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A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces*

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Abstract

We present an extension of the Gromov-Hausdorff metric on the set of compact metric spaces: the Gromov-Hausdorff-Prokhorov metric on the set of compact metric spaces endowed with a finite measure. We then extend it to the non-compact case by describing a metric on the set of rooted complete locally compact length spaces endowed with a boundedly finite measure. We prove that this space with the extended Gromov-Hausdorff-Prokhorov metric is a Polish space. This generalization is needed to define Lévy trees, which are (possibly unbounded) random real trees endowed with a boundedly finite measure.

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1 Introduction

In the present work, we aim to give a topological framework to certain classes of measured metric spaces. The methods go back to ideas from Gromov ([11]), who first considered the so-called Gromov-Hausdorff metric in order to compare metric spaces which might not be subspaces of a common metric space. The classical theory of the Gromov-Hausdorff metric on the space of compact metric spaces, as well as its extension to locally compact spaces, is exposed in particular in Burago, Burago and Ivanov ([4]). Recently, the concept of Gromov-Hausdorff convergence has found striking applications in the field of probability theory, in the context of random graphs. Evans ([8]) and Evans, Pitman and Winter ([9]) considered the space of real trees, which is Polish when endowed with the Gromov-Hausdorff metric. This has given a framework

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to the theory of continuum random trees, which originated with Aldous ([3]). There are also applications in the context of random maps, where there have been significant developments in the last years.

If, in addition to their metric structure, measures are defined on the spaces in consideration, the theory is not yet firmly established. In his monograph [15], Villani gives an account of the current state of the theory. As mentioned by Villani, there are essentially two approaches for the topology of metric measure spaces. The first is to emphasize the importance of the measures carried by the spaces, and to compare metric measure spaces through the measures defined on them. This approach, which goes back to Gromov, was first described, in the context of compact real trees, in the monograph [8] by Evans. Later, a similar framework was developed in [10] for (Polish) metric spaces, endowed with probability measures, as well as in [13, 14] in connection with mass transportation problems. In order to compare two such metric measure spaces, embeddings of both spaces into some common Polish metric space are considered, and the Prokhorov metric is used to compare the ensuing measures.

The second approach, used in this work, consists in combining the Hausdorff metric with the Prokhorov metric in order to compare both geometric and measure-related features of the spaces. Although it is known that the two approaches coincide when restricted to some classes of metric measure spaces (such as spaces carrying measures with the doubling property), this is not the case in general, and our approach is different from [10] in this respect. This comes at the price of having to restrict ourselves to *length spaces*.

Also, we treat the general case of boundedly finite measures instead of probability measures. The latter were considered in [10] (with the Gromov-Prokhorov approach) and in [2, 12] (with the Gromov-Hausdorff-Prokhorov approach). Roughly speaking, our approach corresponds to vague convergence (or weak-# convergence, see Remark 2.3 below), whereas the classical approach using the Prokhorov metric corresponds to weak convergence. Furthermore, the study of metric spaces with probability measures in [2, 10, 12] relies crucially on Strassen's theorem relating the Prokhorov distance between probability measures and the existence of good couplings (see for instance Corollary 11.6.4 in [6]). In the general case of boundedly finite measures considered here, such a correspondence no longer holds. Therefore, in most of the proofs, we are making use of the properties of ε-nets and finite approximations to the measures to replace Strassen's theorem.

This work was motivated by applications in the setting of weighted Lévy trees (which are in particular *length spaces*), see Abraham, Delmas and Hoscheit ([1]). We give an hint of those applications by stating that the construction of a weighted tree coded in a continuous function with compact support is measurable with respect to the Gromov-Hausdorff-Prokhorov topologies described in this paper. This construction allows us to define real tree-valued random variables using continuous random processes on $\mathbb{R}$, in particular the Lévy trees of [7] that describe the genealogy of the so-called critical or sub-critical continuous-state branching processes that become a.s. extinct. These trees have an intrinsic metric, and carry a totally finite measure $\mathbf{m}$. This measure can be renormalized to a probability measure that can be interpreted as the uniform measure on the leaves of the tree. The construction can be generalized to *super-critical* continuous-state branching processes which can live forever; in that case the corresponding genealogical tree is infinite (in the sense that it has infinite diameter) with positive probability. The construction of $\mathbf{m}$ can also be carried over from the sub-critical and critical case, but the ensuing measure is no longer totally finite a.s., but only *boundedly finite*. This paper gives an appropriate framework to handle such tree-valued random variables and also tree-valued Markov processes as in [1].

In the following sections, we describe several properties of the Gromov-Hausdorff-Prokhorov metric, d^c_{GHP}, on the set $\mathbb{K}$ of (isometry classes of) compact metric spaces, with a distinguished element called the root and endowed with a finite measure. Theorem 2.5 ensures that $(\mathbb{K}, d^c_{\mathrm{GHP}})$ is a Polish metric space. We extend those results by considering the Gromov-Hausdorff-Prokhorov metric, d_{GHP}, on the set $\mathbb{L}$ of (isometry classes of) rooted locally compact, complete length spaces, endowed with a boundedly finite measure. Theorem 2.9 ensures that $(\mathbb{L}, d_{\mathrm{GHP}})$ is also a Polish metric space. The proof of the completeness of $\mathbb{L}$ relies on a pre-compactness criterion given in Theorem 2.11. It should be noted that some natural examples of metric measure spaces (such as continuum random trees) do not satisfy doubling estimates on the measures, but are length spaces. In this respect, the pre-compactness criterion might prove a useful complement to Theorem 27.32 in [15]. The methods used are similar to the methods used in [4] to derive properties about the Gromov-Hausdorff topology of the set of locally compact complete length spaces.

The structure of the paper is as follows. Section 2 collects the main results of the paper. The application to real trees is given in Section 3. The proofs of the results in the compact case are given in Section 4. The proofs of the results in the locally compact case are given in Section 5.

2 Main results

2.1 Rooted weighted metric spaces

Let (X, d^X) be a Polish metric space. The diameter of $A \in \mathcal{B}(X)$ is given by

$$\mathrm{diam}\ (A) = \sup\{d^X(x, y);\ x, y \in A\}.$$

For $A, B \in \mathcal{B}(X)$, we set

$$d^X_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0;\ A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\},$$

the Hausdorff distance between A and B, where

$$A^\varepsilon = \{x \in X;\ \inf_{y\in A} d^X(x, y) < \varepsilon\} \tag{2.1}$$

is the ε-halo set of A. If X is compact, then the space of compact subsets of X, endowed with the Hausdorff metric, is compact (see theorem 7.3.8 in [4]). To state a pre-compactness criterion, we will need the notion of ε-nets.

Definition 2.1. *Let (X, d^X) be a metric space, and $\varepsilon > 0$. A subset $A \subset X$ is an ε-net of $B \subset X$ if*

$$A \subset B \subset A^\varepsilon.$$

Notice that, for any $\varepsilon > 0$, compact metric spaces admit finite ε-nets and locally compact spaces admit locally finite ε-nets.

Let $\mathcal{M}_f(X)$ denote the set of all nonnegative finite Borel measures on X. If $\mu, \nu \in \mathcal{M}_f(X)$, we set

$$d^X_{\mathrm{P}}(\mu, \nu) = \inf\{\varepsilon > 0;\ \mu(A) \le \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \le \mu(A^\varepsilon) + \varepsilon \ \text{ for any closed set } A\},$$

the Prokhorov metric between μ and ν. It is well-known (see [5] Appendix A.2.5) that $(\mathcal{M}_f(X), d^X_{\mathrm{P}})$ is a Polish metric space, and that the topology generated by d^X_{P} is exactly the topology of weak convergence (convergence against continuous bounded functionals). When there is no ambiguity on the metric space (X, d^X), we may write d, d_{H}, and d_{P} instead of d^X, d^X_{H} and d^X_{P}. In the case where we consider different metrics on the same space, in order to stress that the metric is d^X, we will write $d_{\mathrm{H}}^{d^X}$ and $d_{\mathrm{P}}^{d^X}$ for the corresponding Hausdorff and Prokhorov metrics. If $\Phi : X \to X'$ is a Borel map between two Polish metric spaces and if μ is a Borel measure on X, we will note $\Phi_*\mu$ the image measure on X' defined by $\Phi_*\mu(A) = \mu(\Phi^{-1}(A))$, for any Borel set $A \subset X$.

In this paper, in order to generalize the properties of probability measures on Polish metric spaces, we will be interested in metric spaces carrying boundedly finite measures, in the following sense:

Definition 2.2. *Let (X, d) be a metric space. A Borel measure is* boundedly finite *if it is finite on all bounded Borel sets.*

The set of all boundedly finite nonnegative Borel measures on X will be noted $\mathcal{M}(X)$. Let $\varnothing$ be a distinguished element of X, which we will call the root. We will often consider the closed ball of radius r centered at $\varnothing$

$$X^{(r)} = \{x \in X; d^X(\varnothing, x) \le r\}, \tag{2.2}$$

and for $\mu \in \mathcal{M}(X)$ its restriction $\mu^{(r)}$ to $X^{(r)}$

$$\mu^{(r)}(dx) = \mathbf{1}_{X^{(r)}}(x)\ \mu(dx). \tag{2.3}$$

Remark 2.3. *The Prokhorov metric could be extended to boundedly finite measures in the following way. If $\mu, \nu \in \mathcal{M}(X)$, we define a generalized Prokhorov metric between μ and ν:*

$$d_{\mathrm{gP}}^X(\mu, \nu) = \int_0^\infty \mathrm{e}^{-r} \left(1 \wedge d_{\mathrm{P}}^X \left(\mu^{(r)}, \nu^{(r)}\right)\right)\ dr. \tag{2.4}$$

It is not difficult to check that d_{gP}^X is well defined (see Lemma 2.8 in a more general framework) and is a metric. Furthermore $(\mathcal{M}(X), d_{\mathrm{gP}}^X)$ is a Polish metric space, and the topology generated by d_{gP}^X is exactly the topology of weak-# convergence (convergence against continuous bounded functionals with bounded support), see [5] Appendix A.2.6. Notice that, when X is locally compact in addition of being Polish, then the weak-# convergence coincides with the classical vague convergence (convergence against continuous bounded functionals with compact *support) on $\mathcal{M}_f(X)$. However, this generalized Prokhorov distance is not well-suited to combination with the Hausdorff distance. Therefore, we will use another approach to compare two Polish metric spaces, endowed with boundedly finite measures, by first comparing balls of finite radius using the Gromov-Hausdorff-Prokhorov metric on compact metric spaces carrying finite measures,* then *integrating over all radii. The topology on boundedly finite measures we recover will however be the same as the one defined by d_{gP}.*

Definition 2.4.

- *A rooted weighted metric space $\mathcal{X} = (X, d, \varnothing, \mu)$ is a metric space (X, d) with a distinguished element $\varnothing \in X$, called the root, and a boundedly finite Borel measure μ.*
- *Two rooted weighted metric spaces $\mathcal{X} = (X, d, \varnothing, \mu)$ and $\mathcal{X}' = (X', d', \varnothing', \mu')$ are said to be GHP-isometric if there exists an isometric one-to-one map $\Phi : X \to X'$ such that $\Phi(\varnothing) = \varnothing'$ and $\Phi_*\mu = \mu'$. In that case, Φ is called a GHP-isometry.*

Notice that if (X, d) is compact, then a boundedly finite measure on X is finite and belongs to $\mathcal{M}_f(X)$. We will now use a procedure due to Gromov ([11]) to compare any two compact rooted weighted metric spaces, even if they are not subspaces of the same Polish metric space.

2.2 Gromov-Hausdorff-Prokhorov metric for compact spaces

For convenience, we recall the definition of the Gromov-Hausdorff metric, see for example Definition 7.3.10 in [4]. Let (X, d) and (X', d') be compact metric spaces. The Gromov-Hausdorff distance between (X, d) and (X', d') is given by

$$d^c_{\mathrm{GH}}((X, d), (X', d')) = \inf_{\varphi,\varphi',Z} d^Z_{\mathrm{H}}(\varphi(X), \varphi'(X')), \tag{2.5}$$

where the infimum is taken over all isometric embeddings $\varphi : X \hookrightarrow Z$ and $\varphi' : X' \hookrightarrow Z$ into some common Polish metric space (Z, d^Z). Note that equation (2.5) actually defines a metric on the set of *isometry classes* of compact metric spaces.

Now, let us introduce the Gromov-Hausdorff-Prokhorov metric on the class of compact metric spaces. Let $\mathcal{X} = (X, d, \varnothing, \mu)$ and $\mathcal{X}' = (X', d', \varnothing', \mu')$ be two compact rooted weighted metric spaces, and define

$$d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{X}') = \inf_{\Phi,\Phi',Z} \left(d^Z(\Phi(\varnothing), \Phi'(\varnothing')) + d^Z_{\mathrm{H}}(\Phi(X), \Phi'(X')) + d^Z_{\mathrm{P}}(\Phi_*\mu, \Phi'_*\mu') \right), \tag{2.6}$$

where the infimum is taken over all isometric embeddings $\Phi : X \hookrightarrow Z$ and $\Phi' : X' \hookrightarrow Z$ into some common Polish metric space (Z, d^Z).

Note that equation (2.6) actually defines a *semimetric*, since $d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{X}') = 0$ if $\mathcal{X}$ and $\mathcal{X}'$ are GHP-isometric. Therefore, we will consider $\mathbb{K}$, the set of GHP-isometry classes of compact rooted weighted metric spaces and identify a compact rooted weighted metric space with its class in $\mathbb{K}$. Then the function d^c_{GHP} is finite on $\mathbb{K}^2$.

Theorem 2.5. *(i) The function d^c_{GHP} defines a metric on $\mathbb{K}$.*

(ii) The space $(\mathbb{K}, d^c_{GHP})$ is a Polish metric space.

This metric (the compact Gromov-Hausdorff-Prokhorov metric) extends the Gromov-Hausdorff metric on compact metric spaces, see [4] section 7, as well as the Gromov-Hausdorff-Prokhorov metric on compact metric spaces endowed with a probability measure, see [12]. See also [10] for another approach.

We end this section by a pre-compactness criterion on $\mathbb{K}$.

Theorem 2.6. *Let $\mathcal{A}$ be a subset of $\mathbb{K}$, such that*

(i) We have $\sup_{(X,d,\varnothing,\mu)\in\mathcal{A}} \mathrm{diam}\,(X) < +\infty$.

(ii) For every $\varepsilon > 0$, there exists a finite integer $N(\varepsilon) \geq 1$, such that for $(X, d, \varnothing, \mu) \in \mathcal{A}$, there is an ε-net of X with cardinal less than $N(\varepsilon)$.

(iii) We have $\sup_{(X,d,\varnothing,\mu)\in\mathcal{A}} \mu(X) < +\infty$.

Then, $\mathcal{A}$ is relatively compact: every sequence in $\mathcal{A}$ admits a sub-sequence that converges in the d^c_{GHP} topology.

Note that we could have defined a Gromov-Hausdorff-Prokhorov metric without reference to any root. However, the introduction of the root is necessary to define the Gromov-Hausdorff-Prokhorov metric for locally compact spaces, see next section.

2.3 Gromov-Hausdorff-Prokhorov metric for locally compact spaces

To consider an extension to non-compact weighted rooted metric spaces, we will consider complete and locally compact length spaces.

We recall that a metric space (X, d) is a length space if for every $x, y \in X$, we have

$$d(x, y) = \inf L(\gamma),$$

where the infimum is taken over all rectifiable curves $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$, and where $L(\gamma)$ is the length of the rectifiable curve γ. We recall that (X, d) is a length space if and only of it satisfies the mid-point condition (see Theorem 2.4.16 in [4]): for all $\varepsilon > 0$, $x, y \in X$, there exists $z \in X$ such that

$$|2d(x,z) - d(x,y)| + |2d(y,z) - d(x,y)| \le \varepsilon.$$

Definition 2.7. *Let $\mathbb{L}$ be the set of GHP-isometry classes of rooted, weighted, complete and locally compact length spaces and identify a rooted, weighted, complete and locally compact length spaces with its class in $\mathbb{L}$.*

If $\mathcal{X} = (X, d, \varnothing, \mu) \in \mathbb{L}$ then for $r \ge 0$ we will consider its restriction to the closed ball of radius r centered at $\varnothing$, $\mathcal{X}^{(r)} = (X^{(r)}, d^{(r)}, \varnothing, \mu^{(r)})$, where $X^{(r)}$ is defined by (2.2), where the metric $d^{(r)}$ is the restriction of d to $X^{(r)}$, and where the measure $\mu^{(r)}$ is defined by (2.3). Recall that the Hopf-Rinow theorem implies that if (X, d) is a complete and locally compact length space, then every closed bounded subset of X is compact. In particular if $\mathcal{X}$ belongs to $\mathbb{L}$, then $\mathcal{X}^{(r)}$ belongs to $\mathbb{K}$ for all $r \ge 0$.

We state a regularity lemma of d^c_{GHP} with respect to the restriction operation.

Lemma 2.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two elements of $\mathbb{L}$. Then the function defined on $\mathbb{R}_+$ by $r \mapsto d^c_{GHP}\left(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)}\right)$ is càdlàg.*

This implies that the following function (inspired by (2.4)) is well defined on $\mathbb{L}^2$:

$$d_{\mathrm{GHP}}(\mathcal{X}, \mathcal{Y}) = \int_0^\infty \mathrm{e}^{-r} \left(1 \wedge d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)}\right)\right) \, dr.$$

Theorem 2.9. *(i) The function d_{GHP} defines a metric on $\mathbb{L}$.*

(ii) The space $(\mathbb{L}, d_{GHP})$ is a Polish metric space.

The next result implies that d^c_{GHP} and d_{GHP} define the same topology on $\mathbb{K} \cap \mathbb{L}$.

Proposition 2.10. *Let $(\mathcal{X}_n, n \in \mathbb{N})$ and $\mathcal{X}$ be elements of $\mathbb{K} \cap \mathbb{L}$. Then the sequence $(\mathcal{X}_n, n \in \mathbb{N})$ converges to $\mathcal{X}$ in $(\mathbb{K}, d^c_{GHP})$ if and only if it converges to $\mathcal{X}$ in $(\mathbb{L}, d_{GHP})$.*

Finally, we give a pre-compactness criterion on $\mathbb{L}$ which is a generalization of the well-known compactness theorem for compact metric spaces, see for instance Theorem 7.4.15 in [4].

Theorem 2.11. *Let $\mathcal{C}$ be a subset of $\mathbb{L}$, such that for every $r \ge 0$,*

(i) For every $\varepsilon > 0$, there exists a finite integer $N(r, \varepsilon) \ge 1$, such that for any $(X, d, \varnothing, \mu) \in \mathcal{C}$, there is an ε-net of $X^{(r)}$ with cardinal less than $N(r, \varepsilon)$.

(ii) We have $\sup_{(X,d,\varnothing,\mu)\in\mathcal{C}} \mu(X^{(r)}) < +\infty$.

Then $\mathcal{C}$ is relatively compact: every sequence in $\mathcal{C}$ admits a sub-sequence that converges in the d_{GHP} topology.

3 Application to real trees coded by functions

A metric space (T, d) is a real tree (or $\mathbb{R}$-tree) if the following properties are satisfied:

(i) For every $s, t \in T$, there is a unique isometric map $f_{s,t}$ from $[0, d(s,t)]$ to T such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s,t)) = t$.

(ii) For every $s, t \in T$, if q is a continuous injective map from $[0,1]$ to T such that $q(0) = s$ and $q(1) = t$, then $q([0,1]) = f_{s,t}([0, d(s,t)])$.

Note that real trees are always length spaces and that complete real trees are the only complete connected spaces that satisfy the so-called *four-point condition:*

$$\forall x_1, x_2, x_3, x_4 \in X,\ d(x_1, x_2) + d(x_3, x_4) \leq (d(x_1, x_3) + d(x_2, x_4)) \vee (d(x_1, x_4) + d(x_2, x_3)). \tag{3.1}$$

We say that a real tree is rooted if there is a distinguished vertex $\varnothing$, the *root* of T.

Definition 3.1. *We denote by $\mathbb{T}$ the set of (GHP-isometry classes of) rooted, weighted, complete and locally compact real trees, in short w-trees.*

We deduce the following corollary from Theorem 2.9 and the four-point condition characterization of real trees.

Corollary 3.2. *The set $\mathbb{T}$ is a closed subset of $\mathbb{L}$ and $(\mathbb{T}, d_{GHP})$ is a Polish metric space.*

Let f be a continuous non-negative function defined on $[0, +\infty)$, such that $f(0) = 0$, with compact support. We set

$$\sigma^f = \sup\{t; f(t) > 0\},$$

with the convention $\sup \varnothing = 0$. Let d^f be the non-negative function defined by

$$d^f(s,t) = f(s) + f(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} f(u).$$

It can be easily checked that d^f is a semi-metric on $[0, \sigma^f]$. One can define the equivalence relation associated with d^f by $s \sim t$ if and only if $d^f(s,t) = 0$. Moreover, when we consider the quotient space

$$T^f = [0, \sigma^f]_{/\sim}$$

and, noting again d^f the induced metric on T^f and rooting T^f at $\varnothing^f$, the equivalence class of 0, it can be checked that the space $(T^f, d^f, \varnothing^f)$ is a rooted compact real tree. We denote by p^f the canonical projection from $[0, \sigma^f]$ onto T^f, which is extended by $p^f(t) = \varnothing^f$ for $t \geq \sigma^f$. Notice that p^f is continuous. We define $\mathbf{m}^f$, a Borel measure on T^f as the image measure on T^f of the Lebesgue measure on $[0, \sigma^f]$ by p^f. We consider the (compact) w-tree $\mathcal{T}^f = (T^f, d^f, \varnothing^f, \mathbf{m}^f)$.

We have the following elementary result (see Lemma 2.3 of [7] when dealing with the Gromov-Hausdorff metric instead of the Gromov-Hausdorff-Prokhorov metric). For a proof, see [1].

Proposition 3.3. *Let f, g be two compactly supported, non-negative continuous functions with $f(0) = g(0) = 0$. Then*

$$d^c_{GHP}(\mathcal{T}^f, \mathcal{T}^g) \leq 6\|f - g\|_\infty + |\sigma^f - \sigma^g|. \tag{3.2}$$

This result and Proposition 2.10 ensure that the map $f \mapsto \mathcal{T}^f$ (defined on the space of continuous functions with compact support vanishing at 0, with the uniform topology) taking values in $(\mathbb{T} \cap \mathbb{K}, d^c_{\mathrm{GHP}})$ or $(\mathbb{T}, d_{\mathrm{GHP}})$ is measurable.

4 Gromov-Hausdorff-Prokhorov metric for compact spaces

4.1 Proof of (i) of Theorem 2.5

In this section, we will prove that d^c_{GHP} defines a metric on $\mathbb{K}$.

First, we will prove the following technical lemma, which is a generalization of Remark 7.3.12 in [4]. Let $\mathcal{X} = (X, d^X, \varnothing^X, \mu^X)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \mu^Y)$ be two elements of $\mathbb{K}$. We will use the notation $X \sqcup Y$ for the disjoint union of the sets X and Y. We will abuse notations and note $X, \mu^X, \varnothing^X$ and $Y, \mu^Y, \varnothing^Y$ the images of $X, \mu^X, \varnothing^X$ and of $Y, \mu^Y, \varnothing^Y$ respectively by the canonical embeddings $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$.

Lemma 4.1. *Let $\mathcal{X} = (X, d^X, \varnothing^X, \mu^X)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \mu^Y)$ be two elements of $\mathbb{K}$. Then*

$$d^c_{GHP}(\mathcal{X}, \mathcal{Y}) = \inf_d \left\{ d(\varnothing^X, \varnothing^Y) + d^d_H(X, Y) + d^d_P(\mu^X, \mu^Y) \right\}, \tag{4.1}$$

where the infimum is taken over all metrics d on $X \sqcup Y$ such that the canonical embeddings $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ are isometries.

Proof. We only have to show that

$$\inf_d \left\{ d(\varnothing^X, \varnothing^Y) + d^d_{\mathrm{H}}(X, Y) + d^d_{\mathrm{P}}(\mu^X, \mu^Y) \right\} \leq d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{Y}), \tag{4.2}$$

since the other inequality is obvious. Let (Z, d^Z) be a Polish space and Φ^X and Φ^Y be isometric embeddings of X and Y in Z. Let $\delta > 0$. We define the following function on $(X \sqcup Y)^2$:

$$d(x, y) = \begin{cases} d^Z(\Phi^X(x), \Phi^Y(y)) + \delta & \text{if } x \in X,\ y \in Y, \\ d^X(x, y) & \text{if } x, y \in X, \\ d^Y(x, y) & \text{if } x, y \in Y. \end{cases} \tag{4.3}$$

It is obvious that d is a metric on $X \sqcup Y$, and that the canonical embeddings of X and Y in $X \sqcup Y$ are isometric. Furthermore, by definition, we have $d(\varnothing^X, \varnothing^Y) = d^Z(\Phi^X(\varnothing^X), \Phi^Y(\varnothing^Y)) + \delta$. Concerning the Hausdorff distance between X and Y, we get that

$$d^d_{\mathrm{H}}(X, Y) \leq d^Z_{\mathrm{H}}(\Phi^X(X), \Phi^Y(Y)) + \delta.$$

Finally, let us compute the Prokhorov distance between μ^X and μ^Y. Let $\varepsilon > 0$ be such that $d^Z_{\mathrm{P}}(\Phi^X_* \mu^X, \Phi^Y_* \mu^Y) < \varepsilon$. Let A be a closed subset of $X \sqcup Y$. By definition,

$$\begin{aligned} \mu^X(A) = \mu^X(A \cap X) &= \Phi^X_* \mu^X(\Phi^X(A \cap X)) \\ &< \Phi^Y_* \mu^Y(\{z \in Z,\ d^Z(z, \Phi^X(A \cap X)) < \varepsilon\}) + \varepsilon \\ &= \Phi^Y_* \mu^Y(\{z \in \Phi^Y(Y),\ d^Z(z, \Phi^X(A \cap X)) < \varepsilon\}) + \varepsilon \\ &\leq \mu^Y(\{y \in Y,\ d(y, A \cap X) < \varepsilon + \delta\}) + \varepsilon \\ &\leq \mu^Y(\{y \in X \sqcup Y,\ d(y, A) < \varepsilon + \delta\}) + \varepsilon. \end{aligned}$$

The same result holds for (X, Y) replaced by (Y, X) and therefore we get $d^d_{\mathrm{P}}(\mu^X, \mu^Y) < \varepsilon + \delta$. This implies

$$d^d_{\mathrm{P}}(\mu^X, \mu^Y) \leq d^Z_{\mathrm{H}}(\Phi^X_* \mu^X, \Phi^Y_* \mu^Y) + \delta.$$

Eventually, we get

$$\begin{aligned} &d(\varnothing^X, \varnothing^Y) + d^d_{\mathrm{H}}(X, Y) + d^d_{\mathrm{P}}(\mu^X, \mu^Y) \\ &\qquad \leq d^Z(\Phi^X(\varnothing^X), \Phi^Y(\varnothing^Y)) + d^Z_{\mathrm{H}}(\Phi^X(X), \Phi^Y(Y)) + d^Z_{\mathrm{H}}(\Phi^X_* \mu^X, \Phi^Y_* \mu^Y) + 3\delta. \end{aligned}$$

Thanks to (2.6) and since $\delta > 0$ is arbitrary, we get (4.2). □

We now prove that d^c_{GHP} does indeed satisfy all the axioms of a metric (as is done in [4] for the Gromov-Hausdorff metric and in [12] in the case of probability measures on compact metric spaces). The symmetry and positiveness of d^c_{GHP} being obvious, let us prove the triangle inequality and positive definiteness.

Lemma 4.2. *The function d^c_{GHP} satisfies the triangle inequality on $\mathbb{K}$.*

Proof. Let $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3$ be elements of $\mathbb{K}$. Let us assume that $d^c_{\mathrm{GHP}}(\mathcal{X}_i, \mathcal{X}_2) < r_i$ for $i \in \{1,3\}$. With obvious notations, for $i \in \{1,3\}$, we consider, as in Lemma 4.1, metrics d_i on $X_i \sqcup X_2$. Let us then consider $Z = X_1 \sqcup X_2 \sqcup X_3$, on which we define

$$d(x,y) = \begin{cases} d_i(x,y) & \text{if } x,y \in (X_i \sqcup X_2)^2 \text{ for } i \in \{1,3\}, \\ \inf_{z\in X_2}\{d_1(x,z) + d_3(z,y)\} & \text{if } x \in X_1, y \in X_3. \end{cases} \tag{4.4}$$

The function d is in fact a metric on Z, and the canonical embeddings are isometries, since they are isometries for d_1 and d_3. By definition, we have

$$d^d_{\mathrm{H}}(X_1, X_3) = \left(\sup_{x_1\in X_1} \inf_{x_3 \in X_3} d(x_1,x_3)\right) \vee \left(\sup_{x_3\in X_3} \inf_{x_1 \in X_1} d(x_1,x_3)\right).$$

We notice that

$$\begin{aligned} \sup_{x_1\in X_1} \inf_{x_3 \in X_3} d(x_1,x_3) &= \sup_{x_1\in X_1} \inf_{x_2\in X_2,\ x_3 \in X_3} d_1(x_1,x_2) + d_3(x_2,x_3) \\ &\le d^{d_1}_{\mathrm{H}}(X_1,X_2) + \inf_{x_2\in X_2,\ x_3 \in X_3} d_3(x_2,x_3) \\ &\le d^{d_1}_{\mathrm{H}}(X_1,X_2) + d^{d_3}_{\mathrm{H}}(X_2,X_3). \end{aligned}$$

Thus, $d^d_{\mathrm{H}}(X_1,X_3) \le d^{d_1}_{\mathrm{H}}(X_1,X_2) + d^{d_3}_{\mathrm{H}}(X_2,X_3)$.

As far as the Prokhorov distance is concerned, for $i \in \{1,3\}$, let ε_i be such that $d^{d_i}_{\mathrm{P}}(\mu_i, \mu_2) < \varepsilon_i$. Then, if $A \subset Z$ is closed,

$$\begin{aligned} \mu_1(A) = \mu_1(A \cap X_1) &< \mu_2(\{x \in X_1 \sqcup X_2,\ d_1(x, A\cap X_1) < \varepsilon_1\}) + \varepsilon_1 \\ &\le \mu_2(A^{\varepsilon_1} \cap X_2) + \varepsilon_1 \\ &< \mu_3(\{x \in X_3 \sqcup X_2,\ d_3(x, A^{\varepsilon_1}\cap X_2) < \varepsilon_3\}) + \varepsilon_1 + \varepsilon_3 \\ &\le \mu_3(A^{\varepsilon_1+\varepsilon_3}) + \varepsilon_1 + \varepsilon_3, \end{aligned}$$

where $A^\varepsilon = \{z \in Z,\ d(z,A) < \varepsilon\}$, for $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_1 + \varepsilon_3$. A similar result holds with (μ_1, μ_3) replaced by (μ_3, μ_1). We deduce that $d^d_{\mathrm{P}}(\mu_1,\mu_3) < \varepsilon_1 + \varepsilon_3$, which implies that $d^d_{\mathrm{P}}(\mu_1,\mu_3) \le d^{d_1}_{\mathrm{P}}(\mu_1,\mu_2) + d^{d_3}_{\mathrm{P}}(\mu_2,\mu_3)$.

By summing up all the results, we get

$$d(\varnothing_1, \varnothing_3) + d^d_{\mathrm{H}}(X_1,X_3) + d^d_{\mathrm{P}}(\mu_1,\mu_3) \le \sum_{i\in\{1,3\}} d^{d_i}(\varnothing_i, \varnothing_2) + d^{d_i}_{\mathrm{H}}(X_i,X_2) + d^{d_i}_{\mathrm{P}}(\mu_i,\mu_2).$$

Then use the definition (2.6) and Lemma 4.1 to get the triangle inequality

$$d^c_{\mathrm{GHP}}(\mathcal{X}_1, \mathcal{X}_3) \le d^c_{\mathrm{GHP}}(\mathcal{X}_1, \mathcal{X}_2) + d^c_{\mathrm{GHP}}(\mathcal{X}_2, \mathcal{X}_3).$$

□

This proves that d^c_{GHP} is a semi-metric on $\mathbb{K}$. We now prove the positive definiteness.

Lemma 4.3. *Let $\mathcal{X}, \mathcal{Y}$ be two elements of $\mathbb{K}$ such that $d^c_{GHP}(\mathcal{X}, \mathcal{Y}) = 0$. Then $\mathcal{X} = \mathcal{Y}$ (as GHP-isometry classes of rooted weighted compact metric spaces).*

Proof. Let $\mathcal{X} = (X, d^X, \varnothing^X, \mu^X)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \mu^Y)$ in $\mathbb{K}$ such that $d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{Y}) = 0$. According to Lemma 4.1, we can find a sequence of metrics $(d^n, n \ge 1)$ on $X \sqcup Y$, such that

$$d^n(\varnothing^X, \varnothing^Y) + d^n_{\mathrm{H}}(X,Y) + d^n_{\mathrm{P}}(\mu^X, \mu^Y) < \varepsilon_n, \tag{4.5}$$

for some positive sequence $(\varepsilon_n, n \ge 1)$ decreasing to 0, where d^n_{H} and d^n_{P} stand for $d^{d^n}_{\mathrm{H}}$ and $d^{d^n}_{\mathrm{P}}$. For any $k \ge 1$, let S_k be a finite $(1/k)$-net of X, containing the root. Since X is compact, we get by Definition 2.1 that S_k is in fact an $(\frac{1}{k} - \delta)$-net of X for some $\delta > 0$. Let $N_k + 1$ be the cardinal of S_k. We will write

$$S_k = \{x_{0,k} = \varnothing^X, x_{1,k}, ..., x_{N_k,k}\}.$$

Let $(V_{i,k}, 0 \leq i \leq N_k)$ be Borel subsets of X with diameter less than $1/k$, that is

$$\sup_{x,x' \in V_{i,k}} d^X(x, x') < 1/k,$$

such that $\bigcup_{0 \leq i \leq N_k} V_{i,k} = X$ and for all $0 \leq i, i' \leq N_k$, we have $V_{i,k} \bigcap V_{i',k} = \varnothing$ and $x_{i,k} \in V_{i,k}$ if $V_{i,k} \neq \varnothing$. We set

$$\mu_k^X(dx) = \sum_{i=0}^{N_k} \mu^X(V_{i,k}) \delta_{x_{i,k}}(dx),$$

where $\delta_{x'}(dx)$ is the Dirac measure at x'. Notice that

$$d_{\mathrm{H}}^X(X, S_k) \leq \frac{1}{k} \quad \text{and} \quad d_{\mathrm{P}}^X(\mu_k^X, \mu^X) \leq \frac{1}{k} \cdot$$

We set $y_{0,k} = y_{0,k}^n = \varnothing^Y$. By (4.5), we get that for any $k \geq 1, 0 \leq i \leq N_k$, there exists $y_{i,k}^n \in Y$ such that $d^n(x_{i,k}, y_{i,k}^n) < \varepsilon_n$. Since Y is compact, the sequence $(y_{i,k}^n, n \geq 1)$ is relatively compact, hence admits a converging sub-sequence. Using a diagonal argument, and without loss of generality (by considering the sequence instead of the sub-sequence), we may assume that for $k \geq 1, 0 \leq i \leq N_k$, the sequence $(y_{i,k}^n, n \geq 1)$ converges to some $y_{i,k} \in Y$.

For any $y \in Y$, we can choose $x \in X$ such that $d^n(x, y) < \varepsilon_n$ and i, k such that $d^X(x, x_{i,k}) < \frac{1}{k} - \delta$. Then, we get

$$d^Y(y, y_{i,k}^n) = d^n(y, y_{i,k}^n) \leq d^n(y, x) + d^X(x, x_{i,k}) + d^n(x_{i,k}, y_{i,k}^n) \leq \frac{1}{k} - \delta + 2\varepsilon_n.$$

Thus, the set $\{y_{i,k}^n, 0 \leq i \leq N_k\}$ is a $(2\varepsilon_n + 1/k - \delta)$-net of Y, and the set $S_k^Y = \{y_{i,k}, 0 \leq i \leq N_k\}$ is an $1/k$-net of Y.

If $k, k' \geq 1$ and $0 \leq i \leq N_k, 0 \leq i' \leq N_{k'}$, then we have

$$\begin{aligned} d^Y(y_{i,k}, y_{i',k'}) &\leq d^Y(y_{i,k}^n, y_{i,k}) + d^Y(y_{i,k}^n, y_{i',k'}^n) + d^Y(y_{i',k'}^n, y_{i',k'}) \\ &\leq d^Y(y_{i,k}^n, y_{i,k}) + d^Y(y_{i',k'}^n, y_{i',k'}) + 2\varepsilon_n + d^X(x_{i,k}, x_{i',k'}), \end{aligned}$$

and, since the terms $d(y_{i,k}^n, y_{i,k})$ and $d(y_{i',k'}^n, y_{i',k'})$ can be made arbitrarily small, we deduce

$$d(y_{i,k}, y_{i',k'}) \leq d(x_{i,k}, x_{i',k'}).$$

The reverse inequality is proven using similar arguments, so that the above inequality is in fact an equality. Therefore the map defined by $\Phi(x_{i,k}) = (y_{i,k})$ from $\cup_{k \geq 1} S_k$ onto $\cup_{k \geq 1} S_k^Y$ is a root-preserving isometry. By density, this map can be extended uniquely to an isometric one-to-one root-preserving embedding from X to Y which we still denote by Φ. Hence the metric spaces X and Y are root-preserving isometric.

As far as the measures are concerned, we set

$$\mu_k^{Y,n} = \sum_{i=0}^{N_k} \mu^X(V_{i,k}) \delta_{y_{i,k}^n} \quad \text{and} \quad \mu_k^Y = \sum_{i=0}^{N_k} \mu^X(V_{i,k}) \delta_{y_{i,k}}.$$

By construction, we have $d_{\mathrm{P}}^n(\mu_k^{Y,n},\mu_k^X) \leq \varepsilon_n$. We get

$$\begin{aligned} d_{\mathrm{P}}^Y(\mu_k^Y,\mu^Y) = d_{\mathrm{P}}^n(\mu_k^Y,\mu^Y) &\leq d_{\mathrm{P}}^Y(\mu_k^Y,\mu_k^{Y,n}) + d_{\mathrm{P}}^n(\mu_k^{Y,n},\mu_k^X) + d_{\mathrm{P}}^X(\mu_k^X,\mu^X) + d_{\mathrm{P}}^n(\mu^X,\mu^Y) \\ &< d_{\mathrm{P}}^Y(\mu_k^Y,\mu_k^{Y,n}) + \varepsilon_n + \frac{1}{k} + \varepsilon_n. \end{aligned}$$

Furthermore, as n goes to infinity, we have that $d_{\mathrm{P}}^Y(\mu_k^Y,\mu_k^{Y,n})$ converges to 0, since the $y_{i,k}^n$ converge towards the $y_{i,k}$. Thus, we actually have

$$d_{\mathrm{P}}^Y(\mu_k^Y,\mu^Y) \leq 1/k.$$

This implies that $(\mu_k^Y, k \geq 1)$ converges weakly to μ^Y. Since by definition $\mu_k^Y = \Phi_*\mu_k^X$ and since Φ is continuous, by passing to the limit, we get $\mu^Y = \Phi_*\mu^X$. This gives that $\mathcal{X}$ and $\mathcal{Y}$ are GHP-isometric. □

This proves that the function d_{GHP}^c defines a metric on $\mathbb{K}$.

4.2 Proof of Theorem 2.6 and of (ii) of Theorem 2.5

The proof of Theorem 2.6 is very close to the proof of Theorem 7.4.15 in [4], where only the Gromov-Hausdorff metric is involved. It is in fact a simplified version of the proof of Theorem 2.11, and is thus left to the reader.

We are left with the proof of (ii) of Theorem 2.5. It is in fact enough to check that if $(\mathcal{X}_n, n \in \mathbb{N})$ is a Cauchy sequence, then it is relatively compact.

First notice that if (Z, d^Z) is a Polish metric space, then for any closed subsets A, B, we have $d_{\mathrm{H}}^Z(A,B) \geq |\mathrm{diam}\,(A) - \mathrm{diam}\,(B)|$, and for any $\mu,\nu \in \mathcal{M}_f(Z)$, we have $d_{\mathrm{P}}^Z(\mu,\nu) \geq |\mu(Z) - \nu(Z)|$. This implies that for any elements of $\mathbb{K}$, $\mathcal{X} = (X, d^X, \varnothing^X, \mu)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \nu)$,

$$d_{\mathrm{GHP}}^c(\mathcal{X},\mathcal{Y}) \geq |\mathrm{diam}\,(X) - \mathrm{diam}\,(Y)| + |\mu(X) - \nu(Y)|\,. \tag{4.6}$$

Furthermore, using the definition of the Gromov-Hausdorff metric (2.5), we clearly have

$$d_{\mathrm{GHP}}^c(\mathcal{X},\mathcal{Y}) \geq d_{\mathrm{GH}}^c((X,d^X),(Y,d^Y)). \tag{4.7}$$

We deduce that if $\mathcal{A} = (\mathcal{X}_n, n \in \mathbb{N})$ is a Cauchy sequence, then (4.6) implies that conditions (i) and (iii) of Theorem 2.6 are fulfilled. Furthermore, thanks to (4.7), the sequence $((X_n, d^{X_n}), n \in \mathbb{N})$ is a Cauchy sequence for the Gromov-Hausdorff metric. Then point (2) of Proposition 7.4.11 in [4] readily implies condition (ii) of Theorem 2.6.

5 Extension to locally compact length spaces

5.1 First results

First, let us state two elementary lemmas. Let $(X, d, \varnothing)$ be a rooted metric space. Recall notation (2.2). We set

$$\partial_r X = \{x \in X;\ d(\varnothing^x, x) = r\}.$$

Lemma 5.1. *Let $(X, d, \varnothing)$ be a complete rooted length space and $r, \varepsilon > 0$. Then we have, for all $\delta > 0$*

$$X^{(r+\varepsilon)} \subset (X^{(r)})^{\varepsilon+\delta}.$$

Proof. Let $x \in X^{(r+\varepsilon)} \backslash X^{(r)}$ and $\delta > 0$. There exists a rectifiable curve γ defined on $[0,1]$ with values in X such that $\gamma(0) = \varnothing$ and $\gamma(1) = x$, such that $L(\gamma) < d(\varnothing, x) + \delta \leq r + \varepsilon + \delta$. There exists $t \in (0,1)$ such that $\gamma(t) \in \partial_r X$. We can bound $d(\gamma(t), x)$ by the length of the fragment of γ joining $\gamma(t)$ and x, that is the length of γ minus the length of the fragment of γ joining $\varnothing$ to $\gamma(t)$. The latter being equal to or larger than $d(\varnothing^X, \gamma(t)) = r$, we get

$$d(\gamma(t), x) \leq L(\gamma) - r < \varepsilon + \delta.$$

Since $\gamma(t) \in X^{(r)}$, we get $x \in \left(X^{(r)}\right)^{\varepsilon+\delta}$. This ends the proof. □

Lemma 5.2. *Let $\mathcal{X} = (X, d, \varnothing, \mu) \in \mathbb{L}$. For all $\varepsilon > 0$ and $r > 0$, we have*

$$d^c_{GHP}(\mathcal{X}^{(r)}, \mathcal{X}^{(r+\varepsilon)}) \leq \varepsilon + \mu(X^{(r+\varepsilon)} \setminus X^{(r)}).$$

Proof. The identity map is an obvious root-preserving embedding $X^{(r)} \hookrightarrow X^{(r+\varepsilon)}$. Then, we have

$$d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}, \mathcal{X}^{(r+\varepsilon)}) \leq d_{\mathrm{H}}(X^{(r)}, X^{(r+\varepsilon)}) + d_{\mathrm{P}}(\mu^{(r)}, \mu^{(r+\varepsilon)}).$$

Thanks to Lemma 5.1, we have $d_{\mathrm{H}}(X^{(r)}, X^{(r+\varepsilon)}) \leq \varepsilon$.

Let $A \subset X$ be closed. We have obviously $\mu^{(r)}(A) \leq \mu^{(r+\varepsilon)}(A)$. On the other hand, we have

$$\mu^{(r+\varepsilon)}(A) \leq \mu^{(r)}(A) + \mu(A \cap (X^{(r+\varepsilon)} \setminus X^{(r)})) \leq \mu^{(r)}(A) + \mu(X^{(r+\varepsilon)} \setminus X^{(r)}).$$

This proves that $d_{\mathrm{P}}(\mu^{(r)}, \mu^{(r+\varepsilon)}) \leq \mu(X^{(r+\varepsilon)} \setminus X^{(r)})$, which ends the proof. □

It is then straightforward to prove Lemma 2.8.

Proof of Lemma 2.8. Let $\mathcal{X} = (X, d^X, \varnothing^X, \mu^X)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \mu^Y)$ be two elements of $\mathbb{L}$. Using the triangle inequality twice and Lemma 5.2, we get for $r > 0$ and $\varepsilon > 0$,

$$\begin{aligned}|d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)}) - d^c_{\mathrm{GHP}}(\mathcal{X}^{(r+\varepsilon)}, \mathcal{Y}^{(r+\varepsilon)})| &\leq d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}, \mathcal{X}^{(r+\varepsilon)}) + d^c_{\mathrm{GHP}}(\mathcal{Y}^{(r)}, \mathcal{Y}^{(r+\varepsilon)}) \\ &\leq 2\varepsilon + \mu^X(X^{(r+\varepsilon)} \setminus X^{(r)}) + \mu^Y(Y^{(r+\varepsilon)} \setminus Y^{(r)}).\end{aligned}$$

As ε goes down to 0, the expression above converges to 0, so that we get right-continuity of the function $r \mapsto d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)})$.

We write $\mathcal{X}^{(r-)}$ for the compact metric space $X^{(r)}$ rooted at $\varnothing^X$ along with the induced metric and the restriction of μ to the *open* ball $\{x \in X;\ d^X(\varnothing^X, x) < r\}$. We define $\mathcal{Y}^{(r-)}$ similarly. Similar arguments as above yield, for $r > \varepsilon > 0$,

$$\begin{aligned}&|d^c_{\mathrm{GHP}}(\mathcal{X}^{(r-)}, \mathcal{Y}^{(r-)}) - d^c_{\mathrm{GHP}}(\mathcal{X}^{(r-\varepsilon)}, \mathcal{Y}^{(r-\varepsilon)})| \\ &\quad\leq d^c_{\mathrm{GHP}}(\mathcal{X}^{(r-)}, \mathcal{X}^{(r-\varepsilon)}) + d^c_{\mathrm{GHP}}(\mathcal{Y}^{(r)}, \mathcal{Y}^{(r-\varepsilon)}) \\ &\quad\leq 2\varepsilon + \mu^X(\{x \in X,\ r - \varepsilon < d^X(\varnothing^X, x) < r\}) + \mu^Y(\{y \in Y,\ r - \varepsilon < d^Y(\varnothing^Y, y) < r\}).\end{aligned}$$

As ε goes down to 0, the expression above also converges to 0, which shows the existence of left limits for the function $r \mapsto d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)})$. □

The next result corresponds to (i) in Theorem 2.9.

Proposition 5.3. *The function d_{GHP} is a metric on $\mathbb{L}$.*

Proof. The symmetry and positivity of d_{GHP} are obvious. The triangle inequality is not difficult either, since d^c_{GHP} satisfies the triangle inequality and the map $x \mapsto 1 \wedge x$ is non-decreasing and sub-additive.

We need to check that d_{GHP} is definite positive. To that effect, let $\mathcal{X} = (X, d^X, \varnothing^X, \mu)$ and $\mathcal{Y} = (Y, d^Y, \varnothing^Y, \nu)$ be two elements of $\mathbb{L}$ such that $d_{\mathrm{GHP}}(\mathcal{X}, \mathcal{Y}) = 0$. We want to prove that $\mathcal{X}$ and $\mathcal{Y}$ are GHP-isometric. We follow the spirit of the proof of Lemma 4.3.

By definition, we get that for almost every $r > 0$, $d^c_{\text{GHP}}(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)}) = 0$. Let $(r_n,\ n \geq 1)$ be a sequence such that $r_n \uparrow \infty$ and such that for $n \geq 1$, $d^c_{\text{GHP}}(\mathcal{X}^{(r_n)}, \mathcal{Y}^{(r_n)}) = 0$. Since d^c_{GHP} is a metric on $\mathbb{K}$, there exists a GHP-isometry $\Phi^n : X^{(r_n)} \to Y^{(r_n)}$ for every $n \geq 1$. Since all the $X^{(r)}$ are compact, we may consider, for $n \geq 1$ and for $k \geq 1$, a finite $1/k$-net of $X^{(r_n)}$ containing the root:

$$S^n_k = \{x^n_{0,k} = \varnothing^X, x^n_{1,k}, ..., x^n_{N^n_k,k}\}.$$

Then, if $k \geq 1$, $n \geq 1$, $0 \leq i \leq N^n_k$, the sequence $(\Phi^j(x^n_{i,k}),\ j \geq n)$ is bounded since the Φ^j are isometries. Using a diagonal procedure, we may assume without loss of generality, that for every $k \geq 1$, $n \geq 1$, $0 \leq i \leq N^n_k$, the sequence $(\Phi^j(x^n_{i,k}),\ j \geq n)$ converges to some limit $y^n_{i,k} \in Y$. We define the map Φ on $S := \bigcup_{n\geq 1,\ k\geq 1} S^n_k$ taking values in Y by

$$\Phi(x^n_{i,k}) = y^n_{i,k}.$$

Notice that Φ is an isometry and root-preserving as $\Phi(\varnothing^X) = \varnothing^Y$ (see the proof of Lemma 4.3). The set $\Phi(S^n_k)$ is obviously a $2/k$-net of $Y^{(r_n)}$, so that $\Phi(S)$ is a dense subset of Y. Therefore, the map Φ can be uniquely extended into a one-to-one root preserving isometry from X to Y, which we will still denote by Φ. It remains to prove that Φ is a GHP-isometry, that is, such that $\nu = \Phi_*\mu$.

For $n \geq 1$, $k \geq 1$, let $(V^n_{i,k},\ 0 \leq i \leq N^n_k)$ be Borel subsets of $X^{(r_n)}$ with diameter less than $1/k$, such that $\bigcup_{0\leq i\leq N_k} V^n_{i,k} = X^{(r_n)}$ and such that for all $0 \leq i, i' \leq N_k$, we have $V^n_{i,k} \bigcap V^n_{i',k} = \varnothing$ and $x^n_{i,k} \in V^n_{i,k}$ if $V^n_{i,k} \neq \varnothing$. We then define the following measures:

$$\mu^n_k = \sum_{i=0}^{N^n_k} \mu(V^n_{i,k})\delta_{x^n_{i,k}} \quad \text{and} \quad \nu^n_k = \sum_{i=0}^{N^n_k} \mu(V^n_{i,k})\delta_{y^n_{i,k}}.$$

Let $A \subset X$ be closed. We obviously have $\mu^n_k(A) \leq \mu^{(r_n)}(A^{1/k})$, and $\mu^{(r_n)}(A) \leq \mu^n_k(A^{1/k})$ that is

$$d^X_{\text{P}}(\mu^n_k, \mu^{(r_n)}) \leq \frac{1}{k}\cdot \tag{5.1}$$

For any $n \geq 1$, $k \geq 1$, we have by construction $\nu^n_k = \Phi_*\mu^n_k$ and $\nu^{(r_n)} = \Phi^j_*\mu^{(r_n)}$ for any $j \geq n \geq 1$. We can then write, for $j \geq n$,

$$\begin{aligned} d^Y_{\text{P}}(\nu^n_k, \nu^{(r_n)}) &= d^Y_{\text{P}}(\Phi_*\mu^n_k, \Phi^j_*\mu^{(r_n)}) \\ &\leq d^Y_{\text{P}}(\Phi_*\mu^n_k, \Phi^j_*\mu^n_k) + d^Y_{\text{P}}(\Phi^j_*\mu^n_k, \Phi^j_*\mu^{(r_n)}) \\ &\leq d^Y_{\text{P}}(\Phi_*\mu^n_k, \Phi^j_*\mu^n_k) + \frac{1}{k}, \end{aligned}$$

where for the last inequality we used $d^Y_{\text{P}}(\Phi^j_*\mu^n_k, \Phi^j_*\mu^{(r_n)}) = d^X_{\text{P}}(\mu^n_k, \mu^{(r_n)})$ and (5.1). Since the two measures $\Phi_*\mu^n_k$ and $\Phi^j_*\mu^n_k$ have the same masses distributed on a finite number of atoms, and the atoms $\Phi^j(x^n_{i,k})$ of $\Phi^j_*\mu^n_k$ converge towards the atoms $y^n_{i,k}$ of $\Phi_*\mu^n_k$, we deduce that

$$\lim_{j\to+\infty} d^Y_{\text{P}}(\Phi_*\mu^n_k, \Phi^j_*\mu^n_k) = 0.$$

Hence, $(\nu^n_k, k \geq 1)$ converges weakly towards $\nu^{(r_n)}$. According to (5.1), the sequence $(\mu^n_k, k \geq 1)$ converges weakly to $\mu^{(r_n)}$. Since we have $\nu^n_k = \Phi_*\mu^n_k$ and Φ is continuous, we get $\nu^{(r_n)} = \Phi_*\mu^{(r_n)}$ for any $n \geq 1$, and thus $\nu = \Phi_*\mu$. This ends the proof. □

We are now ready to prove Proposition 2.10. Note that we will not use (ii) of Theorem 2.9 in this section as it is not yet proved.

Proof of Proposition 2.10. By construction, the convergence in $\mathbb{K}\cap\mathbb{L}$ for the d_{GHP} metric implies the convergence for the d^c_{GHP} metric. We only have to prove that the converse is also true.

Let $\mathcal{X} = (X, d^X, \varnothing, \mu)$ and $\mathcal{X}_n = (X_n, d^{X_n}, \varnothing_n, \mu_n)$ be elements of $\mathbb{K}\cap\mathbb{L}$ and $(\varepsilon_n, n \in \mathbb{N})$ be a positive sequence converging towards 0 such that, for all $n \in \mathbb{N}$,

$$d^c_{\mathrm{GHP}}(\mathcal{X}_n, \mathcal{X}) < \varepsilon_n.$$

Using Lemma 4.1, we consider a metric d^n on the disjoint union $X_n \sqcup X$, such that we have for $n \in \mathbb{N}$, and writing d^n_{H} and d^n_{P} respectively for $d^{d^n}_{\mathrm{H}}$ and $d^{d^n}_{\mathrm{P}}$,

$$d^n(\varnothing_n, \varnothing) + d^n_{\mathrm{H}}(X_n, X) + d^n_{\mathrm{P}}(\mu_n, \mu) < \varepsilon_n.$$

If $x_n \in X_n^{(r)}$, by definition of the Hausdorff metric, there exists $x \in X$ such that $d^n(x_n, x) \le d^n_{\mathrm{H}}(X_n, X)$. Then

$$d^n(\varnothing, x) \le d^n(\varnothing, \varnothing_n) + d^n(\varnothing_n, x_n) + d^n(x_n, x) \le d^n(\varnothing_n, \varnothing) + r + d^n_{\mathrm{H}}(X_n, X) < r + \varepsilon_n.$$

We get that x belongs to $X^{(r+\varepsilon'_n)}$ for some $\varepsilon'_n < \varepsilon_n$ and thus, according to Lemma 5.1, it belongs to $(X^{(r)})^{\varepsilon_n}$, since X is a complete length space. Therefore we have $X_n^{(r)} \subset (X^{(r)})^{\varepsilon_n}$. Similar arguments yield $X^{(r)} \subset (X_n^{(r)})^{\varepsilon_n}$. We deduce that

$$d^n_{\mathrm{H}}(X_n^{(r)}, X^{(r)}) \le \varepsilon_n. \tag{5.2}$$

If $A \subset X_n \sqcup X$ is closed, we may compute

$$\begin{aligned} \mu_n^{(r)}(A) = \mu_n(A \cap X_n^{(r)}) &\le \mu(A^{\varepsilon_n} \cap (X_n^{(r)})^{\varepsilon_n}) + \varepsilon_n \\ &\le \mu^{(r)}(A^{\varepsilon_n}) + \mu((X_n^{(r)})^{\varepsilon_n} \setminus X^{(r)}) + \varepsilon_n \\ &\le \mu^{(r)}(A^{\varepsilon_n}) + \mu(X^{(r+2\varepsilon_n)} \setminus X^{(r)}) + \varepsilon_n, \end{aligned}$$

since $(X_n^{(r)})^{\varepsilon_n} \subset (X^{(r)})^{2\varepsilon_n} \subset X^{(r+2\varepsilon_n)}$. Similarly,

$$\begin{aligned} \mu^{(r)}(A) &\le \mu(A \cap X^{(r-2\varepsilon_n)}) + \mu(X^{(r)} \setminus X^{(r-2\varepsilon_n)}) \\ &\le \mu_n(A^{\varepsilon_n} \cap (X^{(r-2\varepsilon_n)})^{\varepsilon_n}) + \mu(X^{(r)} \setminus X^{(r-2\varepsilon_n)}) + \varepsilon_n \\ &\le \mu_n^{(r)}(A^{\varepsilon_n}) + \mu(X^{(r)} \setminus X^{(r-2\varepsilon_n)}) + \varepsilon_n, \end{aligned}$$

since $(X_n^{(r-2\varepsilon_n)})^{\varepsilon_n} \subset X^{(r)}$. Hence, we finally deduce

$$d^n_{\mathrm{P}}(\mu_n^{(r)}, \mu^{(r)}) \le \varepsilon_n + \mu(X^{(r+2\varepsilon_n)} \setminus X^{(r-2\varepsilon_n)}).$$

This and (5.2) yield

$$d^c_{\mathrm{GHP}}(\mathcal{X}_n^{(r)}, \mathcal{X}^{(r)}) \le 3 d^c_{\mathrm{GHP}}(\mathcal{X}_n, \mathcal{X}) + \mu(X^{(r+2\varepsilon_n)} \setminus X^{(r-2\varepsilon_n)}).$$

Therefore, if $\mu(\partial_r X) = 0$, we have $\lim_{n\to+\infty} d^c_{\mathrm{GHP}}(\mathcal{X}_n^{(r)}, \mathcal{X}^{(r)}) = 0$. Since μ is by definition a finite measure, the set $\{r > 0,\ \mu(\partial_r X) \neq 0\}$ is at most countable. By dominated convergence, we get $\lim_{n\to+\infty} d_{\mathrm{GHP}}(\mathcal{X}_n, \mathcal{X}) = 0$. □

In order to prove Theorem 2.11 on the pre-compactness criterion, we will approximate the elements of a sequence in $\mathcal{C}$ by nets of small radius. The following lemma guarantees that we can construct such nets in a consistent way. We use the convention that $X^{(r)} = \varnothing$ if $r < 0$. In the sequel, if $r > 0$ and $k \ge 0$, we will often use the notation $A_{r,k}(X)$ for the annulus $X^{(r)} \setminus X^{(r-2^{-k})}$.

Lemma 5.4. *If $\mathcal{X} = (X, \varnothing, d, \mu) \in \mathbb{L}$ satisfies condition (i) of Theorem 2.11, then for any $k, \ell \in \mathbb{N}$, there exists a 2^{-k}-net of the annulus $A_{\ell 2^{-k},k}(X) = X^{(\ell 2^{-k})} \setminus X^{((\ell-1)2^{-k})}$ with at most $N(\ell 2^{-k}, 2^{-k-1})$ elements.*

Proof. Let S' be a finite 2^{-k-1}-net of $X^{(\ell 2^{-k})}$ of cardinal at most $N(\ell 2^{-k}, 2^{-k-1})$. Let S'' be the set of elements x in $S' \cap A_{(\ell-1)2^{-k},k+1}(X)$ such that there exists at least one element, say y_x, in $A_{\ell 2^{-k},k}(X)$ at distance at most 2^{-k-1} of x. The set

$$\left(S' \cap A_{\ell 2^{-k},k}\right) \bigcup \{y_x,\ x \in S''\}$$

is obviously a 2^{-k}-net of $A_{\ell 2^{-k},k}(X)$, and its cardinal is bounded by $N(\ell 2^{-k}, 2^{-k-1})$. □

5.2 Proof of Theorem 2.11

Note that we will not use (ii) of Theorem 2.9 in this section as it is not yet proved.

The proof will be divided in several parts. The idea, as in [4], is to construct an abstract limit space, along with a measure, and to check that we can get a convergence (up to extraction). Let $(\mathcal{X}_n, n \in \mathbb{N})$ be a sequence in $\mathcal{C}$, with $\mathcal{X}_n = (X_n, d^{X_n}, \varnothing_n, \mu_n)$. For $\ell, k \in \mathbb{N}$, we will write ℓ_k for $\ell 2^{-k}$.

5.2.1 Construction of the limit space.

Let $\ell, k \in \mathbb{N}$. Recall that, by Lemma 5.4, we can consider $\mathfrak{S}^n_{\ell_k,k}$ a 2^{-k-1}-net of the annulus $A_{\ell_k,k}(X_n)$ with at most $N(\ell_k, 2^{-k-2})$ elements. In order to have a finer sequence of nets, we will consider

$$S^n_{\ell_k,k} = \bigcup_{0 \le k' \le k} \left(A_{\ell_k,k}(X_n) \cap \mathfrak{S}^n_{\lceil \ell_k 2^{k'} \rceil 2^{-k'},k'} \right).$$

By construction $S^n_{\ell_k,k}$ is a 2^{-k-1}-net of $A_{\ell_k,k}(X_n)$ with cardinal at most

$$\bar{N}(\ell_k, 2^{-k-2}) = \sum_{k'=0}^{k} N(\lceil \ell_k 2^{k'} \rceil 2^{-k'}, 2^{-k'-2}).$$

Let $U_{\ell_k,k} = \{(k, \ell, i); 0 \le i \le \bar{N}(\ell_k, 2^{-k-2})\}$ and $U = \bigcup_{k \in \mathbb{N}, \ell \in \mathbb{N}} U_{\ell_k,k}$. We number the elements of $S^n_{\ell_k,k}$ in such a way that

$$S^n_{\ell_k,k} \cup \{\varnothing_n\} = \{x^n_u,\ u = (k, \ell, i), u \in U_{\ell_k,k}\}, \tag{5.3}$$

where $(x^n_u, u \in U)$ is some sequence in X_n and $x^n_{(k,\ell,0)} = \varnothing_n$. Notice that $S^n_{\ell_k,k}$ is empty for ℓ_k large if X_n is bounded. For $u, u' \in U$, we set

$$d^n_{u,u'} = d^{X_n}(x^n_u, x^n_{u'}).$$

Notice that the sequence $(d^n_{u,u'}, n \in \mathbb{N})$ is bounded. Thus, without loss of generality (by considering the sequence instead of the sub-sequence), we may assume that for all $u, u' \in U$, the sequence $(d^n_{u,u'}, n \ge 1)$ converges in $\mathbb{R}$ to some limit $d_{u,u'}$. We then consider an abstract space, $X' = \{x_u, u \in U\}$. On this space, the function d defined by $(x_u, x_{u'}) \mapsto d_{u,u'}$ is a semi-metric. We then consider the quotient space $X'/\sim$, where $x_u \sim x_{u'}$ if $d_{u,u'} = 0$. We will denote by x_u the equivalent class containing x_u. Notice that $d_{u,u'} = 0$ for any $u = (k, \ell, 0)$ and $u' = (k', \ell', 0)$ elements of U and let $\varnothing$ denote their equivalence class. Finally, we let X be the completion of $X'/\sim$ with respect to the metric d, so that $(X, d, \varnothing)$ is a rooted complete metric space.

5.2.2 Approximation by nets

We set

$$\begin{aligned} U^+_{\ell_k,k} &= \bigcup_{0\leq j\leq \ell} U_{j2^{-k},k},\\ S^{n,+}_{\ell_k,k} &= \bigcup_{0\leq j\leq \ell} S^n_{j2^{-k},k} = \{x^n_u, u\in U^+_{\ell_k,k}\},\\ S^+_{\ell_k,k} &= \{x_u, u\in U^+_{\ell_k,k}\}. \end{aligned}$$

By construction $S^{n,+}_{\ell_k,k}$ is a 2^{-k-1}-net of $X_n^{(\ell_k)}$ and $S^{n,+}_{\ell_k,k}\subset S^{n,+}_{\ell'_{k'},k'}$ as well as $S^+_{\ell_k,k}\subset S^+_{\ell'_{k'},k'}$ for any $k\leq k'$ and $\ell_k\leq \ell'_{k'}$.

Remark 5.5. *Also, $v\in U\backslash U^+_{\ell_k,k}$, either $x^n_v=\varnothing_n$ or $d^{X_n}(\varnothing_n,x^n_v)>\ell_k$ and either $x_v=\varnothing$ or $d(\varnothing,x_v)\geq \ell_k$. Notice that the former inequality is strict but the latter is not.*

A correspondence R between two sets A and B is a subset of $A\times B$ such that the projection of R on A (resp. B) is A (resp. B). It is clear that the set defined by

$$\mathcal{R}^{n,+}_{\ell_k,k} = \{(x^n_u,x_u), u\in U^+_{\ell_k,k}\} \tag{5.4}$$

is a correspondence between $S^{n,+}_{\ell_k,k}$ and $S^+_{\ell_k,k}$. The distorsion $\delta_n(\ell_k,k)$ of this correspondence is defined by

$$\delta_n(\ell_k,k) = \sup\{|d^{X_n}(x^n_u,x^n_{u'}) - d(x_u,x_{u'})|;\ u,u'\in U^+_{\ell_k,k}\}. \tag{5.5}$$

Notice that for $k\leq k'$ and $\ell_k\leq \ell'_{k'}$, we have

$$\delta_n(\ell_k,k)\leq \delta_n(\ell'_{k'},k'). \tag{5.6}$$

Since $U^+_{\ell_k,k}$ is finite, for all $\ell,k\in\mathbb{N}$, we have by construction $\lim_{n\to+\infty}\delta_n(\ell_k,k)=0$.

Lemma 5.6. *The set $S^+_{\ell_k,k}$ is a 2^{-k}-net of $X^{(\ell_k)}$.*

Proof. Let $x\in X^{(\ell_k)}$. There exists $v=(k',\ell',j)\in U$ such that $d(x,x_v)<2^{-k-3}$. Notice that $d(\varnothing,x_v)<\ell_k+2^{-k-3}$. We may choose n large enough, so that $\delta_n(\ell_k\vee\ell'_{k'},k\vee k')<2^{-k-3}$. As $x^n_v\in S^{n,+}_{\ell_k\vee\ell'_{k'},k\vee k'}$, we have $|d^{X_n}(\varnothing_n,x^n_v)-d(\varnothing,x_v)|<2^{-k-3}$ and thus $d^{X_n}(\varnothing_n,x^n_v)<\ell_k+2^{-k-2}$. Thanks to Lemma 5.1 and since X_n is a length space, we get that x^n_v belongs to $(X_n^{(\ell_k)})^{2^{-k-2}}$. As $S^{n,+}_{\ell_k,k}$ is a 2^{-k-1}-net of $X_n^{(\ell_k)}$, there exists $u\in U^+_{\ell_k,k}$ such that $d^{X_n}(x^n_u,x^n_v)<2^{-k-1}+2^{-k-2}$. Furthermore, we have that x^n_u and x^n_v belongs to $S^{n,+}_{\ell_k\vee\ell'_{k'},k\vee k'}$. We deduce that

$$d(x,x_u)\leq d(x,x_v)+d(x_v,x_u)\leq 2^{-k-3}+\delta_n(\ell_k\vee\ell'_{k'},k\vee k')+d^{X_n}(x^n_u,x^n_v)<2^{-k}.$$

This gives the result. □

We give an immediate consequence of this approximation by nets.

Lemma 5.7. *The metric space (X,d) is a length space.*

Proof. The proof of this lemma is inspired by the proof of Theorem 7.3.25 in [4]. We will check that (X,d) satisfies the mid-point condition.

Let $k\in\mathbb{N}$ and $x,x'\in X$. According to Lemma 5.6, there exists $\ell\in\mathbb{N}$ large enough and $u,u'\in U^+_{\ell_k,k}$ such that $d(x,x_u)<2^{-k}$ and $d(x',x_{u'})<2^{-k}$. For n large enough, we get that $\delta_n(\ell_k,k)<2^{-k}$. Since (X_n,d^{X_n}) is a length space, there exists $z\in X_n$ such that

$$|2d^{X_n}(z,x^n_u)-d^{X_n}(x^n_u,x^n_{u'})|+|2d^{X_n}(z,x^n_{u'})-d^{X_n}(x^n_u,x^n_{u'})|\leq 2^{-k}.$$

There exists $u'' \in U^+_{\ell_k,k}$ such that $d^{X_n}(x^n_{u''}, z) \leq 2^{-k}$. Then, we deduce that

$$\begin{aligned}|2d(x_{u''},x) - d(x,x')| + &|2d(x_{u''},x') - d(x,x')| \\ &\leq 4d(x,x_u) + 4d(x',x_{u'}) + |2d(x_{u''},x_u) - d(x_u,x_{u'})| \\ &\quad + |2d(x_{u''},x_{u'}) - d(x_u,x_{u'})| \\ &\leq 8\cdot 2^{-k} + 6\delta_n(\ell_k,k) + |2d^{X_n}(x^n_{u''},x^n_u) - d^{X_n}(x^n_u,x^n_{u'})| \\ &\quad + |2d^{X_n}(x^n_{u''},x^n_{u'}) - d^{X_n}(x^n_u,x^n_{u'})| \\ &\leq 19\cdot 2^{-k}.\end{aligned}$$

Since k is arbitrary, we get that (X,d) satisfies the mid-point condition and is therefore a length space. □

5.2.3 Approximation of the measures

Let $(V^n_u, u \in U_{\ell_k,k})$ be Borel subsets of $A_{\ell_k,k}(X_n)$ with diameter less than 2^{-k} such that $\bigcup_{u\in U_{\ell_k,k}} V^n_u = A_{\ell_k,k}(X_n)$ and for all $u, u' \in U_{\ell_k,k}$, we have $V^n_u \bigcap V^n_{u'} = \varnothing$ and $x^n_u \in V^n_u$ as soon as $V^n_u \neq \varnothing$. We set $U_{\infty,k} = \bigcup_{\ell\in\mathbb{N}} U_{\ell_k,k}$ and we consider the following approximation of the measure μ_n:

$$\mu_{n,k} = \sum_{u\in U_{\infty,k}} \mu_n(V^n_u)\delta_{x^n_u}.$$

Notice that $\mu^{(\ell_k)}_{n,k} = \sum_{u\in U_{\ell_k,k}} \mu_n(V^n_u)\delta_{x^n_u}$. The measures $\mu_{n,k}$ are boundedly finite Borel measures on X_n. It is clear that the sequence $(\mu_{n,k}, k \in \mathbb{N})$ converges in the weak-# sense towards μ_n as k goes to infinity, since we have for any $r \in \mathbb{N}$, $d^{d^{X_n}}_{\mathrm{P}}(\mu^{(r)}_{n,k}, \mu^{(r)}_n) \leq 2^{-k}$. On the limit space X, we define

$$\nu_{n,k} = \sum_{u\in U_{\infty,k}} \mu_n(V^n_u)\delta_{x_u} \quad \text{and} \quad \nu^{\{\ell_k\}}_{n,k} = \sum_{u\in U_{\ell_k,k}} \mu_n(V^n_u)\delta_{x_u}.$$

Notice that $\nu^{\{\ell_k\}}_{n,k} \leq \nu^{(\ell_k)}_{n,k}$ but they may be distinct as $\nu^{(\ell_k)}_{n,k}$ may have some atoms on $\partial_{\ell_k} X$ which are in $S^+_{(\ell+1)_k,k}$ but not in $S^+_{\ell_k,k}$, as indicated in Remark 5.5.

Let us show that the sequence $(\nu_{n,k}, k \in \mathbb{N})$ converges, up to an extraction, towards a boundedly finite measure ν on X. For $m \in 2^{-k}\mathbb{N}$, we have

$$\begin{aligned}\nu_{n,k}(X^{(m)}) = \sum_{u\in U_{\infty,k}} \mu_n(V^n_u)\mathbf{1}_{\{d(x_u,\varnothing)\leq m\}} &\leq \sum_{u\in U_{\infty,k}} \mu_n(V^n_u)\mathbf{1}_{\{d^{X_n}(x^n_u,\varnothing_n)\leq m+\delta_n(m,k)\}} \\ &\leq \mu_n(X_n^{(m+\delta_n(m,k)+2^{-k})}),\end{aligned} \tag{5.7}$$

where for the first inequality we used (5.5). Recall that for all $\ell, k \in \mathbb{N}$, we have $\lim_{n\to+\infty} \delta_n(\ell_k,k) = 0$. We define $\eta_k = \delta_{n_k}(k,k)$. Using a diagonal argument, there exists a sub-sequence $(n_k, k \in \mathbb{N})$ such that

$$\eta_k \leq 2^{-k}. \tag{5.8}$$

By (5.6), we have $\delta_{n_k}(m,k) \leq \eta_k$ for $k \geq m$. Thanks to property (ii) of Theorem 2.11, we get that $\mu_{n_k}(X_{n_k})^{(m+\delta_{n_k}(m,k)+2^{-k})}$ is uniformly bounded in $k \in \mathbb{N}$ for m fixed. From the classical pre-compactness criterion for weak-# convergence of boundedly finite measures on a Polish metric space (see Appendix 2.6 of [5]), we deduce that there exists an extraction of the sub-sequence $(n_k, k \in \mathbb{N})$, which we still note $(n_k, k \in \mathbb{N})$, such that $(\nu_{n_k,k}, k \in \mathbb{N})$ converges in the weak-# sense towards some boundedly finite measure ν

on X. This implies the weak convergence of the finite measures $(\nu^{(r)}_{n_k,k}, k \in \mathbb{N})$ towards $\nu^{(r)}$ as soon as $\nu(\partial_r X) = 0$. Since ν is boundedly finite, the set

$$A_\nu = \{r \geq 0;\ \nu(\partial_r X) > 0\} \tag{5.9}$$

is at most countable. Thus, we have $\lim_{n\to+\infty} d_{\mathrm{P}}(\nu^{(r)}_{n_k,k}, \nu^{(r)}) = 0$ for almost every $r > 0$.

5.2.4 Convergence in the Gromov-Hausdorff-Prokhorov metric.

We set $\mathcal{X} = (X, d, \varnothing, \nu)$. Notice that $\mathcal{X} \in \mathbb{L}$ thanks to Lemma 5.7. We will prove that $d_{\mathrm{GHP}}(\mathcal{X}_{n_k}, \mathcal{X})$ converges to 0.

Let $r > 0$. For any $k \in \mathbb{N}$, set $\ell = \lceil 2^k r\rceil$ and recall $\ell_k = 2^{-k}\lceil 2^k r\rceil$. We set

$$\mathcal{Y}^n_k = (S^{n,+}_{\ell_k,k}, d^{X_n}, \varnothing_n, \mu^{(\ell_k)}_{n,k}), \quad \mathcal{Z}^n_k = (S^+_{\ell_k,k}, d, \varnothing, \nu^{\{\ell_k\}}_{n,k}) \quad \text{and} \quad \mathcal{W}^n_k = (X^{(\ell_k)}, d, \varnothing, \nu^{\{\ell_k\}}_{n,k}).$$

The triangle inequalities give

$$d^c_{\mathrm{GHP}}(\mathcal{X}^{(r)}_n, \mathcal{X}^{(r)}) \leq B^1_n + B^2_n + B^3_n + B^4_n + B^5_n + B^6_n, \tag{5.10}$$

with

$$B^1_n = d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(r)}_n, \mathcal{X}^{(\ell_k)}_n\right), \quad B^2_n = d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(\ell_k)}_n, \mathcal{Y}^n_k\right), \quad B^3_n = d^c_{\mathrm{GHP}}\left(\mathcal{Y}^n_k, \mathcal{Z}^n_k\right),$$
$$B^4_n = d^c_{\mathrm{GHP}}\left(\mathcal{Z}^n_k, \mathcal{W}^n_k\right), \quad B^5_n = d^c_{\mathrm{GHP}}\left(\mathcal{W}^n_k, \mathcal{X}^{(\ell_k)}\right), \quad B^6_n = d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(\ell_k)}, \mathcal{X}^{(r)}\right).$$

Lemma 5.2 implies that

$$B^1_n = d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(r)}_n, \mathcal{X}^{(\ell_k)}_n\right) \leq 2^{-k} + \mu_n(X^{(\ell_k)}_n \setminus X^{(r)}_n). \tag{5.11}$$

Since $S^{n,+}_{\ell_k,k}$ is a 2^{-k-1}-net of $X^{\ell_k}_n$ and by definition of $\mu_{n,k}$,

$$d^{d^{X_n}}_{\mathrm{H}}(X^{(\ell_k)}_n, S^{n,+}_{\ell_k,k}) \leq 2^{-k-1} \quad \text{and} \quad d^{d^{X_n}}_{\mathrm{P}}(\mu^{(\ell_k)}_n, \mu_{n,k}\mathbf{1}_{S^{n,+}_{\ell_k,k}}) \leq 2^{-k}.$$

By considering the identity map from $S^{n,+}_{\ell_k,k}$ to $X^{(\ell_k)}$, we deduce that

$$B^2_n = d^c_{\mathrm{GHP}}\left(\mathcal{X}^{(\ell_k)}_n, \mathcal{Y}^n_k\right) \leq 2^{-k+1}. \tag{5.12}$$

Recall the correspondence (5.4). It is easy to check that the function defined on $\left(S^{n,+}_{\ell_k,k} \sqcup S^+_{\ell_k,k}\right)^2$ by

$$d_n(y,z) = \begin{cases} d^{X_n}(y,z) & \text{if } y,z \in S^{n,+}_{\ell_k,k}, \\ d(y,z) & \text{if } y,z \in S^+_{\ell_k,k}, \\ \inf\{d^{X_n}(y,y') + d(z,z') + \frac{1}{2}\delta_n(\ell_k,k);\ (y',z') \in \mathcal{R}^{n,+}_{\ell_k,k}\} & \text{if } y \in S^{n,+}_{\ell_k,k}, z \in S^+_{\ell_k,k} \end{cases} \tag{5.13}$$

is a metric. For this particular metric, we easily have $d_n(\varnothing_n, \varnothing) \leq \frac{1}{2}\,\delta_n(\ell_k,k)$ as well as

$$d^{d_n}_{\mathrm{H}}(S^{n,+}_{\ell_k,k}, S^+_{\ell_k,k}) \leq \frac{1}{2}\delta_n(\ell_k,k) \quad \text{and} \quad d^{d_n}_{\mathrm{P}}(\mu^{(\ell_k)}_{n,k}, \nu^{\{\ell_k\}}_{n,k}) \leq \frac{1}{2}\delta_n(\ell_k,k).$$

We deduce that

$$B^3_n = d^c_{\mathrm{GHP}}\left(\mathcal{Y}^n_k, \mathcal{Z}^n_k\right) \leq \frac{3}{2}\delta_n(\ell_k,k). \tag{5.14}$$

Since $S^+_{\ell_k,k}$ is a 2^{-k}-net of X^{ℓ_k}, thanks to Lemma 5.6,

$$B^4_n = d^c_{\mathrm{GHP}}\left(\mathcal{Z}^n_k, \mathcal{W}^n_k\right) \leq 2^{-k}. \tag{5.15}$$

Concerning B_n^5, we only need to bound the Prokhorov distance between $\nu_{n,k}^{\{\ell_k\}}$ and $\nu_{n,k}^{(\ell_k)}$. Recall that $\nu_{n,k}^{\{\ell_k\}} \leq \nu_{n,k}^{(\ell_k)}$ and that $\nu_{n,k}^{(\ell_k)}$ may differ only on $\partial_{\ell_k} X$. If A is closed,

$$\nu_{n,k}^{\{\ell_k\}}(A) \leq \nu_{n,k}^{(\ell_k)}(A) \quad \text{and} \quad \nu_{n,k}^{(\ell_k)}(A) \leq \nu_{n,k}^{\{\ell_k\}}(A) + \nu_{n,k}(\partial_{\ell_k} X).$$

Recall (5.9). Let $\rho(r) \geq r+3$ such that $\rho(r) \notin A_\nu$ and

$$\varepsilon_{n,k} = 2 d_{\mathrm{P}}(\nu_{n,k}^{(\rho(r))}, \nu^{(\rho(r))}). \tag{5.16}$$

Since $\ell_k \leq r + 2^{-k}$,

$$\nu_{n,k}(\partial_{\ell_k} X) \leq \nu((\partial_{\ell_k} X)^{\varepsilon_{n,k}}) + \varepsilon_{n,k} \leq \nu(X^{(r+2^{-k}+\varepsilon_{n,k})} \backslash X^{(r-2\varepsilon_{n,k})}) + \varepsilon_{n,k}.$$

We deduce that

$$B_n^5 = d_{\mathrm{GHP}}^c\left(\mathcal{W}_k^n, \mathcal{X}^{(\ell_k)}\right) \leq \nu(X^{(r+2^{-k}+\varepsilon_{n,k})} \backslash X^{(r-2\varepsilon_{n,k})}) + \varepsilon_{n,k}. \tag{5.17}$$

Lemma 5.2 and the fact that X is a length space gives

$$B_n^6 = d_{\mathrm{GHP}}^c\left(\mathcal{X}^{(\ell_k)}, \mathcal{X}^{(r)}\right) \leq 2^{-k} + \nu(X^{(\ell_k)} \backslash X^{(r)}). \tag{5.18}$$

Putting (5.11), (5.12), (5.14), (5.15), (5.17), (5.18) in (5.10), we get

$$\begin{aligned} d_{\mathrm{GHP}}^c(\mathcal{X}_n^{(r)}, \mathcal{X}^{(r)}) &\leq 5 \cdot 2^{-k} + \mu_n(X_n^{(\ell_k)} \backslash X_n^{(r)}) \\ &\quad + \frac{3}{2}\delta_n(\ell_k, k) + \nu(X^{(r+2^{-k}+\varepsilon_{n,k})} \backslash X^{(r-2\varepsilon_{n,k})}) + \varepsilon_{n,k} + \nu(X^{(\ell_k)}\ X^{(r)}). \end{aligned} \tag{5.19}$$

We give a more precise upper bound for $\mu_n(X_n^{(\ell_k)} \backslash X_n^{(r)})$. Using arguments similar to those used to get (5.7), we find

$$\begin{aligned} \mu_n(X_n^{(\ell_k)} \backslash X_n^{(r)}) &\leq \mu_n(X_n^{(\ell_k)}) - \mu_n(X_n^{(\ell_k - 2^{-k})}) \\ &\leq \nu_{n,k}(X^{(\ell_k + \delta_n(\ell_k,k) + 2^{-k})}) - \nu_{n,k}(X^{(\ell_k - \delta_n(\ell_k,k) - 4\cdot 2^{-k})}). \end{aligned}$$

For $k \geq r+1$, we have $\delta_n(\ell_k, k) \leq \delta_n(k,k)$ thanks to (5.6). Then, using the sub-sequence $(n_k, k \in \mathbb{N})$ defined at the end of Section 5.2.3 with (5.8),

$$\begin{aligned} \mu_{n_k}(X_{n_k}^{(\ell_k)} \backslash X_{n_k}^{(r)}) &\leq \nu_{n_k,k}(X^{(\ell_k + 2\cdot 2^{-k})}) - \nu_{n_k,k}(X^{(\ell_k - 5\cdot 2^{-k})}) \\ &\leq \nu(X^{(\ell_k + 2\cdot 2^{-k} + \varepsilon_{n_k,k})}) - \nu(X^{(\ell_k - 5\cdot 2^{-k} - \varepsilon_{n_k,k})}) + 2\varepsilon_{n_k,k}. \end{aligned}$$

Note that the sub-sequence $(n_k, k \in \mathbb{N})$ does not depend on r: it is the same for all $r \geq 0$. Using (5.19), we get for $k \geq r+1$:

$$d_{\mathrm{GHP}}^c(\mathcal{X}_{n_k}^{(r)}, \mathcal{X}^{(r)}) \leq 5 \cdot 2^{-k} + \frac{3}{2}\eta_k + 2\nu(X^{(\ell_k + 2^{-k} + \varepsilon_{n,k})} \backslash X^{(\ell_k - 5\cdot 2^{-k} - 2\varepsilon_{n,k})}) + 3\varepsilon_{n_k,k}.$$

As $\lim_{k\to+\infty} \ell_k = r$ and $\lim_{k\to+\infty} \varepsilon_{n_k,k} = 0$, we get using (5.8), that for $r \notin A_\nu$,

$$\lim_{k\to+\infty} d_{\mathrm{GHP}}^c(\mathcal{X}_{n_k}^{(r)}, \mathcal{X}^{(r)}) = 0.$$

By dominated convergence, we get that $\lim_{k\to+\infty} d_{\mathrm{GHP}}(\mathcal{X}_{n_k}, \mathcal{X}) = 0$. Thus we have a converging sub-sequence in $\mathcal{C}$.

5.3 Proof of (ii) of Theorem 2.9

We need to prove that the metric space $(\mathbb{L}, d_{\text{GHP}})$ is separable and complete.

Lemma 5.8. *The metric space* $(\mathbb{L}, d_{GHP})$ *is separable.*

Proof. We can notice that the set $\mathbb{K} \cap \mathbb{L}$ is dense in $(\mathbb{L}, d_{\text{GHP}})$, since for $\mathcal{X} \in \mathbb{L}$ and for all $r > 0$ we have $\mathcal{X}^{(r)} \in \mathbb{K}$ and $d_{\text{GHP}}(\mathcal{X}^{(r)}, \mathcal{X}) \leq \mathrm{e}^{-r}$. Every element of $\mathbb{K}$ can be approximated in the d^c_{GHP} topology by a sequence of metric spaces with finite cardinal, rational edge-lengths and rational weights. Hence, $(\mathbb{K} \cap \mathbb{L}, d^c_{\text{GHP}})$ is separable, being a subspace of a separable metric space. According to Proposition 2.10, $(\mathbb{K} \cap \mathbb{L}, d_{\text{GHP}})$ is also separable. As $\mathbb{K} \cap \mathbb{L}$ is dense in $(\mathbb{L}, d_{\text{GHP}})$, we deduce that $(\mathbb{L}, d_{\text{GHP}})$ is separable. □

Lemma 5.9. *The metric space* $(\mathbb{L}, d_{GHP})$ *is complete.*

Proof. Let $(\mathcal{X}_n, n \in \mathbb{N})$, with $\mathcal{X}_n = (X_n, d^{X_n}, \varnothing_n, \mu_n)$, be a Cauchy sequence in $(\mathbb{L}, d_{\text{GHP}})$. It is enough to prove that it is relatively compact. Thus, we need to prove it satisfies condition (i) and (ii) of Theorem 2.11.

Assume there exists $r_0 \in \mathbb{R}_+$ such that $\sup_{n\in\mathbb{N}} \mu_n(X_n^{(r_0)}) = +\infty$. By considering a sub-sequence, we may assume that $\lim_{n\to+\infty} \mu_n(X_n^{(r_0)}) = +\infty$. This implies that for any $r \geq r_0$, $\lim_{n\to+\infty} \mu_n(X_n^{(r)}) = +\infty$. Thus, we have for any $m \in \mathbb{N}$,

$$\lim_{n\to+\infty} \int_0^{+\infty} \mathrm{e}^{-r} \ \left(1 \wedge \left|\mu_n(X_n^{(r)}) - \mu_m(X_m^{(r)})\right|\right) \ dr \geq \mathrm{e}^{-r_0}\,.$$

Then use (4.6) to get that $(\mathcal{X}_n, n \in \mathbb{N})$ is not a Cauchy sequence. Thus, if $(\mathcal{X}_n, n \in \mathbb{N})$ is a Cauchy sequence, then (ii) of Theorem 2.11 is satisfied.

Let $g_{n,m}(r) = d^c_{\text{GH}}((X_n^{(r)}, d^{X_n^{(r)}}), (X_m^{(r)}, d^{X_m^{(r)}}))$. On the one hand, use (4.7) to get

$$\lim_{\min(n,m)\to+\infty} \int_0^{+\infty} \mathrm{e}^{-r} \ (1 \wedge g_{n,m}(r)) \ dr = 0. \tag{5.20}$$

On the other hand, using (4.7) and Lemma 5.2, and arguing as in the proof of Lemma 2.8, we get that for any $r, \varepsilon \geq 0$,

$$|g_{n,m}(r) - g_{n,m}(r+\varepsilon)| \leq 2\varepsilon.$$

This implies that the functions $g_{n,m}$ are 2-Lipschitz. We deduce from (5.20), that for all $r \geq 0$, $\lim_{\min(n,m)\to+\infty} g_{n,m}(r) = 0$. Thus the sequence $((X_n^{(r)}, d^{X_n^{(r)}}), n \in \mathbb{N})$ is a Cauchy sequence for the Gromov-Hausdorff metric. Then point (2) of Proposition 7.4.11 in [4] readily implies condition (i) of Theorem 2.11. □

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