Symmetric exclusion as a model of non-elliptic dynamical random conductances

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Abstract

We consider a finite range symmetric exclusion process on the integer lattice in any dimension. We interpret it as a non-elliptic time-dependent random conductance model by setting conductances equal to one over the edges with end points occupied by particles of the exclusion process and to zero elsewhere. We prove a law of large numbers and a central limit theorem for the random walk driven by such a dynamical field of conductances using the Kipnis-Varhadan martingale approximation. Unlike the tagged particle in the exclusion process, which is in some sense similar to this model, this random walk is diffusive even in the one-dimensional nearest-neighbor symmetric case.

1 Introduction

1.1 Model and results

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Denote by $\xi = \{\xi(z); z \in \mathbb{Z}^d\}$ the elements of Ω. For $\xi \in \Omega$ and $y, z \in \mathbb{Z}^d$, define $\xi^{y,z} \in \Omega$ as

$$\xi^{y,z}(x) = \begin{cases} \xi(z), & x = y \\ \xi(y), & x = z \\ \xi(x), & x \neq z, y, \end{cases}$$

that is, $\xi^{y,z}$ is obtained from ξ by exchanging the occupation variables at y and z. Fix $R \geq 1$. Consider the transition kernel $p(z, y)$ of a translation-invariant, symmetric, irreducible random walk with range size R. Hence, for $y, z \in \mathbb{Z}^d$ such that $|y - z|_1 \leq R$, $p(0, y - z) = p(z, y) = p(y, z) > 0$, and $\sum_{y \in \mathbb{Z}^d} p(0, y) = 1$. Due to translation invariance we will denote $p(x) := p(0, x)$.

Let $\{(\xi_t, X_t); t \geq 0\}$ be the Markov process on the state space $\Omega \times \mathbb{Z}^d$ with generator given by

$$\begin{aligned} Lf(\xi, x) = &\sum_{y,z \in \mathbb{Z}^d} p(z - y)\big[f(\xi^{y,z}, x) - f(\xi, x)\big] \\ &+ \sum_{y \in \mathbb{Z}^d} c_{x,y}(\xi)\big[f(\xi, y) - f(\xi, x)\big], \end{aligned} \quad (1.1)$$

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for any local function $f : \Omega \times \mathbb{Z}^d \to \mathbb{R}$, with

$$c_{x,y}(\xi) = \begin{cases} \xi(x)\xi(y) & \text{if } |x-y|_1 \leq R, \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

We interpret the dynamics of the process $\{(\xi_t, X_t); t \geq 0\}$ as follows. Checking the action of L over functions f which do not depend on z, we see that $\{\xi_t; t \geq 0\}$ has a Markovian evolution, which corresponds to the well known *symmetric exclusion process* on $\mathbb{Z}^d$, see e.g. [6]. Conditioned on a realization of $\{\xi_t; t \geq 0\}$, the process $\{X_t; t \geq 0\}$ is a continuous time random walk among the field of dynamical random conductances

$$\{c_{x,y}(\xi_t) = \xi_t(x)\xi_t(y)\mathbb{1}_{\{|x-y|_1 \leq R\}} ; x, y \in \mathbb{Z}^d, t \geq 0\}. \tag{1.3}$$

Our main results are the following law of large numbers and functional central limit theorem for the random walk X_t.

Theorem 1.1 (LLN). *Assume that the exclusion process ξ_t starts from the Bernoulli product measure ν_ρ of density $\rho \in [0,1]$. Then X_t/t converges a.s. and in L^1 to 0.*

Theorem 1.2 (Annealed functional CLT). *Under the assumptions of Theorem 1.1, the process $(\epsilon X_{t/\epsilon^2})$ converges in distribution, as ϵ goes to zero, to a non-degenerate Brownian motion in the Skorohod topology.*

1.2 Motivation

The study of random walks in random media represents one of the main research areas within the field of disordered systems of particles. The aim is to understand the motion of a particle in an inhomogeneous medium. This is clearly interesting for applied purposes and has turned out to be a very challenging mathematical program. Much work has been done in this direction in recent years. We refer to [7, 8] for recent overviews of rigorous results on the subject.

One of the easiest models of a random walk in random media is represented by a random walk among (time-independent) random conductances. This model turned out to be relatively simple due to the reversibility properties of the walker. In fact, the behavior of such random walks has been recently analyzed and understood in quite great generality. See [3] for a recent overview and references therein. When considering a field of dynamical random conductances, the mentioned reversibility of the random walk is lost, and other types of techniques are needed. In the recent paper [1], annealed and quenched invariance principles for a random walk in a field of time-dependent random conductances have been derived by assuming fast enough space-time mixing conditions and uniform ellipticity for the field. In particular, the uniform ellipticity, which guarantees heat kernel estimates, is a crucial assumption in their approach even for the annealed statement (ellipticity plays a fundamental role also in the analysis of other random walks in random environments). The model we consider represents a "solvable" example of non-elliptic time-dependent random conductances with strong space-time correlations. Moreover, it strengthens the connection between particle systems theory and the theory of random walks in random media. To overcome the loss of ellipticity we use the "good" properties of the symmetric exclusion in equilibrium.

The proof of our results rely on the martingale approximation method developed by Kipnis and Varhadan [5] for additive functionals of reversible Markov processes. In the original paper [5], the authors apply their method to study a tagged particle in the exclusion process. Indeed, this latter problem has some similarities with our model and our proof is essentially an adaptation of their proof. However, unlike the tagged particle behavior, our random walk is always diffusive even in the one-dimensional nearest-neighbor case.

2 Proofs of the LLN and of the invariance principle

2.1 The environment from the position of the walker

Consider the process $\{\eta_t; t \geq 0\}$ with values in Ω, defined by $\eta_t = \tau_{X_t}\xi_t$, where τ_y denotes the shift operator on Ω (i. e. $\eta_t(z) = \xi_t(z + X_t)$). The process $\{\eta_t; t \geq 0\}$ is usually called the *environment seen by the random walk*. For $\eta \in \Omega$, the process $\{\eta_t; t \geq 0\}$ is also Markovian with generator:

$$\mathcal{L}_{ew}f(\eta) = \sum_{z,y} p(z-y)\big[f(\eta^{y,z}) - f(\eta)\big] + \sum_{y} c_{0,y}(\eta)\big[f(\tau_y\eta) - f(\eta)\big]$$

$$=: \mathcal{L}_{se}f(\eta) + \mathcal{L}_{rc}f(\eta), \tag{2.1}$$

for any local function $f : \Omega \to \mathbb{R}$. The choice of the subindexes in the generators above is just for notational convenience: "ew", "se" and "rc", stand for, "Environment from the point of view of the Walker", "Symmetric Exclusion" and "Random Conductances", respectively.

For any two functions $f, g : \Omega \to \mathbb{R}$, we denote the inner product in $L^2(\nu_\rho)$ by

$$\langle f, g \rangle_{\nu_\rho} := \int_\Omega d\nu_\rho f(\eta)g(\eta),$$

where ν_ρ is the Bernoulli product measure of density $\rho \in [0,1]$. In particular, it is well known that the family $\{\nu_\rho : \rho \in (0,1)\}$ fully characterizes the set of extremal invariant measures for the symmetric exclusion process, and $\mathcal{L}_{se}$ is self-adjoint in $L^2(\nu_\rho)$ (see [6]). The next lemma shows that the same statement holds for the environment as seen by the walker. Before proving it, we define the Dirichlet forms associated to the generators involved in (2.1) as

$$D_a(f) := \langle f, -\mathcal{L}_a f \rangle_{\nu_\rho} \text{ with } a \in \{ew, se, rc\}, \tag{2.2}$$

for a function $f \in L^2(\nu_\rho)$. It follows by a standard computation (cf. [4], Prop. 10.1 P.343) that

$$D_{ew}(f) = D_{se}(f) + D_{rc}(f) = \frac{1}{2}\sum_{z,y} \int d\nu_\rho \, p(z-y)\big[f(\eta^{y,z}) - f(\eta)\big]^2$$

$$+ \frac{1}{2}\sum_{y} \int d\nu_\rho \, c_{0,y}(\eta)\big[f(\tau_y\eta) - f(\eta)\big]^2. \tag{2.3}$$

Lemma 2.1. *The process η_t is reversible and ergodic with respect to the the Bernoulli product measure ν_ρ.*

Proof. We first show that $\mathcal{L}_{ew}$ is self-adjoint in $L^2(\nu_\rho)$, namely, $\langle f, \mathcal{L}_{ew}g \rangle_{\nu_\rho} = \langle \mathcal{L}_{ew}f, g \rangle_{\nu_\rho}$, with $f, g \in L^2(\nu_\rho)$.

By translation invariance, we have

$$\langle f, \mathcal{L}_{rc}g \rangle_{\nu_\rho} = \sum_{y} \int d\nu_\rho \, f(\eta)\left[g(\tau_y\eta) - g(\eta)\right]c_{0,y}(\eta)$$

$$= \sum_{y}\left(\int d\nu_\rho \, f(\tau_{-y}\eta)g(\eta)c_{0,-y}(\eta) - \int d\nu_\rho \, f(\eta)g(\eta)c_{0,y}(\eta)\right) \tag{2.4}$$

$$= \sum_{y}\left(\int d\nu_\rho \, f(\tau_y\eta)g(\eta)c_{0,y}(\eta) - \int d\nu_\rho \, f(\eta)g(\eta)c_{0,y}(\eta)\right) = \langle \mathcal{L}_{rc}f, g \rangle_{\nu_\rho}.$$

Together with the fact that $\mathcal{L}_{se}$ is also self-adjoint, we get

$$\langle f, \mathcal{L}_{ew}g \rangle_{\nu_\rho} = \langle f, \mathcal{L}_{se}g \rangle_{\nu_\rho} + \langle f, \mathcal{L}_{rc}g \rangle_{\nu_\rho} = \langle \mathcal{L}_{se}f, g \rangle_{\nu_\rho} + \langle \mathcal{L}_{rc}f, g \rangle_{\nu_\rho} = \langle \mathcal{L}_{ew}f, g \rangle_{\nu_\rho}.$$

To show the ergodicity, we prove that any harmonic function h such that $\mathcal{L}_{ew}h = 0$ is ν_ρ-a. s. constant.

Indeed, $\mathcal{L}_{ew}h = 0$ implies that $D_{se}(h) = -D_{rc}(h)$. Since the Dirichlet forms are non-negative, $D_{se}(h) = 0 = D_{rc}(h)$. On the other hand, $\mathcal{L}_{se}$ is reversible and ergodic, hence h must be ν_ρ-a. s. constant. $\square$

2.2 Proof of Theorem 1.1

We now express the position of the random walk X_t in terms of the process η_t. For $y \in \mathbb{Z}^d$, let J_t^y denote the number of spatial shifts by y of the process η_t up to time t. Then

$$X_t = \sum_y y J_t^y. \tag{2.5}$$

By compensating the process J_t^y by its intensity $\int_0^t c_{0,y}(\eta_s)ds$, it is standard to check that

$$M_t^y := J_t^y - \int_0^t ds\, c_{0,y}(\eta_s) \quad \text{and} \quad (M_t^y)^2 - \int_0^t ds\, c_{0,y}(\eta_s) \tag{2.6}$$

are martingales with stationary increments vanishing at $t = 0$.

Next, define

$$M_t := \sum_y y M_t^y \quad \text{and} \quad \phi(\eta_s) := \sum_y y c_{0,y}(\eta_s), \tag{2.7}$$

by combining (2.5) and (2.6), we obtain

$$X_t = M_t + \int_0^t ds\, \phi(\eta_s), \tag{2.8}$$

from which we obtain the law of large numbers in Theorem 1.1. Indeed, due to Lemma 2.1, (2.8) expresses X_t as a sum of a zero-mean martingale with stationary and ergodic increments, plus the term $\int_0^t ds\, \phi(\eta_s)$ which, by the ergodic theorem, converges when divided by t to its average

$$\mathbb{E}_{\nu_\rho}[\phi(\eta)] = \sum_{|y|_1 \leq R} y \int d\nu_\rho\, \eta(0)\eta(y) = \rho^2 \sum_{|y|_1 \leq R} y = 0.$$

2.3 Proof of Theorem 1.2

In this section we prove the functional CLT for the process X_t. To this aim we will use again the representation in equation (2.8) and the well known Kipnis-Varhadan method [5] for additive functionals of reversible Markov processes applied to $\int_0^t ds\, \phi(\eta_s)$.

To recall briefly the Kipnis-Varadhan method, we first introduce the Sobolev spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ associated to the generator $\mathcal{L}_{ew}$. Let $\mathcal{D}(\mathcal{L}_{ew})$ be the domain of this generator. Consider in $\mathcal{D}(\mathcal{L}_{ew})$, the equivalence relation $\sim_1$ defined as $f \sim_1 g$ if $\|f - g\|_1 = 0$, where $\|\cdot\|_1$ is the semi-norm given by

$$\|f\|_1^2 := \langle f, -\mathcal{L}f \rangle_{\nu_\rho}. \tag{2.9}$$

Define the space $\mathcal{H}_1$ as the completion of the normed space $(\mathcal{D}(\mathcal{L}_{ew})|_{\sim_1}, \|\cdot\|_1)$. It can be checked that $\mathcal{H}_1$ is a Hilbert space with inner product $\langle f, g \rangle_1 := \langle f, -\mathcal{L}_{ew}g \rangle_{\nu_\rho}$. For $f \in L^2(\nu_\rho)$, let

$$\|f\|_{-1} := \sup \left\{ \frac{\langle f, g \rangle_{\nu_\rho}}{\|g\|_1} : g \in \mathcal{D}(\mathcal{L}_{ew}), \|g\|_1 \neq 0 \right\}. \tag{2.10}$$

Consider $\mathcal{G}_{-1} := \{f \in L^2(\nu_\rho) : \|f\|_{-1} < \infty\}$. As for the $\|\cdot\|_1$ norm, define the equivalence relation $\sim_{-1}$, and let $\mathcal{H}_{-1}$ be the completion of the normed space $(\mathcal{G}_{-1}|_{\sim_1}, \|\cdot\|_{-1})$. $\mathcal{H}_{-1}$ is the dual of $\mathcal{H}_1$ and also a Hilbert space.

Denote by $\cdot$ the Euclidean scalar product in $\mathbb{R}^d$ and fix an arbitrary vector l in $\mathbb{R}^d$. Theorem 1.8 in [5] states that, if $\mathcal{L}_{ew}$ is self-adjoint and $\phi \cdot l \in \mathcal{H}_{-1}$, then there exists a square integrable martingale $\tilde{M}_t^l$ and an error term E_t^l such that

$$\int_0^t ds\, \phi(\eta_s) \cdot l = \tilde{M}_t^l + E_t^l, \tag{2.11}$$

and $|E_t^l|/\sqrt{t}$ converges to zero in $L^2(\nu_\rho)$. In particular, the martingale $\tilde{M}_t^l$ from (2.11) is obtained as the limit as $\lambda \to 0$ of the martingale

$$\tilde{M}_t(\lambda, l) := f_\lambda^l(\eta_t) - f_\lambda^l(\eta_0) - \int_0^t ds\, \mathcal{L}_{ew} f_\lambda^l(\eta_s), \tag{2.12}$$

where f_λ^l is the solution of the resolvent equation

$$(\lambda I - \mathcal{L}) f_\lambda^l = \phi \cdot l. \tag{2.13}$$

Moreover,

$$\mathbb{E}_{\nu_\rho}[\tilde{M}_1(\lambda, l)^2] = \|f_\lambda^l\|_1^2. \tag{2.14}$$

In Lemma 2.2 below we prove a crucial estimate. By (2.10), this implies that $\phi \cdot l \in \mathcal{H}_{-1}$. In view of what we have said above, Theorem 1.8 in [5] implies the decomposition in equation (2.11).

Lemma 2.2. *For any function $f \in \mathcal{D}(\mathcal{L}_{ew})$ and any vector l in $\mathbb{R}^d$, there exists a constant $K = K(l, R) > 0$ such that,*

$$|\langle \phi \cdot l, f \rangle_{\nu_\rho}| \leq K\, D_{ew}(f)^{1/2}. \tag{2.15}$$

Proof. Recall (2.7) and estimate

$$
\begin{aligned}
\left| \langle \phi \cdot l, f \rangle_{\nu_\rho} \right| &= \left| \int d\nu_\rho \sum_y (y \cdot l) c_{0,y}(\eta) f(\eta) \right| = \left| \frac{1}{2} \int d\nu_\rho \sum_y (y \cdot l) \left[c_{0,y}(\eta) - c_{0,-y}(\eta) \right] f(\eta) \right| \\
&= \left| \frac{1}{2} \int d\nu_\rho \sum_y (y \cdot l) c_{0,y}(\eta) \left[f(\tau_y \eta) - f(\eta) \right] \right| \\
&\leq \frac{1}{2} \left(\sum_y (y \cdot l)^2 c_{0,y}(\eta) \right)^{1/2} \left(\int d\nu_\rho \sum_y c_{0,y}(\eta) \left[f(\tau_y \eta) - f(\eta) \right]^2 \right)^{1/2} \\
&\leq K\, D_{rc}(f)^{1/2} \leq K\, D_{ew}(f)^{1/2},
\end{aligned}
\tag{2.16}
$$

where we have used translation invariance, $c_{0,y}(\eta)^2 = c_{0,y}(\eta)$, Cauchy-Schwarz, the finite range assumption on $p(\cdot)$, and the representation of the Dirichlet forms from (2.3), respectively. $\square$

As a consequence of (2.8) and (2.11), we have that for any vector l in $\mathbb{R}^d$,

$$X_t \cdot l = M_t \cdot l + \tilde{M}_t^l + o\left(\sqrt{t}\right). \tag{2.17}$$

Hence, we can approximate X_t by a sum of two martingales $M_t + \tilde{M}_t$. Note that $\tilde{M}_t = \left(\tilde{M}_t^{e_1}, \cdots, \tilde{M}_t^{e_d} \right)$, with $\{e_i\}_{i=1}^d$ denoting the canonical basis of $\mathbb{R}^d$. Since the sum of two

martingales is again a martingale, the functional CLT for X_t follows immediately from the standard functional CLT for martingales provided that we prove the non-degeneracy of the covariance matrix of the martingale given by $M_t + \tilde{M}_t$. Roughly speaking, we have to prove that M_t and $\tilde{M}_t$ do not cancel each other. This is the content of the next proposition which concludes the proof of Theorem 1.2.

Proposition 2.3. *The sum of the two martingales $M_t + \tilde{M}_t$ is a non-degenerate martingale.*

Proof. For $z, y \in \mathbb{Z}^d$ with $p(z - y) > 0$, let $I_t^{y,z}$ denote the total number of swaps of states of the exclusion process from site y to x up to time t. Similarly to (2.6), by compensating the process $I_t^{y,z}$ by its intensity, it is standard to check that

$$N_t^{y,z} := I_t^{y,z} - p(z - y)t \quad \text{and} \tag{2.18}$$

$$(N_t^{y,z})^2 - p(z - y)t \tag{2.19}$$

are martingales.

In particular, the martingales $\{M_t^y \,|\, y \in \mathbb{Z}^d\}$ (recall (2.6)) and $\{N_t^{y,z} \,|\, y, z \in \mathbb{Z}^d, p(z - y) > 0\}$ are jump processes which do not have common jumps. Therefore they are orthogonal, namely, the product of two such martingales is still a martingale.

On the other hand, since

$$f_\lambda^l(\eta_t) - f_\lambda^l(\eta_0) = \sum_{y,z} \int_0^t \left[f_\lambda^l(\eta_{s-}^{y,z}) - f_\lambda^l(\eta_{s-}) \right] dI_s^{y,z}$$
$$+ \sum_y \int_0^t \left[f_\lambda^l(\tau_y \eta_{s-}) - f_\lambda^l(\eta_{s-}) \right] dJ_s^y,$$

by (2.6) and (2.18), we have that the martingale from (2.12) can be expressed as

$$\tilde{M}_t(\lambda, l) = \sum_{y,z} \int_0^t \left[f_\lambda^l(\eta_s^{y,z}) - f_\lambda^l(\eta_s) \right] dN_s^{y,z}$$
$$+ \sum_y \int_0^t \left[f_\lambda^l(\tau_y \eta_s) - f_\lambda^l(\eta_s) \right] dM_s^y. \tag{2.20}$$

Since $M_t, \tilde{M}_t$ are mean-zero square integrable martingales with stationary increments, to prove that $M_t + \tilde{M}_t$ is a non-degenerate martingale, we show that for any vector $l \in \mathbb{R}^d \setminus \{0\}$,

$$\mathbb{E}_{\nu_\rho} \left[\left(M_1 \cdot l + \tilde{M}_1^l \right)^2 \right] > 0. \tag{2.21}$$

The idea is that, using the orthogonality above and equation (2.20), below we can express the second moment in (2.21) as the limit as λ going to zero of the Dirichlet form $D_{se}(f_\lambda^l)$ plus another non-negative term. Then, (2.21) will follow by properly bounding this Dirichlet form (see (2.24) and the paragraph right after it).

Indeed, by (2.20), the orthogonality and the form of the quadratic variations of M_t^y

and $N_t^{y,z}$ (see (2.6) and (2.19)), and (2.3), we have that

$$
\begin{aligned}
\mathbb{E}_{\nu_\rho}\left[\left(M_1 \cdot l + \tilde{M}_1^l\right)^2\right] &= \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho}\left[\left(M_1 \cdot l + \tilde{M}_1(\lambda, l)\right)^2\right] \\
&= \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho}\left[\left(\int_0^1 \sum_{y,z} \left[f_\lambda^l(\eta_s^{y,z}) - f_\lambda^l(\eta_s)\right] dN_s^{y,z}\right)^2\right] \\
&\quad + \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho}\left[\left(\int_0^1 \sum_y \left\{(y \cdot l) + \left[f_\lambda^l(\tau_y \eta_s) - f_\lambda^l(\eta_s)\right]\right\} dM_s^y\right)^2\right] \\
&= \lim_{\lambda \to 0} 2 D_{se}(f_\lambda^l) \\
&\quad + \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho}\left[\sum_y c_{0,y}(\eta) \left\{(y \cdot l) + \left[f_\lambda^l(\tau_y \eta) - f_\lambda^l(\eta)\right]\right\}^2\right].
\end{aligned}
$$
(2.22)

Hence, to conclude (2.21), we argue as follows. Assume that there exists a constant $K = K(l, R) > 0$ such that

$$
|\langle \phi \cdot l, f_\lambda^l \rangle_{\nu_\rho}| \leq K \, D_{se}(f_\lambda^l)^{1/2}.
$$
(2.23)

Then

$$
D_{ew}(f_\lambda^l) \leq |\langle \phi \cdot l, f_\lambda^l \rangle_{\nu_\rho}| \leq K \, D_{se}(f_\lambda^l)^{1/2},
$$
(2.24)

where the first inequality follows by $D_{ew}(f_\lambda^l) \leq D_{ew}(f_\lambda^l) + \lambda |\langle f_\lambda^l, f_\lambda^l \rangle_{\nu_\rho}| = |\langle \phi \cdot l, f_\lambda^l \rangle_{\nu_\rho}|$.

In view of (2.24) and (2.22), if $D_{ew}(f_\lambda^l)$ stays positive in the limit as $\lambda \to 0$, the same holds for $D_{se}(f_\lambda^l)$ and the variance is positive. On the other hand, if $D_{ew}(f_\lambda^l)$ vanishes, then (recall (2.14)), $\mathbb{E}_{\nu_\rho}[\tilde{M}_1(\lambda, l)^2] = D_{ew}(f_\lambda^l) \to 0$ and the limiting variance is $\mathbb{E}_{\nu_\rho}\left[(M_1 \cdot l)^2\right] > 0$.

It remains to show the claim in (2.23). For an arbitrary f, we can estimate

$$
\begin{aligned}
|\langle \phi \cdot l, f \rangle_{\nu_\rho}| &= \left|\frac{1}{2} \int d\nu_\rho \sum_y (y \cdot l) \left[c_{0,y}(\eta) - c_{0,-y}(\eta)\right] f(\eta)\right| \\
&= \left|\frac{1}{2} \sum_{|y|_1 \leq R} \int d\nu_\rho (y \cdot l) \eta(0) \left[\eta(y) - \eta(-y)\right] f(\eta)\right| \\
&\leq \frac{1}{2} \sum_{|y|_1 \leq R} |y \cdot l| \left|\int d\nu_\rho \left[\eta(y) - \eta(-y)\right] f(\eta)\right|.
\end{aligned}
$$
(2.25)

Note that due to the irreducibility of $p(\cdot)$, for any $y \in \mathbb{Z}^d$ with $|y|_1 \leq R$, we can write

$$
\eta(y) - \eta(-y) = \sum_{i=1}^n [\eta(z_i) - \eta(z_{i-1})]
$$

for some sequence $(z_0 = y, z_1, \ldots, z_n = -y)$, with $p(z_i - z_{i-1}) > 0$ for $i = 1, \ldots, n$. Moreover

$$
\begin{aligned}
\left|\int d\nu_\rho \left[\eta(z_i) - \eta(z_{i-1})\right] f(\eta)\right| &= \left|\int d\nu_\rho \, \eta(z_{i-1}) \left[f(\eta^{z_{i-1}, z_i}) - f(\eta)\right]\right| \\
&\leq \rho^{1/2} \left(\int d\nu_\rho \left[f(\eta^{z_{i-1}, z_i}) - f(\eta)\right]^2\right)^{1/2} \leq p(z_i - z_{i-1})^{-1/2} D_{se}(f)^{1/2}.
\end{aligned}
$$
(2.26)

Combining (2.25) and (2.26), we obtain (2.23) which concludes the proof. $\qquad \square$

2.4 Concluding remarks

Remark 2.4 (On the tagged particle in symmetric exclusion)**.** *In the original paper by Kipnis-Varhadan, the authors used their general theorem to show the diffusivity of a tagged particle in the symmetric exclusion process in any dimension. An exceptional case is when the symmetric exclusion is nearest-neighbor and one-dimensional. This has been shown to be sub-diffusive [2] due to the "traffic jam" created by the other particles in the system. In particular, in this latter context, the analogous two martingales involved in (2.17) do annihilate each other and the crucial estimate in (2.23) does not hold.*

Remark 2.5 (Particle systems as non-elliptic dynamical random conductances)**.** *The model we introduced is an example of time-dependent random conductances, non-elliptic from below, but bounded from above, since $c_{x,y}(\xi_t) \in \{0,1\}$. In a similar fashion, we can interpret more general particle systems as models of non-elliptic dynamical random conductances, even unbounded from above. This can be done by considering a particle system $\xi_t \in \mathbb{N}^{\mathbb{Z}^d}$ and again setting $c_{x,y}(\xi_t) = \xi_t(x)\xi_t(y)$ (e.g. a Poissoinian field of independent random walks), provided that the particle system has "well behaving" space-time correlations and good spectral properties.*

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