Joint convergence of several copies of different patterned random matrices[*]

Riddhipratim Basu[†]   Arup Bose[‡]   Shirshendu Ganguly[§]
Rajat Subhra Hazra[¶]

Abstract

We study the joint convergence of independent copies of several patterned matrices in the non-commutative probability setup. In particular, joint convergence holds for the well known Wigner, Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant matrices. We also study some properties of the limits. In particular, we show that copies of Wigner becomes asymptotically free with copies of any of the above other matrices.

Keywords: Random matrices; free probability; joint convergence; patterned matrices; Toeplitz matrix; Hankel matrix; Reverse Circulant matrix; Symmetric Circulant matrix; Wigner matrix.
AMS MSC 2010: Primary 60B20, Secondary 60B10; 46L53; 46L54.
Submitted to EJP on April 24, 2012, final version accepted on September 28, 2012.
Supersedes arXiv:1108.4099.

1 Introduction

The spectrum of large N limit of random matrices has played a crucial role in various disciplines, for example, in the description of excitation of large nuclei, in statistical inference and in telecommunications. The two most important random matrices studied in the literature are the Wigner and the Sample Covariance matrices. Various properties of these matrices are now well known. In particular, for the Wigner matrix, the empirical spectral distribution of the eigenvalues converges to the semi-circular distribution, whereas the local behavior at the edge is governed by the Tracy-Widom law and the bulk by the Dyson Sine Kernel. Similar features of the Sample Covariance matrix are also known.

Another aspect of random matrices which has found applications in operator algebras and telecommunications is the trace of non-commutative polynomials of random

---

[†]Department of Statistics University of California, Berkeley. E-mail: riddhipratim@stat.berkeley.edu
[‡]Statistics and Mathematics Unit, Indian Statistical Institute, Kolkata, India. E-mail: bosearu@gmail.com
[§]Department of Mathematics, University of Washington, Seattle. E-mail: sganguly@math.washington.edu
[¶]Institut für Mathematik, Universität Zürich, Zürich, Switzerland. E-mail: rajatmaths@gmail.com

matrices. The limit behavior of these for Wigner matrices may be described by the notion of freeness introduced by Voiculescu. This notion also helped in the description of the limiting spectrum of the product and of the sum of random matrices. It is also helpful in understanding the spectrum of block random matrices. The combinatorial properties of freeness have been extended to operator valued freeness (also called freeness with amalgamation) and in particular this is useful in describing the behavior of polynomials of random band matrices and rectangular random matrices (see Shlyakhtenko [34] and Benaych-Georges [4, 5]).

In comparison to the theory of Wigner and Sample Covariance matrices not much is known about the limit behavior of other patterned matrices such as the Toeplitz and the Hankel. The study of polynomials of independent copies of a single patterned matrix was initiated in Bose et al. [9]. In this case the description of the asymptotic eigenvalue distribution of the polynomials depend on some "typical" positions of the random matrices in the polynomial. These positions are described in an appropriate manner by freeness for the Wigner matrices, freeness with amalgamation for the band matrices and the rectangular matrices, half independence for the reverse circulant matrices and independence for the symmetric circulant matrices.

In this article we extend the study of traces of non-commutative polynomials to multiple copies of different patterned matrices. In the next subsection we discuss some of the existing results and the main contribution of the present article.

1.1 Overview

A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$; φ is a *state* if for $a \geq 0$ we have $\varphi(a) \geq 0$ and it is *tracial* if $\varphi(ab) = \varphi(ba)$ for all a, b. Elements of $\mathcal{A}$ will be called *variables*.

The connection between large dimensional random matrices (matrices whose elements are random variables) and non-commutative probability spaces is well known and deep. Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $L(\mu) := \bigcap_{p \geq 1} L^p(X, \mu)$ be the algebra of random variables with finite moments of all orders. Set

$$\mathcal{A}_n := Mat_n(L(\mu)) \tag{1.1}$$

as the space of $n \times n$ complex random matrices with entries coming from $L(\mu)$. Then $(\mathcal{A}_n, \varphi_j)$, $j = 1, 2$ are non-commutative probability spaces where

$$\varphi_1(A) = \frac{1}{n} \operatorname{Tr}(A) \text{ and } \varphi_2(A) = \frac{1}{n} \operatorname{E}[\operatorname{Tr}(A)]. \tag{1.2}$$

The *joint distribution* of a family $(a_i)_{i \in I}$ of variables in $(\mathcal{A}, \varphi)$ is the collection of *joint moments* $\{\varphi(a_{i_1} \cdots a_{i_k})\}$, $k \in \mathbb{N}$ and $i_1, \cdots, i_k \in I$. Let $(\mathcal{A}_n, \varphi_n)_{n \geq 1}$ and $(\mathcal{A}, \varphi)$ be non-commutative probability spaces and let $(a_{i,n}; i \in I) \subset \mathcal{A}_n$ for each n, $(a_i; i \in I) \subset \mathcal{A}$. Then $(a_{i,n}; i \in I)$ *converges in distribution* to $(a_i; i \in I)$ if all joint moments converge. Equivalently, for all $p \in \mathbb{C}[X_i, i \in I]$,

$$\lim_n \varphi_n(p(\{a_{i,n}\}_{i \in I})) = \varphi(p(\{a_i\}_{i \in I})). \tag{1.3}$$

Convergence of an $n \times n$ real symmetric matrix A_n with respect to φ_1 and φ_2 demands convergence for each non-negative integer k, respectively of $\varphi_1(A_n^k)$ (almost surely) and $\varphi_2(A_n^k)$.

A related notion of convergence is that of the spectral distribution. If the eigenvalues

of A_n are $\{\lambda_i\}$, then the spectral measure of A_n is defined as

$$L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}. \tag{1.4}$$

If as $n \to \infty$, L_n converges weakly (almost surely) to a measure μ with distribution function F say, then F (or μ) is called the *limiting spectral distribution* (LSD) of $\{A_n\}$. In his pioneering work, Wigner [42] showed that the GUE (Gaussian Unitary Ensemble, Hermitian matrices with i.i.d. complex Gaussian entries with variance $1/n$) converges with respect to φ_2 to the *semi-circular* variable s characterized by the limit moments

$$\varphi(s^k) = \int t^k \frac{1}{2\pi} \sqrt{4 - t^2} \; \mathbf{1}_{|t|\leq 2} dt.$$

The probability law with density $(2\pi)^{-1}\sqrt{4 - t^2}\mathbf{1}_{|t|\leq 2}$ having the above moments is called the *semi-circle law*. This result was extended in many directions for Gaussian Orthogonal Ensemble (GOE) and Gaussian Symplectic Ensemble (GSE) and in fact for i.i.d. entries with finite second moment. See Bai and Silverstein [2] for a detailed treatment.

Voiculescu [39] introduced the notion of freeness in the context of free groups. It played the role of independence in non-commutative probability spaces. Unital subalgebras $\{\mathcal{A}_i\}_{i\in I} \subset \mathcal{A}$ are said to be *free* if $\varphi(a_1 \cdots a_n) = 0$ whenever $\varphi(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for all j.

The notions of freeness and of convergence as in (1.3) together yield an obvious and natural notion of *asymptotically free*. Voiculescu [40] showed that if we take k independent Hermitian random matrices $\{W_{i,n}\}_{1\leq i\leq k}$ distributed as GUE then they are asymptotically free. In other words, for any polynomial $\mathbb{P}$ in k variables,

$$\mathrm{E}\left[\frac{1}{n}\mathrm{Tr}(\mathbb{P}(W_{1,n}, \ldots, W_{k,n}))\right] \to \tau(\mathbb{P}(s_1, \ldots, s_k)) \text{ as } n \to \infty,$$

where $(s_1, \ldots, s_k)$ is a collection of free (and semi-circular) variables in some non-commutative probability space $(\mathcal{A}, \tau)$. Asymptotic freeness of GUE has been a key feature in the development of free probability and its various applications. Voiculescu [40] also showed the asymptotic freeness of GUE and diagonal constant matrices. Later, Voiculescu [41] improved the result to asymptotic freeness of GUE and general $n \times n$ deterministic matrices $\{D_{i,n}\}$ (having LSD) and satisfying

$$\sup_n \|D_{i,n}\| < \infty \text{ for each } i, \tag{1.5}$$

where $\|\cdot\|$ denotes the operator norm. This inclusion of constant matrices had important implications in the factor theory of von Neumann algebras. Dykema [20] established a similar result for a family of independent Wigner matrices

(symmetric matrices with i.i.d. real entries having uniformly bounded moments) and block-diagonal constant matrices with bounded block size. The results were also shown to hold with respect to φ_1 almost surely (see Hiai and Petz [24, 25] for details). For general results on freeness between Wigner and deterministic matrices we refer to Anderson et al. [1]. Various other extensions to Wishart ensembles, GOE, GSE are also available. See Capitaine and Casalis [14], Capitaine and Donati-Martin [15], Collins et al. [18], Ryan [32], Schultz [33] and Voiculescu [41].

Freeness is present elsewhere too and one important place is the Haar distributed matrices. It is well known that any unitary invariant matrix (in particular GUE) can be written as UDU^* where D is a diagonal matrix and U is Haar distributed on the space of unitary matrices and independent of D. Voiculescu [40] showed that $\{U, U^*\}$ and D are asymptotically free.

Hiai and Petz [24] showed that the Haar unitaries and general deterministic matrices satisfying

(1.5) are almost surely asymptotically free. Collins [16] showed that general deterministic matrices and Haar measure on unitary group are asymptotically free almost surely provided the deterministic matrices jointly converge. The case for orthogonal and symplectic groups were dealt with in Collins and Śniady [17].

An operator valued extension of freeness is freeness with amalgamation and may be described by replacing the base field $\mathbb{C}$ and the state $\varphi : \mathcal{A} \to \mathbb{C}$ respectively by a subalgebra $\mathcal{B} \subset \mathcal{A}$ and a linear map $E : \mathcal{A} \to \mathcal{B}$ (satisfying certain properties). See Speicher [36] for the description of operator valued free probability and its combinatorial properties. This notion was used by Shlyakhtenko [34] to describe the spectrum of non-commutative polynomials of band Winger matrices. It may also be used to characterize singular values of polynomials of rectangular random matrices (see Benaych-Georges [4, 5]) and spectrum of block matrices (see Rashidi Far et al. [31]).

One of the important applications of these in random matrix theory was the study of the spectrum of $W_n + P_n$ where W_n is a Wigner matrix and P_n is another suitable matrix, independent of W_n. The spectrum of this perturbation has been of interest for a long time (see Fulton [22]). Suppose the spectral measure of P_n weakly converges to μ_P. Then the spectral measure of $W_n + P_n$ converges weakly, almost surely and in expectation, to the *free convolution* of μ_P and the semicircular law whenever μ_P has compact support or P_n satisfy (1.5). These results were derived using asymptotic freeness results between deterministic (or random) matrices and Wigner matrix. Pastur and Vasilchuk [30] extended these results for unbounded perturbations (possibly random) using the analytic machinery of Stieltjes transform. It is to be noted that this result on the sum does not yield asymptotic freeness between the matrices.

The special case where P_n has finite rank has received considerable amount of interest recently. In this case, the limit measure is still the semi-circular law but the behavior at the edge has some interesting properties. See Benaych-Georges et al. [6], Capitaine et al. [12, 13], Féral and Péché [21] and Péché [29].

One relevant question is whether this asymptotic freeness persists for some other types of matrices. Consider the class of *patterned matrices*. These are matrices where, along with symmetry, some other assumptions are imposed on the structure. Important examples are the Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant. The spectrum of these matrices were studied in Bose and Sen [8], Bryc et al. [11], Hammond and Miller [23]. Generally speaking the Stieltjes transform does not seem to be a convenient tool to study these matrices due to the strong dependence among the rows and columns. Bose et al. [9] showed that under suitable assumptions on the pattern, there is joint convergence of i.i.d. copies of a *single* pattern matrix as dimension goes to infinity. One important consequence is that in the limit other kinds of non-free independence may arise. In particular, Symmetric Circulants are commutative and Reverse Circulants are asymptotically half independent. As yet, no description of independence is available for the Toeplitz and Hankel matrices.

As a more general goal, we investigate the joint convergence of multiple independent copies of these matrices, including the Wigner. Inter alia, we address the asymptotic freeness of the Wigner matrices and patterned matrices.

In Theorem 3.1, we provide sufficient conditions for the joint convergence holds. We deal with only real symmetric matrices as the structure of many of these matrices change if one takes complex entries. One of the basic *necessary* assumptions on the pattern matrices is *Property B*, which states that the maximum number of times any entry is repeated in a row remains uniformly bounded across all rows as $n \to \infty$. All the above five matrices satisfy Property B. Under Property B and some moment as-

sumptions on the entries, we show that if a criteria (Condition 3.1) holds for one copy each of any subcollection of matrices, then the joint convergence holds for multiple copies. This Condition 3.1 is satisfied by all the five matrices. We use the method of moments and the so called volume method to prove these results. See Bose and Sen [8], Bryc et al. [11] for the use of volume method for convergence of spectral measure of patterned matrices. As an application of Theorem 3.1, the following holds: if $\mathbb{P}$ is a symmetric polynomial in any two of the following scaled matrices: Wigner, Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant with uniformly bounded entries then the spectral measure L_n of the matrix $\mathbb{P}$ converges to a non-random measure μ on $\mathbb{R}$ weakly almost surely.

In Theorem 3.5, we show that any collection of Wigner matrices is free of the other four matrices. As already discussed, Wigner and deterministic matrices are asymptotically free. By the results of Collins [16] and Collins and Śniady [17] the results are true for general deterministic matrices which converge jointly. To the best of our knowledge these results directly do not imply the freeness result Theorem 3.5. This is because, the existing results need some conditions on the behavior of the trace of the matrices as pointed out in Remark 3.6 of Collins [16]. The condition in Collins [16] (equation (3.4) therein) was studied in Capitaine and Casalis [14]. It was shown that under the technical condition on the random matrices (see Condition C and C' in Capitaine and Casalis [14]) there is asymptotic freeness between Wigner and other random matrices. Although the Theorems of Capitaine and Casalis [14] are for GUE, it is expected that the results would be true for real entries or GOE. In other available criteria for freeness, condition (1.5) appears (see Anderson et al. [1] and Theorem 22.2.4 of Speicher [37]). This is not applicable in our situation as it is known from the works of Bose and Sen [8], Bryc et al. [11] that the spectral norms of

random Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant are unbounded.

Instead of attempting to check/modify the technical sufficient condition of Capitaine and Casalis [14] we extend the volume method to derive Theorem 3.5. This technique is similar in spirit to those in Chapter 22 of Nica and Speicher [27]. However, we bypass the detailed properties of the permutation group and the Weingarten functions. It is quite feasible that the techniques of Collins [16] and Capitaine and Casalis [14] may be extended to prove Theorem 3.5. Incidentally, if we take the Wigner with complex entries then Theorem 3.5 holds for any patterned matrix satisfying Property B and having an LSD.

The use of random matrix theory and free probability in CDMA (Code Division Multiple Access) and MIMO (multiple input and multiple output) systems was shown in many articles.

See Couillet et al. [19], Oraby [28], Rashidi Far et al. [31] and Tulino and Verdú [38]. For a MIMO system with n_1 transmitter antennae and n_2 receiver antennae, the received signal is represented in terms of equation $\mathbb{Y}_n = \mathbb{H}\mathbb{A}_n + \mathbb{B}_n$ where $\mathbb{A}_n$ is an n_1-dimensional vector depending on n and $\mathbb{B}_n$ is a noise signal and $\mathbb{H}$ is the channel matrix which generally has a block structure as below and $\mathbb{Y}_n$ is an n_2 dimensional vector.

$$\mathbb{H} = \begin{bmatrix} C_1 & C_2 & \dots & C_L & \mathbf{0} & \dots & & \dots & \mathbf{0} \\ \mathbf{0} & C_1 & C_2 & \dots & C_L & \mathbf{0} & & & \vdots \\ \vdots & \mathbf{0} & C_1 & C_2 & \dots & C_L & \mathbf{0} & & \\ & & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & & \dots & \mathbf{0} & C_1 & C_2 & \dots & C_L \end{bmatrix}.$$

One of the main issues in the study of a MIMO system is the eigenvalue distribution of $\mathbb{H}\mathbb{H}^*$ since this is linked to the capacity of the channel. Here $\{C_i\}$ can be Wigner matrices or more general matrices. It may also happen that some of the blocks are Toeplitz or Hankel or any other structured matrices. Studying the spectral properties of such matrices boils down to studying the joint convergence of different patterned matrices. The results of this article can be used for studying such systems. We refer the readers to the recent article by Male [26] which applies similar results for the MIMO system.

Finally we point out that we could not obtain full characterization of the joint limits if one of the matrices is not Wigner. It is known that in a complex unital algebra only two notions of independence of subalgebras may arise: freeness and classical independence (see Speicher [35]). Although Reverse Circulant limit shows half independence, this notion is only for variables of an algebra and not for subalgebras (see Bose et al. [10]). For other matrices like Toeplitz and Hankel nothing is known yet about the joint convergence.

In Section 2 we recall definitions of pattern matrices and express the trace in terms of circuits and words (equivalently pair-partitions). In Section 3 we state our main results on joint convergence of patterned matrices including those mentioned earlier as well as Theorem 3.4 on the contribution of certain monomials depending on the structure of the matrices. We also discuss the properties of the sum of two random matrices in the limit. The final Section 4 is dedicated to the proofs.

2 Some basic definitions and notation

2.1 Patterned matrices, link function, trace formula and words

Patterned matrices are defined via

link functions. A link function L is defined as a function $L : \{1, 2, .., n\}^2 \to \mathbb{Z}^d, n \geq 1$. For our purposes $d = 1$ or 2. Although L depends on n, to avoid complexity of notation we suppress the n and consider $\mathbb{N}^2$ as the common domain. We also assume that L is symmetric in its arguments, that is, $L(i, j) = L(j, i)$.

Let $\{x(i)\}$ and $\{x(i, j)\}$ be a sequence of real random variables, referred to as the *input sequence*. The sequence of matrices $\{A_n\}$ under consideration will be defined by

$$A_n \equiv ((a_{i,j}))_{1 \leq i,j, \leq n} \equiv ((x(L(i, j)))).$$

Some important matrices we shall discuss in this article are:

(W_n) Wigner matrix: $L : \mathbb{N}^2 \to \mathbb{Z}^2$ where $L(i, j) = (\min(i, j), \max(i, j))$.

(T_n) Toeplitz matrix: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = |i - j|$.

(H_n) Hankel matrix: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = i + j$.

(RC_n) Reverse Circulant: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = (i + j) \mod n$.

(SC_n) Symmetric Circulant: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = n/2 - |n/2 - |i - j||$.

It is now well known that the limiting spectral distribution (LSD) of the above matrices exist. Bose et al. [9] reviewed the results on LSD of the above matrices. For various results on Wigner matrices we refer to the excellent exposition by Anderson et al. [1].

The L function for all the five matrices defined above satisfy the following property. This property was introduced by Bose and Sen [8] and shall be crucial to us. (For any set S, $\#S$ or $|S|$ will denote the number of elements in S).

Property B: We say a link function L satisfies *Property B* if,

$$\Delta(L) = \sup_n \sup_{t \in \mathbb{Z}^d_{\geq}} \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n, L(k,l) = t\} < \infty. \tag{2.1}$$

In particular, $\Delta(L) = 2$ for T_n, SC_n and $\Delta(L) = 1$ for W_n, H_n and RC_n.

Consider h different type of patterned matrices where type j has p_j independent copies, $1 \leq j \leq h$. The different link functions shall be referred to as *colors* and different independent copies of the matrices of any given color shall be referred to as *indices*. Let $\{X^j_{i,n}, 1 \leq i \leq p_j\}$ be $n \times n$ symmetric patterned matrices with link functions $L_j, j = 1, 2, \cdots, h$. Let $X^j_i(L_j(p,q))$ denote the (p,q)-th entry of $X^j_{i,n}$. We suppress the dependence on n to simplify notation. Two natural assumptions on the link function and the input sequence are:

A1. All link functions $\{L_j, j = 1, 2, \cdots, h\}$ satisfy *Property B*, that is,

$$\max_{1 \leq j \leq h} \sup_{n \geq 1} \sup_t \sup_{1 \leq p \leq n} \#\{q : 1 \leq q \leq n, L_j(p,q) = t\} \leq \Delta < \infty.$$

A2. Input sequences $\{X^j_i(k) : k \in \mathbb{Z} \text{ or } \mathbb{Z}^2\}$ are real random variables independent across i, j and k with mean zero and variance 1 and the moments are uniformly bounded, that is,

$$\sup_{1 \leq j \leq h} \sup_{1 \leq i \leq p_j} \sup_{n \geq 1} \sup_{1 \leq p,q \leq n} \mathrm{E}\left[|X^j_i(L_j(p,q))|^k\right] \leq c_k < \infty.$$

We consider $\{\frac{1}{\sqrt{n}} X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}$ as elements of $\mathcal{A}_n$ given in (1.1) and investigate the joint convergence with respect to the normalized tracial states φ_1 or φ_2 (as in (1.2)). The sequence of matrices jointly converge if and only if for all monomials q,

$$\varphi_d\left(q\left(\frac{1}{\sqrt{n}}\{X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}\right)\right)$$

converge to a limit as $n \to \infty$ for either $d = 1$ or $d = 2$. For $d = 1$, the convergence is in the almost sure sense. The case of $h = 1$ and $p_1 = 1$ (a single patterned matrix) was dealt in Bose and Sen [8] and $h = 1$ and $p_1 > 1$ (i.i.d. copies of a single patterned matrix) was dealt in Bose et al. [9]. In particular, convergence holds for i.i.d. copies of any one of the five patterned matrices. The starting point in showing this was the trace formula. The related concepts of circuits, matchings and words will be extended below to multiple copies of several matrices.

Since our primary aim is to show convergence for every monomial, we shall from now on, fix an arbitrary monomial q of length k. We generally denote the colors and indices present in q by $(c_1, c_2, \cdots, c_k)$ and $(t_1, t_2, \cdots, t_k)$ respectively. Then we may write,

$$q\left(\frac{1}{\sqrt{n}}\{X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}\right) = \frac{1}{n^{k/2}} Z_{c_1,t_1} Z_{c_2,t_2} \cdots Z_{c_k,t_k}, \tag{2.2}$$

where $Z_{c_m,t_m} = X^{c_m}_{t_m}$ for $1 \leq m \leq k$.

From (2.2) we get,

$$
\begin{aligned}
\widetilde{\mu_n}(q) &:= \frac{1}{n} \operatorname{Tr} \left[\frac{1}{n^{k/2}} Z_{c_1,t_1} Z_{c_2,t_2} \cdots Z_{c_k,t_k} \right] \\
&= \frac{1}{n^{1+k/2}} \sum_{j_1,j_2,\cdots,j_k} [Z_{c_1,t_1}(L_{c_1}(j_1,j_2)) Z_{c_2,t_2}(L_{c_2}(j_2,j_3)) \cdots Z_{c_k,t_k}(L_{c_k}(j_k,j_1))] \\
&= \frac{1}{n^{1+k/2}} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,n\} \\ \pi(0)=\pi(k)}} \prod_{i=1}^{k} Z_{c_i,t_i}(L_{c_i}(\pi(i-1),\pi(i))) \\
&= \frac{1}{n^{1+k/2}} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,n\} \\ \pi(0)=\pi(k)}} \mathbf{Z}_\pi \quad \text{say.}
\end{aligned}
\tag{2.3}
$$

Also define,

$$
\overline{\mu_n} = \mathrm{E}[\widetilde{\mu_n}]. \tag{2.4}
$$

Keeping in mind that we seek to show the existence of the limits in (2.3) and (2.4) as $n \to \infty$, we now develop some appropriate notions. In particular these help us to show that certain terms in these sums are negligible in the limit.

Any map $\pi : \{0,\cdots,k\} \to \{1,\cdots,n\}$ with $\pi(0) = \pi(k)$ will be called a *circuit*. Its dependence on k and n will be suppressed. Observe that $\widetilde{\mu_n}$ and $\overline{\mu_n}$ involve sums over circuits. Any value $L_{c_i}(\pi(i-1),\pi(i))$ is called an *L-value* of π. If an L-value is repeated e times in π then π is said to have an *edge* of order e. Due to independence and mean zero of the input sequences,

$$
\mathrm{E}[\mathbf{Z}_\pi] = 0 \quad \text{if } \pi \text{ has any edge of order one.} \tag{2.5}
$$

If all L-values appear more than once then we say the circuit is *matched* and only these circuits are relevant due to the above.

A circuit is said to be *color matched* if all the L-values are repeated within the same color. A circuit is said to be *color* and *index* matched if in addition, all the L-values are also repeated within the same index.

We can define an equivalence relation on the set of color and index matched circuits, extending the ideas of Bose et al. [9] and Bose and Sen [8]. We say $\pi_1 \sim \pi_2$ if and only if their matches take place at the same colors and at the same indices. Or,

$$
c_i = c_j, t_i = t_j \quad \text{and} \quad L_{c_i}(\pi_1(i-1),\pi_1(i)) = L_{c_j}(\pi_1(j-1),\pi_1(j))
$$
$$
\Longleftrightarrow
$$
$$
c_i = c_j, t_i = t_j \quad \text{and} \quad L_{c_i}(\pi_2(i-1),\pi_2(i))) = L_{c_j}(\pi_2(j-1),\pi_2(j)).
$$

An equivalence class can be expressed as a colored and indexed word w: each word is a string of letters in alphabetic order of their first occurrence with a subscript and a superscript to distinguish the index and the color respectively. The i-th position of w is denoted by $w[i]$. Any i is a *vertex* and it is *generating* (or *independent*) if either $i = 0$ or $w[i]$ is the position of the first occurrence of a letter. By abuse of notation we also use $\pi(i)$ to denote a vertex.

For example, if

$$
q = X_1^1 X_2^1 X_1^2 X_1^2 X_2^2 X_2^2 X_2^1 X_1^1 = Z_{1,1} Z_{1,2} Z_{2,1} Z_{2,1} Z_{2,2} Z_{2,2} Z_{1,2} Z_{1,1},
$$

then $a_1^1 b_2^1 c_1^2 c_1^2 d_2^2 d_2^2 b_2^1 a_1^1$ is *one* colored and indexed word corresponding to q. Any colored and indexed word uniquely determines the monomial it corresponds to. A colored and

indexed (matched) word is *pair-matched* if all its letters appear exactly twice. We shall see later that under *Property B*, only such circuits and words survive in the limits of (2.3) and (2.4).

Now we define some useful subsets of the circuits. For a colored and indexed word w, let

$$\Pi_{CI}(w) = \{\pi : w[i] = w[j] \Leftrightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i))) = (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j)))\}. \quad (2.6)$$

Also define

$$\Pi_{CI}^*(w) = \{\pi : w[i] = w[j] \Rightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i)) = (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j)))\}. \quad (2.7)$$

Every colored and indexed word has a corresponding non-indexed version which is obtained by dropping the indices from the letters (i.e. the subscripts). For example, $a_1^1 b_2^1 c_1^2 c_1^2 d_2^2 d_2^2 b_2^1 a_1^1$ yields $a^1 b^1 c^2 c^2 d^2 d^2 b^1 a^1$. For any monomial q, dropping the indices amounts to replacing, for every j, the independent copies X_i^j by a single X^j with link function L_j. In other words it corresponds to the case where $p_j = 1$ for $1 \le j \le h$.

Let $\psi(q)$ be the monomial obtained by dropping the indices from q. For example,

$$\text{if } q = Z_{1,1} Z_{1,2} Z_{2,1} Z_{2,1} Z_{2,2} Z_{2,2} Z_{1,2} Z_{1,1} \text{ then } \psi(q) = Z_1 Z_1 Z_2 Z_2 Z_2 Z_2 Z_1 Z_1.$$

(2.6) and (2.7) get mapped to the following subsets of non-indexed colored word w' via ψ:

$$\Pi_C(w) = \{\pi : w[i] = w[j] \Leftrightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\},$$

$$\Pi_C^*(w) = \{\pi : w[i] = w[j] \Rightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\}.$$

Since pair-matched words are going to be crucial, let us define:

$$CIW(2) = \{w : w \text{ is indexed and colored pair-matched corresponding to } q\}$$
$$CW(2) = \{w : w \text{ is non-indexed colored pair-matched corresponding to } \psi(q)\}.$$

For $w \in CIW(2)$, let us consider the word obtained by dropping the indices of w. This defines an injective mapping into $CW(2)$ and we continue to denote this mapping by ψ.

For any $w \in CW(2)$ and $w' \in CIW(2)$, we define (*whenever the limits exist*),

$$p_C(w) = \lim_{n \to \infty} \frac{1}{n^{1+k/2}} |\Pi_C^*(w)| \quad \text{and} \quad p_{CI}(w') = \lim_{n \to \infty} \frac{1}{n^{1+k/2}} |\Pi_{CI}^*(w')|.$$

3 Main results

Our first result is on the joint convergence of several patterned random matrices and is analogous to Proposition 1 of Bose et al. [9] who considered the case $h = 1$. Recall the quantity Δ introduced in Assumption (A1).

Theorem 3.1. *Let* $\{\frac{1}{\sqrt{n}} X_{i,n}^j, 1 \le i \le p_j\}_{1 \le j \le h}$ *be a sequence of real symmetric patterned random matrices satisfying Assumptions* (A1) *and* (A2). *Fix a monomial q of length k and assume that, for all*

$w \in CW(2)$,

$$p_C(w) = \lim_{n \to \infty} \frac{1}{n^{1+k/2}} |\Pi_C^*(w)| \quad exists. \quad (3.1)$$

Then,

1. *for all* $w \in CIW(2)$, $p_{CI}(w)$ *exists and* $p_{CI}(w) = p_C(\psi(w))$,

2. *we have*

$$\lim_{n\to\infty} \overline{\mu_n}(q) = \sum_{w\in CIW(2)} p_{CI}(w) = \alpha(q) \text{(say)} \qquad (3.2)$$

with

$$|\alpha(q)| \leq \begin{cases} \frac{k!\Delta^{k/2}}{(k/2)!2^{k/2}} & \text{if } k \text{ is even and each index appears even number of times} \\ 0 & \text{otherwise.} \end{cases}$$

3. $\lim_{n\to\infty} \widetilde{\mu_n}(q) = \alpha(q)$ *almost surely.*

As a consequence if (3.1) *holds for every q then* $\{\frac{1}{\sqrt{n}} X_{i,n}^j, 1 \leq i \leq p_j\}_{1\leq j\leq h}$ *converges jointly in both the states φ_1 and φ_2 and the limit is independent of the input sequence.*

Remark 3.2. *(i) Theorem 3.1 asserts that if the joint convergence holds for $p_j = 1, j = 1, 2, \cdots, h$ (that is if condition (3.1) holds), then the joint convergence continues to hold for $p_j \geq 1$. There is no general way of checking (3.1). However, see the next theorem.*

(ii) Under the conditions of Theorem 3.1, if the monomial q yields a symmetric matrix, then the corresponding LSD exists almost surely and is symmetric. This is easy to see since we have almost sure convergence of the empirical moments of q. These moments determine a unique distribution due to the bound on $\alpha(q)$ given in Theorem 3.1. The limit distribution is symmetric since all its odd moments are zero.

The moment conditions may be reduced to some extent depending on the monomial when the input sequences are independent and identically distributed. The proof would use truncation coupled with an application of the bounded Lipschitz metric and Hoffman-Wielandt inequality. For such truncation arguments we refer to Bose and Sen [8] or Section 2.1.5 of Anderson et al. [1]. Sometimes the rank inequality can also be used, for example, see Bai and Silverstein [2]. In particular, if we consider a sum of two random matrices with i.i.d. input sequences it is enough to assume that the second moment is finite. The arguments are exactly similar to the proof of Theorem 2 of Bose and Sen [8] and are omitted. Detailed result on sum of random matrices is discussed later in Corollary 3.7.

Theorem 3.3. *Suppose Assumption (A2) holds. Then $p_C(w)$ exists for all monomials q and for all $w \in CW(2)$, for any two of the following matrices at a time: Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant.*

Theorem 3.1 and Theorem 3.3 shows that if $\mathbb{P}$ is a symmetric polynomial in any of the two matrices Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant then the spectral measure of $\mathbb{P}$ converges almost surely

if the input sequence satisfies Assumption (A2).

In general the value of $p_C(w)$ cannot be computed for arbitrary pair-matched word. In the two tables, we provide some examples. As seen in the two tables, $p_C(w)$ equals one for certain words. We now identify a class of such words. This has ramifications later in the study of freeness.

If for a $w \in CW(2)$, sequentially deleting all double letters of the same color each time leads to the empty word then we call w a *colored Catalan word*. Just as

the set of Catalan words of length m are in bijection with the

set of non-crossing pair partitions $NC_2(m)$, the colored Catalan words are in bijection will the following

set of pair partitions (denoted by $NC_2^{(p)}(m)$)and this correspondence will be useful in

Table 1: $p_C(w)$ for colored words corresponding to monomials $q = q(T, H)$

Monomial	Word	$p_C(w)$
TTHH	aabb	1
THTH	abab	2/3
TTTTHH	aabbcc	1
	abbacc	1
	ababcc	2/3
HHHHTT	aabbcc	1
	abbacc	1
	ababcc	0
TTHTTH	aabccb	1
	abcbac	1/2
	abcabc	1/2
HHTHHT	aabccb	1
	abcbac	1/2
	abcabc	0

Table 2: $p_C(w)$ for colored words corresponding to monomials $q = q(H, R)$ and $q(H, S)$

Monomial	Word	$p_C(w)$	Monomial	Word	$p_C(w)$
RRHH	aabb	1	SSHH	aabb	1
RHRH	abab	0	SHSH	abab	2/3
RRRRHH	aabbcc	1	SSSSHH	aabbcc	1
	abbacc	1		abbacc	1
	ababcc	0		ababcc	1
HHHHRR	aabbcc	1	HHHHSS	aabbcc	1
	abbacc	1		abbacc	1
	ababcc	0		ababcc	0
RRHRRH	aabccb	1	HHHSHS	aabcbc	1/2
	abcbac	0		abbcac	1/2
	abcabc	2/3		abcabc	0
HHRHHR	aabccb	1	HHSHHS	aabccb	1
	abcbac	0		abcbac	1/2
	abcabc	1/2		abcabc	0

Section 4.4. For $p = (p(1), p(2), \ldots, p(m))$ integers (colors), denote

$$NC_2^{(p)}(m) = \{\pi \in NC_2(m) : p(\pi(r)) = p(r) \text{ for all } r = 1, \ldots, m\}.$$

In the non-colored and non-indexed situation, Bose and Sen [8] established that $p(w) = 1$ for the five matrices for all Catalan words w. Banerjee and Bose [3] introduced the following condition
on the link function which guarantees this.

Consider the following boundedness property of the number of matches between rows across all pairs of columns.

Property P: A link function L satisfies *Property P* if

$$M^* = \sup_n \sup_{i,j} \#\{1 \le k \le n : L(k, i) = L(k, j)\} < \infty. \tag{3.3}$$

Note that the five matrices satisfy *Property P*.

It is not hard to see that colored Catalan words are in one-one correspondence with non-crossing colored pair-partitions. Thus freeness and semi-circularity may be described for our limits in the language of words: if the limit satisfies $p_C(w) = 0$ for all words which are not colored Catalan, then the limit is free. *In addition*, if $p_C(w) = 1$ for

all colored Catalan words, then the limits are also semicircular, which is precisely what happens for Wigner matrices. For the other four matrices, the limit is neither semicircular nor free but $p_C(w) = 1$ for all colored Catalan words as Theorem 3.4 shows. This extends the main result of Banerjee and Bose [3] to multiple copies of colored matrices.

Theorem 3.4. *(i) Suppose X and Y satisfy Assumption* (A1) *and Assumption* (A2). *Consider any monomial in X and Y of length $2k$. Then*

$$|\Pi_C^*(w)| \geq n^{1+k} \text{ for any colored Catalan word } w.$$

As a consequence, $p_C(w) \geq 1$ for any colored Catalan word w.

(ii) Suppose the link functions satisfy Property B and Property P and the input satisfies Assumption (A2). *Then for any colored Catalan word, $p_C(w) = 1$.*

It is well known that independent Wigner matrices are asymptotically free and also they are asymptotically free of any class of deterministic matrices $\{D_{i,n}\}_{1 \leq i \leq p}$ which satisfy (1.5) (see Theorem 5.4.5 of Anderson et al. [1]). Moreover, the deterministic matrices can be replaced by random matrices $\{A_n\}$ when $\sup_n \|A_n\| < \infty$ (see Speicher [37]) or when they satisfy the sufficient condition (Condition C) of Capitaine and Casalis [14].

These results cannot be used here since the spectral norm of Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant are unbounded as $n \to \infty$. Nevertheless, using the notions of circuits and words we are able to show freeness in a relatively simple way.

Theorem 3.5. *Suppose $\{W_{i,n}, 1 \leq i \leq p, A_{i,n}, 1 \leq i \leq p\}$ are independent matrices satisfying assumptions* (A2) *where $W_{i,n}$ are Wigner matrices and $A_{i,n}$ are any of Toeplitz, Hankel, Symmetric Circulant or Reverse Circulant matrices. Then the collection $\{W_{i,n}, 1 \leq i \leq p\}$ is asymptotically free of the collection $\{A_{i,n}, 1 \leq i \leq p\}$.*

Remark 3.6. *Incidentally, the freeness between GUE and other patterned matrices is much easier to establish. Indeed, it can be shown that GUE and any patterned matrices (having Property B, satisfying* (A2) *and having LSD) are asymptotically free. We provide a brief proof of this assertion at the end of Section 4.*

3.1 Sum of patterned random matrices

The following result on sum of two patterned matrices essentially follows from Theorem 3.1.

Corollary 3.7. *Let A and B be two independent patterned matrices satisfying Assumptions* (A1) *and* (A2). *Suppose $p_C(w)$ exists for every q and every w. Then LSD for $\frac{A+B}{\sqrt{n}}$ exists in the almost sure sense, is symmetric and does not depend on the underlying distribution of the input sequences of A and B. Moreover, if either LSD of $\frac{A}{\sqrt{n}}$ or LSD of $\frac{B}{\sqrt{n}}$ has unbounded support then LSD of $\frac{A+B}{\sqrt{n}}$ also has unbounded support.*

Proof. The assumptions imply that LSDs of
of $\frac{A}{\sqrt{n}}$ and $\frac{B}{\sqrt{n}}$ exist. By Theorem 3.1, $\{\frac{A}{\sqrt{n}}, \frac{B}{\sqrt{n}}\}$ converge jointly and hence $\lim_{n\to\infty} \frac{1}{n^{k/2+1}} E(\text{Tr}(A + B)^k = \beta_k$ exists for all $k > 0$. Now let us fix k. Let Q_k be the set of monomials such that $(A + B)^k = \sum_{q \in Q_k} q(A, B)$. Hence

$$\frac{1}{n} \text{Tr}(\frac{A+B}{\sqrt{n}})^k = \frac{1}{n^{1+k/2}} \sum_{q \in Q_k} \text{Tr}(q(A, B)) = \sum_{q \in Q_k} \overline{\mu_n}(q)$$

where $\overline{\mu_n}(q)$ is as in Section 2. By (3) of Theorem 3.1, $\overline{\mu_n}(q) \to \alpha(q)$, almost surely and hence,

$$\beta_k = \lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}(\frac{A+B}{\sqrt{n}})^k = \sum_{q\in Q_k} \alpha(q) \text{ almost surely.}$$

Using (2) of Theorem 3.1, we have

$$\beta_{2k} = \sum_{q\in Q_{2k}} \alpha(q) \le |Q_{2k}| \frac{(2k)!}{k!2^k}\Delta^k = 2^{2k}\frac{(2k)!}{k!2^k}\Delta^k.$$

Now by using Stirling's formula, $\beta_{2k} \le (Ck)^k$ for some constant C. Hence $\sum_k \beta_{2k}^{-1/2k} = \infty$ and *Carleman's Condition* is satisfied implying that the LSD exists.

To prove symmetry of the limit, let $q \in Q_{2k+1}$. Then from (2) of Theorem 3.1, it follows that $\alpha(q) = 0$. Hence $\beta_{2k+1} = \sum_{q\in Q_{2k+1}} \alpha(q) = 0$ and the distribution is symmetric.

To prove unboundedness, without loss of generality let us assume that LSD $\mathcal{L}_A$ of $\frac{A}{\sqrt{n}}$ has unbounded support. Let us denote by $\beta_{2k}(A)$ the $(2k)$th moment of $\mathcal{L}_A$. Since L^p norm converges to essential supremum as $p \to \infty$, it follows that $(\beta_{2k}(A))^{1/2k} \to \infty$ as $k \to \infty$. Also, $\beta_{2k}(A) = \alpha(q_{2k})$ where $q_{2k}(A,B) = A^{2k}$ and $q_{2k} \in Q_{2k}$. Since $\alpha(q)$ is non-negative for all q, it implies $\beta_{2k} \ge \beta_{2k}(A)$. So $\lim_{k\to\infty}(\beta_{2k})^{1/2k} = \infty$ and hence the LSD of $\frac{A+B}{\sqrt{n}}$ has unbounded support. □

As pointed out earlier in Remark 3.2 (ii), the above result continues to hold if the two input sequences are i.i.d. with finite second moments. Also, in particular, all conclusions in Proposition 3.7 hold when A and B are any two of Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant matrices. It does not seem easy to identify the LSD for these sums. Some simulation results are given below.

When one of the matrix is Wigner, Theorem 3.5 implies that the limit is the free convolution of the semicircular law and the corresponding LSD. This result about the sum when one of them is Wigner also follows from the results of Pastur and Vasilchuk [30]. It also follows from the work of Biane [7] that any free convolution with the semi-circular law is continuous and the density can be expressed in terms of Stieltjes transform of the LSD. Unfortunately, the Stieltjes transform of the LSD of the Toeplitz and Hankel are not known.
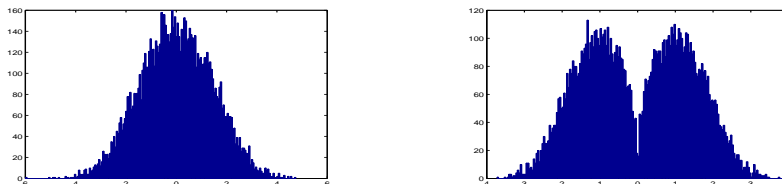
Figure 1: (i) (left) Histogram plot of empirical distribution of Reverse Circulant+ Symmetric Circulant ($n = 500$) with entries $N(0,1)$ (ii) (right) Histogram plot of empirical distribution of Reverse Circulant+Hankel ($n = 500$) with $N(0,1)$ entries.
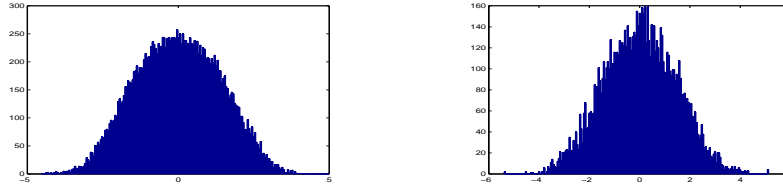
Figure 2: (i) (left) Histogram plot of empirical distribution of Toeplitz+Hankel($n = 1000$) with entries $N(0,1)$ (ii) (right) Histogram plot of empirical distribution of Toeplitz+Symmetric Circulant ($n = 500$) with $N(0,1)$ entries.

4 Proofs

To simplify the notational aspects in all our proofs we restrict ourselves to $h = 2$.

4.1 Proof of Theorem 3.1

(1) We first show that

$$\Pi_C^*(w) = \Pi_{CI}^*(w) \ \text{ for all } \ w \in CIW(2). \tag{4.1}$$

Let $\pi \in \Pi_{CI}^*(w)$. As q is fixed,

$$\psi(w)[i] = \psi(w)[j] \ \Rightarrow \ w[i] = w[j]$$
$$\Rightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i))) \ = \ (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j))) \quad (\text{as } \pi \in \Pi_{CI}^*(w)).$$

This implies $L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))$. Hence $\pi \in \Pi_C^*(\psi(w))$.

Now conversely, let $\pi \in \Pi_C^*(\psi(w))$. Then we have

$$
\begin{aligned}
w[i] &= w[j] \\
\Rightarrow \psi(w)[i] &= \psi(w)[j] \\
\Rightarrow L_{c_i}(\pi(i-1), \pi(i)) &= L_{c_j}(\pi(j-1), \pi(j)) \\
\Rightarrow Z_{c_i,t_i}(L_{c_i}(\pi(i-1), \pi(i))) &= Z_{c_j,t_j}(L_{c_j}(\pi(j-1), \pi(j))).
\end{aligned}
$$

as $w[i] = w[j] \Rightarrow c_i = c_j$ and $t_i = t_j$. Hence $\pi \in \Pi_{CI}^*(w)$.
So (4.1) is established. As a consequence,

$$p_{CI}(w) = \lim_{n \to \infty} \frac{1}{n^{1+k/2}} |\Pi_{CI}^*(w)| = p_C(\psi(w)).$$

Hence by (4.1) $p_{CI}(w)$ exists for all $w \in CIW(2)$ and $p_{CI}(w) = p_C(\psi(w))$, proving (1).

(2) Recall that $\mathbf{Z}_\pi = \prod_{j=1}^k Z_{c_j,t_j}(L_{c_i}(\pi(j-1), \pi(j))$ and using (2.4) and (2.5)

$$\overline{\mu}_n(q) = \frac{1}{n^{1+k/2}} \sum_{w:\, w \text{ matched}} \sum_{\pi \in \Pi_{CI}(w)} \mathrm{E}(\mathbf{Z}_\pi). \tag{4.2}$$

By using Assumption (A2)

$$\sup_\pi \mathrm{E}\,|\mathbf{Z}_\pi| < K < \infty. \tag{4.3}$$

By using often used arguments of Bose and Sen [8] and of Bryc et al. [11], for any colored and indexed matched word w which is matched but is not pair-matched,

$$\lim_{n\to\infty} \frac{1}{n^{1+k/2}} \Big| \sum_{\pi\in\Pi_{CI}(w)} \mathrm{E}(\mathbf{Z}_\pi) \Big| \leq \frac{K}{n^{1+k/2}} |\Pi_{CI}(w)| \to 0. \tag{4.4}$$

By using (4.4), and the fact that $\mathrm{E}(\mathbf{Z}_\pi) = 1$ for every color index pair-matched word (use Assumption (A2)), calculating the limit in (4.2) reduces to calculating

$\lim \frac{1}{n^{1+k/2}} \sum_{w:\, w\in CIW(2)} |\Pi_{CI}(w)|$.

Now consider any $w \in CIW(2)$. Observe that any circuit in $\Pi^*_{CI}(w) - \Pi_{CI}(w)$ must have an edge of order four. Hence by (4.4),

$$\lim_{n\to\infty} \frac{|\Pi^*_{CI}(w) - \Pi_{CI}(w)|}{n^{1+k/2}} = 0.$$

As a consequence, since there are finitely many words,

$$\lim_{n\to\infty} \overline{\mu}_n(q) = \lim_{n\to\infty} \sum_{w\in CIW(2)} \frac{|\Pi_{CI}(w)|}{n^{1+k/2}} = \lim_{n\to\infty} \sum_{w\in CIW(2)} \frac{|\Pi^*_{CI}(w)|}{n^{1+k/2}} = \sum_{w\in CIW(2)} p_{CI}(w) = \alpha(q). \tag{4.5}$$

To complete the proof of (2), we note that, if either k is odd or some index appears an odd number of times in q then for that q, $CIW(2)$ is empty and hence, $\alpha(q) = 0$. Now suppose that k is even and every index appears an even number of times. Then

$$|CIW(2)| \leq |CW(2)| \leq \frac{k!}{(k/2)!2^{k/2}}.$$

The first inequality above follows from the fact mentioned earlier that ψ is an injective map from $CIW(2)$ to $CW(2)$.

The second inequality follows by observing that the total number of *colored* pair matched word of length k is less than the number of pair matched words of length k.

Now note that $p_{CI}(w) \leq \Delta^{k/2}$. Combining all these, we get $|\alpha(q)| \leq \frac{k!\Delta^{k/2}}{(k/2)!2^{k/2}}$.

(3) Now we claim that

$$\mathrm{E}[(\widetilde{\mu_n}(q) - \overline{\mu}_n(q))^4] = O(n^{-2}).$$

Observe that,

$$\mathrm{E}[(\widetilde{\mu_n}(q) - \overline{\mu}_n(q))^4] = \frac{1}{n^{2k+4}} \sum_{\pi_1,\pi_2,\pi_3,\pi_4} \mathrm{E}\Big(\prod_{j=1}^4 (\mathbf{Z}_{\pi_j} - E(\mathbf{Z}_{\pi_j}))\Big). \tag{4.6}$$

We say $(\pi_1,\pi_2,\pi_3,\pi_4)$ are "jointly matched" if each L-value occurs at least twice across all circuits (among same color) and they are said to be "cross matched" if each circuit has at least one L^* value which occurs in some other circuit.

If $(\pi_1,\pi_2,\pi_3,\pi_4)$ are not jointly matched then without loss of generality there exists at least one L-value in π_1 which does not occur anywhere else. Using $\mathrm{E}(\mathbf{Z}_{\pi_1}) = 0$ and independence,

$$\mathrm{E}\Big(\prod_{j=1}^4 (\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j}))\Big) = \mathrm{E}\Big(\mathbf{Z}_{\pi_1} \prod_{j=2}^4 (\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j}))\Big) = 0. \tag{4.7}$$

Again, if $(\pi_1, \pi_2, \pi_3, \pi_4)$ are jointly matched but not cross matched, then without loss of generality, assume π_1 is only self matched. Then by independence,

$$E(\prod_{j=1}^{4}(\mathbf{Z}_{\pi_j} - E(\mathbf{Z}_{\pi_j}))) = E[\mathbf{Z}_{\pi_1} - E(\mathbf{Z}_{\pi_1})] \, E[\prod_{j=2}^{4}(\mathbf{Z}_{\pi_j} - E(\mathbf{Z}_{\pi_j}))] = 0. \qquad (4.8)$$

So we are left with circuits that are jointly matched and cross matched with respect to q. Let Q_q be the number of such circuits.

We claim that $Q_q = O(n^{2k+2})$. Since the circuits are jointly matched there are at most $2k$ distinct L values among all the four circuits. Let u be the number of distinct L values (of a single color) in the circuits. Clearly, for a fixed choice of matches among those distinct L values (number of such choices is bounded in n), the number of jointly matched and cross matched circuits are $O(n^{u+4})$, so the number of such circuits with $u \le 2k-2$ is $O(n^{2k+2})$. Hence it suffices to prove that for a fixed choice of matches among $u = 2k-1$ or $u = 2k$ distinct L-values occurring across all four circuits, the number of jointly matched and cross matched circuits is $O(n^{2k+2})$.

We consider only the case $u = 2k-1$ and the other case is dealt in a similar way. Since $u = 2k-1$, it follows that every L-value occurs exactly twice across all four circuits. Since π_1 is not self matched, there is an L value in π_1 which does not occur anywhere else in π_1. We consider the following dynamic construction of $(\pi_1, \pi_2, \pi_3, \pi_4)$. Since the circuit is cross matched, there exists an L value which is assigned to a single edge, say $L(\pi_1(i_* - 1), \pi(i_*))$. First choose one of the n possible values for the initial value $\pi_1(0)$, and continue filling in the values of $\pi_1(i), i = 1, 2, ..., i_* - 1$. Then, starting at $\pi_1(k) = \pi_1(0)$, sequentially choose the values of $\pi_1(k-1), \pi_1(k-2), ..., \pi_1(i_*)$, thus completing the entire circuit π_1. At every stage there are n ways to choose a vertex if there is no L-match of the edge being constructed with the previously constructed edges, otherwise there are at most $\Delta(L_1, L_2)$ choices. So there are $O(n)$ choices for at most $2k-2$ distinct L values and hence the number of jointly matched and cross matched circuits for $u = 2k-1$ is $O(n^{2k-2+4})$, as required.

By Assumption (A2), $E[\prod_{j=1}^{4}(\mathbf{Z}_{\pi_j} - E(\mathbf{Z}_{\pi_j}))]$ is uniformly bounded over all $(\pi_1, \pi_2, \pi_3, \pi_4)$ by K, say. By this and (4.6)–(4.8), it follows that

$$E[(\widetilde{\mu_n}(q) - \overline{\mu}_n(q))^4] = O(\frac{n^{2k+2}}{n^{2k+4}}) = O(n^{-2}). \qquad (4.9)$$

Now using Borel-Cantelli Lemma, $\widetilde{\mu_n}(q) - \overline{\mu}_n(q) \to 0$ almost surely as $n \to \infty$ and this completes the proof.

4.2 Proof of Theorem 3.3

Condition (3.1) which needs to be verified (only for even degree monomials), crucially depends on the type of the link function and hence we need to deal with every example differently. Since we are dealing with only two link functions, we simplify the notation. Let X and Y be patterned matrices with link function L_1 and L_2 respectively with independent input sequences satisfying Assumptions (A1) and (A2). Let $q(X, Y)$ be any monomial such that both X and Y occur an even number of times in q. Let $deg(q) = 2k$ and let the number of times X and Y occurs in the monomial be k_1 and k_2 respectively. Note that we have $k = k_1 + k_2$. Then it is enough to show that (3.1) holds for every pair-matched colored word w of length $2k$ corresponding to q.

Let X and Y be any of the two following matrices: Wigner (W_n), Toeplitz (T_n), Hankel (H_n), Reverse Circulant (RC_n) and Symmetric Circulant (SC_n). The case when both X and Y are of the same pattern was dealt in Bose et al. [9].

Proof of Theorem 3.3 is immediate once we establish the following Lemma.

Lemma 4.1. *Let X and Y be any of the matrices, W_n, T_n, H_n, RC_n and SC_n, satisfying Assumption (A2). Let $w \in CW(2)$ corresponding to a monomial q of length $2k$. Then there exists a (finite) index set I independent of n and $\{\Pi^*_{C,l}(w) : l \in I\} \subset \Pi^*_C(w)$ such that*

(1) $\quad \Pi^*_C(w) = \cup_{l \in I} \Pi^*_{C,l}(w)$, *and* $p_{C,l}(w) := \lim_{n \to \infty} \frac{|\Pi^*_{C,l}(w)|}{n^{1+k}}$ *exists for all* $l \in I$,

(2) $\quad$ *for* $l \neq l'$ *we have,*

$$|\Pi^*_{C,l}(w) \cap \Pi^*_{C,l'}(w)| = \mathrm{o}(n^{1+k}). \tag{4.10}$$

Assuming Lemma 4.1, $|\Pi^*_C(w)| = |\cup_{l \in I} \Pi^*_{C,l}(w)|$ for some finite index set I and

$$p_C(w) = \lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi^*_C(w)| = \sum_{l \in I} \lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi^*_{C,l}(w)| = \sum_{l \in I} p_{C,l}(w). \tag{4.11}$$

The proof of this lemma treats each pair of matrices separately. Since the arguments are similar for the different pairs, we do not provide the detailed proof for each case but only a selection of the arguments in most cases.

The set S of all generating vertices of w is split into the three classes $\{0\} \cup S_X \cup S_Y$ where

$$S_X = \{i \wedge j : c_i = c_j = X, \ w[i] = w[j]\}, \ \ S_Y = \{i \wedge j : c_i = c_j = Y, \ w[i] = w[j]\}.$$

For every $i \in S - \{0\}$, let j_i denote the index such that $w[j_i] = w[i]$. Let $\pi \in \Pi^*_C(w)$.

(i) *Toeplitz and Hankel:* Let X and Y be respectively the Toeplitz (T) and the Hankel (H) matrix. Observe that,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)| \text{ for all } i \in S_T$$

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \text{ for all } i \in S_H.$$

Let I be $\{-1, 1\}^{k_1}$ and $l = (l_1, ..., l_{k_1}) \in I$. Let $\Pi^*_{C,l}(w)$ be the subset of $\Pi^*_C(w)$ such that,

$$\pi(i-1) - \pi(i) = l_i(\pi(j_i - 1) - \pi(j_i)) \qquad \text{for all } i \in S_T,$$

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \qquad \text{for all } i \in S_H.$$

Now clearly,

$$\Pi^*_C(w) = \bigcup_l \Pi^*_{C,l}(w) \text{ (not a disjoint union).}$$

Now let us define,

$$v_i = \frac{\pi(i)}{n} \qquad \text{and} \qquad U_n = \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}. \tag{4.12}$$

Then,

$$|\Pi^*_{C,l}(w)| = \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \leq i \leq 2k, \ v_{i-1} - v_i = l_i(v_{j_i-1} - v_{j_i}) \ \forall i \in S_T$$
$$\text{and } v_{i-1} + v_i = v_{j_i-1} + v_{j_i} \ \forall i \in S_H, \ v_0 = v_{2k}\}.$$

Let us denote $\{v_i : i \in S\}$ by v_S. It can easily be seen from the above equations (other than $v_0 = v_{2k}$) that each of the $\{v_i : i \notin S\}$ can be written uniquely as an integer linear combination $L^l_i(v_S)$. Moreover, $L^l_i(v_S)$ only contains $\{v_j : j \in S, \ j < i\}$ with non-zero coefficients. Clearly,

$$|\Pi^*_{C,l}(w)| = \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \leq i \leq 2k, v_0 = v_{2k}, v_i = L^l_i(v_S) \ \forall i \notin S\}. \tag{4.13}$$

Any integer linear combinations of elements of U_n is again in U_n if and only if it is between 0 and 1. Hence,

$$|\Pi_{C,l}^*(w)| = \#\{v_S : v_i \in U_n \ \forall i \in S, v_0 = L_{2k}^l(v_S), 0 \le L_i^l(v_S) < 1 \ \forall i \notin S\}. \qquad (4.14)$$

From (4.14) it follows that, $\frac{|\Pi_{C,l}^*(w)|}{n^{1+k}}$ is nothing but the Riemann sum for the function $I(0 \le L_i^l(v_S) < 1, i \notin S, v_0 = L_{2k}^l(v_S))$ over $[0,1]^{k+1}$ and converges to the integral and hence

$$p_{C,l}(w) = \lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi_{C,l}^*(w)| = \int\limits_{[0,1]^{k+1}} I\left(0 \le L_i^l(v_S) < 1, i \notin S, v_0 = L_{2k}^l(v_S)\right) dv_S.$$

$$(4.15)$$

This shows part (1) of Lemma 4.1. For part (2) let $l \ne l'$. Without loss of generality, let us assume that, $l_{i_1} = -l'_{i_1}$. Let $\pi \in \Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)$. Then $\pi(i_1 - 1) = \pi(i_1)$ and hence $L_{i_1-1}^l(v_S) = v_{i_1}$. It now follows along the lines of the preceding arguments that

$$\lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)| \le \int \cdots \int\limits_{[0,1]^{k+1}} I(v_i = L_{i_1-1}^l(v_S)) dv_S. \qquad (4.16)$$

$L_{i_1-1}^l(v_S)$ contains $\{v_j : j \in S, \ j < i_1\}$ and hence $\{L_{i_1-1}^l(v_S) = v_i\}$ is a k-dimensional subspace of $[0,1]^{k+1}$ and hence has Lebesgue measure 0.

(ii) *Hankel and Reverse Circulant:* Let X and Y be Hankel (H) and Reverse Circulant (RC) respectively. Then

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \quad \text{for all } i \in S_H, \qquad (4.17)$$

$$(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n \quad \text{for all } i \in S_{RC}. \qquad (4.18)$$

Clearly, as all the $\pi(i)$ are between 1 and n, relation (4.18) implies $(\pi(i-1) + \pi(i)) - (\pi(j_i - 1) + \pi(j_i)) = a_i n$ where $a_i \in \{0, 1, -1\}$

Let $a = (a_1, ..., a_{k_2}) \in I = \{-1, 0, 1\}^{k_2}$. Let $\Pi_{C,a}^*(w)$ be the subset of $\Pi_C^*(w)$ such that,

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \ \forall i \in S_H \text{ and}$$

$$(\pi(i-1) + \pi(i)) - (\pi(j_i - 1) + \pi(j_i)) = a_i n \quad \forall i \in S_{RC}.$$

Now clearly,

$$\Pi_C^*(w) = \bigcup_a \Pi_{C,a}^*(w) \text{ (a disjoint union)}.$$

Now we get that,

$$|\Pi_{C,a}^*(w)| = \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \le i \le 2k, \ v_{i-1} + v_i = (v_{j_i-1} + v_{j_i}) + a_i \ \forall i \in S_{RC}$$
$$\text{and} \ v_{i-1} + v_i = v_{j_i-1} + v_{j_i} \ \forall i \in S_H, \ v_0 = v_{2k}\}.$$

Other than $v_0 = v_{2k}$, each $\{v_i : i \notin S\}$ can be written uniquely as an affine linear combination $L_i^a(v_S) + b_i^{(a)}$ for some integer $b_i^{(a)}$. Moreover, $L_i^a(v_S)$ only contains $\{v_j : j \in S, \ j < i\}$ with non-zero coefficients. Arguing as in the previous case,

$$|\Pi_{C,a}^*(w)| = \#\{v_S : v_i \in U_n \ \forall i \in S, v_0 = L_{2k}^a(v_S) + b_{2k}^{(a)}, 0 \le L_i^a(v_S) + b_i^{(a)} < 1 \forall i \notin S\}. \qquad (4.19)$$

This is again a Riemann sum and hence as before,

$$p_{C,a}(w) = \lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi_{C,a}^*(w)| = \int\limits_{[0,1]^{k+1}} I\left(0 \le L_i^a(v_S) + b_i^{(a)} < 1, i \notin S, v_0 = L_{2k}^a(v_S) + b_{2k}^{(a)}\right) dv_S$$

and the proof of this case is complete.

(iii) *Hankel and Symmetric Circulant:* Let X and Y be Hankel (H) and Symmetric Circulant (SC) respectively. Note that

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \ \ \forall i \in S_H \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i-1) - \pi(j_i)|| \ \ \forall i \in S_S.$$

It can be easily seen from the second equation above that either $|\pi(i-1) - \pi(i)| = |\pi(j_i-1) - \pi(j_i)|$ or $|\pi(i-1) - \pi(i)| + |\pi(j_i-1) - \pi(j_i)| = n$. There are six cases for each Symmetric Circulant match $[i, j_i]$, and with $v_i = \pi(i)/n$, these are:

1. $v_{i-1} - v_i - v_{j_i-1} + v_{j_i} = 0$.
2. $v_{i-1} - v_i + v_{j_i-1} - v_{j_i} = 0$.
3. $v_{i-1} - v_i + v_{j_i-1} - v_{j_i} = 1$.
4. $v_{i-1} - v_i - v_{j_i-1} + v_{j_i} = 1$.
5. $v_i - v_{i-1} + v_{j_i-1} - v_{j_i} = 1$.
6. $v_i - v_{i-1} + v_{j_i} - v_{j_i-1} = 1$.

Now we can write $\Pi_C^*(w)$ as the (not disjoint) union of 6^{k_2} possible $\Pi_{C,l}^*(w)$ where l denotes the combination of cases (1)–(6) above that is satisfied in the k_2 matches of Symmetric Circulant. For each $\pi \in \Pi_{C,l}^*(w)$, each $\{v_i : i \notin S\}$ can be written uniquely as an affine integer combination of v_S. As in the previous two pairs of matrices in (i) and (ii), $\lim_{n\to\infty} \frac{1}{n^{1+k}} |\Pi_{C,l}^*(w)|$ exists as an integral.

Now (4.10) can be checked case by case. As a typical case suppose Case 1 and Case 3 hold. Then $\pi(i-1) - \pi(i) = n/2$ and $v_{i-1} - v_i = 1/2$. Since i is generating and v_{i-1} is a linear combination of $\{v_j : \ j \in S, \ j < i\}$, this implies a non-trivial linear relation between the independent vertices v_S. This, in turn implies that the number of circuits π satisfying the above conditions is $o(n^{1+k})$.

(iv) *Toeplitz and Symmetric Circulant:* Let X and Y be Toeplitz (T) and Symmetric Circulant (SC) respectively. Again note that,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i-1) - \pi(j_i)| \ \ \forall i \in S_T \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i-1) - \pi(j_i)|| \ \ \forall i \in S_{SC}. \quad (4.20)$$

Now, (4.20) implies either $|\pi(i-1) - \pi(i)| = |\pi(j_i-1) - \pi(j_i)|$ or $|\pi(i-1) - \pi(i)| + |\pi(j_i - 1) - \pi(j_i)| = n$.

There are six cases for each Symmetric Circulant match as in Case (iii) above and two cases for each Toeplitz match.

As before we can write $\Pi_C^*(w)$ as the (not disjoint) union of $2^{k_1} \times 6^{k_2}$ possible $\Pi_{C,l}^*(w)$ where l denotes a combination of cases (1)–(6) for all SC matches (as in Case (iii)) and a combination of cases (1)–(2) for all T matches. As before, for each $\pi \in \Pi_{C,l}^*(w)$, each of the $\{v_i : i \notin S\}$ can be written uniquely as an affine integer combination of v_S. As earlier, $\lim_{n\to\infty} \frac{1}{n^{1+k}} |\Pi_{C,l}^*(w)|$ exists as an integral.

Now, (4.10) is again checked case by case. Suppose $l \neq l'$ and $\pi \in \Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)$. For $l \neq l'$, there must be one Toeplitz or Symmetric Circulant match such that two of the possible cases in (1)–(2) or in (1)–(6) occur simultaneously. Here we just deal with a typical pair Case (1) and Case (2) for the Toeplitz match. Then we have $\pi(i-1) - \pi(i) = 0$ and hence $v_{i-1} - v_i = 0$. Since i is generating and v_{i-1} is a linear combination of $\{v_j : \ j \in S, \ j < i\}$, this implies there exist a non-trivial relation between the independent vertices v_S. This, in turn implies that the number of circuits π satisfying the above conditions in $o(n^{1+k})$. Now suppose the Symmetric Circulant match happens for both

case (1) and case (2). Then again we have $v_i = v_{i-1}$ and we can argue as before to conclude that (4.10) holds.

(v) *Toeplitz and Reverse Circulant:* Let X and Y be Toeplitz (T) and Reverse Circulant (RC) respectively. Note,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)| \quad \text{for all } i \in S_T,$$

$$(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n \quad \text{for all } i \in S_{RC}.$$

Clearly, as all the $\pi(i)$ are between 1 and n, $(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n$ implies $(\pi(i-1) + \pi(i)) - (\pi(j_i - 1) + \pi(j_i)) = a_i n$ where $a_i \in \{0, 1, -1\}$

Let the number of Toeplitz and Reverse Circulant matches be k_1, and k_2 respectively and let us denote $S_T = \{i_1, i_2, ..., i_{k_1}\}$, $S_{RC} = \{i_{k_1+1}, i_{k_1+2}, ..., i_{k_1+k_2}\}$.
Let $l = (c, a) = (c_{i_1}, ..., c_{i_{k_1}}, a_{i_{k_1+1}}, ..., a_{i_{k_1+k_3}}) \in I = \{-1, 1\}^{k_1} \times \{-1, 0, 1\}^{k_3}$.
Let $\Pi_{C,l}^*(w)$ be the subset of $\Pi_C^*(w)$ such that,

$$\pi(i-1) - \pi(i) = c_i(\pi(j_i - 1) - \pi(j_i)) \quad \forall i \in S_T$$

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) + a_i n \quad \forall i \in S_{RC}.$$

Now clearly,

$$\Pi_C^*(w) = \bigcup_{l \in I} \Pi_{C,l}^*(w),$$

and translating this in the language of v_i's, we get

$$\begin{aligned}
|\Pi_{C,l}^*(w)| &= \#\{(v_0, ..., v_{2k}) : v_i \in U_n \,\forall 0 \le i \le 2k, \; v_{i-1} + v_i = (v_{j_i-1} + v_{j_i}) + a_i \;\; \forall i \in S_{RC} \\
&\quad \text{and} \quad v_{i-1} - v_i = c_i(v_{j_i-1} - v_{j_i}) \;\; \forall i \in S_T, \;\; v_0 = v_{2k}\}.
\end{aligned}$$

As in the previous cases, $\lim_{n \to \infty} \frac{|\Pi_{C,l}^*(w)|}{n^{1+k}}$ exists. It remains to show that, $\lim_{n \to \infty} \frac{|\Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)|}{n^{1+k}} = 0$ for $l \ne l'$. If $l = (c, a) \ne l' = (c', a')$, then either $c \ne c'$ or $a \ne a'$. If $c = c'$, then clearly $\Pi_{C,l}^*(w)$ and $\Pi_{C,l'}^*(w)$ are disjoint. Let $c \ne c'$. Without loss of generality, we assume $c_{i_1} = -c_{i_1}$. Then clearly, for every $\pi \in \Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)$ we have $v_{i_1-1} = v_{i_1}$, which gives a non-trivial relation between $\{v_j : j \in S\}$. That in turn implies the required limit is 0.

(vi) *Reverse Circulant and Symmetric Circulant:* Let X and Y be Reverse Circulant (RC) and Symmetric Circulant (SC) respectively. Then

$$\pi(i-1) + \pi(i) \mod n = \pi(j_i - 1) + \pi(j_i) \mod n \;\; \forall i \in S_{RC} \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i - 1) - \pi(j_i)|| \;\; \forall i \in S_{SC}.$$

As before, the latter equation implies either $|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)|$ or $|\pi(i-1) - \pi(i)| + |\pi(j_i - 1) - \pi(j_i)| = n$.

There are now three cases for each Reverse Circulant match:

1. $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = 0$.
2. $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = 1$.
3. $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = -1$.

Also, there are six cases for each Symmetric Circulant match as in Case (iii).

As before we can write $\Pi_C^*(w)$ as the union of $3^{k_1} \times 6^{k_2}$ possible $\Pi_{C,l}^*(w)$. Hence arguing in a similar manner, $\lim_{n \to \infty} \frac{1}{n^{1+k}} |\Pi_{C,l}^*(w)|$ exists as an integral. Now, to check (4.10), case by case. Suppose $l \ne l'$ and $\pi \in \Pi_{C,l}^*(w) \bigcap \Pi_{C,l'}^*(w)$. Since $l \ne l'$, there must be one Reverse Circulant or Symmetric Circulant match such that two of the possible

cases (1)–(3) or (1)–(6) (which appear in Case (iii)) occur simultaneously. It is easily seen that such an occurrence is impossible for a Reverse Circulant match.

So we assume there is a Symmetric Circulant match and deal with one such typical match. Suppose then we have both case (1) and case (2). Then again we have $v_i = v_{i-1}$ and as a consequence (4.10) holds.

(vii) *Wigner and Hankel:* Let X and Y be Wigner (W) and Hankel (H) respectively. Observe that,

$$(\pi(i-1), \pi(i)) = \begin{cases} (\pi(j_i - 1), \pi(j_i)) & \text{(Constraint } C1) \\ (\pi(j_i), \pi(j_i - 1)) & \text{(Constraint } C2, \text{ for all } i \in S_W). \end{cases} \tag{4.21}$$

Also, $\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i)$ for all $i \in S_H$. So for each Wigner match there are two constraints and hence there are 2^{k_1} choices. Let λ be a typical choice of k_1 constraints and $\Pi^*_{C,\lambda}(w)$ be the subset of $\Pi^*_C(w)$ where the above relations hold. Hence

$$\Pi^*_C(w) = \bigcup_\lambda \Pi^*_{C,\lambda}(w) \quad \text{(not a disjoint union)}.$$

Now using equation (4.12) we have,

$$\begin{aligned} |\Pi^*_{C,\lambda}(w)| &= \#\{(v_0, v_1 \ldots v_{2k}) : 0 \le v_i \le 1, v_0 = v_{2k}, v_{i-1} + v_i = v_{j_i - 1} + v_{j_i}, \ i \in S_H \\ & \quad v_{i-1} = v_{j_i - 1}, v_i = v_{j_i}, (C1), v_{i-1} = v_{j_i}, v_i = v_{j_i - 1}(C2), i \in S_W\}. \end{aligned}$$

It can be seen from the above equations that each v_j, $j \notin S$ can be written (not uniquely) as a linear combination L^λ_j of elements in v_S. Hence as before,

$$\begin{aligned} |\Pi^*_{C,\lambda}(w)| &= \#\{v_S : v_i = L^\lambda_i(v_S), v_0 = v_{2k}, \text{ for } i \notin S, , v_{i-1} + v_i = v_{j_i - 1} + v_{j_i}, \ i \in S_H \\ & \quad v_{i-1} = v_{j_i - 1}, v_i = v_{j_i}, (C1), v_{i-1} = v_{j_i}, v_i = v_{j_i - 1}(C2), i \in S_W\}. \end{aligned}$$

So the limit of $|\Pi^*_{C,\lambda}(w)|/n^{1+k}$ exists and can be expressed as an appropriate Riemann integral.

Now we show (4.10). Without loss of generality assume λ_1 is a C_1 constraint and λ_2 is a C_2 constraint. For any $\pi \in \Pi^*_{C,\lambda_1}(w) \bigcap \Pi^*_{C,\lambda_2}(w)$ we note that for $i \in S$,

$$(\pi(j_i), \pi(j_i - 1)) = (\pi(i-1), \pi(i)) = (\pi(j_i - 1), \pi(j_i)),$$

which implies $\pi(i) = \pi(i-1)$. Now i is a generating vertex. But $\pi(i) = \pi(i-1)$ and hence is fixed, having chosen the first $i-1$ vertices. This lowers the order by a power of n and hence the claim follows.

(vii) Wigner and other matrices: Since the other cases such as Wigner and Toeplitz and Wigner and Reverse Circulant follow by similar and repetitive arguments we refrain from presenting a proof for them.

4.3 Proof of Theorem 3.4

Let w be a colored word of length $2k$ for a monomial $q = q(X, Y)$. Let w' be obtained from w by a cyclic permutation, that is, there exists l such that $w'[i] = w[(i+l) \bmod 2k]$. Note that w' is a colored word for the monomial q' obtained from q by the same cyclic permutation. We have the following lemma.

Lemma 4.2. $|\Pi^*_C(w)| = |\Pi^*_C(w')|$ *and* $p_C(w) = p_C(w')$.

Proof of Lemma 4.2. Let $\pi \in \Pi^*_C(w)$. Let $\pi'(i) = \pi((i+l) \bmod 2k))$. Clearly, $\pi'(0) = \pi'(2k)$. Also

$$w'[i] = w'[j] \Rightarrow L^*(\pi'(i-1), \pi'(i)) = L^*(\pi'(j-1), \pi'(j))$$

where L^* is equal to L_1 or L_2 according as $w'[i] = w'[j]$ is an X match or a Y match. Hence, $\pi' \in \Pi_C^*(w')$.

As w can also be obtained from w' by another cyclic permutation, it follows that the map $\pi \to \pi'$ is a bijection between $\Pi_C^*(w)$ and $\Pi_C^*(w')$. Hence $|\Pi_C^*(w)| = |\Pi_C^*(w')|$ and $p_C(w) = p_C(w')$. □

Proof of Theorem 3.4. (i) We use induction on the length of the word.

If $k = 1$ then $q = XX$ or $q = YY$. The only colored Catalan word is aa (drop superscript for ease). In either case, $\pi(0) = i, \pi(1) = j, \pi(2) = i$ is a circuit in $\Pi_C^*(w)$ for $1 \leq i \leq n, 1 \leq j \leq n$. Hence, $|\Pi_C^*(w)| \geq n^2$ and the result is true for $k = 1$.

Now let us assume that the claim holds for all monomials q of length less than $2k$ and all Catalan words corresponding to q. By Lemma 4.2, without loss of generality we assume that $w = aaw_1$ where w_1 is a Catalan word of length $(2k - 2)$. Now let $\pi' \in \Pi_C^*(w_1)$. For fixed j, $1 \leq j \leq n$, define π by

$$\pi(0) = \pi'(0) \tag{4.22}$$

$$\pi(1) = j \tag{4.23}$$

$$\pi(j) = \pi'(j - 2), \qquad j \geq 2. \tag{4.24}$$

Clearly π is a circuit and $\pi(0) = \pi(2)$ implies $L(\pi(0), \pi(1)) = L(\pi(1), \pi(2))$. Hence $\pi \in |\Pi_C^*(w)|$ and so, $|\Pi_C^*(w)| \geq n|\Pi_C^*(w_1)| \geq n^{k+1}$ and hence (i) is proved.

(ii) We shall now show that $p_C(w) \leq 1$ for matrices whose link functions satisfy *Property B* and *Property P*. The proof is same as the proof of Theorem 2(ii) of Banerjee and Bose [3], with appropriate changes to add color and index. We indicate the changes while keeping the notation similar to theirs for easy comparison. The proof uses $(2k+1)$-tuple π which are not necessarily

circuits, that is, $\pi(0) = \pi(2k)$ is not assumed. Let w be a colored Catalan word. Define

$$C'(w) = \{\pi : w[i] = w[j] \Rightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\}$$

$$\Gamma_{i,j}(w) = \{\pi \in C(w) : \pi(0) = i, \pi(2k) = j\}, \ (1 \leq i, j \leq n), \ \gamma_{i,j}(w) = |\Gamma_{i,j}(w)|.$$

Clearly, $|\Pi_C^*(w)| = \sum_{i=1}^n \gamma_{i,i}(w)$. Now consider the following statement $\mathbf{S}_k'$ for all $k \geq 1$:

$\mathbf{S}_k'$: For any colored Catalan w of length $(2k)$, there exists $M_k > 0$ such that

$$\gamma_{i,j}(w) \leq M_k n^{k-1} \text{ for all } i \neq j \text{ and } \frac{1}{n} \sum_{i=1}^n \left| \frac{\gamma_{i,i}(w)}{n^k} - 1 \right| = O(1/n).$$

The proof of $\mathbf{S}_k'$ easily follows by repeating the steps of the proof of Theorem 2(ii) of Banerjee and Bose [3] and changing the set $C(w)$ there by $C'(w)$ and using *Property B* and *Property P*. To avoid repetitive arguments we skip the details. Once the validity of $\mathbf{S}_k'$ is asserted, one gets $p_C(w) \leq 1$ and the result now follows using part (i). □

4.4 Proof of Theorem 3.5

We need the following development for describing freeness.

Let S_n be the group of permutations of $(1, 2, \ldots n)$.

Definition 4.3. *Let $\mathcal{A}$ be an algebra. Let $\psi_k : \mathcal{A}^k \longrightarrow C$ $k > 0$ be multi linear functions. For $\alpha \in S_n$, let $c_1, c_2, \ldots c_r$ be the cycles of α. Then define*

$$\psi_\alpha[A_1, A_2, \ldots, A_n] = \psi_{c_1}[A_1, A_2, \ldots, A_n]\psi_{c_2}[A_1, A_2, \ldots, A_n] \ldots \psi_{c_r}[A_1, A_2, \ldots, A_n]$$

where
$$\psi_c[A_1, A_2, \ldots, A_n] = \psi_p\left(A_{i_1}A_{i_2}\ldots A_{i_p}\right) \quad \text{if } c = (i_1, i_2 \ldots i_p).$$

Freeness is intimately tied to non-crossing partitions. We describe the relevant portion of this relation in brief below. See Theorem 14.4 of Nica and Speicher [27] for more details. Let $NC_2(m)$ be the set of non-crossing pair-partitions of $\{1, 2, \ldots, m\}$. A typical pair-partition π will be written in the form $\{(r, \pi(r)), \; r = 1, 2, \ldots, m\}$. Recall that, $p = (p(1), p(2), \ldots, p(m))$ integers (also can be referred to as colors), let

$$NC_2^{(p)}(m) = \{\pi \in NC_2(m) : p(\pi(r)) = p(r) \text{ for all } r = 1, \ldots, m\}.$$

Suppose $d_1, d_2, \ldots, d_m, s_1, s_2, \ldots s_m$ are elements in some non-commutative probability space $(\mathcal{B}, \varphi)$. Suppose $\{s_1, s_2, \ldots s_m\}$ are free and each s_i follows the semicircular law. Then the collections $\{s_1, s_2, \ldots s_m\}$ and $\{d_1, d_2, \ldots d_m\}$ are free if and only if,

$$
\begin{aligned}
\varphi(s_{p(1)}d_1 \ldots s_{p(m)}d_m) &= \sum_{\pi \in NC(m)} k_\pi[s_{p(1)}, \ldots s_{p(m)}] . \varphi_{\pi\gamma}[d_1, \ldots, d_m] \\
&= \sum_{\pi \in NC_2^{(p)}(m)} \varphi_{\pi\gamma}[d_1, \ldots, d_m], \quad\quad (4.25)
\end{aligned}
$$

where $\gamma \in S_m$ is the cyclic permutation with one cycle and $\gamma = (1, 2, \ldots, m-1, m)$. Here k_n denotes the free cumulants and k_π for a partition π is defined along the same lines as Definition 4.3.

We shall also drop the suffix C from $p_C(w)$, $\Pi_C(w)$, $\Pi_C^*(w)$ etc. for simplicity. Fix a monomial q of Wigner (W) and any other patterned matrix (A) of length $2k$. To prove freeness we show that the limiting variables satisfy the relation (4.25). We have already remarked that freeness is intimately tied to non-crossing partitions but freeness in the limit can also be roughly described in terms of colored words in the following manner.

1. If for a colored word the pair-partitions corresponding to the Wigner matrix cross, then $p(w) = 0$.

2. If the pair-partition corresponding to the letters of matrix A cross with any pair-partition of W then also $p(w) = 0$.

For example, $p(w_1w_2w_1w_2a_1a_1) = 0$ and $p(w_1a_1w_1a_1) = 0$. This is essentially the main content of Lemma 4.4 given below.

We will discuss in detail the proof of Theorem 3.5 for $p = 1$ and indicate how the results continue to hold for $p \geq 1$.

We need a few preliminary Lemmata to prove the main result. We first use these Lemmata to prove Theorem 3.5 and then provide the proofs of the Lemmata.

We now concentrate only on (colored) pair-matched words. For a word w the pair $(i, j) 1 \leq i < j \leq 2k$ is said to be a match if $w[i] = w[j]$. A match (i, j) is said to be a W match or an A match according as $w[i] = w[j]$ is a Wigner or an A letter.

We define $w_{(i,j)}$ to be the word of length $j - i + 1$ as

$$w_{(i,j)}[k] = w[i - 1 + k] \text{ for all } 1 \leq k \leq j - i + 1.$$

Let $w_{(i,j)^c}$ be the word of length $t + i - j - 1$ obtained by removing $w_{(i,j)}$ from w, that is,

$$w_{(i,j)^c}[r] = \begin{cases} w[r] & \text{if } r < i, \\ w[r + j - i + 1] & \text{if } r \geq i. \end{cases}$$

Note that in general these subwords may not be matched. If (i, j) is a W match, we will call $w_{(i,j)}$ a *Wigner string* of length $(j - i + 1)$. For instance, for the monomial

$WAAAAWWW$, $w = abbccadd$ is a word and $abbcca$ and dd are Wigner strings of length six and two respectively. For any word w, we define the following two classes:

$$\Pi^*_{(C2)}(w) = \{\pi \in \Pi^*(w) : (i,j) \, W \text{ match} \Rightarrow (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))\} \quad (4.26)$$
$$\Pi^*_{(i,j)}(w) = \{\pi \in \Pi^*(w) : (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))\}. \quad (4.27)$$

Note that the condition appearing above involves $C2$ constraint defined in (4.21) and

$$\Pi^*_{(C2)}(w) = \bigcap_{(i,j): \, W\, match} \Pi^*_{(i,j)}(w). \quad (4.28)$$

It is well known that if we have a collection of only Wigner matrices then $p(w) \neq 0$ if and only if all the constraints in the word are $C2$ constraints. See for example Bose and Sen [8]. We need the following crucial extension in the present setup.

Lemma 4.4. *For a colored pair-matched word w of length $2k$ with $p(w) \neq 0$ we have:*

(a) Every Wigner string is a colored pair-matched word;

(b) For any (i,j) which is a W match we have

$$\lim_{n \to \infty} \frac{|\Pi^*(w) - \Pi^*_{(i,j)}(w)|}{n^{1+k}} = 0. \quad (4.29)$$

(c)

$$\lim_{n \to \infty} \frac{|\Pi^*(w) - \Pi^*_{(C2)}(w)|}{n^{1+k}} = 0. \quad (4.30)$$

Note that (c) and (b) are equivalent by (4.28) and as the number of pairs (i,j) is finite.

Lemma 4.5. *Suppose X_n has LSD and they satisfy Assumption A1 and A2, then for any $l \geq 1$ and integers $(k_1, k_2, \ldots, k_l)$, we have*

$$\mathrm{E}\left[\prod_{i=1}^{l}(\frac{1}{n}\mathrm{Tr}(\left(\frac{X_n}{\sqrt{n}}\right)^{k_i}))\right] - \prod_{i=1}^{l}\mathrm{E}\left[\frac{1}{n}\mathrm{Tr}(\left(\frac{X_n}{\sqrt{n}}\right)^{k_i})\right] \to 0 \text{ as } n \to \infty.$$

Assuming the above lemmas we now prove Theorem 3.5.

Proof of Theorem 3.5. We take a single copy of W and A to show the result. For multiple copies the proof is essentially remains same modulo some notation. Let q be a typical monomial, $q = WA^{q(1)}WA^{q(2)}\ldots WA^{q(m)}$ of length $2k$, where the $q(i)$'s may equal 0. So, $k = m/2 + (q(1) + q(2) + \ldots + q(m))/2$. From Theorem 3.3, for every such monomial q, $\frac{1}{n^{k+1}}\mathrm{Tr}(q)$ converges to say $\varphi(sa^{q(1)}\ldots sa^{q(m)})$, where s follows the semicircular law and a is the marginal limit of A, and φ is the appropriate functional defined on the space of non-commutative polynomial algebra generated by a and s. It is enough to prove that φ satisfies (4.25).

Let us expand the expression for

$$\lim_{n\to\infty} \frac{1}{n^{1+k}} \, \mathrm{E}[\mathrm{Tr}(WA^{q(1)}WA^{q(2)}\ldots WA^{q(m)})]$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{\substack{i(1),i(2),\ldots i(m) \\ j(1),j(2),\ldots j(m)=1}}^{n} \mathrm{E}[w_{i(1)j(1)} a^{q(1)}_{j(1)i(2)} w_{i(2)j(2)} a^{q(2)}_{j(2)i(3)} \ldots w_{i(m)j(m)} a^{q(m)}_{j(m)i(1)}]$$

(4.31)

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{w\in CW(2)} \sum_{\pi\in\Pi^*(w)} \mathrm{E}[\mathbb{X}_\pi]$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{w\in CW(2)} \sum_{\pi\in\Pi^*_{(C2)}(w)} \mathrm{E}[\mathbb{X}_\pi] \text{ (by Lemma 4.4 (c) and Assumption (A2)).}$$ (4.32)

Colored pair-matched words of length $2k$ are in bijection with the set of pair-partitions on $\{1,2,\ldots,2k\}$ (denoted by $\mathcal{P}_2(2k)$). Now each such word w induces σ_w a pair-partition of $\{1,2,\ldots m\}$ that is induced by only the Wigner matches i.e $(a,b) \in \sigma_w$ iff (a,b) is a Wigner match. So given any pair-partition σ of $\{1,2,\ldots,m\}$, we denote by $[\sigma]_W$ the class of all w which induce the partition σ. So the sum in (4.32) can be written as,

$$\lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{w\in[\sigma]_W} \sum_{\pi\in\Pi^*_{(C2)}(w)} E[\mathbb{X}_\pi].$$ (4.33)

By $C2$ constraint imposed on the class $\Pi^*_{(C2)}(w)$, if (r,s) is a W match then $(i(r),j(r)) = (j(s),i(s))$ (or, equivalently in terms of π we have, $(\pi(r-1),\pi(r)) = (\pi(s),\pi(s-1))$).

Therefore, we have the following string of equalities. Let tr be the normalized trace. The equality in (4.34) follows from (4.31) and (4.32). The steps in (4.35), (4.36) and (4.37) follow easily from calculations similar to Proposition 22.32 of Nica and Speicher [27]. The last step follows from the fact that the number of cycles of $\sigma\gamma$ is equal to $1 + m/2$ if and only if $\sigma \in NC_2(m)$. The notation $\mathrm{tr}_{\sigma\gamma}$ is given in Definition 4.3.

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \, \mathrm{E}[\mathrm{Tr}(WA^{q(1)}WA^{q(2)}\ldots WA^{q(m)})]$$

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\ldots i(m) \\ j(1),j(2),\ldots j(m)=1}}^{n} \prod_{(r,s)\in\sigma} \delta_{i(r)j(s)}\delta_{i(s)j(r)} \, \mathrm{E}[a^{q(1)}_{j(1)i(2)} \ldots a^{q(m)}_{j(m)i(1)}]$$ (4.34)

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\ldots i(m) \\ j(1),j(2),\ldots j(m)=1}}^{n} \prod_{(r,s)\in\sigma} \delta_{i(r)j(s)}\delta_{i(s)j(r)} \, \mathrm{E}[a^{q(1)}_{j(1)i(\gamma(1))} \ldots a^{q(m)}_{j(m)i(\gamma(m))}]$$

(4.35)

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\ldots i(m) \\ j(1),j(2),\ldots j(m)=1}}^{n} \prod_{r=1}^{m} \delta_{i(r)j(\sigma(r))} \, \mathrm{E}[a^{q(1)}_{j(1)i(\gamma(1))} \ldots a^{q(m)}_{j(m)i(\gamma(m))}]$$ (4.36)

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{j(1),j(2),\ldots j(m)=1}^{n} \mathrm{E}[a^{q(1)}_{j(1)j(\sigma\gamma(1))} \ldots a^{q(m)}_{j(m)j(\sigma\gamma(m))}]$$ (4.37)

$$= \sum_{\sigma\in NC_2(m)} \lim_{n\to\infty} \mathrm{E}\left(\mathrm{tr}_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \ldots, A^{(q_m)}])\right).$$

Now it follows from Lemma 4.5 that,

$$\sum_{\sigma \in NC_2(m)} \lim_{n \to \infty} \mathrm{E}\left(\mathrm{tr}_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \ldots, A^{(q_m)})]\right) = \sum_{\sigma \in NC_2(m)} \lim_{n \to \infty} (\mathrm{E}\,\mathrm{tr})_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \ldots, A^{(q_m)})]$$

$$= \sum_{\sigma \in NC_2(m)} \varphi_{\sigma\gamma}[a^{(q_1)}, a^{(q_2)}, \ldots, a^{(q_m)})].$$

This shows (4.25) and hence freeness in the limit.

The above method can be easily extended to plug in more independent copies of W and A. The following details will be necessary.

1. The extension of Lemmata 4.4 and 4.5. Note that these extensions can be easily obtained using the injective mapping ψ described in Section 3 and used in Theorem 3.1.

2. When we consider several independent copies of the Wigner matrix the product in (4.36) gets replaced by

$$\prod_{r=1}^{m} \delta_{i(r)j(\sigma(r))} \delta_{p(r)p(\sigma(r))}.$$

Here $(p(1), p(2), \ldots, p(m))$ denotes the colors corresponding to the independent Wigner matrices. The calculations are similar to Theorem 22.35 of Nica and Speicher [27].

The rest are some algebraic details, which we skip. □

Having proved the Theorem we now come back to the proof of Lemma 4.4 and 4.5. The next Lemma turns out to be the most essential ingredient in proving Lemma 4.4 and it points out the behavior of a colored pair-matched word which contains a Wigner string inside it.

Lemma 4.6. *For any colored pair-matched word w and a Wigner string $w_{(i,j)}$ which is a pair-matched word and satisfies equation (4.29)*

$$p(w) = p(w_{(i,j)})p(w_{(i,j)^c}). \tag{4.38}$$

Further, if $w_{(i+1,j-1)}$ and $w_{(i,j)^c}$ satisfy (4.30) then so does w.

Proof. Given any $\pi_1 \in \Pi^*(w_{(i+1,j-1)})$ and $\pi_2 \in \Pi^*(w_{(i,j)^c})$ construct π as:

$$\pi = (\pi_2(0), \ldots \pi_2(i-1), \pi_1(0), \ldots \pi_1(j-i-1) = \pi_1(0), \pi_2(i-1), \ldots (2k-j+i-1)) \in \Pi^*_{(i,j)}(w).$$

Conversely, from any $\pi \in \Pi^*_{(i,j)}(w)$ one can construct π_1 and π_2 by reversing the above construction.

So we have

$$|\Pi^*_{(i,j)}(w)| = |\Pi^*(w_{(i+1,j-1)})||\Pi^*(w_{(i,j)^c})|. \tag{4.39}$$

Let $|w_{(i+1,j-1)}| = 2l_1$ and $|w_{(i,j)^c}| = 2l_2$ and note that $(1 + l_1) + (1 + l_2) = k + 1$.

Now using the fact that $w_{(i,j)}$ satisfies (4.29) and dividing equation (4.39) by n^{k+1} we get in the limit,

$$p(w) = p(w_{(i+1,j-1)})p(w^c_{(i,j)}).$$

Now we claim that

$$|\Pi^*(w_{(i,j)})| = n|\Pi^*(w_{(i+1,j-1)})|. \tag{4.40}$$

Now given $\pi \in \Pi^*(w_{(i,j)})$, one can always get a $\pi' \in \Pi^*(w_{(i+1,j-1)})$, where the $\pi(i-1)$ is arbitrary and hence $\frac{|\Pi^*(w_{(i,j)})|}{n} \leq |\Pi^*(w_{(i+1,j-1)})|$. Also given a $\pi' \in \Pi^*(w_{(i+1,j-1)})$ one can choose $\pi(i-1)$ in n ways and also assign $\pi(j) = \pi(i-1)$ or $\pi(i)$, making j a dependent vertex. So we get that, $|\Pi^*(w_{(i,j)})| \geq n|\Pi^*(w_{(i+1,j-1)})|$. This shows (4.40). So from (4.40) it follows that

$$p(w_{(i,j)}) = p(w_{(i+1,j-1)}),$$

whenever $w_{(i,j)}$ is a Wigner string.

Also note that from the first construction,

$$|\Pi^*_{(C2)}(w)| = |\Pi^*_{(C2)}(w_{(i+1,j-1)})||\Pi^*_{(C2)}(w_{(i,j)^c})|.$$

Now suppose $w_{(i+1,j-1)}$ and $w_{(i,j)^c}$ satisfy (4.30). So we have that

$$|\Pi^*(w_{(i+1,j-1)})| = |\Pi^*_{(C2)}(w_{(i+1,j-1)})| + o(n^{l_1+1}) \text{ and } |\Pi^*(w_{(i,j)^c})| = |\Pi^*_{(C2)}(w_{(i,j)^c})| + o(n^{l_2+1}).$$

Multiplying these and using the fact (from (4.39))
$|\Pi^*(w)| = |\Pi^*(w_{(i+1,j-1)})||\Pi^*(w_{(i,j)^c})| + o(n^{k+1})$, the result follows. $\qquad\square$

We now give a proof of Lemma 4.4.

Proof of Lemma 4.4. We use induction on the length l of the Wigner string. Let w be a pair-matched colored word of length $2k$ with $p(w) \neq 0$. First suppose the Wigner string is of length 2, that is, $l = 2$. We may without loss of generality assume them in the starting position. So we for any $\pi \in \Pi^*(w)$ with above property we have

$$(\pi(0), \pi(1)) = \begin{cases} (\pi(1), \pi(2)) \\ (\pi(2), \pi(1)). \end{cases}$$

In the first case $\pi(0) = \pi(1) = \pi(2)$ and so $\pi(1)$ is not
a generating vertex and this lowers the number of generating vertices (which is not possible as $p(w) \neq 0$). Hence, the only possibility is $(\pi(0), \pi(1)) = (\pi(2), \pi(1))$ and the circuit is complete for the Wigner string and so it is a pair-matched word, proving part (a). Also, as a result of the above arguments only $C2$ constraints survive, which shows (b).

Now suppose the result holds for all Wigner strings of length strictly less than l. Consider a Wigner string of length l, say $w_{(1,l)}$ (we assume it to start from the first position). We break the proof into two cases I and II. In case I, we suppose that the Wigner string has a Wigner string of smaller order and use induction hypothesis and Lemma 4.6 to show the result. In Case II, we assume that there is no Wigner string inside. So there is a string of letters coming from matrix A after a Wigner letter. We show that this string is pair-matched and the last Wigner letter before the l-th position is essentially at the first position. This also implies that the string within a Wigner string do not cross a Wigner letter.

Case I: Suppose that $w_{(1,l)}$ contains a Wigner string of length less than l at the position (p, q) with $1 \leq p < q \leq l$. Since $w_{(p,q)}$ is a Wigner string, by Lemma 4.6 we have,

$$p(w) = p(w_{(p,q)})p(w_{(p,q)^c}) \neq 0.$$

So by induction hypothesis and the fact that both $p(w_{(p,q)})$ and $p(w_{(p,q)^c})$ are not equal to zero we have, $w_{(p,q)}$ and $w_{(p,q)^c}$ are pair-matched words and they also satisfy (4.29). So $w_{(1,l)}$ is a pair-matched word, as it is made up of $w_{(p,q)}$ and $w_{(p,q)^c}$ which are pair-matched. Also from second part of Lemma 4.6, we have $w_{(1,l)}$ satisfies part (b) and (c).

Case II: Suppose there is no Wigner string in the first l positions. Consider the last Wigner letter in the first $l-1$ positions, say at position j_0. Since there is no Wigner string of smaller length, $\pi(j_0)$ is a generating vertex. Also, as j_0 is the last Wigner letter, the positions from j_0 to $l-1$ are all letters coming from the matrix A.

Now we use the structure of the matrix A.

Subcase II(i): Suppose A is a Toeplitz matrix. Let $s_i = (\pi(j_0+i) - \pi(j_0+i-1))$ with $i = 1, 2, \ldots, l-1-j_0$. Now consider the following equation

$$s_1 + s_2 \ldots + s_{l-1-j_0} = (\pi(l-1) - \pi(j_0)). \tag{4.41}$$

If for any j, $w[j]$ is the first appearance of that letter, then consider s_j to be an independent variable (can be chosen freely). Then due to the Toeplitz link function, if $w[k] = w[j]$, where $k > j$, then $s_k = \pm s_j$. Since $(1,l)$ is a W match, $\pi(l-1)$ is either $\pi(0)$ or $\pi(1)$ and hence $\pi(l-1)$ is not a generating vertex. Note that (4.41) is a constraint on the independent variables unless $s_1 + \ldots + s_{l-1-j_0} = 0$. If this is non-zero, this non-trivial constraint lowers the number of independent variables and hence the limit contribution will be zero, which is not possible as $p(w) \neq 0$. So we must have,

$$\pi(l-1) = \pi(j_0) \quad \text{and} \quad j_0 = 1.$$

This also shows $(\pi(l), \pi(l-1)) = (\pi(0), \pi(1))$ and hence $w_{(1,l)}$ is a colored word. As $s_1 + \ldots + s_{l-1-j_0} = 0$, all the independent variables occur twice with different signs in the left side, since otherwise it would again mean a non-trivial relation among them and thus would lower the order. Hence we conclude that the Toeplitz letters inside the first l positions are also pair-matched. Since the $C2$ constraint is satisfied at the position $(1,l)$, part (b) also holds.

Subcase II(ii): Suppose A is a Hankel matrix. We write, $t_i = (\pi(j_0+i) + \pi(j_0+i-1))$ and consider

$$-t_1 + t_2 - t_3 \ldots (-1)^{l-j_0-1} t_{l-j_0-1} = (-1)^{l-j_0-1}(\pi(l-1) - \pi(j_0)). \tag{4.42}$$

Now again as earlier, the t_i's are independent variables, and so this implies that again to avoid a non-trivial constraint which would lower the order, both sides of the

equation (4.42) have to vanish, which automatically leads to the conclusion that $\pi(l-1) = \pi(j_0) = \pi(1)$. So $j_0 = 1$ and again the Wigner paired string of length l is pair-matched. Part (b) also follows as the $C2$ constraint holds.

Subcase II(iii): A is Symmetric or Reverse Circulant. Note that they have link functions which are quite similar to Toeplitz and Hankel respectively, the proofs are very similar to the above two cases and hence we skip them. $\qquad \square$

Proof of Lemma 4.5. We first show that,

$$\mathrm{E}\left[\prod_{i=1}^{l}\left(\mathrm{tr}\,\frac{X_n^{k_i}}{n^{k_i/2}} - \mathrm{E}\left[\mathrm{tr}\,\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right] = \mathrm{O}(\frac{1}{n}) \text{ as } n \to \infty, \tag{4.43}$$

where tr denotes the normalized trace. To prove (4.43), we see that,

$$\mathrm{E}\left[\prod_{i=1}^{l}\left(\mathrm{tr}\,\frac{X_n^{k_i}}{n^{k_i/2}} - \mathrm{E}\left[\mathrm{tr}\,\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right] = \frac{1}{n^{\sum_{i=1}^{l} k_i/2+l}} \sum_{\pi_1, \pi_2, ..\pi_l} \mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i} - \mathrm{E}(X_{\pi_i})))]. \tag{4.44}$$

If the circuit π_i is not jointly matched with the other circuits then $\mathrm{E}\,X_{\pi_i} = 0$ and

$$\mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i} - \mathrm{E}(X_{\pi_i})))] = \mathrm{E}[X_{\pi_i}(\prod_{j\neq i}(X_{\pi_i} - \mathrm{E}(X_{\pi_i})))] = 0.$$

If any of the circuits is self matched i.e. it has no cross matched edge then

$$\mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i} - \mathrm{E}(X_{\pi_i})))] = \mathrm{E}[X_{\pi_i} - \mathrm{E}(X_{\pi_i})]\,\mathrm{E}[(\prod_{j\neq i}(X_{\pi_i} - \mathrm{E}(X_{\pi_i})))] = 0.$$

Now total number of circuits $\{\pi_1, \pi_2, \ldots \pi_l\}$ where each edge appears at least twice and one edge at least thrice is $\leq C n^{\sum_{i=1}^{l} k_i/2 + l - 1}$, by Property B. Hence using Assumption (A2) such terms in (4.44) are of the order $\mathrm{O}(\frac{1}{n})$. Now consider rest of

the terms where all the edges appear exactly twice. As a consequence $\sum_{i=1}^{l} k_i$ is even. Also number of partitions of $\frac{1}{2}\sum_{i=1}^{l} k_i$ into l circuits is independent of n. We need to consider only $\{\pi_1, \pi_2, \ldots \pi_l\}$ which are jointly matched but not self matched.

If we prove that for such a partition the number of circuits is less than $C n^{\sum_{i=1}^{l} k_i + l - 1}$ we are done since the number of such partitions is independent of n and (4.3).

Since π_1 is not self matched we can without loss of generality assume that the edge value for $(\pi(0), \pi(1))$ occurs exactly once in π_1. So construct π_1 as follows. First choose $\pi_1(0) = \pi_1(k_1)$ and then choose the remaining vertices in the order $\pi_1(k_1), \pi_1(k_1 - 1) \ldots \pi_1(1)$. One sees that we loose one degree of freedom as in this way the edge $(\pi(0), \pi(1))$ is determined and we cannot choose it arbitrarily.

The result now follows from (4.43) by using induction. For $l = 2$ expanding and using the fact that expected normalized trace of the powers of $X_n/\sqrt{n}$ converges we get,

$$\mathrm{E}\left[\prod_{i=1}^{2}\left(\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right]$$
$$= \mathrm{E}\left[\left(\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}}\right]\right)\left(\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}}\right]\right)\right]$$
$$= \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}} \operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}}\right] - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1}}\right]\mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2}}\right] \to 0 \text{ as } n \to \infty.$$

So the result holds for $l = 2$. Now suppose it is true for all $2 \leq m < l$. We expand

$$\lim_{n\to\infty} \mathrm{E}[\prod_{i=1}^{l}(\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i}) - \mathrm{E}(\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})))] = 0$$

to get

$$\lim_{n\to\infty} \sum_{m=1}^{l}(-1)^m \sum_{i_1 < i_2 \ldots < i_m} \mathrm{E}[\prod_{j=1}^{m} \operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_{i_j}})] \prod_{i\notin\{i_1,i_2,\ldots i_m\}} \mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})] = 0.$$

Now using the result for products of smaller order successively,

$$\lim_{n\to\infty}(-1)^l \mathrm{E}[\prod_{j=1}^{l} \operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_j})] = \lim_{n\to\infty} \sum_{m<l}(-1)^m \sum_{i_1 < i_2 \ldots < i_m} \mathrm{E}[\prod_{j=1}^{m} \operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_{i_j}})] \prod_{i\notin\{i_1,i_2,\ldots i_m\}} \mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})].$$

Now every term in right side is by induction hypothesis $\lim_{n\to\infty} \prod_{i=1}^{l} \mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})]$ and from this the Lemma follows. $\qquad\square$

Proof Remark 3.6. We just briefly sketch the arguments as the proof is quite similar to the previous section but much easier. Note that if W is

a centered GUE with variance $1/n$ then,

$$\mathrm{E}[W_{ij}W_{kl}] = \frac{1}{n}\delta_{il}\delta_{jk}. \tag{4.45}$$

This equation (4.45) provides the $C2$ constraint in the proof of Theorem 3.5. So following the steps in the proof of Theorem 3.5 we have

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}\,\mathrm{E}[\mathrm{Tr}(WA^{q(1)}WA^{q(2)}\ldots WA^{q(m)})]$$

$$= \lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{\sigma\in\mathcal{P}_2(m)}\sum_{\substack{i(1),i(2),\ldots i(m)\\ j(1),j(2),\ldots j(m)=1}}^{n}\prod_{(r,s)\in\sigma}\delta_{i(r)j(s)}\delta_{i(s)j(r)}\,\mathrm{E}[a_{j(1)i(2)}^{q(1)}\ldots a_{j(m)i(1)}^{q(m)}]$$

$$= \sum_{\sigma\in NC_2(m)}\lim_{n\to\infty}\mathrm{E}\left(\mathrm{tr}_{\sigma\gamma}[A^{(q_1)},A^{(q_2)},\ldots,A^{(q_m)})]\right).$$

Now the result follows by applying Lemma 4.5 which holds under Property B and existence of LSD. $\qquad\square$

References

[1] G. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*. Cambridge University Press, 2010. MR-2760897

[2] Z. Bai and J. W. Silverstein. *Spectral analysis of large dimensional random matrices*. Springer Series in Statistics. Springer, New York, second edition, 2010. ISBN 978-1-4419-0660-1. DOI: 10.1007/978-1-4419-0661-8. MR-2567175

[3] S. Banerjee and A. Bose. Noncrossing partitions, Catalan words and the semicircular law. *To appear in J. Theoret. Probab.*, 2011. DOI: 10.1007/s10959-011-0365-4.

[4] F. Benaych-Georges. Rectangular random matrices, entropy, and Fisher's information. *J. Operator Theory*, 62(2):371–419, 2009. ISSN 0379-4024. MR-2552088

[5] F. Benaych-Georges. Rectangular random matrices, related convolution. *Probab. Theory Related Fields*, 144(3-4):471–515, 2009. ISSN 0178-8051. DOI: 10.1007/s00440-008-0152-z. MR-2496440

[6] F. Benaych-Georges, A. Guionnet, and M. Maida. Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices. *Electron. J. Probab.*, 16:no. 60, 1621–1662, 2011. ISSN 1083-6489. DOI: 10.1214/EJP.v16-929. MR-2835249

[7] P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.,* 46:.3, 705–718, 1997. MR-1488333

[8] A. Bose and A. Sen. Another look at the moment method for large dimensional random matrices. *Electron. J. Probab.*, 13:no. 21, 588–628, 2008. ISSN 1083-6489. MR-2399292

[9] A. Bose, R. S. Hazra, and K. Saha. Convergence of joint moments for independent random patterned matrices. *Ann. Probab.*, Vol. 39, No. 4, 1607–1620, 2011. DOI: 10.1214/10-AOP597. MR-2857252

[10] A. Bose, R. S. Hazra, and K. Saha. Half independence and half cumulants. *Elect. Comm. Probab.*, Vol. 16, 405–422, 2011. MR-2831080

[11] W. Bryc, A. Dembo, and T. Jiang. Spectral measure of large random Hankel, Markov and Toeplitz matrices. *Ann. Probab.*, 34(1):1–38, 2006. ISSN 0091-1798. DOI: 10.1214/009117905000000495. MR-2206341

[12] M. Capitaine, C. Donati-Martin, and D. Féral. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *Ann. Probab.*, 37(1):1–47, 2009. ISSN 0091-1798. DOI: 10.1214/08-AOP394. MR-2489158

[13] M. Capitaine, C. Donati-Martin, D. Féral, and M. Février. Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices. *Electron. J. Probab.*, 16:no. 64, 1750–1792, 2011. ISSN 1083-6489. DOI: 10.1214/EJP.v16-934. MR-2835253

[14] M. Capitaine and M. Casalis. Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Application to beta random matrices. *Indiana Univ. Math. J.*, 53(2):397–431, 2004. ISSN 0022-2518. DOI: 10.1512/iumj.2004.53.2325. MR-2060040

[15] M. Capitaine and C. Donati-Martin. Strong asymptotic freeness for Wigner and Wishart matrices. *Indiana Univ. Math. J.*, 56(2):767–803, 2007. ISSN 0022-2518. DOI: 10.1512/iumj.2007.56.2886. MR-2317545

[16] B. Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, (17):953–982, 2003. ISSN 1073-7928. DOI: 10.1155/S107379280320917X. MR-1959915

[17] B. Collins and P. Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Comm. Math. Phys.*, 264(3):773–795, 2006. ISSN 0010-3616. DOI: 10.1007/s00220-006-1554-3.

[18] B. Collins, A. Guionnet, and E. Maurel-Segala. Asymptotics of unitary and orthogonal matrix integrals. *Adv. Math.*, 222(1):172–215, 2009. ISSN 0001-8708. DOI: 10.1016/j.aim.2009.03.019. MR-2531371

[19] R. Couillet and M. Debbah and J. Silverstein. A deterministic equivalent for the analysis of correlated MIMO multiple access channels. *IEEE Trans. Inform. Theory* 57, no. 6, 3493–3514, 2011. MR-2817033

[20] K. Dykema. On certain free product factors via an extended matrix model. *J. Funct. Anal.*, 112(1):31–60, 1993. ISSN 0022-1236. DOI: 10.1006/jfan.1993.1025. MR-1207936

[21] D. Féral and S. Péché. The largest eigenvalue of rank one deformation of large Wigner matrices. *Comm. Math. Phys.*, 272(1):185–228, 2007. ISSN 0010-3616. DOI: 10.1007/s00220-007-0209-3.

[22] W. Fulton. Eigenvalues of sums of Hermitian matrices (after A. Klyachko). *Astérisque*, (252):Exp. No. 845, 5, 255–269, 1998. ISSN 0303-1179. Séminaire Bourbaki. Vol. 1997/98. MR-1685640

[23] C. Hammond and S. J. Miller. Distribution of eigenvalues for the ensemble of real symmetric Toeplitz matrices. *J. Theoret. Probab.*, 18(3):537–566, 2005. ISSN 0894-9840. DOI: 10.1007/s10959-005-3518-5. MR-2167641

[24] F. Hiai and D. Petz. Asymptotic freeness almost everywhere for random matrices. *Acta Sci. Math. (Szeged)*, 66(3-4):809–834, 2000. ISSN 0001-6969. MR-1804226

[25] F. Hiai and D. Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. ISBN 0-8218-2081-8. MR-1746976

[26] C. Male. The norm of polynomials in large random and deterministic matrices. To appear in Prob. Theory Related Fields. 2012.

[27] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-85852-6; 0-521-85852-6. DOI: 10.1017/CBO9780511735127. MR-2266879

[28] T. Oraby. The spectral laws of Hermitian block-matrices with large random blocks. *Electron. Comm. Probab.*, 12:465–476, 2007. ISSN 1083-589X. MR-2365648

[29] S. Péché. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields*, 134(1):127–173, 2006. ISSN 0178-8051. DOI: 10.1007/s00440-005-0466-z.

[30] L. Pastur and V. Vasilchuk. On the law of addition of random matrices. *Comm. Math. Phys.*, 214(2):249–286, 2000. ISSN 0010-3616. DOI: 10.1007/s002200000264. MR-1796022

[31] R. Rashidi Far, T. Oraby, W. Bryc, and R. Speicher. On slow-fading MIMO systems with nonseparable correlation. *IEEE Trans. Inform. Theory*, 54(2):544–553, 2008. ISSN 0018-9448. MR-2444540

[32] Ø. Ryan. On the limit distributions of random matrices with independent or free entries. *Comm. Math. Phys.*, 193(3):595–626, 1998. ISSN 0010-3616. DOI: 10.1007/s002200050340. MR-1624843

[33] H. Schultz. Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases. *Probab. Theory Related Fields*, 131(2):261–309, 2005. ISSN 0178-8051. DOI: 10.1007/s00440-004-0366-7. MR-2117954

[34] D. Shlyakhtenko. Random Gaussian band matrices and freeness with amalgamation. *Internat. Math. Res. Notices*, (20):1013–1025, 1996. ISSN 1073-7928. DOI: 10.1155/S1073792896000633. MR-1422374

[35] R. Speicher On universal products. *Fields Institute Communications,* Vol. 12, AMS, pp. 257-266,1997. MR-1426844

[36] R. Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. *Mem. Amer. Math. Soc.*, 132(627):x+88, 1998. ISSN 0065-9266. MR-1407898

[37] R. Speicher. Free probability theory. *To appear in Handbook on Random Matrix Theory*, 2010. MR-2216214

[38] A. M. Tulino and S. Verdú. *Random matrices and wireless communications,* volume vol. 1 of *Fundations and Trends in Communications and Information Theory.* Now Publishers Inc., 2004.

[39] D. Voiculescu. Addition of certain noncommuting random variables. *J. Funct. Anal.,* 66(3):323–346, 1986. ISSN 0022-1236. DOI: 10.1016/0022-1236(86)90062-5. MR-0839105

[40] D. Voiculescu. Limit laws for random matrices and free products. *Invent. Math.,* 104(1):201–220, 1991. ISSN 0020-9910. DOI: 10.1007/BF01245072. MR-1094052

[41] D. Voiculescu. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices*, (1):41–63, 1998. ISSN 1073-7928. DOI: 10.1155/S107379289800004X. MR-1601878

[42] E. P. Wigner (1958). On the distribution of the roots of certain symmetric matrices. *Ann. of Math.*, (2), 67, 325–327. MR-0095527