^{(1,-)}$ we deduce that, a.s. for each fixed t,

$$X_t \geq tA\{b_\delta(t)\} + b_\delta(t)\left(Z_t - tN\{b_\delta(t)\}\right) + Y_t^{(1)} + Y_t^{(2,-)}.$$

Here the $Y's$ and Z are independent,

$$EY_t^{(1)} = 0, VarY_t^{(1)} = t \int_{|x| \leq b_\delta(t)} x^2 \Pi(dx),$$

and

$$P\{Y_t^{(2,-)} = 0\} = \exp -tM\{b_\delta(t)\}.$$

It follows from (11) that $tM\{b_\delta(t)\} = o\left(tA\{b_\delta(t)\}/b_\delta(t)\right)$ as $t \downarrow 0$, and $tA\{b_\delta(t)\} = \delta b_\delta(t)$, so for all sufficiently small t we have

$$P(Y_t^{(2,-)} \neq 0) \leq 1/\delta.$$

So for such t Chebychev's inequality gives

$$\begin{aligned} P(X_t &\leq \frac{\delta}{2} b_\delta(t)) \leq \frac{1}{\delta} + \\ & P\left(tA\{b_\delta(t)\} + b_\delta(t)\left(Z_t - tN\{b_\delta(t)\}\right) + Y_t^{(1)} \leq \frac{\delta}{2} b_\delta(t), Y_t^{(2-)} = 0\right) \\ &\leq \frac{1}{\delta} + P\left(b_\delta(t)\left(Z_t - tN\{b_\delta(t)\}\right) + Y_t^{(1)} \leq -\frac{\delta}{2} b_\delta(t)\right) \\ &\leq \frac{1}{\delta} + \frac{4\{Var[Y_t^{(1)} + b_\delta(t) Z_t]\}}{\{\delta b_\delta(t)\}^2}. \end{aligned} \tag{52}$$

Next we note that for all small enough t

$$\begin{aligned} Var[Y_t^{(1)} + b_\delta(t) Z_t] &= t \int_{|x| \leq b_\delta(t)} x^2 \Pi(dx) + \{b_\delta(t)\}^2 tN\{b_\delta(t)\} \\ &\leq t \left(\int_{|x| \leq b_\delta(t)} x^2 \Pi(dx) + \{b_\delta(t)\}^2 T\{b_\delta(t)\} \right) \\ &= tU(b_\delta(t)) \leq 3tb_\delta(t) A(b_\delta(t)) = 3\delta \{b_\delta(t)\}^2, \end{aligned}$$

where we have used (12). Putting this into (52) gives

$$P(X_t \le \frac{\delta}{2} b_\delta(t)) \le \frac{1}{\delta} + \frac{12\delta\{b_\delta(t)\}^2}{\{\delta b_\delta(t)\}^2} = \frac{13}{\delta}.$$

Finally, using (51) we deduce from this that, for arbitrary $K > 0$ and small enough t

$$P(X_t \le KtL_1(t)) \le P(X_t \le \frac{\delta}{2} b_\delta(t)) \le \frac{13}{\delta}.$$

Letting $t \downarrow 0$, then $\delta \to \infty$, we see that (9) holds with $d(t) = tL_1(t)$.

Finally (8) clearly implies (7). On the other hand if (7) holds the above proof shows that with $b(t) = b_1(t)$ as defined in (44) we have $xA(b(x)) = b(x)$ for $0 < x \le x_1$, and by (50)

$$\frac{X_t}{b(t)} \ge \frac{X_t}{tL_1(t)} \xrightarrow{P} \infty \text{ as } t \downarrow 0. \qquad (53)$$

Since $P(\Delta_t^{(1)} \le b(t)) = \exp -tM(b(t))$ and

$$tM(b(t)) = \frac{b(t)M(b(t))}{A(b(t))} \cdot \frac{tA(b(t))}{b(t)} = \frac{b(t)M(b(t))}{A(b(t))} \to 0,$$

(8) follows from (53). ■

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