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Ergodicity of self-attracting motion*

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Abstract

We study the asymptotic behaviour of a class of self-attracting motions on $\mathbb{R}^d$. We prove the decrease of the free energy related to the system and mix it together with stochastic approximation methods. We finally obtain the (limit-quotient) ergodicity of the self-attracting diffusion with a speed of convergence.

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1 Introduction

1.1 Statement of the problem

This text is devoted to study the asymptotic behaviour of a Brownian motion, interacting with its own passed trajectory, so-called "self-interacting motion". Namely, we fix an interaction potential function $W : \mathbb{R}^d \to \mathbb{R}$, and consider the stochastic differential equation

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \left(\frac{1}{t}\int_0^t \nabla W(X_t - X_s)\,\mathrm{d}s\right)\mathrm{d}t, \tag{1.1}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion, with an initial condition of given X_0 (with the condition of continuity at $t = 0$). This equation can be rewritten using the normalized occupation measure μ_t:

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s,$$

where δ_x is the Dirac measure concentrated at the point x. Using this convention, the equation (1.1) becomes

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_t(X_t)\,\mathrm{d}t, \tag{1.2}$$

where $*$ stands for the convolution.

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Similar problems have already been studied since the 90's, for instance by Durrett and Rogers [8], or Benaïm, Ledoux and Raimond [2], initially to modelize the evolution of polymers. The first time-continuous self-interacting processes have been introduced by Durrett and Rogers [8] under the name of "Brownian polymers". They are solutions to SDEs of the form

$$\mathrm{d}X_t = \mathrm{d}B_t + \left(\int_0^t f(X_t - X_s)\,\mathrm{d}s \right) \mathrm{d}t \tag{1.3}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion and f a given function. We remark that, in the latter equation, the drift term is given by the non-normalized measure $t\mu_t$ and not by μ_t as for the process we will study here. As the process $(X_t, t \geq 0)$ evolves in an environment changing with its past trajectory, this SDE defines a self-interacting diffusion, which can be either self-repelling or self-attracting, depending on the function f. In any dimension, Durrett & Rogers obtained that the upper limit of $|X_t|/t$ does not exceed a deterministic constant whenever f has a compact support. Nevertheless, very few results are known as soon as the interaction is not self-attracting.

Self-interacting diffusions, with dependence on the (convoled) empirical measure $(\mu_t, t \geq 0)$, were first considered by Benaïm, Ledoux & Raimond [2]. A great difference between these diffusions and Brownian polymers is that the drift term is divided by t. This implies that the far away (in time) interaction is less important than the close interaction (the interaction is not "uniform in time" anymore). Benaïm et al. have shown in [2, 3] that, in a compact manifold, the asymptotic behaviour of μ_t can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. Afterwards, one can go further in this study and give sufficient conditions for the a.s. convergence of the empirical measure. It happens that, when the interaction is symmetric, μ_t converges a.s. to a local minimum of a nonlinear free energy functional (each local minimum having a positive probability to be chosen), this free energy being a Lyapunov function for the deterministic flow. Part of these results have recently been generalized to $\mathbb{R}^d$ (see [9]) assuming a confinement potential satisfying some conditions — these hypotheses being required since in general the process can be transient, and is thus very difficult to analyze. In these works, no rate of convergence is obtained. Most of the results on the topic are summarized in a recent survey of Pemantle [12], which also includes self-interacting random walks.

Coming back to the process introduced by Durrett & Rogers, most of the results obtained have in common that the drift may overcome the noise, so that the randomness of the process is "controlled". To illustrate that, let us mention, for the same model of Durrett & Rogers, the case of a repulsive and compactly supported function f, that was conjectured in [8] and has been solved very recently by Tarrès, Tóth and Valkó [16]:

Conjecture 1.1 (Durrett & Rogers [8]). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an odd function of compact support, such that $xf(x) \geq 0$. Then, for the process X defined by* (1.3)*, the quotient X_t/t converges a.s. to 0.*

In (1.1), the drift term is divided by t, and so it is bounded for a compactly supported interaction W. As for the process of the conjecture, the interaction potential is in general not strong enough for the process (1.1) to converge or to be positive recurrent, and the behaviour is then very difficult to analyze. In particular, it is hard to predict the relative importance of the drift term (in competition with the Brownian motion) in the evolution.

On the other hand, in our case of uniformly convex W, the interaction potential is attractive enough to compare the diffusion (a bit modified) to an Ornstein-Uhlenbeck process, which gives an access to its ergodic behaviour.

Another problem, related to the one considered in this paper, is the diffusion corresponding to MacKean and Vlasov's PDE. Namely, consider the Markov process defined by the SDE

$$\mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \nu_t(Y_t)\,\mathrm{d}t, \tag{1.4}$$

where ν_t stands for the *law* of Y_t, and W is a smooth strictly uniformly convex function.

The asymptotic behaviour of Y has been intensively studied these last years, by Carrillo, MacCann & Villani [5], Bolley, Guillin & Villani [4], or Cattiaux, Guillin & Malrieu [7] for instance. It turns out that, under some assumptions, the laws ν_t converge to the limit measure ν^*. This measure is characterized as a fixed point of a map $\Pi : \nu \mapsto \Pi(\nu)$ associating to a measure ν the probability measure

$$\Pi(\nu)(\mathrm{d}x) := \frac{1}{Z} e^{-W*\nu(x)}\mathrm{d}x,$$

which is the stationary measure of the process, with ν_t in the right-hand side of (1.4) replaced by ν and $Z = Z(\nu)$ is the normalization constant.

In particular, Carrillo, MacCann & Villani [5] have shown, using some mass transport tools, that the relative free energy corresponding to ν_t with respect to the limit measure ν^* decreases exponentially fast to 0. Then Talagrand's inequality allows to compare the relative free energy to the Wasserstein distance in case of uniform convexity of the interaction potential W, and so they have obtained the decrease to 0 of the quadratic Wasserstein distance between ν_t and ν^*.

We remark that a huge difference between the preceding Markov process and the (non-Markov) self-interacting diffusion is that the asymptotic σ-algebra is in general not trivial for the non-Markov process. Nevertheless, we will use a similar mass transport method to show the convergence of the empirical measure μ_t.

1.2 Main results

Our results are analogous to those of Carrillo et al. [5]: under some assumptions imposed on the interaction potential W, we show that the empirical measure μ_t almost surely converges to an equilibrium state, which is unique up to translation:

Theorem 1.2 (Main result)**.** *Suppose that $W \in C^2(\mathbb{R}^d)$ and:*

1. *spherical symmetry: $W(x) = W(|x|)$;*
2. *uniform convexity: denoting by $\mathbb{S}^{d-1}$ the $(d-1)-$dimensional sphere,*

$$\exists C_W > 0 : \quad \forall x \in \mathbb{R}^d, \forall v \in \mathbb{S}^{d-1}, \qquad \left.\frac{\partial^2 W}{\partial v^2}\right|_x \geq C_W; \tag{1.5}$$

3. *W has at most a polynomial growth: there exists some polynomial P such that*

$$\forall x \in \mathbb{R}^d \quad |W(x)| + |\nabla W(x)| + \|\nabla^2 W(x)\| \leq P(|x|); \tag{1.6}$$

Then, there exists a unique (deterministic) symmetric density $\rho_\infty : \mathbb{R}^d \to \mathbb{R}_+$, such that almost surely, there exists a random c_∞ such that

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s \xrightarrow[t\to+\infty]{*-weakly} \rho_\infty(x - c_\infty)\,\mathrm{d}x.$$

Moreover, there exists $a > 0$ such that the speed of convergence of μ_t toward $\rho_\infty(\cdot + c_\infty)$ for the Wasserstein distance is at least $\exp\{-a\sqrt[k+1]{\log t}\}$, where k is the degree of P.

Remark 1.3. *The assumption* (1) *corresponds to the physical assumption of the interaction force between two particles being directed along the line joining them, and to the third Newton's law (that is the equality between the action and the reaction forces). The symmetry assumption cannot be omitted, as shows an example in the appendix.*

Remark 1.4. *We will suppose in the following, without any loss of generality (it suffices to add a positive constant to P), that $P \geq 1$ is of degree $k \geq 2$ and such that for all $x, y \in \mathbb{R}^d$, we have $P(|x-y|) \leq P(|x|)P(|y|)$. Indeed, we choose $P(|x|) = A_1 + A_2|x|^k$, where A_1, A_2 are two positive constants large enough. This will be used in §2.2 and §2.3.*

The origin of the following remark will be clear after the discussion in §2.4.

Remark 1.5. *The density ρ_∞ is the same limit density as in the result of [5], uniquely defined (among the centered densities) by the following property: ρ_∞ is a positive function, proportional to $e^{-W*\rho_\infty}$.*

We can also consider the same drifted motion in presence of an external potential V. For this, the following result is a generalization of Theorem 1.2 (where we replace C_W by C in the notation):

Theorem 1.6. *Let X be the solution to the equation*

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \left(\nabla V(X_t) + \frac{1}{t}\int_0^t \nabla W(X_t - X_s)\mathrm{d}s\right)\mathrm{d}t. \tag{1.7}$$

Suppose, that $V \in C^2(\mathbb{R}^d)$ and $W \in C^2(\mathbb{R}^d)$, and:

1. *spherical symmetry: $W(x) = W(|x|)$;*
2. *V and W are convex, $\lim_{|x|\to+\infty} V(x) = +\infty$, and either V or W is uniformly convex:*

$$\exists C > 0: \quad \forall x \in \mathbb{R}^d, \forall v \in \mathbb{S}^{d-1},\ \left.\frac{\partial^2 V}{\partial v^2}\right|_x \geq C \quad \text{or} \quad \forall x \in \mathbb{R}^d, \forall v \in \mathbb{S}^{d-1},\ \left.\frac{\partial^2 W}{\partial v^2}\right|_x \geq C;$$

3. *V and W have at most a polynomial growth: for some polynomial P, we have $\forall x \in \mathbb{R}^d$*

$$|V(x)| + |W(x)| + |\nabla V(x)| + |\nabla W(x)| + \|\nabla^2 V(x)\| + \|\nabla^2 W(x)\| \leq P(|x|). \tag{1.8}$$

Then there exists a unique density $\rho_\infty : \mathbb{R}^d \to \mathbb{R}_+$, such that almost surely

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s \xrightarrow[t\to+\infty]{*-weakly} \rho_\infty(x)\,\mathrm{d}x.$$

As the proof of the latter Theorem coincides with the proof of Theorem 1.2 almost identically, we do not present it here. It suffices to add V in the arguments below. Moreover, if V is symmetric with respect to some point q, then the corresponding density ρ_∞ is also symmetric with respect to the same point q.

The proof of Theorem 1.2 is split into two parts. We start by introducing a natural "reference point" for a measure μ:

Definition 1.7. *Consider a measure μ on $\mathbb{R}^d$, decreasing fast enough for $W * \mu$ to be defined. The* center *of μ is the point $c_\mu = c(\mu)$ such that $\nabla W * \mu(c_\mu) = 0$, or equivalently, the point where the convolution $W * \mu$ (the potential generated by μ) takes its minimal value. Also, we define the* centered *measure μ^c as the translation of the measure μ, bringing c_μ to the origin:*

$$\mu^c(A) = \mu(A + c_\mu). \tag{1.9}$$

Remark 1.8. *This notion of center had been previously introduced by Raimond in [13], as it has been pointed out to us. Indeed, to study the linear attracting d-dimensional case of Brownian polymers, Raimond has defined the center and proved that the process remains close to $c_t = c(\mu_t)$ (and that c_t converges a.s.). The role of c_t will be slightly different here.*
A sufficient condition for the existence of the center is that W is convex, and it is unique as soon as W is strictly convex.

Remark 1.9. *The assumption* (2) *in Theorem 1.2 and the definition of the center imply that*

$$|(\nabla W * \mu_t(X_t), X_t - c_t)| \geq C_W |X_t - c_t|^2. \tag{1.10}$$

The first part of the proof of Theorem 1.2 consists in proving the convergence of the centered occupation measures:

Theorem 1.10. *Under the assumptions of Theorem 1.2, for some symmetric density function $\rho_\infty : \mathbb{R}^d \to \mathbb{R}_+$, we have almost surely*

$$\mu_t^c \xrightarrow[t\to+\infty]{*-weakly} \rho_\infty(x)\,\mathrm{d}x.$$

The second is the convergence of centers:

Theorem 1.11. *Under the assumptions of Theorem 1.2, almost surely the centers $c_t := c(\mu_t)$ converge to some (random) limit c_∞.*

It is clear that the two latter theorems imply the main result. Let us sketch their proofs.

1.3 Physical interpretation

In this part, we will explain non-rigorously the different steps needed to prove Theorem 1.2 (and so this will give a brief outline of the paper). Our main tools are the following:

- comparison of $|X_t - c_t|$ with the absolute value of an Ornstein-Uhlenbeck process,
- discretization of the process and of the dynamical system,
- decrease of the free energy.

We also give a physical interpretation, leading to the result.

1.3.1 Existence and uniqueness of X

First, a standard remark is Markovianization: the behaviour of the pair (X_t, μ_t) is Markovian. The reader will find it, together with some other standard remarks, in §2.1.1. Unfortunately, the Markov process (X_t, μ_t) is infinite-dimensional and, in general (except for the case of a polynomial interaction W), we do not manage to reduce to a finite-dimensional process. This is why we do not use this information directly in order to obtain interesting properties on μ_t, as the state space is then too large.

After these remarks, we discuss the global existence and uniqueness for the solutions of (1.2) in §2.1.4.

1.3.2 Discretization

A next step is discretization: we choose a deterministic sequence of times $T_n \to +\infty$, with $T_n \gg T_{n+1} - T_n \gg 1$, and consider the behaviour of the measures μ_{T_n}. As $T_n \gg T_{n+1} - T_n$, it is natural to expect (and we will give the corresponding statement) that the empirical measures μ_t on the interval $[T_n, T_{n+1}]$ almost do not change and thus stay close to μ_{T_n}. So we can approximate, on this interval, the solution X_t of (1.2) by the solution of the same equation with $\mu_t \equiv \mu_{T_n}$:

$$\mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_{T_n}(Y_t)\,\mathrm{d}t, \quad t \in [T_n, T_{n+1}],$$

in other words, by a Brownian motion in a potential $W * \mu_{T_n}$ that does not depend on time.

On the other hand, the series of general term $T_{n+1} - T_n$ increases. So, using Birkhoff Ergodic Theorem[1], we see that the (normalized) distribution $\mu_{[T_n, T_{n+1}]}$ of values of X_t on these intervals becomes (as n increases) close to the equilibrium measures $\Pi(\mu_{T_n})$ for a Brownian motion in the potential $W * \mu_{T_n}$, where (see §3.1)

$$\Pi(\mu)(\mathrm{d}x) := \frac{1}{Z(\mu)} e^{-W*\mu(x)}\,\mathrm{d}x, \quad Z(\mu) := \int_{\mathbb{R}^d} e^{-W*\mu(x)}\,\mathrm{d}x.$$

As

$$\mu_{T_{n+1}} = \frac{T_n}{T_{n+1}}\,\mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}}\,\mu_{[T_n, T_{n+1}]},$$

we then have

$$\mu_{T_{n+1}} \approx \frac{T_n}{T_{n+1}}\,\mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}}\,\Pi(\mu_{T_n}) = \mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}}(\Pi(\mu_{T_n}) - \mu_{T_n}),$$

and

$$\frac{\mu_{T_{n+1}} - \mu_{T_n}}{T_{n+1} - T_n} \approx \frac{1}{T_{n+1}}(\Pi(\mu_{T_n}) - \mu_{T_n}).$$

This could motivate us to approximate the behaviour of the measures μ_t by trajectories of the flow (on the infinite-dimensional space of measures)

$$\dot{\mu} = \frac{1}{t}(\Pi(\mu) - \mu), \tag{1.11}$$

or after a logarithmic change of variable $\theta = \log t$,

$$\mu' = \Pi(\mu) - \mu. \tag{1.12}$$

In fact, it is not a priori clear that the flow defined by (1.12) exists, as the space of measures is infinite-dimensional. Though the flow can be shown to be well defined on a subspace of exponentially decreasing measures, we prefer to avoid all these problems by working directly with the discretization model in §3.1. Nevertheless, this flow serves very well in motivating the considered functions and lemmas describing their behaviour, as the Euler method applied to (1.12) corresponds to the previous discretization procedure.

1.3.3 Physical interpretation: gas re-distribution

Before proceeding further, let us give a physical interpretation to the flow (1.12), predicting its asymptotic behaviour. Namely, note that a Brownian motion drifted by some potential V,

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla V(X_t)\mathrm{d}t,$$

[1] see for instance [14], chap. X

can be thought as movement of gas particles under this potential, and the stationary probability measure, $m = \frac{1}{Z(V)} e^{-V} \mathrm{d}x$, is the density with which the gas becomes distributed after some time passes. So, in dimension one, a discrete approximation of the flow (1.12) can be seen as follows. We take a tube, filled with W-interacting gas and separated in a plenty of very small cells (see Fig. 1).

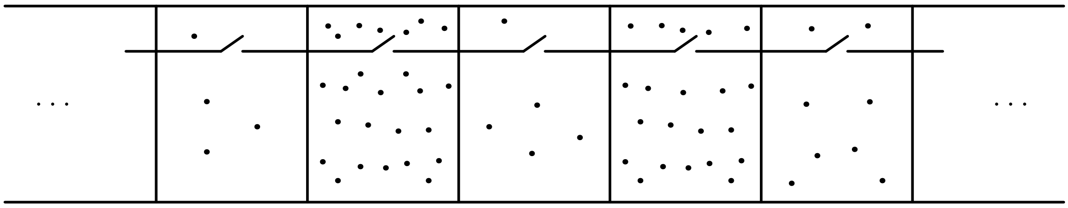

Figure 1: Gas: phase “separation”

Each unit of time, small parts (of proportion ε) of gas in these cells are separated, allowed to travel along the tube, and are proposed to equilibrate in the potential generated. This part of all the gas being small, its auto-interaction is negligible, thus their new distribution is governed by the field $V := W * \mu$ generated by the major part of the particles staying fixed to their cells. The small part is then equilibrated to its weight ε times $\Pi(\mu)$.

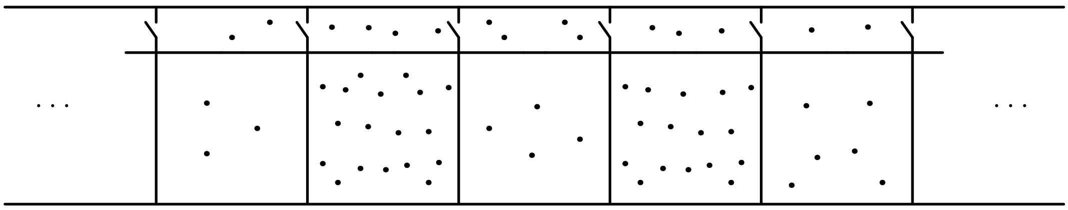

Figure 2: Gas: phase “re-distribution”

Then, it is separated again by the cells, thus the distribution after such step becomes

$$(1-\varepsilon)\mu + \varepsilon\Pi(\mu) = \mu + \varepsilon(\Pi(\mu) - \mu).$$

On the other hand, this procedure does not require any work (in the physical sense) to be done: the only actions are opening and closing the doors. So, due to the general principle, one can expect that the system will tend to its equilibrium. And a tool allowing to show that it is the case is the *free energy*, that we recall in the next paragraph.

We conclude by noticing that the same physical interpretation can be considered for the problem in any dimension d, by placing in $\mathbb{R}^{d+1}$ two close parallel walls (corresponding to the tube in dimension one), and placing the cells along them.

1.3.4 Free energy functional

A tool allowing to show the convergence of trajectories of (1.12) is the *free energy* that, due to a general physical principle, should not increase along the trajectories as long as we do not do any work.

Namely, consider an absolutely continuous probability measure $\mu = \mu(x)\mathrm{d}x$ (by an abuse of notation, we denote the measure and its density by the same letter). Imagine $\mu(x)$ as the density of a gas, particles of which implement the Brownian motion $\sqrt{2}\mathrm{d}B_t$, as well as interact with the potential $W(x-y)$. Then, one defines the *free energy* of μ as the sum of its "entropy" $\mathcal{H}$ and "potential energy":

$$\mathcal{F}(\mu) := \mathcal{H}(\mu) + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mu(x)W(x-y)\mu(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.13}$$

where the entropy of the measure μ is

$$\mathcal{H}(\mu) := \int_{\mathbb{R}^d} \mu(x)\log\mu(x)\mathrm{d}x. \tag{1.14}$$

Then, as we have already said, a general physical principle says that, as we are doing no work on the system, the free energy should decrease, and the system should tend to its minimum.

Indeed, the free energy $\mathcal{F}$ is a Lyapunov function for the flow (1.12) (when it is defined, though it is defined only for measures that are absolutely continuous with respect to the Lebesgue measure, and otherwise $\mathcal{F}(\mu) = +\infty$). This can be seen by joining two statements. First, the probability measure $m = Z(V)^{-1}e^{-V} = \Pi(\mu)$ is (what corresponds to the same physical principle) the unique global minimum of the free energy

$$\mathcal{F}_V(\mu) := \mathcal{H}(\mu) + \int_{\mathbb{R}^d} V(x)\mu(x)\,\mathrm{d}x,$$

of a non-interacting Brownian motion in the exterior potential $V = W * \mu$ (see §1.3.3 and Lemma 2.14 in §2.4). The second one is the inequality

$$\partial_{m-\mu}\mathcal{F}|_\mu \le \mathcal{F}_{W*\mu}(m) - \mathcal{F}_{W*\mu}(\mu), \tag{1.15}$$

where $m = \Pi(\mu)$. On one hand, it can be easily seen by an explicit computation, noticing that the entropy part is convex. On the other hand, such a differentiation corresponds to replacing some small parts of the gas distributed with respect to the measure μ by the one distributed with respect to the measure m, and in the right-hand side we have the corresponding free energies of these small parts in the potential, generated by the main part of the gas.

Then, differentiating the function $\mathcal{F}$ along the trajectories of the flow (1.12), one finds for the solution $\mu(\theta)$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{F}(\mu(\theta)) \le \mathcal{F}_{W*\mu(\theta)}(\Pi_{W*\mu(\theta)}(\mu(\theta))) - \mathcal{F}_{W*\mu(\theta)}(\mu(\theta)) \le 0,$$

with the equality if and only if $\mu(\theta) = \Pi(\mu(\theta))$.

Finally (and we recall these arguments in §3.1), the fixed points of Π are exactly the translation images of the density ρ_∞, that is the centered global minimum of the functional $\mathcal{F}$. So, roughly speaking, the function $\mathcal{F}$ is the Lyapunov function of the flow (1.12). The words "roughly speaking" here refer to that these arguments are non-rigorous: we avoided showing that the flow is indeed well-defined, the free energy functional is defined only for absolutely continuous measures, etc. Though all of this serves well as a motivation to (rigorous) lemmas of free energy behaviour used in this paper.

We conclude this paragraph by indicating that for the dynamics in presence of an exterior potential V (the case of Theorem 1.6) one has to replace the free energy function by

$$\mathcal{F}_{V,W}(\mu) := \mathcal{H}(\mu) + \int_{\mathbb{R}^d} V(x)\mu(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mu(x)W(x-y)\mu(y)\,\mathrm{d}x\,\mathrm{d}y.$$

and, instead of $\mathcal{F}_{W*\mu}$, consider $\mathcal{F}_{V+W*\mu}$ for the energy of "small parts".

1.3.5 Conclusion

We are now ready to conclude the sketches of the proofs of Theorems 1.10 and 1.11 (as it was already mentioned, they immediately imply Theorem 1.2).

Namely, we consider the discretized Euler-like evolution of the flow (1.11), defined by the rule

$$\tilde{\mu}_{T_{n+1}} = \tilde{\mu}_{T_n} + \frac{\Delta T_n}{T_{n+1}} (\Pi(\tilde{\mu}_{T_n}) - \tilde{\mu}_{T_n}). \tag{1.16}$$

For the measures $\tilde{\mu}_{T_n}$ defined by this procedure, we obtain (using discrete rigorous analogues of informal arguments of the previous paragraph) some estimates on the speed with which their free energy decreases. This allows us to estimate distances from these measures to the set of translates of ρ_∞ (because they are the only minima of $\mathcal{F}$).

Now, for the true random trajectory μ_t, we estimate the distance from the centered measures μ_t^c to the equilibrium point. To do this at some moment t, we choose an earlier moment t', replace the measure $\mu_{t'}$ by a close smooth measure $\tilde{\mu}_{t'}$, and consider some deterministic discrete iterates by (1.16). On one hand, for this new trajectory, the free energy is defined (as we have chosen a smooth approximation). So we control the decrease of energy and hence the distance from the centered measure $\tilde{\mu}_t^c$ to ρ_∞. On the other hand, an accurate computation allows us to control the distance between the random measure μ_t and the approximating deterministic image $\tilde{\mu}_t$ of its smooth perturbation. The sum of these distances then estimates (gives an upper bound) the distance from μ_t^c to ρ_∞, and the obtained estimate tends to 0 as $t \to +\infty$. This concludes the proof of Theorem 1.10.

Finally, to prove Theorem 1.11, one first computes the speed of drift of the center c_t, and then shows that the series of general term $|c_{T_{n+1}} - c_{T_n}|$ converges, and the oscillations $osc_{[T_n,T_{n+1}]} c_t$ tend to zero. This implies the existence of the limit of c_t as $t \to +\infty$.

1.4 Outline

To conclude this introduction, we indicate how the rest of this paper is organized.

At the beginning of Section 2, we show the existence and uniqueness of solutions to (1.2) starting at any positive moment $r > 0$. After that, the rest of Section 2 is devoted to the presentation of some crucial preliminary computations which are at the basis of our proofs. Indeed, we will compare the centered process $(|X_t - c_t|, t \geq 0)$ with the absolute value of an Ornstein-Uhlenbeck process. Then, we will show that the map Π, restricted to a well-chosen subset of probability measures, is Lipschitz. These results are essential to prove our main theorem. We end Section 2 by introducing a new transport metric, similar to the Wasserstein distance (so that the space of probability measures equipped with the weak* topology is complete), but allowing to control the integration of W with respect to any probability measure. We also introduce the free energy functional corresponding to our process. Most of the material there is not new, except for the combination of stochastic approximation of the empirical measure (see [2]) with free energy functionals (see [5]) and the achieving of a bound on the convergence rate.

Section 3 consists in the proofs of our main results. Let us now describe the main steps of the proof of Theorem 1.10, which is postponed to §3.1. Actually, instead of proving this result, we will show the following stronger statement (which is actually named Theorem 3.1 and located in Section 3.1):

Theorem 1.12. *There exists a constant $a > 0$ such that almost surely, we have for t large enough $W_2(\mu_t^c, \rho_\infty) = O(\exp\{-a \sqrt[k+1]{\log t}\})$, where k is the degree of the polynomial*

P and W_2 is the quadratic Wasserstein distance (defined in §2.3), that is the minimal L^2-distance taken over all the couplings between μ_t^c and ρ_∞.

Note that this result gives in particular a minoration of the speed of convergence of the centered empirical measure μ_t^c toward ρ_∞. For a better understanding, we will decompose the proof of this statement into some intermediate propositions. Remark that all the probability measures considered here are centered (with respect to the same center). Let us now explain the strategy of the proof.
First, note that the empirical measure μ_t of X_t evolves very slowly, as it has been explained in §1.3.2. Indeed, let us choose an appropriate interval $[T_n, T_{n+1})$. On this interval, fix the empirical measure μ_t at μ_{T_n}. We then construct a new process Y, coupled with X (the coupling is such that X and Y are driven by the same Brownian motion), such that for all $t \in [T_n, T_{n+1})$, we have

$$\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t - \nabla W * \mu_{T_n}(Y_t)\mathrm{d}t.$$

This new process has two advantages. First, it is Markovian (and its invariant probability measure is $\Pi(\mu_{T_n})(\mathrm{d}x) = \frac{1}{Z}e^{-W*\mu_{T_n}(x)}\mathrm{d}x$), and so is easier than X to study. Second, its evolution is very close to the evolution of the desired X. Indeed, we will use Y to prove Proposition 3.2, asserting that the transport distance between the empirical measure on $[T_n, T_{n+1}]$, denoted by $\mu_{[T_n,T_{n+1}]}$, and the probability measure $\Pi(\mu_{T_n})$ (both measures being centered in c_{T_n}) is controlled by $T_n^{-\frac{1}{3}\min(8C_W, 1/5d)}$ and so, this distance vanishes as $n \to +\infty$. This will be done in §3.1.1.
After that, we also remark that if a.s. the empirical measure μ_t converges weakly* to μ_∞, then for t large enough, the process X shall be very close to Z, defined by

$$\mathrm{d}Z_t = \sqrt{2}\mathrm{d}B_t - \nabla W * \mu_\infty(Z_t)\mathrm{d}t.$$

The process Z is obviously Markovian and the limit-quotient theorem applies (see [14]):

$$\frac{1}{t}\int_0^t \delta_{Z_s}\mathrm{d}s \underset{t\to+\infty}{\longrightarrow} \Pi(\mu_\infty) \quad \text{a.s.}$$

for the weak* convergence of measures. So when the limit μ_∞ exists, it satisfies $\mu_\infty = \Pi(\mu_\infty)$. This explains, in a slightly different way of §1.3.2, the idea of introducing the dynamical system $\dot{\mu} = \Pi(\mu) - \mu$ (after the time-shift $t \mapsto e^t$ in order to work with a time-homogeneous system) defined on the set of probability measures that are integrable for the polynomial P. As noticed previously, instead of considering the latter dynamical system, we will work with its discretized version, with the knots chosen at the moments T_n. We will then prove, in Proposition 3.5, that the transport distance between the deterministic trajectory induced by the smoothened (discrete) dynamical system and the (centered) random trajectory μ_{T_n} is controlled and decreases to 0. This will be done in §3.1.2.
Next, it remains to show that the free energy (defined in §2.4) between this (centered) deterministic trajectory and the set of translates of ρ_∞ goes to 0. As the free energy is controlled by the quadratic Wasserstein distance W_2, this implies that the transport distance between the two previous quantities decreases, as asserted in Proposition 3.6. The §3.1.3 is devoted to the proof of this result.
To conclude, we only have to put all the pieces together and use the triangle inequality: $W_2(\mu_t^c, \rho_\infty)$ is upper bounded by the sum of three distances, involving the flow Φ_n induced by the discretization of the dynamical system $\dot{\mu} = \Pi(\mu) - \mu$ on the interval $[T_n, T_{n+1})$, for n large enough. The first term of the summation bound will be $W_2(\mu_t^c, \Pi(\mu_{T_n}^c))$, the second one $W_2(\Pi(\mu_{T_n}^c), \Phi_n^n(\mu_{T_n}^c))$ and the third one $W_2(\Phi_n^n(\mu_{T_n}^c), \rho_\infty)$.

Next, Section 3.2 presents the proof of Theorem 1.11. Indeed, the previous decrease estimates will allow us to show the convergence of the center, after having made the appropriate choice $T_n = n^{3/2}$.

Finally, we have gathered in two appendices:

1. a discussion of the existence and uniqueness of solution to (1.2) starting at $r = 0$,
2. a counter-example, showing the need of the symmetry for the (convex) potential W in order to obtain the convergence of the center. (Indeed, without this hypothesis on W, the convergence of the centered empirical measure still holds true.)

2 Preliminaries

As usual, we denote by $\mathcal{M}(\mathbb{R}^d)$ the space of signed (bounded) Borel measures on $\mathbb{R}^d$ and by $\mathcal{P}(\mathbb{R}^d)$ its subspace of probability measures. We will need the following measure space:

$$\mathcal{M}(\mathbb{R}^d; P) := \{\mu \in \mathcal{M}(\mathbb{R}^d); \int_{\mathbb{R}^d} P(|y|)\,|\mu|(\mathrm{d}y) < +\infty\}, \tag{2.1}$$

where $|\mu|$ is the variation of μ (that is $|\mu| := \mu^+ + \mu^-$ with (μ^+, μ^-) the Hahn-Jordan decomposition of μ: $\mu = \mu^+ - \mu^-$). Belonging to this space will enable us to always check the integrability of P (and therefore of W and its derivatives thanks to the domination condition (1.6)) with respect to the (random) measures to be considered. We endow this space with the dual weighted supremum norm (or dual P-norm) defined for $\mu \in \mathcal{M}(\mathbb{R}^d; P)$ by

$$||\mu||_P := \sup_{\varphi \in \mathcal{C}(\mathbb{R}^d); |\varphi| \leq P} \left| \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu \right| = \int_{\mathbb{R}^d} P(|y|)\,|\mu|(\mathrm{d}y), \tag{2.2}$$

where $\mathcal{C}(\mathbb{R}^d)$ is the set of continuous functions $\mathbb{R}^d \to \mathbb{R}$. We recall that $P(|x|) \geq 1$, so that $\|\mu\|_P \geq |\mu(\mathbb{R}^d)|$. This norm naturally arises in the approach to ergodic results for time-continuous Markov processes of Meyn & Tweedie [11]. It also makes $\mathcal{M}(\mathbb{R}^d; P)$ a Banach space.
Next, we consider $\mathcal{P}(\mathbb{R}^d; P) = \mathcal{M}(\mathbb{R}^d; P) \cap \mathcal{P}(\mathbb{R}^d)$. We remark that both $\mathcal{M}(\mathbb{R}^d; P)$ and $\mathcal{P}(\mathbb{R}^d; P)$ contain any probability measure with an exponential tail and, in particular, any compactly supported measure. For any $\kappa > 0$, we also define

$$\mathcal{P}_\kappa(\mathbb{R}^d; P) := \{\mu \in \mathcal{P}(\mathbb{R}^d; P)\,;\, ||\mu||_P = \int_{\mathbb{R}^d} P(|x|)\mu(\mathrm{d}x) \leq \kappa\}. \tag{2.3}$$

2.1 Existence and uniqueness of solutions

2.1.1 Markovian form; local existence and uniqueness

First step in studying the trajectories of (1.2) is to pass to the couple (X_t, μ_t). A standard remark is that the behaviour of this couple is infinite-dimensional Markovian (and in general, except for W being polynomial, cannot be reduced to a finite-dimensional Markov process). This reduction is easily implied by the identity

$$\mu_{t+s} = \frac{t}{t+s}\mu_t + \frac{1}{t+s}\int_t^{t+s} \delta_{X_u}\,\mathrm{d}u. \tag{2.4}$$

Note that the second term in the right-hand side of (2.4) can be written as $\frac{s}{t+s}\mu_{[t,t+s]}$, where $\mu_{[t,t+s]}$ is the empirical measure during the time interval $[t, t+s]$:

$$\mu_{[t_1,t_2]} := \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \delta_{X_u}\,\mathrm{d}u.$$

Now, passing μ_t to the left-hand side of (2.4), dividing by s and passing to the limit as $s \to 0$, we obtain the following SDE for the couple (X_t, μ_t):

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_t(X_t)\,\mathrm{d}t, \\ \dot{\mu}_t = \frac{1}{t}(-\mu_t + \delta_{X_t}). \end{cases} \tag{2.5}$$

For any $t_0 > 0$, the local existence and uniqueness of solutions to (2.5), in a neighbourhood of t_0, is implied by well-known arguments: see Theorem 11.2 of [15].

However, in order to study the asymptotic behaviour of solutions to (1.2), we should first show the global existence of these solutions, in other words, that they do not explode in a finite time. This will be done in §2.1.2.

Note also that the equation (2.5) clearly has a singularity at $t = 0$. To avoid this singularity, sometimes the equation (2.5) is considered with an initial condition (X_r, μ_r) at some positive time $r > 0$ (and thus for $t \in [r, +\infty)$). The case $r = 0$ is studied in the appendix. After the time-shift $s = t - r$, the system (2.5) transforms to

$$\begin{cases} \mathrm{d}X_s = \sqrt{2}\,\mathrm{d}B_s - \nabla W * \mu_s(X_s)\,\mathrm{d}s, \\ \dot{\mu}_s = \frac{1}{s+r}(-\mu_s + \delta_{X_s}). \end{cases} \tag{2.6}$$

In fact, we can restrict our consideration to such situations only (as, anyway, we are investigating the asymptotic behaviour of solutions at infinity), but it is interesting to show that the equation (1.2) has indeed existence and uniqueness of solutions for any initial value problem $X_0 = x_0$. It is done in the appendix.

2.1.2 Center-drift estimates

A natural "reference point" that one can associate to a measure μ is the equilibrium point $c_\mu = c(\mu)$ of the potential it generates with W, defined by the equation $\nabla W * \mu(c_\mu) = 0$ (see Definition 1.7, §1.2), that we refer to as the *center* of the measure μ. Also, it will be convenient to consider the *centered measure* μ^c, obtained from μ by the translation that shifts the center to the origin.

Note that the implicit function theorem allows to estimate (on an interval of existence of solution (X_t, μ_t) to (2.5)) the derivative $\dot{c}_t$ of $c_t := c_{\mu_t}$. In particular, we will see that c_t is a function of class C^1 on this interval. Indeed, the function $(x,t) \mapsto \nabla W * \mu_t(x)$ is C^1-smooth:

$$\begin{aligned} \mathrm{d}(\nabla W * \mu_t)(x) &= \nabla^2 W * \mu_t(x)\,\mathrm{d}x + \nabla W * \dot{\mu}_t(x)\,\mathrm{d}t \\ &= \nabla^2 W * \mu_t(x)\,\mathrm{d}x + \frac{1}{t}\nabla W * (-\mu_t + \delta_{X_t})(x)\,\mathrm{d}t, \end{aligned}$$

and for any (x,t) we have $\nabla^2 W * \mu_t(x) \geq C_W I > 0$. The implicit function theorem thus implies that c_t is a function of t of class C^1 (on the interval of existence of solution), and that

$$\begin{aligned} \dot{c}_t &= -\left(\frac{\partial}{\partial x}\nabla W * \mu_t(x)\mid_{x=c_t}\right)^{-1} \frac{\partial}{\partial t}(\nabla W * \mu_t)(c_t) = -\frac{1}{t}\left(\nabla^2 W * \mu_t(c_t)\right)^{-1} \nabla W * \delta_{X_t}(c_t) \\ &= \frac{1}{t}\left(\nabla^2 W * \mu_t(c_t)\right)^{-1} \nabla W(X_t - c_t). \end{aligned}$$

This implies that the projection of the center drift velocity on the line from c_t to X_t is directed towards X_t, as $\nabla W(X_t - c_t)$ is positively proportional to $X_t - c_t$ and

$$\left(\left(\nabla^2 W * \mu_t(c_t)\right)^{-1} \nabla W(X_t - c_t), X_t - c_t\right) > 0.$$

This also immediately gives an upper bound on the drift speed:

$$|\dot{c}_t| \leq \frac{1}{t} \cdot \frac{P(|X_t - c_t|)}{C_W}. \tag{2.7}$$

2.1.3 Law of X-center distances: Ornstein-Uhlenbeck estimate

To continue our study, first we would like to obtain an estimate on the behaviour of the distance $|X_t - c_t|$. Namely, we are going to compare it with (the absolute value of) the Ornstein-Uhlenbeck process, and to obtain exponential-decrease bounds on its occupation measure in §2.2.1.

Proposition 2.1. *The process (X_t) can be considered as the first element of the pair (X_t, Z_t) of processes such that*

i) $|X_t - c_t| \le 2 + Z_t$,

ii) Z_t *is the absolute value of a* $3d$*-dimensional Ornstein-Uhlenbeck process.*

Proof. From

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla W * \mu_t(X_t)\,\mathrm{d}t \\ \dot{c}_t = \frac{1}{t}\left(\nabla^2 W * \mu_t(c_t)\right)^{-1} \nabla W(X_t - c_t) \end{cases}$$

one obtains that the difference $|X_t - c_t|$, while it is positive, satisfies the SDE

$$\begin{aligned} \mathrm{d}|X_t - c_t| &= \sqrt{2}\left(\frac{X_t - c_t}{|X_t - c_t|}, \mathrm{d}B_t\right) + \frac{d-1}{|X_t - c_t|}\mathrm{d}t \\ &- \left(\frac{X_t - c_t}{|X_t - c_t|}, \nabla W * \mu_t(X_t) + \frac{1}{t}(\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(X_t - c_t)\right)\mathrm{d}t. \end{aligned}$$

In the same way, we define the desired Z_t, which shall satisfy the equation

$$\mathrm{d}Z_t = \sqrt{2}\mathrm{d}\gamma_t - \left(\frac{C_W}{2} Z_t - \frac{3d-1}{Z_t}\right)\mathrm{d}t, \tag{2.8}$$

where γ is also a Brownian motion. Take a one-dimensional standard Brownian motion β independent of the Brownian motion B and let γ be defined as

$$\mathrm{d}\gamma_t = \alpha(|X_t - c_t|)\left(\frac{X_t - c_t}{|X_t - c_t|}, \mathrm{d}B_t\right) + \sqrt{1 - \alpha^2(|X_t - c_t|)}\mathrm{d}\beta_t, \tag{2.9}$$

where $\alpha : [0, +\infty) \to [0, 1]$ is a C^∞-function which is identically zero in some neighbourhood of 0 and $\alpha(r) = 1$ for any $r \ge 1$. The process Z is then defined by (2.8).

We point out that, as B and β are independent, B is a d-dimensional Brownian motion while β is 1-dimensional. It follows (by Itô's formula) that Z defined by (2.8) is the absolute value of a $3d$-dimensional Ornstein-Uhlenbeck process.

On the other hand, for any t, either $|X_t - c_t| \le 2 + Z_t$ (and there is nothing else to do), or $|X_t - c_t| > 2 + Z_t$ and then both $|X_t - c_t|$ and Z_t share exactly the same Brownian component (as $\alpha \equiv 1$), with the inequality between the drift terms of $2 + Z_t$ and $|X_t - c_t|$:

$$\begin{aligned} -\frac{C_W}{2} Z_t + \frac{3d-1}{Z_t} &\ge -C_W|X_t - c_t| + \frac{d-1}{|X_t - c_t|} \ge \\ &\ge -\left(\nabla W * \mu_t|_{X_t}, \frac{X_t - c_t}{|X_t - c_t|}\right) + \frac{d-1}{|X_t - c_t|} - \\ &\quad - \left(\frac{1}{t}(\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(X_t - c_t), \frac{X_t - c_t}{|X_t - c_t|}\right), \end{aligned} \tag{2.10}$$

because $|X_t - c_t| \ge \frac{d-1}{3d-1} Z_t$. A comparison theorem concludes the proof. □

2.1.4 Global existence

Proposition 2.2. *For any $r > 0$ and for any initial condition (X_r, μ_r), the solution to* (2.5) *exists (and is unique) on the whole interval* $[r, +\infty)$.

Proof. As we already have the local existence and uniqueness, it suffices to check that the solution X_t cannot explode in a finite time (this impossibility will imply that the measures μ_t, as the normalized occupation measures of X_t, also stay in a compact domain –for the P-norm– for any bounded interval of time).

Let us introduce the increasing sequence of stopping times $\tau_0 = 0$ and

$$\tau_n := \inf\{t \geq \tau_{n-1} : |X_t| > n\}.$$

We use the comparison of $X_t - c_t$ with the Ornstein-Uhlenbeck process Z_t (see §2.1.3):

$$|X_{\min(t,\tau_n)} - c_{\min(t,\tau_n)}| \leq 2 + Z_{\min(t,\tau_n)}.$$

As Z is globally defined for any bounded interval of time, letting n go to infinity, we deduce that $X_t - c_t$ does not explode in a finite time. To conclude, we use the inequality (2.7) to show the global existence of c_t. We have:

$$|\dot{c}_t| \leq \frac{1}{t}\frac{P(|X_t - c_t|)}{C_W} \leq \frac{1}{t}\frac{P(2 + Z_t)}{C_W} \leq \frac{1}{t}\frac{P(2)}{C_W}P(Z_t).$$

Any trajectory of Z being bounded on any finite interval of time, the integral $\int_r^t \frac{P(Z_s)}{s}\mathrm{d}s$ is finite for any $t \geq r$. So, there exists a global strong solution $(X_t, t \geq 0)$. ☐

2.2 Exponential tails estimates

2.2.1 Estimates for the centered empirical measure

We shall now estimate the behaviour of the centered measures μ_t^c. Namely, we are going to prove that these measures are exponentially decreasing. For shortness and simplicity, we introduce the following sets

Definition 2.3. *Let $\alpha, C > 0$ be given. Define*

$$K_{\alpha,C}^0 := \{\mu \in \mathcal{P}(\mathbb{R}^d); \quad \forall r > 0,\, \mu(\{y; |y| > r\}) < Ce^{-\alpha r}\}, \tag{2.11}$$

$$K_{\alpha,C} := \{\mu \in \mathcal{P}(\mathbb{R}^d); \quad \mu^c \in K_{\alpha,C}^0\}. \tag{2.12}$$

Also, for general positive measures, we denote the spaces defined by the same relations by $\tilde{K}_{\alpha,C}^0$ and $\tilde{K}_{\alpha,C}$.

For the following, we need one easy lemma, that will be useful to show the exponential decrease of μ_t.

Lemma 2.4. *Let Z be the absolute value of the $3d$-dimensional Ornstein-Uhlenbeck process with Gaussian stationary measure $\mathrm{d}\gamma_{OU}(x) = e^{-C_W|x|^2/2}\mathrm{d}x$. Then, there exists $C_1 > 0$, such that for almost any trajectory Z_t, one has almost surely*

$$\exists T : \forall t \geq T,\, \forall r > 0 \quad \frac{1}{t}|\{s \leq t : Z_s > r\}| < C_1 e^{-r}.$$

Proof. Note that the function $f(x) = e^{|x|}$ is γ_{OU}-integrable. Hence, by the limit quotient (ergodic) theorem, we have almost surely when $t \to \infty$:

$$\frac{1}{t}\int_0^t f(Z_s)\mathrm{d}s \to \int f(x)\mathrm{d}\gamma_{OU}(x) =: I.$$

Thus for all t large enough, $\frac{1}{t}\int_0^t e^{|Z_s|}\mathrm{d}s \le I+1$. Applying Chebychev's inequality, we see that for all $r>0$,

$$\frac{1}{t}\left|\{s\le t : Z_s > r\}\right| < (I+1)e^{-r}. \quad \square$$

The main result of this subsection is the following, showing that the measure μ_t belongs to the set $K_{\alpha,C}$.

Proposition 2.5. *There exist two constants $\alpha, C > 0$ such that a.s. at any sufficiently large time t, we have $\mu_t \in K_{\alpha,C}$.*

To prove this proposition, we need two intermediate lemmas, whose proofs are postponed.

Lemma 2.6. *There exist $\alpha_0, C_0 > 0$ such that a.s. for any sufficiently large time t, we have $\mu_{[t/2,t]}(\cdot + c_{t/2}) \in K^0_{\alpha_0,C_0}$.*

Lemma 2.7. *Let $\alpha_0, C_0 > 0$ be fixed. Then there exist $\alpha, C > 0$ such that the following holds. For any given coefficients $0 < \lambda \le 1/2$ and $\mu \in \mathcal{P}(\mathbb{R}^d; P)$, let η and ν be two probability measures such that $\eta(\cdot + c_\mu) \in K^0_{\alpha,C}$ and $\nu(\cdot + c_\mu) \in K^0_{\alpha_0,C_0}$. Letting $\tilde{\mu} = (1-\lambda)\mu + \lambda\nu$, we have*

$$((1-\lambda)\eta + \lambda\nu)(\cdot + c_{\tilde{\mu}}) \in K^0_{\alpha,C}.$$

In other words, this lemma provides an “induction step" for showing that “a big part of the centered measure has exponentially small tails" for a procedure of repetitive mixing with measure having exponential tails (this is not obvious, as the center is shifted by such a procedure).

Proof of Proposition 2.5. First, let us estimate the drift of the center. Namely, taking together (2.7) and Proposition 2.1, we have

$$|\dot{c}_t| \le \frac{1}{tC_W}P(|X_t - c_t|) \le \frac{1}{tC_W}P(2+Z_t) \le \frac{P(2)}{tC_W}P(Z_t),$$

for the corresponding Ornstein-Uhlenbeck trajectory Z_t.

On the other hand, Z is a Harris recurrent process and $P(Z)$ is integrable with respect to the Gaussian measure, thus due to the limit-quotient (or Birkhoff) theorem, almost surely there exists a limit

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t P(Z_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} P(|z|)\,\mathrm{d}\gamma_{OU}(z) =: I.$$

So, almost surely from some time t_1 we have

$$\forall t > t_1, \quad \frac{1}{t}\int_0^t P(Z_s)\,\mathrm{d}s \le I+1.$$

Therefore, after this time we can estimate the displacement of the center between the moments $t/2$ and t: $\forall t > t_1$

$$|c_{t/2} - c_t| \le \int_{t/2}^t |\dot{c}_s|\,\mathrm{d}s \le \int_{t/2}^t \frac{C}{s}P(Z_s)\,\mathrm{d}s \le \frac{C}{t/2}\int_0^t P(Z_s)\,\mathrm{d}s \le 2C(I+1) =: C_3.$$

In fact, the same estimate holds for any t' between $t/2$ and t:

$$|c_{t'} - c_t| \le C_3.$$

This immediately implies that for any $t > t_1$ and $n \in \mathbb{N}$ such that $2^{-n+1}t > t_1$, one has

$$|c_t - c_{t/2^n}| \leq C_3 n.$$

Let us now apply Lemma 2.7. First let us decompose, for any $t \in [t_1, 2t_1]$, the measure μ_{2t} as $\frac{1}{2}\mu_t + \frac{1}{2}\mu_{[t,2t]}$, then the measure μ_{4t} as $\frac{1}{4}\mu_t + \left(\frac{1}{4}\mu_{[t,2t]} + \frac{1}{2}\mu_{[2t,4t]}\right)$, ..., and finally the measure $\mu_{2^n t}$ as $\frac{1}{2^n}\mu_t + \left(\frac{1}{2^n}\mu_{[t,2t]} + \cdots + \frac{1}{2}\mu_{[2^{n-1}t,2^n t]}\right)$. An induction argument, together with Lemma 2.6, immediately shows that in each such decomposition, the second term shifted by the corresponding $c(\mu_{2^j t})$ belongs to $\tilde{K}^0_{\alpha,C}$. The only part left to handle is $\frac{1}{2^n}\mu_t$. But the distance between c_t and $c_{2^n t}$ does not exceed $C_3 n$, and the centered measure μ_t^c is compactly supported. So it is contained in a ball of some (random) radius R that can be chosen uniform over $t \in (t_1, 2t_1)$. Now the measure $\frac{1}{2^n}\mu_t$ is of total weight 2^{-n} and it vanishes outside a ball of radius R. If α is small enough so that $e^{\alpha C_3} < 2$, then for any $r > C_3 n + R$, we have

$$\frac{1}{2^n}\mu_t(|y - c_{2^n t}| > r) \leq \frac{1}{2^n}\mu_t^c(|y| > r - C_3 n) = 0,$$

and for $r \leq C_3 n + R$ and n big enough,

$$\frac{1}{2^n}\mu_t(|y - c_{2^n t}| > r) \leq 2^{-n} < e^{-n\alpha C_3} e^{-\alpha R} \leq e^{-\alpha r}.$$

The middle inequality comes, for n large enough, from a comparison between exponent bases, $e^{\alpha C_3} < 2$, with respect to which a multiplication constant $e^{-\alpha R}$ is minor. Finally, joining the obtained $\frac{1}{2^n}\mu_t(\cdot + c_{2^n t}) \in \tilde{K}^0_{\alpha,1}$ and $\left(\frac{1}{2^n}\mu_{[t,2t]} + \cdots + \frac{1}{2}\mu_{[2^{n-1}t,2^n t]}\right)(\cdot + c_{2^n t}) \in \tilde{K}^0_{\alpha,C}$, we obtain $\mu_{2^n t} \in K_{\alpha,C+1}$. □

Proof of Lemma 2.6. This lemma immediately follows from Lemma 2.4, once we notice that

$$\begin{aligned}
\mu_{[t/2,t]}(|y - c_{t/2}| > r) &= \frac{2}{t}\left|\{s : t/2 < s < t, |X_s - c_{t/2}| > r\}\right| \\
&\leq \frac{2}{t}\left|\{s : t/2 < s < t, |X_s - c_s| > r - |c_{t/2} - c_s|\}\right| \\
&\leq \frac{2}{t}\left|\{s : s < t, Z_s > r - C_3\}\right| \leq C_0 e^{\alpha_0 C_3} \cdot e^{-\alpha_0 r}. \quad \square
\end{aligned}$$

Proof of Lemma 2.7. First, let us estimate the position of the center of $\tilde{\mu}$ in a way that is linear in λ and does not depend on α and C. Indeed, we recall that $c_{\tilde{\mu}}$ is the minimum of the function $W * \tilde{\mu}$. So there exists a constant $C' > 0$ such that, at the point c_μ, the gradient of this function can be bounded as

$$\left|\nabla W * \tilde{\mu}|_{c_\mu}\right| = \left|(1-\lambda)\nabla W * \mu|_{c_\mu} + \lambda \nabla W * \nu|_{c_\mu}\right| \leq \lambda \|\nu(\cdot + c_\mu)\|_P \leq C'\lambda,$$

because the norm $\|\nu(\cdot + c_\mu)\|_P$ is uniformly bounded due to the condition $\nu(\cdot + c_\mu) \in K^0_{\alpha_0,C_0}$.

Now, restricting the function $W * \tilde{\mu}$ to the line joining c_μ and $c_{\tilde{\mu}}$, that is considering

$$f(s) = W * \tilde{\mu}\left(c_\mu + s\frac{c_{\tilde{\mu}} - c_\mu}{|c_{\tilde{\mu}} - c_\mu|}\right),$$

one sees that $|f'(0)| \leq C'\lambda$, $f'(|c_{\tilde{\mu}} - c_\mu|) = 0$ and $f'' \geq C_W$, which implies $|c_{\tilde{\mu}} - c_\mu| \leq \frac{C'}{C_W}\lambda$.

Let us now estimate the measure $((1-\lambda)\eta + \lambda\nu)(|y - c_{\tilde{\mu}}| \geq r)$. Indeed, first note that $\{y : |y - c_{\tilde{\mu}}| \geq r\} \subset \{y : |y - c_\mu| \geq r - C''\lambda\}$, where $C'' = C'/C_W$. Thus, we have by definition of $\tilde{\mu}$

$$\begin{aligned}\tilde{\mu}(|y-c(\tilde{\mu})|\geq r) &\leq \tilde{\mu}(|y-c(\mu)|\geq r-C''\lambda)\\ &\leq (1-\lambda)\eta(|y-c(\mu)|\geq r-C''\lambda)+\lambda\nu(|y|\geq r-C''\lambda)\\ &\leq (1-\lambda)Ce^{-\alpha(r-C''\lambda)}+\lambda C_0e^{-\alpha_0(r-C''\lambda)}\\ &\leq \left(1-\frac{\lambda}{2}\right)Ce^{C''\alpha\lambda-\alpha r}-\lambda\left(\frac{C}{2}e^{-\alpha r}-C_0e^{\alpha_0C''\lambda-\alpha_0 r}\right)\\ &\leq e^{\lambda(C''\alpha-1/2)}Ce^{-\alpha r}-\lambda\left(\frac{C}{2}e^{(\alpha_0-\alpha)r}-C_0e^{\alpha_0C''\lambda}\right)e^{-\alpha_0 r}. \qquad (2.13)\end{aligned}$$

Once α is small enough so that $C''\alpha<1/2$, $\alpha<\alpha_0$ and once C is greater than $2C_0e^{\alpha_0C''}$, the right-hand side of (2.13) is not greater than $Ce^{-\alpha r}$. This concludes the proof. □

2.2.2 Estimates for the centered measure Π

Lemma 2.8. *For any $\kappa>1$, the map Π restricted to $\mathcal{P}_\kappa(\mathbb{R}^d;P)$ is bounded and Lipschitz.*

Proof. First, we need to show that $Z(\mu)$ is bounded from below on $\mathcal{P}_\kappa(\mathbb{R}^d;P)$. For $\mu\in\mathcal{P}_\kappa(\mathbb{R}^d;P)$, the domination condition (1.6) implies that $W*\mu(x)\leq||\mu||_P P(|x|)\leq\kappa P(|x|)$. So we have:

$$Z(\mu)=\int_{\mathbb{R}^d}e^{-W*\mu(x)}\mathrm{d}x\geq\int_{\mathbb{R}^d}e^{-\kappa P(|x|)}\mathrm{d}x.$$

Now, assuming without any loss of generality that $W(0)=0$ and $\nabla W(0)=0$ because of the assumption (1), and using that

$$\begin{aligned}W*\mu(x)=\int_{\mathbb{R}^d}W(x-y)\mu(\mathrm{d}y) &\geq \frac{C_W}{2}\int_{\mathbb{R}^d}|x-y|^2\mu(\mathrm{d}y)\\ &\geq \frac{C_W}{2}\int_{\mathbb{R}^d}\left(\frac{|x|^2}{4}-|y|^2\right)\mu(\mathrm{d}y)\geq\frac{C_W}{2}\left(\frac{|x|^2}{4}-\kappa\right),\end{aligned}$$

we hence have the following bound for $\Pi(\mu)$:

$$||\Pi(\mu)||_P\leq\left(\int_{\mathbb{R}^d}e^{-\kappa P(|x|)}\mathrm{d}x\right)^{-1}\cdot\int_{\mathbb{R}^d}P(|x|)e^{-\frac{C_W}{2}(|x|^2/4-\kappa)}\mathrm{d}x=:C_\kappa. \qquad (2.14)$$

Note that Π is of class C^1 on $\mathcal{P}(\mathbb{R}^d;P)$ endowed with the strong topology. Denote by $\mathcal{M}_0(\mathbb{R}^d;P)$ the set $\{\mu\in\mathcal{M}(\mathbb{R}^d;P):\int_{\mathbb{R}^d}x\,\mathrm{d}\mu(x)=0\}$. As the set of probability measures has no interior point, we have to specify the meaning of C^1: there exists a continuous linear operator $D\Pi(\mu):\mathcal{M}_0(\mathbb{R}^d;P)\to\mathcal{M}_0(\mathbb{R}^d;P)$, continuously depending on μ, such that $\|\Pi(\mu')-\Pi(\mu)-D\Pi(\mu)(\mu-\mu')\|_P=o(\|\mu-\mu'\|_P)$ provided that $\mu'\in\mathcal{P}(\mathbb{R}^d;P)$ and μ' converges toward μ. Indeed, we naturally choose

$$\begin{aligned}D\Pi(\mu)\cdot\nu &:= -(W*\nu)\Pi(\mu)-\frac{DZ(\mu)\cdot\nu}{Z(\mu)^2}e^{-W*\mu}\\ &= -(W*\nu)\Pi(\mu)+\int_{\mathbb{R}^d}W*\nu(y)\frac{e^{-W*\mu(y)}}{Z(\mu)}\mathrm{d}y\,\frac{e^{-W*\mu}}{Z(\mu)}\\ &= -\left(W*\nu-\int_{\mathbb{R}^d}W*\nu(y)\Pi(\mu)(\mathrm{d}y)\right)\Pi(\mu). \qquad (2.15)\end{aligned}$$

Now, note that the norms $\|D\Pi\|$ are uniformly bounded for $\mu\in\mathcal{P}_\kappa(\mathbb{R}^d;P)$ (for any given κ). Indeed, fix $\nu\in\mathcal{M}_0(\mathbb{R}^d;P)$. Since $|W*\nu(x)|\leq||\nu||_P P(|x|)$, we find that

$$\|D\Pi(\mu)\cdot\nu\|_P\leq(1+C_\kappa)\|\nu\|_P\int_{\mathbb{R}^d}P^2(|x|)\Pi(\mu)(\mathrm{d}x).$$

For $\mu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, the same computation used for the bound (2.14) on the norm of $\Pi(\mu)$ enables to control the last integral. Hence, we deduce a bound (call it C'_κ) on the norm of the differential. Thus, Π is Lipschitz as stated. $\square$

We prove now the exponential decrease for the centered measure $\Pi(\mu)$.

Proposition 2.9. *There exist $C_W, C_\Pi > 0$ such that for all $\mu \in \mathcal{P}(\mathbb{R}; P)$, we have $\Pi(\mu)(\cdot + c_\mu) \in K^0_{C_W, C_\Pi}$.*

Proof. Note first that, imposing a condition $C_\Pi \geq e^{2C_W}$, we can restrict ourselves only to $R \geq 2$: for $R < 2$, the estimate is obvious.

The measure $\Pi(\mu)$ has the density $\frac{1}{Z(\mu)} e^{-W*\mu(x)}$. To avoid working with the normalization constant $Z(\mu)$, we will prove a stronger inequality, that is

$$\Pi(\mu)(|x - c_\mu| \geq R) \leq C_\Pi e^{-C_W R} \cdot \Pi(\mu)(|x - c_\mu| \leq 2), \tag{2.16}$$

which is equivalent to

$$\int_{|x-c_\mu| \geq R} e^{-W*\mu(x)} \mathrm{d}x \leq C_\Pi e^{-C_W R} \int_{|x-c_\mu| \leq 2} e^{-W*\mu(x)} \mathrm{d}x.$$

We use the polar coordinates, centered at the center c_μ, and so we want to prove that

$$\int_{\mathbb{S}^{d-1}} \int_R^{+\infty} e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} \mathrm{d}\lambda \mathrm{d}v \leq C_\Pi e^{-C_W R} \int_{\mathbb{S}^{d-1}} \int_0^2 e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} \mathrm{d}\lambda \mathrm{d}v.$$

It suffices to prove such an inequality "directionwise": for all $v \in \mathbb{S}^{d-1}$, for all $R \geq 2$

$$\int_R^{+\infty} e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} \mathrm{d}\lambda \leq C_\Pi e^{-C_W R} \int_0^2 e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} \mathrm{d}\lambda.$$

But from the uniform convexity of W and the definition of the center, the function $f(\lambda) = W * \mu(c_\mu + \lambda v)$ satisfies $f'(0) = 0$ and $\forall r > 0$, $f''(r) \geq C_W$. Hence, f is monotone increasing on $[0, +\infty)$, and in particular,

$$\int_0^2 e^{-f(\lambda)} \lambda^{d-1} \mathrm{d}\lambda \geq e^{-f(2)} \int_0^2 \lambda^{d-1} \mathrm{d}\lambda =: C_1 e^{-f(2)}. \tag{2.17}$$

On the other hand, for all $\lambda \geq 2$, $f'(\lambda) \geq f'(2) \geq 2C_W$, and thus $f(\lambda) \geq 2C_W(\lambda - 2) + f(2)$. Hence,

$$\begin{aligned} \int_R^{+\infty} e^{-f(\lambda)} \lambda^{d-1} \mathrm{d}\lambda &\leq e^{-f(2)} \int_R^{+\infty} \lambda^{d-1} e^{-2C_W(\lambda-2)} \mathrm{d}\lambda \\ &\leq C_2 R^{d-1} e^{-2C_W R} \cdot e^{-f(2)} \leq C_3 e^{-C_W R} \cdot e^{-f(2)}. \end{aligned} \tag{2.18}$$

Comparing (2.17) and (2.18), we obtain the desired exponential decrease. $\square$

2.3 A new transport metric: the $\mathcal{T}_P$-metric

Usually, to estimate the distance between two probability measures, one introduces the quadratic Wasserstein distance. Namely, for $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d; P)$, the quadratic Wasserstein distance is defined as

$$W_2(\mu_1, \mu_2) := \left(\inf\{\mathbb{E}(|\xi_1 - \xi_2|^2)\}\right)^{1/2},$$

where the infimum is taken over all the random variables such that $\{\text{law of } \xi_1\} = \mu_1$ and $\{\text{law of } \xi_2\} = \mu_2$. In our setting, for a measure μ, the corresponding probability measure $\Pi(\mu)$ is defined using the convolution $W * \mu$. So, it would be rather natural to use a distance, looking like the one for the weak* topology, but allowing to control $W * \mu$ for our unbounded function W. Indeed, we are looking for a distance similar to the Wasserstein distance, such that

- we can evaluate expectations as $\mathbb{E}W(\xi_1-\xi_2)$ or $\mathbb{E}|\nabla W(\xi_1-\xi_2)|$,
- the set $\mathcal{P}(\mathbb{R}^d;P)$ equipped with that distance is complete.

As we control W and its derivatives with P, this motivates to introduce the following new metric looking like the Wasserstein distance:

Definition 2.10. *For $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d;P)$, we define the P-translation distance between them as*

$$\mathcal{T}_P(\mu_1,\mu_2):=\inf\left\{\iint_0^1 P(|f(s,\omega)|)|f'_s(s,\omega)|\,\mathrm{d}s\mathrm{d}\mathbb{P}\right\},\tag{2.19}$$

where the infimum is taken over the maps $f:[0,1]\times\Omega\to\mathbb{R}$, where Ω is a probability space, such that $\{$law of $f(0,\cdot)\}=\mu_1$, and $\{$law of $f(1,\cdot)\}=\mu_2$.

We also denote the $\mathcal{T}_P$-distance between two s-shifted measures (shifting s to the origin) by

$$\mathcal{T}_P^{(s)}(\mu_1,\mu_2)=\mathcal{T}_P(\mu_1(\cdot+s),\mu_2(\cdot+s)).\tag{2.20}$$

Remark 2.11. *In dimension one, we have the equivalent definition:*

$$\mathcal{T}_P(\mu_1,\mu_2):=\int_{\mathbb{R}}P(|x|)|\mu_1((-\infty,x])-\mu_2((-\infty,x])|\,\mathrm{d}x.$$

The equivalence comes from a coupling by increasing rearrangement.

We wish to emphasize that $\mathcal{T}_P^{(s)}$ corresponds to the $\mathcal{T}_P$ distance between two probability measures shifted by the same shift s (and this shift does not coincide with the center of the measure in general).

The following lemma will be useful to show the convergence of the empirical measure in the W_2-meaning, as Theorem 3.1 shows.

Lemma 2.12. *There exists a constant $C>0$ such that for any $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d;P)$, we have*

$$W_2^2(\mu_1,\mu_2)\le C\mathcal{T}_P(\mu_1,\mu_2).$$

If moreover μ_1 and μ_2 belong to the set K_{α,C_0}, then there exists $C'>0$ such that

$$\mathcal{T}_P(\mu_1,\mu_2)\le C'W_2^2(\mu_1,\mu_2).$$

Proof. Suppose that $\mu_1,\mu_2\in K_{\alpha,C_0}$. Choose ξ_1,ξ_2 realizing the optimal W_2-transport between them, and let us estimate the $\mathcal{T}_P$-cost of the same transport. Indeed,

$$\mathcal{T}_P(\mu_1,\mu_2)\le\int|\xi_1-\xi_2|P(\max(|\xi_1|,|\xi_2|))\mathrm{d}\mathbb{P}\le W_2(\mu_1,\mu_2)\left(\int P^2(\max(|\xi_1|,|\xi_2|))\mathrm{d}\mathbb{P}\right)^{1/2},$$

where the second inequality is Cauchy-Schwarz. As $\mu_1,\mu_2\in K_{\alpha,C_0}$, we conclude that

$$\int P^2(\max(|\xi_1|,|\xi_2|))\mathrm{d}\mathbb{P}\le\int P^2(r)\,\mathrm{d}\max(0,1-2C_0e^{-\alpha r})=:C'<+\infty.$$

Let now ξ_1,ξ_2 be two random variables corresponding to the $\mathcal{T}_P$-optimal transport of μ_1 to μ_2. We then have (due to the assumptions on P, see Remark 1.4)

$$\begin{aligned}W_2^2(\mu_1,\mu_2)&\le\int|\xi_1-\xi_2|^2\mathrm{d}\mathbb{P}\le\int|\xi_1-\xi_2|\cdot 2\max(\xi_1,\xi_2)\mathrm{d}\mathbb{P}\le\\&\le\int|\xi_1-\xi_2|\cdot\frac{P(\max(\xi_1,\xi_2)/2)}{4}\mathrm{d}\mathbb{P}\le C\mathcal{T}_P(\mu_1,\mu_2).\end{aligned}\tag{2.21}$$

Indeed, the inequality (2.21) is due to the fact that the path between ξ_1 and ξ_2 either stays outside the ball of radius $\frac12\max(|\xi_1|,|\xi_2|)$ centered in 0, in which case we estimate its length from below as $|\xi_1-\xi_2|$, or this path has a part joining the maximum norm vector to this ball, which is of length greater than $\frac12\max(|\xi_1|,|\xi_2|)\ge\frac14|\xi_1-\xi_2|$. □

It is clear from the definition that $\mathcal{T}_P$ is a distance; and also taking into account that for all $x > 0$, $|P'(x)| \leq P(x)$ (see Remark 1.4, it suffices to increase the constant A_1 if necessary), one easily has

$$\|\mu_2\|_P \leq \|\mu_1\|_P + \mathcal{T}_P(\mu_1, \mu_2). \tag{2.22}$$

Thus, the set $\mathcal{P}(\mathbb{R}^d; P)$ is $\mathcal{T}_P$-complete. Indeed, a $\mathcal{T}_P$-Cauchy sequence (μ_n) will have a weak limit μ and it is easy to check that $\|\mu\|_P = \lim_{n\to+\infty} \|\mu_n\|_P < +\infty$. So, $\mu \in \mathcal{P}(\mathbb{R}^d; P)$. Now, we are going to estimate the deviation of trajectories in terms of $\mathcal{T}_P$-metric, a result that will be useful in §3.1.

Lemma 2.13. *For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d; P)$, the following statements hold:*

1. *The map c is locally Lipschitz in the sense of $\mathcal{T}_P$-metric:*

$$|c(\mu) - c(\nu)| \leq \frac{1}{C_W} \min(P(|c(\mu)|), P(|c(\nu)|)) \cdot \mathcal{T}_P(\mu, \nu);$$

2. *For all $v \in \mathbb{R}^d$, we have $\mathcal{T}_P(\mu, \mu(\cdot + v)) \leq |v| P(|v|) \|\mu\|_P$;*
3. *There exists $C_P > 0$ such that for $v = c(\mu) - c(\nu)$, we have*

$$\mathcal{T}_P^{(c)}(\mu, \nu) \leq \sup_{x \geq 0} \frac{P(x + |v|)}{P(x)} \mathcal{T}_P(\mu, \nu) \leq \begin{cases} (1 + C_P|v|) \cdot \mathcal{T}_P(\mu, \nu), & |v| \leq 1 \\ P(|v|) \mathcal{T}_P(\mu, \nu), & \forall |v|; \end{cases}$$

4. *For all $\kappa > 0$, the application $\mu^c : \mathcal{P}_\kappa(\mathbb{R}^d; P) \to \mathcal{P}(\mathbb{R}^d; P)$ is $\mathcal{T}_P$-Lipschitz.*

Proof. (1) Denoting by c_μ (resp. c_ν) the center of μ (resp. ν), we recall that

$$\nabla W * \nu(c_\mu) = \nabla W * \mu(c_\mu) + \nabla W * (\nu - \mu)(c_\mu).$$

Choose any transport $f(\omega, s)$ between μ and ν. We then have

$$\begin{aligned} |\nabla W * \nu(c_\mu)| &= \left| \nabla W * \mu(c_\mu) + \int_0^1 \mathbb{E} \left(\nabla W_{|_{c_\mu - f(\omega,s)}} \right)'_s \, \mathrm{d}s \right| \leq \\ \leq \int_0^1 \mathbb{E} \left| \left(\nabla^2 W_{|_{c_\mu - f(\omega,s)}}, f'_s(\omega, s) \right) \right| \, \mathrm{d}s &\leq \int_0^1 \mathbb{E} \left[P(|c_\mu| + |f(\omega, s)|) |f'_s(\omega, s)| \right] \, \mathrm{d}s \leq \\ \leq \int_0^1 \mathbb{E} \left[P(|c_\mu|) P(|f(\omega, s)|) |f'_s(\omega, s)| \right] \, \mathrm{d}s &= P(|c_\mu|) \int_0^1 \mathbb{E} \left[P(|f(\omega, s)|) |f'_s(\omega, s)| \right] \, \mathrm{d}s. \end{aligned}$$

Passing to the infimum over the transports $f(\omega, s)$, we then have

$$|\nabla W * \nu_{|_{c_\mu}}| \leq P(|c_\mu|) \mathcal{T}_P(\mu, \nu).$$

Join now the points c_μ and c_ν by a line, and recall that W is uniformly convex. The second derivative of $W * \nu$ along this line is then at least $C_W \nu(\mathbb{R}^d)$ and noticing that $\nabla W * \nu(c_\nu) = 0$, we obtain

$$|c_\nu - c_\mu| \leq \frac{P(|c_\mu|)}{C_W} \mathcal{T}_P(\mu, \nu). \tag{2.23}$$

Interverting the roles of μ and ν, we conclude.

(2) We have by definition of $\mathcal{T}_P$ that

$$\mathcal{T}_P(\mu, \mu(\cdot + v)) \leq \int_{\mathbb{R}^d} |v| P(|x| + |v|) \, \mu(\mathrm{d}x) \leq |v| P(|v|) \int_{\mathbb{R}^d} P(|x|) \, \mu(\mathrm{d}x).$$

(3) For any transport $f(s,\omega)$ between $\mu = \{\text{law of } f(0,\omega)\}$ and $\nu = \{\text{law of } f(1,\omega)\}$, the map $f(s,\omega) - v$ is a transport between μ^c and ν^c of price

$$\int_\Omega \int_0^1 P(|f(s,\omega)-v|)|f_s'(s,\omega)| \,\mathrm{d}s\mathrm{d}\mathbb{P}(\omega) \leq \sup_{x\geq 0} \frac{P(x+|v|)}{P(x)} \int_\Omega \int_0^1 P(|f(s,\omega)|)|f_s'(s,\omega)| \,\mathrm{d}s\mathrm{d}\mathbb{P}(\omega).$$

The left-hand side is an upper bound for $\mathcal{T}_P^{(c)}(\mu,\nu)$ and passing in the right-hand side to the infimum over all the possible transports f, we obtain the desired $\sup_{x\geq 0} \frac{P(x+|v|)}{P(x)} \mathcal{T}_P(\mu,\nu)$.

(4) Suppose that $\mu,\nu \in \mathcal{P}_\kappa(\mathbb{R}^d;P)$. Then, by the preceding points, we have

$$\begin{aligned}
\mathcal{T}_P(\mu(\cdot+c_\mu),\nu(\cdot+c_\nu)) &\leq \mathcal{T}_P(\mu(\cdot+c_\mu),\nu(\cdot+c_\mu)) + \mathcal{T}_P(\nu(\cdot+c_\mu),\nu(\cdot+c_\nu)) \\
&\leq P(|c_\mu|)\mathcal{T}_P(\mu,\nu) + |c_\mu - c_\nu| P(|c_\mu - c_\nu|) ||\nu(\cdot + c_\nu)||_P \\
&\leq P(|c_\mu|)\mathcal{T}_P(\mu,\nu) \\
&+ P(|c_\mu - c_\nu|)\frac{1}{C_W} \min(P(|c_\mu|), P(|c_\nu|)) \|\nu\|_P \mathcal{T}_P(\mu,\nu).
\end{aligned}$$

Remark that, as $\mu,\nu \in \mathcal{P}_\kappa(\mathbb{R}^d;P)$, the norms $|c_\mu|$ and $|c_\nu|$ are uniformly bounded, as well as $\|\nu\|_P$, thus

$$\mathcal{T}_P(\mu(\cdot+c_\mu),\nu(\cdot+c_\nu)) \leq \left(P(|c_\mu|) + P(|c_\mu - c_\nu|)\frac{1}{C_W}\min(P(|c_\mu|), P(|c_\nu|))\|\nu\|_P\right)\mathcal{T}_P(\mu,\nu),$$

where $P(|c_\mu|) + P(|c_\mu - c_\nu|)\frac{1}{C_W}\min(P(|c_\mu|), P(|c_\nu|))\|\nu\|_P$ is uniformly bounded by some constant C_κ, which is the Lipschitz constant. ☐

2.4 Free energy functional

In this paragraph, we will establish and prove the rigorous statements corresponding to the non-rigorous physical interpretation of §1.3.4.

First, we recall that the free energy of a measure is defined as the sum of its entropy $\mathcal{H}$ and its potential energy:

$$\mathcal{F}(\mu) = \mathcal{H}(\mu) + \frac{1}{2}\iint \mu(x)W(x-y)\mu(y)\,\mathrm{d}x\mathrm{d}y, \quad \text{where} \quad \mathcal{H}(\mu) = \int \mu(x)\log\mu(x)\,\mathrm{d}x.$$

The free energy of a non-self-interacting gas in an exterior potential V is defined as

$$\mathcal{F}_V(\mu) = \mathcal{H}(\mu) + \int \mu(x)V(x)\mathrm{d}x$$

and the map Π associates to a measure μ the probability measure $\frac{1}{Z}e^{-W*\mu(x)}\mathrm{d}x$ (as soon as $W*\mu$ is well-defined).

The first auxiliary statement implies that, as we mentioned it in §1.3.4, $\Pi(\mu)$ is the unique global minimum of $\mathcal{F}_{W*\mu}$.

Lemma 2.14. *For any potential V such that e^{-V} is integrable, the probability measure $Z^{-1}e^{-V}$ is the unique global minimum of $\mathcal{F}_V$ on $\mathcal{P}(\mathbb{R}^d)$.*

Proof. Let $\mu = Z^{-1}e^{-V}$. Then, for any arbitrary absolutely continuous probability measure ν, letting $\rho(x) = Ze^{V(x)}\nu(x)$ be its density with respect to μ, we see that

$$\begin{aligned}
\mathcal{F}_V(\nu) &= \int_{\mathbb{R}^d} (V(x) + \log\nu(x))\,\nu(\mathrm{d}x) = \int_{\mathbb{R}^d} (\log\rho(x) - \log Z)\,\nu(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \rho(x)\log\rho(x)\,\mu(\mathrm{d}x) - \log Z.
\end{aligned}$$

Thus Jensen's inequality, applied to the convex function $\rho\log\rho$, leads immediately to the conclusion. ☐

Now, we will compare the transport distance, from a centered measure to the global minimum of $\mathcal{F}$, to its free energy functional. Actually, McCann [10] proved the following for the free energy functional:

Proposition 2.15 (McCann)**.** *There exists a centered symmetric density ρ_∞, which is the unique, up to translation, global minimum of $\mathcal{F}$. Moreover, $\mathcal{F}$ is a displacement convex functional, that is for two probability measures μ_0, μ_1 and the Wasserstein-optimal transport between them*

$$\xi_s = (1-s)\xi_0 + s\xi_1,$$

where $\mu_0 = \{\text{law of } \xi_0\}$, $\mu_1 = \{\text{law of } \xi_1\}$, $\mathbb{E}|\xi_0 - \xi_1|^2 = W_2^2(\mu_0, \mu_1)$, one has

$$\mathcal{F}(\{\text{law of } \xi_s\}) \geq (1-s)\mathcal{F}(\mu_0) + s\mathcal{F}(\mu_1).$$

Finally, the transport distance from a centered measure μ to ρ_∞ can be estimated as

$$W_2^2(\mu, \rho_\infty) \leq \frac{2}{C_W}\mathcal{F}(\mu|\rho_\infty), \tag{2.24}$$

where $\mathcal{F}(\mu|\rho_\infty) = \mathcal{F}(\mu) - \mathcal{F}(\rho_\infty)$.

Remark 2.16. *i) The uniqueness of the minimum comes from the strict displacement convexity of the restriction to the space of centered measures.*

ii) The functional $\mathcal{F}$ is not convex in the usual sense, due to the self-interacting part.

iii) Inequality (1.15) *together with Lemma 2.14 immediately imply that the minimum of $\mathcal{F}$ is also a fixed point of Π.*

3 Proofs

3.1 Proof of Theorem 1.10

In fact, instead of showing that the centered empirical measure of the process converges toward a (deterministic) density function, we will prove a stronger statement, controlling the speed of convergence in the sense of the transport distance:

Theorem 3.1. *There exists $a > 0$ such that almost surely, as $t \to +\infty$,*

$$\mathcal{T}_P(\mu_t^c, \rho_\infty) = O\left(\exp\{-a \sqrt[k+1]{\log t}\}\right),$$

where k is the degree of the polynomial P, as well as

$$W_2(\mu_t^c, \rho_\infty) = O\left(\exp\{-a \sqrt[k+1]{\log t}\}\right).$$

The proof of this statement will be decomposed into several propositions. We first present them all, postponing their proofs to the end of this paragraph. Then we deduce from them Theorem 3.1. Finally, we prove these propositions.

Let us explain our strategy to prove this statement. As it was announced in §1.3, we will discretize the random process. Namely, we define the sequence T_n of moments of time as $T_n := n^{3/2}$ and then, $\Delta T_n := T_{n+1} - T_n$ is of order $n^{1/2} = T_n^{1/3}$. Also, for what follows, we will associate to a random trajectory $(X_t, t \geq 0)$ the sequence (L_n) defined by

$$L_n := \max_{0 \leq t \leq T_{n+1}} |X_t - c_{T_n}| \leq C \log T_n. \tag{3.1}$$

An easy conclusion from the Ornstein-Uhlenbeck comparison §2.1.3 and logarithmic drift of the center is that almost surely $L_n \leq C' \log n$ for any n large enough.

Now, let us state the first of the propositions mentioned above. This result allows to estimate the "Euler-method" one-step error in the description of the behaviour of measures μ_t:

Proposition 3.2. *Almost surely there exists n_0 such that for any $n \geq n_0$, we have*

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n})) \leq (\Delta T_n)^{-\beta},$$

where $\beta = \min\left(8C_W, \frac{1}{5d}\right)$.

Associated to the moments of time T_n, consider the following, roughly speaking, Euler-approximation maps for the flow $\dot{m} = \frac{1}{t}(\Pi(m) - m)$, with the knots chosen at the moments T_n:

Definition 3.3. *For any $i \leq j$, define $\Phi_i^j : \mathcal{P}(\mathbb{R}^d, P) \to \mathcal{P}(\mathbb{R}^d, P)$ as*

$$\Phi_i^i = id, \quad \Phi_i^{i+1}(\mu) = \mu + \frac{\Delta T_i}{T_{i+1}}(\Pi(\mu) - \mu), \quad \Phi_i^j = \Phi_{j-1}^j \circ \cdots \circ \Phi_i^{i+1}.$$

Let us first exhibit an invariant set for Φ.

Lemma 3.4. *For any α, C as in Lemma 2.7, corresponding to $\alpha_0 = C_W$ and $C_0 = C_\Pi$ (from Proposition 2.9), if $\mu \in K_{\alpha,C}$ and $i \leq j$, then $\Phi_i^j(\mu) \in K_{\alpha,C}$.*

Proof. This is a direct corollary of Lemma 2.7. □

Denote, for a probability measure μ and for a number $h > 0$, by $\mu^{(h)}$ the "smoothing convolution"

$$\mu^{(h)} := \mu * \left(\frac{1}{vol(\mathcal{U}_h(0))} \cdot \mathbb{1}_{\mathcal{U}_h(0)}\, \mathrm{d}x\right),$$

where $\mathcal{U}_h(0)$ is the ball of radius h in $\mathbb{R}^d$, centered at the origin.

The following proposition allows to compare the deterministic Euler-like behaviour of the smoothened, at some moment T_i, measure with the true random trajectory:

Proposition 3.5. *There exist some constants $A, C_1, C_1', C_2, C_3 > 0$ such that for any $\delta > 0$ small enough, almost surely there exists n_0 for which, for any $j > i \geq n_0$, $i \geq [j^{1-\delta}]$ for j large enough and any $h > 0$, we have*

$$\mathcal{T}_P^{(c_{T_j})}(\Phi_i^j(\mu_{T_i}^{(h)}), \mu_{T_j}) \leq C_1' \cdot n^{(1+\frac{3}{2}A+\frac{\beta}{2})\delta - \frac{\beta}{2}} + C_1 h \left(\frac{T_j}{T_i}\right)^A, \tag{3.2}$$

provided that the right-hand side of (3.2) *does not exceed C_3. Also, under the same condition,*

$$|c(\Phi_i^j(\mu_{T_i}^{(h)})) - c_{T_j})| \leq C_2.$$

Next, we have to show that the deterministic trajectory of an absolutely continuous measure becomes sufficiently close to the set of translates of ρ_∞. To do this, due to the estimate (2.24), it suffices to estimate the free energy:

Proposition 3.6. *Let $\mu \in K_{\alpha,C}$. Then, there exist $a_1, C_4, C_5 > 0$ such that there exists n_0 for which the following statements hold for any $j \geq i \geq n_0$:*

i) $\mathcal{F}(\Phi_i^j(\mu)|\rho_\infty) \leq C_4 + \frac{T_i}{T_j}(\mathcal{F}(\mu|\rho_\infty) - C_4)$,

ii) $\mathcal{F}(\Phi_i^j(\mu)|\rho_\infty) \leq C_5 e^{-a_1 \sqrt[k+1]{\log(T_j/T_i)}}$ *if* $\mathcal{F}(\mu|\rho_\infty) \leq 2C_4$.

Now, modulo these propositions, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Recall from Proposition 3.2 that $\beta = \min(8C_W, (5d)^{-1})$. Note first that the distances $\mathcal{T}_P^{(c_{T_n})}(\mu_t, \mu_{T_n})$ for $t \in [T_n, T_{n+1}]$ are uniformly bounded for n sufficiently big by

$$\frac{L_n^{k+1}\Delta T_n}{T_{n+1}} \leq C\frac{(\log n)^{k+1}}{n} \ll e^{-\sqrt[k+1]{\log n}};$$

where L_n is defined by (3.1). Hence, it suffices to check the estimate for the sequence of moments T_n:

$$\mathcal{T}_P(\mu_{T_n}^c, \rho_\infty) \leq e^{-a\sqrt[k+1]{\log n}}.$$

Now, for any sufficiently large n, choose $i := [n^{1-\delta}]$, where a small $\delta > 0$ will be chosen and fixed (in a way that does not depend on n) later. Then, considering for some $h > 0$ a smoothened convolution $\mu_{T_i}^{(h)}$ and its Euler-image $\Phi_i^n(\mu_{T_i}^{(h)})$, we have by Proposition 3.5

$$\mathcal{T}_P^{(c_{T_n})}(\Phi_i^n(\mu_{T_i}^{(h)}), \mu_{T_n}) \leq C_1' \cdot n^{(1+\frac{3}{2}A+\frac{\beta}{2})\delta-\frac{\beta}{2}} + C_1 h\left(\frac{T_n}{T_i}\right)^A, \tag{3.3}$$

provided that the right-hand side does not exceed C_3. So, for any fixed choice of $\delta < \frac{\beta/2}{1+(3A+\beta)/2}$, the first term in the right-hand side of (3.3) will decrease as a negative power of n and thus faster than $e^{-a\sqrt[k+1]{\log T_n}}$.

Take now $h = \frac{C_3}{C_1}\left(\frac{T_i}{T_n}\right)^{A+1}$. For such a choice of h, the second term in the right-hand side of (3.3) is not greater than $\frac{T_i}{T_n} \sim n^{-\delta}$. So it also decreases quicker than $e^{-a\sqrt[k+1]{\log T_n}}$ and thus $\mathcal{T}_P^{(c_{T_n})}(\mu_{T_n}, \Phi_i^n(\mu_{T_i}^{(h)})) \leq e^{-a\sqrt[k+1]{\log T_n}}$.

Finally, we have to estimate $\mathcal{T}_P^{(c_{T_n})}(\Phi_i^n(\mu_{T_i}^{(h)}), \rho_\infty(\cdot + c_{T_n}))$. To do this, it suffices to estimate the free energy $\mathcal{F}(\Phi_i^n(\mu_{T_i}^{(h)}))$, as

$$\mathcal{T}_P\left((\Phi_i^n(\mu_{T_i}^{(h)}))^c, \rho_\infty\right) \leq C_1' \cdot W_2^2\left((\Phi_i^n(\mu_{T_i}^{(h)}))^c, \rho_\infty\right) \leq C_1' \cdot \mathcal{F}(\Phi_i^n(\mu_{T_i}^{(h)})).$$

Indeed, remember that

$$\mathcal{F}(\mu_{T_i}^{(h)}) = \mathcal{H}(\mu_{T_i}^{(h)}) + \frac{1}{2}\iint \mu_{T_i}^{(h)}(\mathrm{d}x)W(x-y)\mu_{T_i}^{(h)}(\mathrm{d}y).$$

The first term here does not exceed $-\log vol(\mathcal{U}_h(0)) \leq d\cdot|\log(h/d)|$ (as the density of $\mu_{T_i}^{(h)}$ does not exceed $(h/d)^{-d}$), while the second term is bounded. Thus $\mathcal{F}(\mu_{T_i}^{(h)}) \leq C_6 \log n$ for some constant C_6. Hence, from the first part of Proposition 3.6, for $j = \left[\left(\frac{C_6}{C_4}\log n\right)^{2/3} i\right]$, we have

$$\mathcal{F}(\Phi_i^j(\mu_{T_i}^{(h)})|\rho_\infty) \leq C_4 + \frac{T_i}{T_j}\left(\mathcal{F}(\mu_{T_i}^{(h)}|\rho_\infty) - C_4\right) \leq 2C_4.$$

Applying the second part, with $\Phi_i^j(\mu_{T_i}^{(h)})$ as a starting measure, we obtain

$$\mathcal{F}(\Phi_i^n(\mu_{T_i}^{(h)})) = \mathcal{F}(\Phi_j^n \circ \Phi_i^j(\mu_{T_i}^{(h)})) \leq C_5 e^{-a_1\sqrt[k+1]{\log(T_n/T_j)}} \leq e^{-a\sqrt[k+1]{\log T_n}}.$$

Thus, $\mathcal{F}(\Phi_i^n(\mu_{T_i}^{(h)})) \leq e^{-a\sqrt[k+1]{\log T_i}}$ and hence

$$\mathcal{T}_P\left((\Phi_i^n(\mu_{T_i}^{(h)}))^c, \rho_\infty\right) \leq C_1' \cdot W_2^2\left((\Phi_i^n(\mu_{T_i}^{(h)}))^c, \rho_\infty\right) \leq C_1' \cdot e^{-a\sqrt[k+1]{\log T_n}}. \quad \square$$

Let us now prove Propositions 3.2–3.6. Each proposition will be proved in a different paragraph.

3.1.1 One-step error estimate

This section is devoted to the proof of Proposition 3.2.

To estimate the difference between the normalized occupation measure of X_t on $[T_n, T_{n+1}]$, and the measure $\Pi(\mu_t)$, we will first introduce another process, for which $\Pi(\mu_{T_n})$ is the stationary measure: the process with "frozen" measure μ_{T_n}. More precisely, on $[T_n, T_{n+1})$ we consider a process Y with some choice of Y_{T_n}, satisfying

$$\mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_{T_n}(Y_t)\,\mathrm{d}t, \tag{3.4}$$

generated by *the same* Brownian motion B_t as X_t. In other words, the couple (X_t, Y_t) satisfies

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_t(X_t)\,\mathrm{d}t \\ \mathrm{d}Y_t = \sqrt{2}\,\mathrm{d}B_t - \nabla W * \mu_{T_n}(Y_t)\,\mathrm{d}t. \end{cases} \tag{3.5}$$

The following lemma allows to control the difference between them:

Lemma 3.7. *For all $t \in [T_n, T_{n+1}]$, we have*

$$|X_t - Y_t| \le e^{-C_W(t-T_n)}|X_{T_n} - Y_{T_n}| + \frac{\Delta T_n}{T_n C_W} P(2L_n). \tag{3.6}$$

Proof. The process $X_t - Y_t$ is of class C^1. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_t - Y_t) = -(\nabla W * \mu_t(X_t) - \nabla W * \mu_{T_n}(Y_t)).$$

Adding and substracting $\nabla W * \mu_{T_n}(X_t)$, we see that

$$\mathrm{d}(X_t - Y_t) = -\left[\nabla W * (\mu_t - \mu_{T_n})(X_t) - (\nabla W * \mu_{T_n}(Y_t) - \nabla W * \mu_{T_n}(X_t))\right]\mathrm{d}t.$$

The last term can be rewritten as

$$-(\nabla W * \mu_{T_n}(Y_t) - \nabla W * \mu_{T_n}(X_t)) = \frac{1}{T_n}\int_0^{T_n}\int_0^1 \nabla^2 W|_{uY_t+(1-u)X_t-X_s} \cdot (X_t - Y_t)\mathrm{d}u\,\mathrm{d}s.$$

Noting the first term as D_t, and putting a scalar product with $X_t - Y_t$, we see

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X_t - Y_t|^2 &= (D_t, X_t - Y_t) \\ &- \frac{1}{T_n}\int_0^{T_n}\left(\int_0^1 \nabla^2 W|_{(uY_t+(1-u)X_t)-X_s}\mathrm{d}u \cdot (X_t - Y_t), X_t - Y_t\right)\mathrm{d}s \\ &\le (D_t, X_t - Y_t) - C_W|X_t - Y_t|^2. \end{aligned}$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}t}|X_t - Y_t|^2 \le -2C_W|X_t - Y_t|^2 + 2|D_t||X_t - Y_t|$. Redividing by $2|X_t - Y_t|$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|X_t - Y_t| \le |D_t| - C_W|X_t - Y_t|. \tag{3.7}$$

Finally, notice that $|D_t| \le P(2L_n)\frac{\Delta T_n}{T_n}$, as it is the difference between the forces generated at X_t by μ_{T_n} and by $\mu_t = \mu_{T_n} + \frac{t-T_n}{t}\left(\mu_{[T_n,t]} - \mu_{T_n}\right)$. Solving $\dot{u}_t = P(2L_n)\frac{\Delta T_n}{T_n} - C_W u_t$, we obtain the desired estimate for the difference $|X_t - Y_t|$ on the interval $[T_n, T_{n+1}]$. □

For the following (see Proposition 3.9 and Lemma 3.10 below), we assume that the initial distribution of Y_{T_n} is absolutely continuous with respect to $\Pi(\mu_{T_n})$, and we use an estimate on its density. So finally, we define the process Y_t for all t in the following way: for every interval $[T_n, T_{n+1})$ the initial value Y_{T_n} is chosen randomly with respect to the restriction of $\Pi(\mu_{T_n})$ to $\mathcal{U}_2(c_{T_n})$, the ball of radius 2. On each new interval, the choice is independent of X and of all the past. Then, inside the interval (T_n, T_{n+1}), the couple (X_n, Y_n) satisfies (3.5).

Let us compare the occupation measures of the processes X and Y on these intervals of time. Denote by $\mu^Y_{[T_n,T_{n+1}]}$ the occupation measure of Y on the interval $[T_n, T_{n+1}]$. Then, we have the following:

Lemma 3.8. *For any family of choices $Y_{T_n} \in \mathcal{U}_2(c_{T_n})$, we have*

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \mu^Y_{[T_n,T_{n+1}]}) = o(T_n^{-1/5}), \quad \text{as } n \to +\infty,$$

provided that for n large enough $L_n \le C_3' \log n$.

Proof. The measures $\mu_{[T_n,T_{n+1}]}$ and $\mu^Y_{[T_n,T_{n+1}]}$ are both images of the normalized Lebesgue measure $\frac{1}{\Delta T_n}\mathrm{Leb}_{[T_n,T_{n+1}]}$ under the maps $X_\bullet$ and $Y_\bullet$ respectively. So, consider the transport $\xi_s(t) = (1-s)X_t + sY_t$ between them.

Using this transport, we have an estimate

$$\begin{aligned}\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \mu^Y_{[T_n,T_{n+1}]}) &\le \frac{1}{\Delta T_n}\int_{T_n}^{T_{n+1}}\int_0^1 P((1-s)X_t + sY_t - c_{T_n})|X_t - Y_t| \,\mathrm{d}s\,\mathrm{d}t \\ &\le \frac{1}{\Delta T_n}\int_{T_n}^{T_{n+1}} P(\max(|X_t - c_{T_n}|, |Y_t - c_{T_n}|))|X_t - Y_t| \,\mathrm{d}t. \end{aligned} \tag{3.8}$$

By definition of L_n, we have for all $t \in [T_n, T_{n+1}]$, $|X_t - c_{T_n}| \le L_n$ and due to Lemma 3.7,

$$\begin{aligned}|Y_t - X_t| &\le e^{-C_W(t-T_n)}|X_{T_n} - Y_{T_n}| + \frac{\Delta T_n}{T_n C_W}P(2L_n) \\ &\le L_n + 1 + \frac{\Delta T_n}{T_n C_W}P(2L_n) \le L_n + 2,\end{aligned}$$

provided that $L_n \le C' \log n$ and n is sufficiently big. This implies that

$$|Y_t - c_{T_n}| \le |Y_t - X_t| + |X_t - c_{T_n}| \le 2L_n + 2.$$

Now, substituting the obtained estimates to the right-hand side of (3.6), we see that

$$\begin{aligned}&\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \mu^Y_{[T_n,T_{n+1}]}) \le \\ &\le \frac{1}{\Delta T_n}\int_{T_n}^{T_{n+1}} P(2L_n+2)\cdot\left(e^{-C_W(t-T_n)}(L_n+1) + \frac{\Delta T_n}{T_n C_W}P(2L_n)\right)\mathrm{d}t \le \\ &\le \frac{P(2L_n+2)P(2L_n)\Delta T_n}{C_W T_n} + \frac{P(2L_n+2)(L_n+1)}{C_W \Delta T_n} = o(T_n^{-1/5})\end{aligned}$$

(we have used that $\Delta T_n \sim T_n^{1/3}$, and once again the logarithmic growth of L_n). □

Now, we will compare the occupation measure $\mu^Y_{[T_n,T_{n+1}]}$ with $\Pi(\mu_{T_n})$. To do this, we use Proposition 1.2 of Cattiaux & Guillin [6] (see also Wu [18]), stating that the trajectory mean of a function ψ is, with a probability close to 1 that can be exponentially controlled, close to its stationary mean. Namely, this proposition says the following:

Proposition 3.9 (Cattiaux & Guillin [6])**.** *Given a process ξ with stationary measure m and Poincaré constant C_P, an initial measure ν and a function ψ satisfying $|\psi| \le 1$, one has for any $0 < \rho < 1$ and $t > 0$*

$$\mathbb{P}_\nu\left(\frac{1}{t}\int_0^t \psi(\xi_s)\,\mathrm{d}s - \int \psi\,\mathrm{d}m \ge \rho\right) \le \left\|\frac{\mathrm{d}\nu}{\mathrm{d}m}\right\|_{L_2(m)} \exp\left(-\frac{t\rho^2}{8C_P Var_m(\psi)}\right).$$

We will use this proposition with ψ being the indicator function $\psi = \mathbb{1}_M$ of various sets M: it then allows to compare the occupation measure of the set M to its $\Pi(\mu_{T_n})$-measure.

We know that $m = \Pi(\mu_{T_n})$ is the unique stationary measure of the drifted Brownian motion (3.4). Also, the Poincaré constant for this process is $2C_W$ (see [1]).

To proceed, we have to declare the initial measure $\nu = \nu_n$ for Y_{T_n}, and we choose it to be the measure $\Pi(\mu_{T_n})$ restricted to the ball $\mathcal{U}_2(c_{T_n})$ and then normalized accordingly. Then,

$$\left\| \frac{\mathrm{d}\nu_n}{\mathrm{d}\Pi(\mu_{T_n})} \right\|_{L_2(\Pi(\mu_{T_n}))} = \frac{1}{\Pi(\mu_{T_n})(\mathcal{U}_2(c_{T_n}))} \leq c_E = C_1',$$

the latter inequality is due to the exponential tails of $\Pi(\mu_{T_n})$, see (2.16). Having made these choices, we are going to prove the following

Lemma 3.10. *For n large enough, we have almost surely*

$$\mathcal{T}_P^{(c_{T_n})}(\mu^Y_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n})) = O((\Delta T_n)^{-\min\left(8C_W, \frac{1}{5d}\right)}).$$

Proof. The previous estimates imply that the process Y_t on $[T_n, T_{n+1}]$ almost surely for all n sufficiently big stays inside the ball $\mathcal{U}_{R_n}(c_{T_n})$, where $R_n := 3L_n$. Now, take this ball and cut it into some number N_n parts $M_1, \dots, M_{N_n}$ of diameter less than $\varepsilon_n := \frac{2dR_n}{\sqrt[d]{N_n}}$ (by cubic the grid with the step $2R_n/\sqrt[d]{N_n}$, that is decomposing each of the coordinate segments of length $2R_n$ into $\sqrt[d]{N_n}$ parts). We will choose and fix the number N_n later.

For each of these parts, choose

$$\rho_j := \max\left(\frac{1}{N_n^2}, \frac{\Pi(\mu_{T_n})(M_j)}{N_n}\right).$$

Let $\psi_j = \mathbb{1}_{M_j}$. Then, the probability that all the empirical measures $\mu^Y_{[T_n,T_{n+1}]}(M_j)$ are ρ_j-close to their "theoretical" values $\Pi(\mu_{T_n})(M_j)$ is at least

$$1 - 2c_E \sum_{j=1}^{N_n} \exp\left(-\frac{\rho_j^2 \Delta T_n}{16 C_W Var_{\Pi(\mu_{T_n})}(\psi_j)}\right).$$

As the variance $Var_{\Pi(\mu_{T_n})}(\psi_j)$ does not exceed $\Pi(\mu_{T_n})(\psi_j)$, we have a lower bound for the previous probability by

$$1 - 2c_E \sum_{j=1}^{N_n} \exp\left(-\frac{\rho_j \Delta T_n}{16C_W} \cdot \frac{\rho_j}{\Pi(\mu_{T_n})(\psi_j)}\right) \geq 1 - 2N_n c_E \exp\left(-\frac{\Delta T_n}{16 C_W N_n^3}\right),$$

as $\frac{\rho_j}{\Pi(\mu_{T_n})(\psi_j)} \geq \frac{1}{N_n}$ and $\rho_j \geq \frac{1}{N_n^2}$. So, taking $N_n = \sqrt[10]{T_n} \sim (\Delta T_n)^{3/10}$, we see that the series

$$\sum_n N_n \exp\left(-\frac{\Delta T_n}{16 C_W N_n^3}\right) \asymp \sum_n (\Delta T_n)^{3/10} \exp\left(-(\Delta T_n)^{1/10}\right)$$

converges, so almost surely for all n sufficiently big, all the closeness conditions on the occupation measures are satisfied: the measures $\mu^Y_{[T_n,T_{n+1}]}(M_j)$ are a.s. ρ_j-close to $\Pi(\mu_{T_n})(M_j)$.

Now, let us estimate the c_{T_n}-centered distance $\mathcal{T}_P^{(c_{T_n})}(\mu^Y_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n}))$, provided that these conditions are fulfilled. Indeed, first transport inside each M_j the part $\min(\mu^Y_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n}))$: we pay at most $P(3L_n)\varepsilon_n = O\left((\Delta T_n)^{-\frac{1}{5d}}\right)$. Next, bring the exterior part of $\Pi(\mu_{T_n})$ to the ball $\mathcal{U}_{R_n}(c_{T_n})$: due to the exponential decrease estimates, we pay at most

$$\int_{R_n}^{+\infty} P(r)\mathrm{d}(1-Ce^{-C_W r}) \sim R_n^{k+1} e^{-C_W R_n} = O\left((\Delta T_n)^{-8C_W}\right)$$

as $R_n = 3\log T_n$. Finally, let us re-distribute the parts left: we pay at most

$$\begin{aligned}\sum_{j=1}^{N_n} \rho_j R_n P(R_n) &= R_n P(R_n) \sum_{j=1}^{N_n} \max\left(\frac{1}{N_n^2}, \frac{\Pi(\mu_{T_n})(M_j)}{N_n}\right) \\ &\leq R_n P(R_n) \sum_{j=1}^{N_n} \left(\frac{1}{N_n^2} + \frac{\Pi(\mu_{T_n})(M_j)}{N_n}\right) \\ &\leq 2R_n P(R_n) \frac{1}{N_n} = O\left((\Delta T_n)^{-1/5}\right).\end{aligned}$$

Adding these three estimates, we obtain the desired $\mathcal{T}_P^{c_{T_n}}(\mu^Y_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n})) = O\left((\Delta T_n)^{-\beta}\right)$ with $\beta = \min(8C_W, (5d)^{-1})$. □

Putting Lemmas 3.8 and 3.10 together, and recalling that $\Delta T_n \sim T_n^{\frac{1}{3}}$, we conclude that almost surely, for all n sufficiently big,

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n})) \leq T_n^{-\min\left(\frac{8}{3}C_W, \frac{1}{15d}\right)}.$$

Proposition 3.2 is thus proven.

3.1.2 Euler method error control

Let us prove Proposition 3.5 by induction on j. Roughly speaking, the scheme of the proof is the following. The error of the Euler method approximation at any moment T_{j+1} comes from two parts: on one hand, from the (eventually growing) error at the moment T_j, and on the other hand, from the difference between $\mu_{[T_j,T_{j+1}]}$ and $\Pi(\mu_{T_j})$. The first part of the error can be controlled due to the Lipschitz property of the map Π (on the compact set $K_{\alpha,C}$ to which belong all the measures μ of interest here). Indeed, usually under the Lipschitz flow the error grows exponentially with time, but as we have a factor $\frac{1}{t}$ in the right hand side, the error grows exponentially in $\log t$ (the fraction $(\frac{T_j}{T_i})^A$ in the right hand side comes from there). Finally, the second part is estimated using an explicit ergodic theorem – a statement by Cattiaux & Guillin, giving an upper bound for the probability that the distribution of a random trajectory is too far from the stationary measure.

The case $j = i$ is obvious: the only term in the right-hand side is $C_1 h$, being an estimate for the distance to the smoothened convolution:

$$\begin{aligned}\mathcal{T}_P^{(c_{T_i})}(\mu_{T_i}^{(h)}, \mu_{T_i}) &\leq \int_{\mathbb{R}^d}\int_{\mathcal{U}_h(0)} |v| \cdot P\left(\max(|x-c_{T_i}|, |x+v-c_{T_i}|)\right) \frac{\mathrm{d}v}{vol(\mathcal{U}_h(0))}\,\mathrm{d}\mu(x) \\ &\leq \int_{\mathbb{R}^d} P(|x-c_{T_i}|+h)\cdot h\,\mathrm{d}\mu(x) \leq P(h)\|\mu(\cdot + c_{T_i})\|_P \cdot h = C_1 h,\end{aligned}$$

provided that $h \leq 1$ (because the norm $\|\mu^c_{T_i}\|_P$ is bounded due to the exponential tails of μ^c).

Let us now check the step of induction. Namely, assume that the conclusion holds for some $j \geq i$:

$$\mathcal{T}_P^{(c_{T_j})}(\Phi_i^j(\mu_{T_i}^{(h)}), \mu_{T_j}) \leq \sum_{k=i}^{j-1} \frac{\Delta T_k}{T_{k+1}} (\Delta T_k)^{-\beta} \left(\frac{T_j}{T_k}\right)^A + C_1 h \left(\frac{T_j}{T_i}\right)^A,$$

and check it for $j+1$. To do this, first shift the center of the translation distance from $c_{T_{j+1}}$ to c_{T_j}: from Proposition 2.5, we have

$$\mathcal{T}_P^{(c_{T_{j+1}})}(\cdot,\cdot) \le (1 + C_1' \cdot |c_{T_{j+1}} - c_{T_j}|)\mathcal{T}_P^{(c_{T_j})}(\cdot,\cdot),$$

provided that $|c_{T_{j+1}} - c_{T_j}| \le 1$. On the other hand, we have by Lemma 2.13

$$|c_{T_{j+1}} - c_{T_j}| \le \mathrm{Lip}_{K_{\alpha,C}}(c) \cdot \mathcal{T}_P^{(c_{T_j})}(\mu_{T_{j+1}}, \mu_{T_j}) \le c_1' \cdot \frac{\Delta T_j}{T_{j+1}},$$

so finally

$$\mathcal{T}_P^{(c_{T_{j+1}})}(\cdot,\cdot) \le \left(1 + C_1' \cdot \frac{\Delta T_j}{T_{j+1}}\right) \mathcal{T}_P^{(c_{T_j})}(\cdot,\cdot) \le \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \mathcal{T}_P^{(c_{T_j})}(\cdot,\cdot). \tag{3.9}$$

Now, the map Π is Lipschitz on $K_{\alpha,C}$ by Proposition 2.9, so for any two measures ν_1, ν_2, one has

$$\mathcal{T}_P(\Phi_j^{j+1}(\nu_1), \Phi_j^{j+1}(\nu_2)) \le \left(1 + \frac{\Delta T_j}{T_{j+1}}(\mathrm{Lip}_{K_{\alpha,C}}(\Pi) + 1)\right) \mathcal{T}_P(\nu_1,\nu_2) \le \left(\frac{T_{j+1}}{T_j}\right)^{A_2} \mathcal{T}_P(\nu_1,\nu_2).$$

Substituting for ν_1 and ν_2 respectively the translated by c_{T_j} images of measures $\Phi_i^j(\mu_{T_i}^{(h)})$ and μ_{T_j} respectively, we see that

$$\mathcal{T}_P^{(c_{T_j})}(\Phi_i^{j+1}(\mu_{T_i}^{(h)}), \Phi_j^{j+1}(\mu_{T_j})) \le \left(\frac{T_{j+1}}{T_j}\right)^{A_2} \mathcal{T}_P^{(c_{T_j})}(\Phi_i^j(\mu_{T_i}^{(h)}), \mu_{T_j}). \tag{3.10}$$

Now, using this estimate and Proposition 3.2 asserting that

$$\mathcal{T}_P^{(c_{T_j})}(\Phi_j^{j+1}(\mu_{T_j}), \mu_{T_{j+1}}) \le \left(\frac{\Delta T_j}{T_{j+1}}\right)(\Delta T_j)^{-\beta},$$

with $\beta = \min(8C_W, (5d)^{-1})$, we obtain

$$\begin{aligned}
\mathcal{T}_P^{(c_{T_{j+1}})}(\Phi_i^{j+1}(\mu_{T_i}^{(h)}), \mu_{T_{j+1}}) &\le \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \mathcal{T}_P^{(c_{T_j})}(\Phi_i^{j+1}(\mu_{T_i}^{(h)}), \mu_{T_{j+1}}) \le \\
&\le \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \left(\mathcal{T}_P^{(c_{T_j})}(\Phi_i^{j+1}(\mu_{T_i}^{(h)}), \Phi_j^{j+1}(\mu_{T_j})) + \mathcal{T}_P^{(c_{T_j})}(\Phi_j^{j+1}(\mu_{T_j}), \mu_{T_{j+1}})\right) \le \\
&\le \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \left(\left(\frac{T_{j+1}}{T_j}\right)^{A_2} \mathcal{T}_P^{(c_{T_j})}(\Phi_i^j(\mu_{T_i}^{(h)}), \mu_{T_j}) + \frac{\Delta T_j}{T_{j+1}}(\Delta T_j)^{-\beta}\right) = \\
&= \left(\frac{T_{j+1}}{T_j}\right)^{A_1+A_2} \mathcal{T}_P^{(c_{T_j})}(\Phi_i^j(\mu_{T_i}^{(h)}), \mu_{T_j}) + \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \frac{\Delta T_j}{T_{j+1}}(\Delta T_j)^{-\beta}.
\end{aligned} \tag{3.11}$$

Finally, we fix the choice of $A := A_1 + A_2$, and, using the induction assumption, the right-hand side of (3.11) is not greater than

$$\begin{aligned}
&\left(\frac{T_{j+1}}{T_j}\right)^{A} \left(\sum_{k=i}^{j-1} \frac{\Delta T_k}{T_{k+1}}(\Delta T_k)^{-\beta} \left(\frac{T_j}{T_k}\right)^{A} + C_1 h \left(\frac{T_j}{T_i}\right)^{A}\right) + \left(\frac{T_{j+1}}{T_j}\right)^{A_1} \frac{\Delta T_j}{T_{j+1}}(\Delta T_j)^{-\beta} \le \\
&\le \sum_{k=i}^{j-1} \frac{\Delta T_k}{T_{k+1}}(\Delta T_k)^{-\beta} \left(\frac{T_{j+1}}{T_k}\right)^{A} + C_1 h \left(\frac{T_{j+1}}{T_i}\right)^{A} + \left(\frac{T_{j+1}}{T_j}\right)^{A} \frac{\Delta T_j}{T_{j+1}}(\Delta T_j)^{-\beta} = \\
&= \sum_{k=i}^{j} \frac{\Delta T_k}{T_{k+1}}(\Delta T_k)^{-\beta} \left(\frac{T_{j+1}}{T_k}\right)^{A} + C_1 h \left(\frac{T_{j+1}}{T_i}\right)^{A}.
\end{aligned}$$

The induction step is proved.

Now, for j large enough, we choose $i = [j^{1-\delta}]$. Then, we have by the preceding inequalities

$$\mathcal{T}_P^{(c_{T_j})}(\mu_{T_j}, \Phi_i^j(\mu_{T_i}^{(h)})) \le \sum_{k=i}^{j-1} \frac{(\Delta T_k)^{1-\beta}}{T_{k+1}} \left(\frac{T_j}{T_k}\right)^A + C_1 h \left(\frac{T_j}{T_i}\right)^A, \tag{3.12}$$

provided that the right-hand side does not exceed C_3.

Denote by C_1' a generic constant. Let us estimate the first term in the right-hand side:

$$\sum_{k=i}^{j-1} \frac{\Delta T_k}{T_{k+1}} (\Delta T_k)^{-\beta} \left(\frac{T_j}{T_k}\right)^A \le \sum_{k=i}^{j-1} \frac{2}{k} (\Delta T_k)^{-\beta} \left(\frac{T_j}{T_k}\right)^A \le$$
$$\le \sum_{k=i}^{j-1} \frac{2}{i} (\Delta T_i)^{-\beta} \left(\frac{T_j}{T_i}\right)^A \le C_1' \cdot j \frac{(i^{1/2})^{-\beta}}{i} \left(\frac{j^{3/2}}{i^{3/2}}\right)^A \le C_1' \cdot j^{(1+\frac{3}{2}A+\frac{\beta}{2})\delta - \frac{\beta}{2}}. \tag{3.13}$$

3.1.3 Decrease of energy

This section is devoted to the proof of Proposition 3.6. Actually, it will be a corollary of a result showing that the relative free energy $\mathcal{F}(\Phi_i^j(\mu)|\rho_\infty)$ decreases with a speed that can be explicitly controlled.

The decrease of energy can be estimated in the following way. First, we note that the only way for $\mu \in K_{\alpha,C}$ to have big free energy is to have big entropy. Hence, if this energy is sufficiently big, the free energy of the images $\Phi_i^j(\mu)$ decreases, up to an additive constant, as $\frac{T_i \mathcal{F}(\mu)}{T_j}$. Indeed, the self-interaction part stays uniformly bounded, as the set $K_{\alpha,C}$ is a no-exit set for the discretized dynamics. Now, the measure $\Phi_i^j(\mu)$ is a mixture of μ taken with the proportion $\frac{T_i}{T_j}$, and of various $\Pi(\Phi_i^{i'}(\mu))$. The latter ones are "nice" measures (having uniformly bounded densities). Hence, their entropies are uniformly bounded, and hence the entropy $\mathcal{H}(\Phi_i^j(\mu))$ is bounded by $\frac{T_i \mathcal{F}(\mu)}{T_j} + \mathrm{const}$.

The above arguments allow to show that the energy of $\Phi_i^j(\mu)$ becomes less than a uniform constant at the time $T_j = T_i \cdot \mathcal{F}(\mu)$. A finer technique is required to estimate the decrease speed once the free energy is sufficiently small. Namely: let $\varphi_\mu(\cdot) := \mathcal{F}_{W*\mu}(\cdot)$ be the free energy in the μ-generated potential. Then, one can easily see that the energy of a linear combination $(1-\varepsilon)\mu + \varepsilon\Pi(\mu)$ differs from the energy of μ by the difference $\varphi_\mu(\Pi(\mu)) - \varphi_\mu(\mu)$ plus the second-order terms coming from the self-interaction of the replaced part. Now, the measure $\Pi(\mu)$ is the measure with the least free energy in the potential $W * \mu$. Hence, to estimate the decrease speed of $\mathcal{F}(\Phi_i^j(\mu))$, we have to find a lower bound for $\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))$.

This is done with help of the displacement convexity of the functional $\mathcal{F}$. Namely, considering the optimal transport ν_s from μ to $\rho_\infty(\cdot + c_\mu)$, we notice that the free energy $\mathcal{F}(\nu_s|\rho_\infty)$ decreases at least linearly. On the other hand, for small values of s, up to the second order terms, this energy can be once again estimated as the energy of μ minus the difference $\varphi_\mu(\mu) - \varphi_\mu(\nu_s)$. With a well-chosen moment s, we obtain a measure ν_s with a good lower bound for this difference, and hence immediately (as $\Pi(\mu)$ is the global minimum for $\varphi_\mu(\cdot)$) obtain a lower bound for $\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))$. This lower bound will be obtained in terms of $\mathcal{F}(\mu|\rho_\infty)$ only, and thus the free energy of $\Phi_i^j(\mu)$ will be shown to decrease at least as quickly as a solution of some differential equation (see Lemma 3.12 below).

Formalizing the above arguments, we will first state the following lemma:

Lemma 3.11. *For any $\mu \in K_{\alpha,C}$, we have $\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu)) \geq g(\mathcal{F}(\mu|\rho_\infty))$, where*

$$g(E) = \begin{cases} C_7 \frac{E}{|\log E|^k}, & 0 \leq E \leq \varepsilon_0 < 1 \\ \frac{E}{\varepsilon_0} g(\varepsilon_0), & \varepsilon_0 < E \leq \varepsilon_1 \\ E + (g(\varepsilon_1) - \varepsilon_1), & E > \varepsilon_1 \end{cases}$$

is an increasing continuous function, and the constants $C_7, \varepsilon_0, \varepsilon_1$ depend only on α and C.

We postpone its proof, but we use it as a motivation for the next result, which immediately implies Proposition 3.6:

Lemma 3.12. *There exists n_0 such that for any $\mu \in K_{\alpha,C}$ and for any $j \geq i \geq n_0$, we have*

$$\mathcal{F}(\Phi_i^j(\mu)|\rho_\infty) \leq y(T_j),$$

where y is the unique solution to

$$\dot{y} = -\frac{1}{t}\frac{g(y)}{2}, \tag{3.14}$$

with the initial condition $y(T_i) = \max(\mathcal{F}(\mu_{T_i}), 1)$.

Proposition 3.6 is its immediate corollary, as the solution of (3.14) decreases exponentially for big energies y and has the form $y(t) = \exp\left\{-\sqrt[k+1]{\frac{C_7}{2}(k+1)\log\left(\frac{t}{T_0}\right)}\right\}$ for $y \leq \varepsilon_0$ (this situation happens for t large enough).

Finally, we need two easy auxiliary statements for the free energy:

Lemma 3.13. *For any absolutely continuous probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d; P)$, such that their repective free energies are finite, and for all $\lambda \in [0,1]$, we have*

$$\mathcal{F}((1-\lambda)\mu + \lambda\nu|\rho_\infty) \leq \mathcal{F}(\mu|\rho_\infty) - \lambda(\varphi_\mu(\mu) - \varphi_\mu(\nu)) + \\ + \frac{\lambda^2}{2}\iint (\mu-\nu)(x)W(x-y)(\mu-\nu)(y)\,\mathrm{d}x\mathrm{d}y. \tag{3.15}$$

Moreover, for all absolutely continuous probability measure $\mu \in \mathcal{P}(\mathbb{R}^d; P)$, we have

$$\varphi_\mu(\mu) - \varphi_\mu(\nu) = \mathcal{F}(\mu) - \mathcal{F}(\nu) + \frac{1}{2}\iint (\mu-\nu)(x)W(x-y)(\mu-\nu)(y)\,\mathrm{d}x\mathrm{d}y. \tag{3.16}$$

Proof. Note that $\mathcal{H}((1-\lambda)\mu + \lambda\nu) \leq (1-\lambda)\mathcal{H}(\mu) + \lambda\mathcal{H}(\nu) = \mathcal{H}(\mu) - \lambda(\mathcal{H}(\mu) - \mathcal{H}(\nu))$. So, it suffices to prove (3.15) with entropy terms removed from both sides (from both $\mathcal{F}$ and φ_μ in the right-hand side). After this removing, the formula becomes a Taylor expansion for a degree two polynomial. The same holds for (3.16), with a remark that the entropy terms are exactly the same in both sides. □

Corollary 3.14. *For any fixed α, C, there exists C'' such that for all $\mu \in K_{\alpha,C}$, for all $0 < \lambda < 1$, we have*

$$\mathcal{F}((1-\lambda)\mu + \lambda\Pi(\mu)|\rho_\infty) \leq \mathcal{F}(\mu|\rho_\infty) - \lambda(\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))) + C''\lambda^2.$$

Proof. For $\mu \in K_{\alpha,C}$, the integral that is the coefficient before λ^2 in (3.15) is uniformly bounded. □

Let us now prove the previous lemmas.

Proof of Lemma 3.12. Recall that, due to Corollary 3.14, we have for $\mu \in K_{\alpha,C}$,

$$\mathcal{F}((1-\lambda)\mu + \lambda\Pi(\mu)|\rho_\infty) \leq \mathcal{F}(\mu|\rho_\infty) - \lambda\left(\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))\right) + C''\lambda^2.$$

Now note that, if n_0 is chosen sufficiently big, we have for any j:

$$C''\frac{\Delta T_j}{T_{j+1}} \leq \frac{g(y(T_j))}{3}. \tag{3.17}$$

Indeed, the left-hand side of (3.17) decreases as $\frac{1}{j}$, while its right-hand side decreases as $\exp\{-\sqrt[k+1]{\frac{C_7}{2}(k+1)\log T_j}\} \geq \frac{1}{j}$. Now, for every $\check{\mu}_j := \Phi_i^j(\mu)$, we have $\check{\mu}_j \in K_{\alpha,C}$ due to Lemma 3.4 and hence due to Lemma 3.11:

$$\varphi_{\check{\mu}_j}(\check{\mu}_j) - \varphi_{\check{\mu}_j}(\Pi(\check{\mu}_j)) \geq g(\mathcal{F}(\check{\mu}_j|\rho_\infty)).$$

Hence, proving the statement of the lemma by induction on j, we have to deduce from $\mathcal{F}(\check{\mu}_j|\rho_\infty) \leq y(T_j)$ the analogous statement for $\check{\mu}_{j+1}$, given that

$$\mathcal{F}(\check{\mu}_{j+1}|\rho_\infty) \leq y(T_j) - g(y(T_j))\frac{\Delta T_j}{T_{j+1}} + C''\left(\frac{\Delta T_j}{T_{j+1}}\right)^2 \leq y(T_j) - \frac{2}{3}g(y(T_j))\frac{\Delta T_j}{T_{j+1}}.$$

Let $\theta_j = \log T_j$. Then, $\Delta\theta_j := \theta_{j+1} - \theta_j \leq \frac{4}{3}\frac{\Delta T_j}{T_{j+1}}$ for all j large enough. So, once again asking n_0 to be chosen sufficiently big, we have

$$\mathcal{F}(\check{\mu}_{j+1}|\rho_\infty) \leq y(T_j) - \frac{2}{3}\cdot\frac{3}{4}g(y(T_j))\Delta\theta_j = \tilde{y}(\theta_j) - \frac{g(\tilde{y}(\theta_j))}{2}\Delta\theta_j,$$

where $\tilde{y}(\theta) = y(e^\theta)$. We conclude by noticing that $g(y)$ is an increasing function of y. So, as $\tilde{y}(\theta)$ is the solution to the equation $\tilde{y}(\theta)' = -\frac{g(\tilde{y}(\theta))}{2}$, we have

$$\tilde{y}(\theta_j) - \frac{g(\tilde{y}(\theta_j))}{2}\Delta\theta_j \leq \tilde{y}(\theta_{j+1}),$$

hence $\mathcal{F}(\check{\mu}_{j+1}|\rho_\infty) \leq \tilde{y}(\theta_{j+1}) = y(T_{j+1})$, thus proving the induction step. □

Proof of Lemma 3.11. Note first that, for $\mu \in K_{\alpha,C}$, the integral

$$\iint (\mu - \Pi(\mu))(\mathrm{d}x)W(x-y)(\mu - \Pi(\mu))(\mathrm{d}y)$$

is bounded by a uniform constant. Thus, due to Lemma 3.13, $\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))$ admits a lower bound

$$\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu)) \geq \mathcal{F}(\mu|\rho_\infty) - C_\Delta \tag{3.18}$$

where the constant C_Δ is uniform over all $\mu \in K_{\alpha,C}$.

Now, let us give another way to estimate the difference $\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu))$. Indeed, $\Pi(\mu)$ is the global minimiser of $\mathcal{F}$, hence for any probability measure ρ, we have

$$\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu)) \geq \varphi_\mu(\mu) - \varphi_\mu(\rho). \tag{3.19}$$

Recall that the free energy functional $\mathcal{F}$ is displacement convex. Denote by $\xi_s = (1-s)\xi_0 + s\xi_1$, for $0 \leq s \leq 1$, the quadratic Wasserstein optimal transport between $\mu = \{\text{law of } \xi_0\}$ and $\rho_\infty(\cdot + c_\mu) = \{\text{law of } \xi_1\}$. Let $\nu_s = \{\text{law of } \xi_s\}$. Then,

$$\mathcal{F}(\nu_s|\rho_\infty) \geq (1-s)\mathcal{F}(\mu|\rho_\infty).$$

Thus, we have due to Lemma 3.13,

$$\begin{aligned}\varphi_\mu(\mu)-\varphi_\mu(\nu_s) &= \mathcal{F}(\mu|\rho_\infty)-\mathcal{F}(\nu_s|\rho_\infty)+\frac{1}{2}\iint(\nu_s-\mu)(\mathrm{d}x)W(x-y)(\nu_s-\mu)(\mathrm{d}y)\\ &\geq s\mathcal{F}(\mu|\rho_\infty)+\frac{1}{2}\iint(\nu_s-\mu)(\mathrm{d}x)W(x-y)(\nu_s-\mu)(\mathrm{d}y).\end{aligned}$$

Let us now estimate the second term in the right-hand side of this inequality. Indeed, let (η_0,η_1) be an independent copy of (ξ_0,ξ_1). Then

$$\begin{aligned}&\iint W(x-y)(\nu_s-\mu)(\mathrm{d}x)(\nu_s-\mu)(\mathrm{d}y)=\\ &=\mathbb{E}\left[W(\xi_0-\eta_0)-W(\xi_s-\eta_0)-W(\xi_0-\eta_s)+W(\xi_s-\eta_s)\right].\end{aligned}$$

For any fixed L, we can divide this expectation into two parts: the one corresponding to $\max_{i,j\in\{0,1\}}(|\xi_i|,|\eta_j|)>L$ and the one with $|\xi_i|\leq L$ and $|\eta_i|\leq L$ for $i=0,1$. We also remind that $\nu_i\in K_{\alpha,C_2}$ for $i=0,1$ and that P controls W as well as its first and second derivatives. So, there exists a positive constant $\tilde{C}$ such that

$$\begin{aligned}&\left|\mathbb{E}\left[W(\xi_0-\eta_0)-W(\xi_s-\eta_0)-W(\xi_0-\eta_s)+W(\xi_s-\eta_s)\right]\right|\\ \leq\ &\left|\mathbb{E}\left[W(\xi_0-\eta_0)-W(\xi_s-\eta_0)-W(\xi_0-\eta_s)+W(\xi_s-\eta_s)\right]\mathbb{1}_{\left\{\max_{i,j\in\{0,1\}}(|\xi_i|,|\eta_j|)\leq L\right\}}\right|\\ +\ &\int_L^{+\infty}W(2l)\,\mathrm{d}\mathbb{P}_{\max(\xi_0,\xi_1,\eta_0,\eta_1)}(l)\\ \leq\ &\mathbb{E}\left[\max_{|x|\leq 4L}P(|x|)|\xi_0-\xi_s||\eta_0-\eta_s|\mathbb{1}_{\left\{\max_{i,j\in\{0,1\}}(|\xi_i|,|\eta_j|)\leq L\right\}}\right]+4\int_L^{\infty}P(2l)\,\mathrm{d}(1-C_2e^{-\alpha l})^4\\ \leq\ &s^2P(4L)W_2^2(\nu_0,\nu_1)+\tilde{C}P(2L)e^{-\alpha L}.\end{aligned}$$

So, using the already mentioned comparison $W_2^2(\mu,\rho_\infty)\leq\frac{2}{C_W}\mathcal{F}(\mu|\rho_\infty)$, we have

$$\begin{aligned}\varphi_\mu(\mu)-\varphi_\mu(\nu_s) &\geq s\mathcal{F}(\mu|\rho_\infty)-s^2P(4L)W_2^2(\mu,\rho_\infty)-\tilde{C}P(2L)e^{-\alpha L}\\ &\geq s\mathcal{F}(\mu|\rho_\infty)-\frac{2}{C_W}s^2P(4L)\mathcal{F}(\mu|\rho_\infty)-\tilde{C}P(2L)e^{-\alpha L}.\end{aligned}$$

We decide from now on to fix $s=\frac{C_W}{4P(4L)}$, with the choice of L to be fixed later. Then, $s-\frac{2}{C_W}s^2P(4L)=\frac{s}{2}$ and

$$\varphi_\mu(\mu)-\varphi_\mu(\nu_s)\geq\frac{C_W}{8P(4L)}\mathcal{F}(\mu|\rho_\infty)-\tilde{C}P(2L)e^{-\alpha L}. \tag{3.20}$$

For $\mathcal{F}(\mu|\rho_\infty)$ sufficiently small, fixing $L=\frac{2}{\alpha}|\log\mathcal{F}(\mu|\rho_\infty)|$, we have

$$\frac{C_W}{16P(4L)}\mathcal{F}(\mu|\rho_\infty)\geq\tilde{C}P(2L)e^{-\alpha L} \tag{3.21}$$

and hence the right-hand side of (3.20) is estimated from below by

$$\frac{C_W}{16P(4L)}\mathcal{F}(\mu|\rho_\infty)\geq\frac{C_7}{|\log\mathcal{F}(\mu|\rho_\infty)|^k}\mathcal{F}(\mu|\rho_\infty).$$

So taking $g(E):=\frac{C_7}{|\log E|^k}E$ for such values of $E=\mathcal{F}(\mu|\rho_\infty)$, the conclusion of the lemma is satisfied for such E's. Next, fixing ε_0 to be such that (3.21) is satisfied for

$\mathcal{F}(\mu|\rho_\infty) \le \varepsilon_0$, and for any $\mathcal{F}(\mu|\rho_\infty) \ge \varepsilon_0$, choosing the same L as for $\mathcal{F}(\mu|\rho_\infty) = \varepsilon_0$, we have

$$\frac{C_W}{8P(4L)}\mathcal{F}(\mu|\rho_\infty) - \tilde{C}P(2L)e^{-\alpha L} \ge \frac{\mathcal{F}(\mu|\rho_\infty)}{\varepsilon_0} g(\varepsilon_0),$$

what allows to deduce

$$\varphi_\mu(\mu) - \varphi_\mu(\Pi(\mu)) \ge \begin{cases} g(\mathcal{F}(\mu|\rho_\infty)) & \text{if} \quad \mathcal{F}(\mu|\rho_\infty) \le \varepsilon_0, \\ \frac{\mathcal{F}(\mu|\rho_\infty)}{\varepsilon_0} g(\varepsilon_0) & \text{if} \quad \mathcal{F}(\mu|\rho_\infty) > \varepsilon_0. \end{cases} \tag{3.22}$$

Finally, taking the maximum between the right-hand side of (3.18) and (3.22), we obtain the desired conclusion. □

3.2 Proof of Theorem 1.11

To prove that the center of the measure converges a.s., we will show that the sequence (c_{T_n}) converges a.s. (with a well-chosen time-step T_n) as a Cauchy sequence. Our strategy will consist in using some of the latter estimates of §3.1 to show that the series of general term $|c_{T_{n+1}} - c_{T_n}|$ converges a.s. and that the oscillations $\underset{t\in[T_n,T_{n+1}]}{osc} c_t$ go to zero. This will imply the existence of the limit of c_t.

As it has been already shown in (2.7), we have for any $t \in [T_n, T_{n+1}]$

$$|c_t - c_{T_n}| \le \int_{T_n}^t \frac{P(|X_u - c_u|)}{C_W u}\,\mathrm{d}u \le \int_{T_n}^t \frac{P(L_n + |c_u - c_{T_n}|)}{C_W u}\,\mathrm{d}u \le P(L_n + C_3)\frac{\Delta T_n}{T_n}.$$

Thus, almost surely one has $\underset{t\in[T_n,T_{n+1}]}{osc} c_t \to 0$ as $n \to +\infty$. So, to prove Theorem 1.11, it suffices to show that the sequence (c_{T_n}) converges almost surely.

Now, let us estimate the distance $|c_{T_{n+1}} - c_{T_n}|$. Indeed, we have

$$\mu_{T_{n+1}} = \mu_{T_n} + \frac{\Delta T_n}{T_{n+1}}(\mu_{[T_n,T_{n+1}]} - \mu_{T_n}).$$

Translating c_{T_n} to the origin, using the decrease estimates of §3.1 and recalling that $c(\cdot) : K^0_{\alpha,C} \to \mathbb{R}^d$ is a $\mathcal{T}_P$-Lipschitz function, we see that

$$|c_{T_{n+1}} - c_{T_n}| \le \mathrm{Lip}_{K^0_{\alpha,C}}(c) \cdot \frac{\Delta T_n}{T_{n+1}} \cdot \mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \mu_{T_n}).$$

As in §3.1.2, the distance in the right-hand side can be estimated as the sum of two distances:

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \mu_{T_n}) \le \mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \Pi(\mu_{T_n})) + \mathcal{T}_P^{(c_{T_n})}(\Pi(\mu_{T_n}), \mu_{T_n}). \tag{3.23}$$

We already have an estimation of the first term in this sum:

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]}, \Pi(\mu)) \le T_n^{-\min\left(\frac{8}{3}C_W, \frac{1}{15d}\right)}.$$

On the other hand, the limit density ρ_∞ is a fixed point of the map Π. And the map Π being Lipschitz on $K^0_{\alpha,C}$, the second summand in (3.23) can be estimated as

$$\mathcal{T}_P^{(c_{T_n})}(\Pi(\mu_{T_n}), \mu_{T_n}) \le (\mathrm{Lip}_{K^0_{\alpha,C}}(\Pi) + 1) \cdot \mathcal{T}_P(\mu^c_{T_n}, \rho_\infty).$$

The latter distance is already estimated in the proof of Theorem 1.10: almost surely for n large enough, we have

$$\mathcal{T}_P(\mu^c_{T_n}, \rho_\infty) \le \exp\{-a \sqrt[k+1]{\log T_n}\}.$$

Finally, adding the estimates for the first and the second terms in (3.23), we obtain that for all n sufficiently big,

$$\mathcal{T}_P^{(c_{T_n})}(\mu_{[T_n,T_{n+1}]},\mu_{T_n}) \le T_n^{-\min\left(\frac{8}{3}C_W,\frac{1}{15d}\right)} + (\mathrm{Lip}_{K^0_{\alpha,C}}(\Pi)+1)\exp\{-a\sqrt[k+1]{\log T_n}\}$$

and hence

$$|c_{T_{n+1}}-c_{T_n}| \le \mathrm{Lip}_{K^0_{\alpha,C}}(c)\cdot\frac{\Delta T_n}{T_{n+1}}\left(T_n^{-\min\left(\frac{8}{3}C_W,\frac{1}{15d}\right)} + (\mathrm{Lip}_{K^0_{\alpha,C}}(\Pi)+1)\exp\{-a\sqrt[k+1]{\log T_n}\}\right).$$

We choose $T_n = n^{3/2}$ and so $\frac{\Delta T_n}{T_n} \asymp n^{-1}$. Hence

$$\sum_n |c_{T_{n+1}} - c_{T_n}| \le \mathrm{const}\cdot\sum_n \frac{1}{n^{1+\min(4C_W,1/(10d))}} + C_1'\cdot\sum_n \frac{\exp\{-a\sqrt[k+1]{\frac{3}{2}\log n}\}}{n}.$$

Both series in the right-hand side converge, and thus the series of general term $|c_{T_{n+1}} - c_{T_n}|$ converges almost surely. This concludes the proof.

Appendix 1: Singularity at $t = 0$

Let us now prove that a solution to the equation (1.1) with any initial condition at $t = 0$ (where the equation has a singularity) exists and is unique.

Proposition 3.15. *For any x_0 and almost every trajectory B_t of the Brownian motion, a (continuous at $t = 0$) solution X_t to the equation* (1.1) *with the initial condition $X_0 = x_0$ exists on all the interval $[0,+\infty)$ and is unique.*

Proof. As Proposition 2.2 provides us global existence and uniqueness of solutions, starting from any arbitrary positive time $r > 0$, it suffices to check the existence and uniqueness on some interval $[0,\delta)$. For the sake of simplicity of notation, suppose that $x_0 = 0$.

Let $\delta_1 > 0$ be such that for all $0 \le t \le \delta_1$, $|B_t| \le \frac{1}{2}$ and $\delta_1 \sup_{|x|\le 2}|\nabla W(x)| \le \frac{1}{3}$. We work on the trajectories which are staying inside $\mathcal{U}_1(0)$, the unit ball centered in $x_0 = 0$. So, we consider $X_\bullet : [0,\delta_1) \to \mathcal{U}_1(0)$, $t \mapsto X_t$. Denote by μ^X the empirical measure of the process X. Then, the application $\chi : X \mapsto \tilde{X}$ such that

$$\tilde{X}_t = B_t + \int_0^t \nabla W * \mu_s^X(\tilde{X}_s)\mathrm{d}s,$$

is well-defined on this space, and $\tilde{X}_t$ also remains stuck in $\mathcal{U}_1(0)$. Indeed, for any time $t \le \delta_1$, such that the solution $\tilde{X}$ is defined on $[0,t]$ and stays in $\mathcal{U}_1(0)$, we have

$$\begin{aligned}\left|\int_0^t \nabla W * \mu_s^X(\tilde{X}_s)\,\mathrm{d}s\right| &= \left|\int_0^t \frac{1}{s}\int_0^s \nabla W(\tilde{X}_s - X_u)\,\mathrm{d}u\,\mathrm{d}s\right| \\ &\le \int_0^t \frac{1}{s}\int_0^s \sup_{|x|\le 2}|\nabla W(x)|\,\mathrm{d}u\,\mathrm{d}s \le \delta_1 \sup_{|x|\le 2}|\nabla W(x)| \le \frac{1}{3}.\end{aligned}$$

Thus, if there existed a time $t_0 \le \delta_1$ such that $|\tilde{X}_{t_0}| \ge 7/8$ for the first time, then we would have $|\tilde{X}_{t_0}| \le 1/2 + 1/3$, which would contradict the bound $|\tilde{X}_{t_0}| \ge 7/8$. So, $\tilde{X}$ stays in $\mathcal{U}_1(0)$ for any $0 \le t \le \delta_1$.

Let us now show that for $\delta < \delta_1$ sufficiently small, the map χ is a contraction on the space of continuous maps $X_\bullet$ from $[0,\delta]$ to $\mathcal{U}_1(0)$ with $X_0 = 0$. Indeed, consider now two trajectories $X^{(1)}$ and $X^{(2)}$, realizing a coupling with the same Brownian motion, and their respective images (by χ) $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$. Then, denoting by $\mathrm{Lip}(W)$ the Lipschitz constant of ∇W on the ball of radius 2, $\mathrm{Lip}(W) := \sup\{||\nabla^2 W(x)|| : |x| \le 2\}$, we have

$$\begin{aligned}
|\tilde{X}_t^{(1)} - \tilde{X}_t^{(2)}| &= \left| \int_0^t \nabla W * \mu_s^{X^{(1)}}(\tilde{X}_s^{(1)})\, \mathrm{d}s - \int_0^t \nabla W * \mu_s^{X^{(2)}}(\tilde{X}_s^{(2)})\, \mathrm{d}s \right| \\
&= \left| \int_0^t \frac{1}{s} \int_0^s \nabla W(\tilde{X}_s^{(1)} - X_u^{(1)}) - \nabla W(\tilde{X}_s^{(2)} - X_u^{(2)})\, \mathrm{d}u\, \mathrm{d}s \right| \\
&\le \int_0^t \frac{1}{s} \int_0^s |\nabla W(\tilde{X}_s^{(1)} - X_u^{(1)}) - \nabla W(\tilde{X}_s^{(2)} - X_u^{(2)})|\, \mathrm{d}u\, \mathrm{d}s \\
&\le \int_0^t \frac{1}{s} \int_0^s \mathrm{Lip}(W)(|\tilde{X}_s^{(1)} - \tilde{X}_s^{(2)}| + |X_u^{(1)} - X_u^{(2)}|)\, \mathrm{d}u\, \mathrm{d}s \\
&\le t\, \mathrm{Lip}(W)(||\tilde{X}^{(1)} - \tilde{X}^{(2)}||_{C([0,\delta])} + ||X^{(1)} - X^{(2)}||_{C([0,\delta])}),
\end{aligned}$$

where $||X||_{C([0,\delta])}$ is the norm of X on the space $C([0,\delta])$. As $t \le \delta$, we conclude that $||\tilde{X}^{(1)} - \tilde{X}^{(2)}||_{C([0,\delta])} \le \delta\ \mathrm{Lip}(W)(||\tilde{X}^{(1)} - \tilde{X}^{(2)}||_{C([0,\delta])} + ||X^{(1)} - X^{(2)}||_{C([0,\delta])})$. As soon as $\delta\ \mathrm{Lip}(W) < 1$, we have

$$||\tilde{X}^{(1)} - \tilde{X}^{(2)}||_{C([0,\delta])} \le \frac{\delta\ \mathrm{Lip}(W)}{1 - \delta\ \mathrm{Lip}(W)} ||X^{(1)} - X^{(2)}||_{C([0,\delta])}.$$

We choose δ such that $\delta\ \mathrm{Lip}(W) < 1/3$ and then χ is a contraction, as stated, with $\mathrm{Lip}(\chi) \le 1/2$. So, we have obtained existence and uniqueness of the solution on $[0,\delta]$. □

Appendix 2: Non-symmetric counter-example

We end this paper with an example showing that for a non-symmetric interaction potential W, the conclusion of Theorem 1.2 does not hold.

Consider a non-symmetric quadratic interaction potential $W(x) = \frac{1}{2}(x-1)^2$. Then, the averages of the process $(X_t)_t$ defined by (1.1), that is $\frac{1}{t}\int_0^t X_s \mathrm{d}s = c_t - 1$ tends to $+\infty$.

To motivate this behaviour, heuristically, we first note that, for any finite-variance measure ν, the convolution $W * \nu$ equals

$$W * \nu(x) = \frac{1}{2}(x-1)^2 - (x-1)\mathbb{E}(\nu) + \frac{1}{2}\mathbb{E}(\nu^2) = \frac{x^2}{2} - (\mathbb{E}(\nu)+1)x + C_1'.$$

Hence $\Pi(\nu)$ is the Gaussian law $\mathcal{N}(1+\mathbb{E}(\nu), 1)$. Thus, if we consider a trajectory of the approximating flow $\dot{\nu}_t = \frac{1}{t}(\Pi(\nu_t) - \nu_t)$, we have for its mean value

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\nu_t = \frac{1}{t}(\mathbb{E}\nu_t + 1 - \mathbb{E}\nu_t) = \frac{1}{t},$$

and so $\mathbb{E}\nu_t \sim \log t$.

For a formal proof, note that (as the interaction potential is a polynomial of degree 2) the evolution of the couple (X_t, c_t), where $c_t = c(\mu_t) = \mathbb{E}\mu_t + 1$ is Markovian:

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}B_t - (X_t - c_t)\mathrm{d}t, \\ \dot{c}_t = \frac{1}{t}(X_t - c_t + 1). \end{cases}$$

Changing X_t to $Y_t = X_t - c_t$, we obtain

$$\begin{cases} \mathrm{d}Y_t = \mathrm{d}B_t - \left(Y_t + \frac{1}{t}(Y_t+1)\right)\mathrm{d}t, \\ \dot{c}_t = \frac{1}{t}(Y_t + 1). \end{cases}$$

The equation on Y does not contain c_t. So, explicit solution of this system and rigorous justification of the desired properties become an exercise.

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