Electronic Journal of Probability

Vol. 9 (2004), Paper no. 19, pages 575-593.

Journal URL
http://www.math.washington.edu/~ejpecp/

Asymptotic laws for nonconservative self-similar fragmentations

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Abstract We consider a self-similar fragmentation process in which the generic particle of mass x is replaced by the offspring particles at probability rate x^α, with positive parameter α. The total of offspring masses may be both larger or smaller than x with positive probability. We show that under certain conditions the typical mass in the ensemble is of the order $t^{-1/\alpha}$ and that the empirical distribution of masses converges to a random limit which we characterise in terms of the reproduction law.

Keywords: self-similar fragmentation, power laws

AMS subject classification (2000): 60G18, 60J25

Submitted to EJP on July 1, 2004. Final version accepted on July 13, 2004.

1 Introduction

We study the following continuous-time model of particle fragmentation. Each particle in ensemble is characterised by a positive quantity which we call *mass*. The generic particle of mass x lives a random exponentially distributed time with parameter x^α. During the life-time the mass does not vary and at the end the particle splits into fragments of masses $x\xi_j$, where $\{\xi_j\}$ is independent of the lifetime of the particle and follows a given probability distribution called *reproduction law*. Each particle is autonomous, meaning that the splitting probability rate and the descendant fragment masses depend only on the mass of the particle and not on the history of this or other coexisting particles. We are interested in the case $\alpha > 0$, when particles with smaller mass tend to live longer. We refer to the proceedings [13] and the survey [1] for a number of examples arising in physics and chemistry.

The idea of the model was suggested by Kolmogorov in [20], and the first results are due to Filippov [18]. Brennan and Durrett [16, 17] rediscovered an instance of the model in the context of binary interval splitting. In a recent series of papers Bertoin [5, 6, 7] introduced more involved fragmentation processes in which a particle may produce infinitely many generations within an arbitrary time period or may undergo a continuous mass erosion; see also [2, 4, 8, 24, 25, 29] for related examples.

The research so far was mainly focussed on the *conservative* case $\Sigma\,\xi_j = 1$ when the total mass is preserved by each splitting. It has been shown that the particles demonstrate quite a regular long-run behaviour: the typical mass in the ensemble is of the order $t^{-1/\alpha}$ and a scaled empirical distribution of masses converges to a nonrandom limit. Filippov [18] proved the convergence of empirical distributions in probability (see also [7]), while Brennan and Durrett [17] showed convergence with probability one in the binary case. Baryshnikov and Gnedin [3] studied a sequential interval packing problem which may be seen as a binary instance of *dissipative* fragmentation with $\Sigma\,\xi_j \le 1$ and $\mathbb{P}(\Sigma\,\xi_j < 1) > 0$, and proved convergence of the mean measures associated with the empirical distributions.

Conservative or dissipative fragmentations can be treated both as continuous-time interval splitting schemes, similar to discrete-time random recursive constructions (as in [23]), or as state-discretised processes with values in Kingman's partition structures [5, 6, 7, 8, 10]. These approaches fail completely if the reproduction law allows the possibility of mass *creation*, when the total mass of the offspring may exceed the mass of the parent particle. Such 'improper fragmentations' are both physically plausible and useful in the situations where the generalised mass models some nonadditive quantity like, e.g. surface energy by aerosols.

By allocating particle of mass x at $-\log x$ the fragmentation process can be seen as a branching random walk with location-dependent sojourn times. From this viewpoint, a constraint on the sum of masses seems rather odd, which suggests that such a condition is not essential for the asymptotics. In this paper we argue that this intuition is indeed correct, in the sense that the convergence of properly scaled empirical distributions of masses holds under fairly general assumptions on the reproduction law. Though we do require that the individual offspring masses cannot exceed the parent mass, there is no constraint on the total offspring mass. A new feature appearing in the general nonconservative case is that the limit of scaled empirical distributions is not completely deterministic, rather involves a random factor which admits a characterisation by a distributional fixed-point equation. This phenomenon reminds us, of course, of strong limit theorems for branching random walks, see e.g. Biggins [15] and Uchiyama [30]; we shall discuss the connection and the differences later on.

The rest of this work is organised as follows. Notation and basic assumptions are given in Section 2, and Section 3 presents the homogeneous case $\alpha = 0$ which has a simple connection with branching random walks. Then we compute the first moment of power sums and then determine its asymptotics using a contour integral. This yields the convergence of mean measures in Section 5. An alternative approach based on a limit theorem of Brennan and Durrett is presented in Section 6. In Section 7, we consider a remarkable martingale, which plays the same role as the so-called additive martingales in the homogeneous case. The main result of convergence of scaled empirical measures is proved in Section 8. Then we provide some examples, and finally, in Section 10, we sketch the extension of the preceding results to self-similar fragmentations with possibly infinite reproduction measure.

2 Definitions and assumptions on the reproduction law

It will be assumed that $\alpha > 0$ unless explicitly indicated.

The collection $\{\xi_j\}$ of offspring masses of a unit particle is identified with a random sequence of positive real numbers which is either finite or converges to 0. We shall also view $\{\xi_j\}$ as a random set, defined formally as a counting random measure $\Sigma\, \delta_{\xi_j}$ on $]0,1]$. Basically we require that

$$\{\xi_j\} \subset]0,1]\,, \qquad \mathbb{E}\#\{\xi_j\} > 1\,, \qquad \mathbb{P}(1 \in \{\xi_j\}) < 1\,, \qquad \{\xi_j\} \neq \emptyset\,. \tag{1}$$

Many features of the fragmentation process can be expressed in terms of the *structural measure*

$$\sigma(B) = \mathbb{E}\,\#(\{\xi_j\} \cap B)\,, \qquad B \subset]0,1],$$

and its Mellin transform

$$\phi(\beta) = \int_0^1 x^\beta \sigma(\mathrm{d}x) = \mathbb{E} \sum_j \xi_j^\beta$$

which we call the *characteristic function*. In particular, the first three conditions in (1) amount to the assumptions that σ is supported by $]0,1]$, that $\sigma\,[0,1] > 1$ and that $\sigma\{1\} < 1$.

Because $|\phi(\beta)| \le \phi(\Re\,\beta)$, the natural domain of definition of ϕ is a complex halfplane to the right of the convergence abscissa β_a of the integral. If $\beta_a = -\infty$ the halfplane is the whole plane, and otherwise the halfplane may be open or closed. The characteristic function is analytical in the halfplane, strictly decreasing on the real axis, and in view of (1) satisfies $\phi(0) > 1$ and $\phi(\beta) \to \sigma\{1\} < 1$ as $\Re\beta \to \infty$.

It is crucial for our results and will be assumed throughout that there exists the *critical exponent* $\beta^* > 0$ satisfying the equation

$$\phi(\beta) = 1\,. \tag{2}$$

If the critical exponent exists then it is unique and there are no solutions to (2) in the halfplane $\Re\beta > \beta^*$. And if some $\beta \neq \beta^*$ with $\Re\beta = \beta^*$ satisfies (2) then σ is *arithmetic*, meaning that σ is a discrete measure supported by a geometric sequence. Note that in the conservative case $\beta^* = 1$ and in the dissipative case $\beta^* < 1$.

Equation (2) has no real solutions (thus the critical exponent is not defined) only if $\phi(\beta_a+) < 1$. An example of this situation is the measure

$$\sigma(\mathrm{d}x) = c\, \mathbf{1}_{\{x<1/2\}}\, x^{-3/2} \log^{-2} x\; \mathrm{d}x$$

with a suitable choice of c, and $\beta_a = 1/2$.

Further assumptions about the reproduction law will be introduced in a due place. Specifically, the L^2-convergence result in Section 8 requires that

$$\mathbb{E}\left(\sum_j \xi_j^{\beta^*}\right)^2 < \infty\,. \tag{3}$$

Example. Consider a dissipative reproduction law induced by the uniform stick-breaking. Let $U_0, U_1, \ldots$ be i.i.d. uniform, and let

$$\xi_j = (1 - U_j)\prod_{k=0}^{j-1} U_k\,, \qquad j = 1, 2, \ldots$$

meaning that the uniform portion of the mass $1 - U_0$ is lost, and the rest is fragmented by the 'random alms' principle (as Halmos called stick-breaking). Equivalently, the offspring $\{\xi_j\}$ of a unit particle can be seen as the collection of masses of a random Poisson-Dirichlet distribution with parameter 1, upon removing a mass selected by a size-biased pick. Building the characteristic function

$$\phi(\beta) = \sum_{j=1}^{\infty} \frac{1}{(1+\beta)^{j+1}} = \frac{1}{\beta(\beta+1)}$$

we see that the abscissa is at $\beta_a = 0$ and the critical exponent is $\beta^* = (-1+\sqrt{5})/2$.

It should be noted that there is no any substantial constraint on σ imposed by the requirement that σ be a structural measure. Given σ on $]0,1]$, satisfying $\sigma\,[0,1] > 1$ and $\sigma\{1\} < 1$, a possible reproduction law satisfying (1) with this structural measure can be constructed as follows. Decompose $\sigma = \sigma_1 + \sigma_2$ so that σ_1 be probability measure and σ_2 some other measure. Let ξ_1 be a random point with distribution σ_1 and let $\xi_2, \xi_3, \ldots$ be the atoms of a Poisson point process on the unit interval, with intensity measure σ_2. Clearly the point process $\{\xi_j\}$ will have intensity σ.

Let $X(t) = \{X_j(t)\}$ be the set of particles coexisting at time $t \geq 0$. We assume that the process starts with a sole particle of unit mass, that is $X(0) = \{1\}$. Denoting $X^{(y)}$ the fragmentation process that starts with a particle of mass $y > 0$ we have the fundamental self-similarity identity

$$X^{(y)}(t) \stackrel{d}{=} y\,X(ty^{\alpha})\,. \tag{4}$$

We shall find useful to consider the power-sum functionals

$$M(t,\beta) = \sum_j X_j^{\beta}(t)$$

and their means

$$m(t,\beta) = \mathbb{E}M(t,\beta).$$

For shorthand we sometimes refer to the mass of a particle raised to the power β as the *β-mass*, thus $M(t,\beta)$ is the total β-mass of the population existing at time t. Two instances with obvious physical interpretations are the 0-mass equal to the number of particles, and the 1-mass equal to the total mass of the ensemble.

Observe the obvious interchangeability between parameters α and β. In the variables $\widetilde{\xi}_j = \xi_j^\beta$ the fragmentation process has the life-time parameter α/β and differs only by a particle-wise transformation of masses. Thus, for $\beta = \beta^*$ the transformed fragmentation process has the mean-value mass conservation property $\mathbb{E}\,\Sigma\,\widetilde{\xi}_j = 1$. The change of variables with $\beta < 0$ yields mathematically equivalent (though physically curious) process where the individual 'fragments' grow, but the decaying life-times slow down the total increase of mass.

3 The homogeneous case

In this section we briefly inquire into the *homogeneous* case $\alpha = 0$ where each life-time is mean one exponential, as in [5].

We argue that in the homogeneous case each $M(t, \beta)$ is finite (meaning that the series converges absolutely for $\Re\beta > \beta_a$) and the mean total β-mass is given by the formula

$$m(t, \beta) = \exp(-t\psi(\beta)) \tag{5}$$

where and henceforth

$$\psi(\beta) = 1 - \phi(\beta).$$

The formula is valid without any constraint on the fragmentation law $\{\xi_j\}$. For example, when each particle always splits in two then $M(t, 0)$ is a binary Yule process and $m(t, 0) = e^t$, as is well known [28].

The formula is shown by partitioning all particles that ever existed in generations, starting with the sole progenitor (generation 0). Clearly, the mean total β-mass of generation k is $\phi^k(\beta)$. Let $\tau_1, \tau_2, \ldots$ be the epochs of a rate one Poisson process. The probability that a given particle in generation k contributes to $m(t, \beta)$ is $\mathbb{P}(\tau_k \le t < \tau_{k+1})$, thus $m(t, \beta)$ is the same as in the 'generational' model where particles of each generation $1, 2, \ldots$ appear simultaneously at times $\tau_1, \tau_2, \ldots$. But in the generational model the index of generation existing at time t is a Poisson variable with parameter t, thus the mean β-mass at time t is indeed

$$e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!}\, \phi^k(\beta) = \exp(-t\psi(\beta))\,.$$

Under assumption (1), it follows that in the homogeneous case $|m(t, \beta)|$ grows exponentially for $\Re\,\beta < \beta^*$, and decays exponentially for $\Re\,\beta > \beta^*$. Moreover, for $\Re\,\beta > \beta^*$ the total β-mass of all particles that ever existed has a finite mean value $1/\psi(\beta)$, and this also holds for arbitrary $\alpha \ge 0$.

The process $Z(t) = \{Z_j(t)\}$ with $Z_j(t) = -\log X_j(t)$ is a continuous-time *branching random walk*, as studied in [30, 15], and in this context the formula (5) is, of course, well known. It is read from the work of Biggins [14, 15] that for every $\beta > \beta_a$, the process

$$W(t, \beta) \;:=\; \exp(t\psi(\beta)) \sum_j X_j^\beta(t)\,, \qquad t \ge 0$$

is a martingale with càdlàg paths, which converges almost surely and in mean as $t \to \infty$ provided that $\psi(\beta) < \beta\psi'(\beta)$, see [10] for details. Note that this holds in the special case when $\beta = \beta^*$ is the critical exponent, for which there is the identity $W(t, \beta^*) = M(t, \beta^*)$.

When furthermore the reproduction law is not arithmetic, the asymptotic behaviour of the empirical distribution of masses can be described as follows: for every continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support,

$$\lim_{t\to\infty} \sqrt{t}\,\mathrm{e}^{-t(\beta\psi'(\beta)-\psi(\beta))} \sum_j f(t\psi'(\beta) - Z_j(t)) \;=\; \frac{W(\infty, \beta)}{\sqrt{2\pi|\psi''(\beta)|}} \int_{-\infty}^{\infty} f(-z)\mathrm{e}^{\beta z}\mathrm{d}z\,. \tag{6}$$

where $W(\infty, \beta)$ is the terminal value of $W(t, \beta)$.

4 Asymptotics of mean power sums

4.1 Power sums and their means

The above formula (5) for $\alpha = 0$ implies that also in the case $\alpha > 0$ each $m(t, \beta)$ is finite for $t \geq 0$, simply because the mean life-times increase with α. The first-split decomposition with application of (4) shows that $M(t, \beta)$ satisfies the distributional identity

$$M(t, \beta) \stackrel{d}{=} 1(t < \tau) + 1(t \geq \tau) \sum_j \xi_j^\beta\, M_j(\xi_j^\alpha (t - \tau)) \tag{7}$$

where τ is the exponential life-time of the progenitor, and the M_j's are independent replicas of $M(\cdot, \beta)$, which are also independent of τ and $\{\xi_j\}$. Computing expectations we arrive at the integral equation

$$m(t, \beta) = e^{-t} + \int_0^t e^{-s} \int_0^1 m((t - s)x^\alpha, \beta)x^\beta\, \sigma(\mathrm{d}x)\,.$$

Differentiating we see that $m(\cdot, \beta)$ is a solution to the Cauchy problem for the integro-differential equation

$$\partial_t\, m(t, \beta) = -m(t, \beta) + \int_0^1 m(x^\alpha t, \beta)x^\beta\, \sigma(\mathrm{d}x)\,. \tag{8}$$

which must be complemented by the initial value $m(0, \beta) = 1$. Uniqueness of C^∞ solutions for equations of this type is shown in [19].

Equation (8) defines functions m for all $\Re\,\beta > \beta_a$. For the higher derivatives we have

$$\partial_t^k\, m(t, \beta) = \partial_t^k m(0, \beta)\; m(t, k\beta + \alpha),$$

thus m is increasing in t for $\beta < \beta^*$ and decreasing for $\beta > \beta^*$.

Solving (8) in power series is straightforward. Introducing

$$\gamma(n, \beta) = \prod_{k=0}^{n-1} \psi(\beta + \alpha k) \tag{9}$$

(by convention, $\gamma(0, \beta) = 1$), we compute

$$m(t, \beta) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\, \gamma(n, \beta) \tag{10}$$

which is an entire function of $t \in \mathbb{C}$. It is indeed the right solution because from the formula for derivatives it is clear that $m(\cdot, \beta)$ should be C^∞ for $t \geq 0$. For $\alpha = 0$ we have $\gamma(n, \beta) = \psi^n(\beta)$ and (10) becomes (5). See [19] for yet another interesting representation of $m(t, \beta)$, in the form of a generalised Dirichlet series.

4.2 A contour integral

When $\phi(\beta)$ is a rational function, splitting $\gamma(\cdot\,, \beta)$ in linear factors shows that the series (10) represents a generalised hypergeometric function, in which case the $|t| \to \infty$ asymptotic expansions have been thoroughly studied by Mellin-Barnes' contour integral technique, see [22]. This method extends to the more general situation considered here (also see [3]).

Call β *singular* if $\psi(\beta + \alpha n) = 0$ for some integer $n \geq 0$. For singular β the series $m(t, \beta)$ is a polynomial, thus $m(t, \beta) = \gamma(n, \beta)t^n + O(t^{n-1})$ for $t \to \infty$. Analogous asymptotics hold also for nonsingular β, but it is more difficult to justify, because m is then an infinite series which starts alternating from some term. A good heuristic amounts to substituting $m \sim ct^a$ into (8) – the left-hand side is then of the order t^{a-1} while the right-hand side is $o(t^a)$ exactly when $\psi(\beta + a\alpha) = 0$, which suggests that $a = (\beta^* - \beta)/\alpha$ is the right exponent. Although this kind of reasoning can be made precise it gives no idea of the coefficient, see [19].

Assuming β nonsingular we extrapolate the function $\gamma(\cdot\,, \beta)$ from the integer values to arbitrary complex values z (such that $\phi(\alpha z + \beta)$ is defined) by means of the formula

$$\gamma(z, \beta) = \prod_{k=0}^{\infty} \frac{\psi(\beta + \alpha k)}{\psi(\beta + \alpha(k + z))}\,. \tag{11}$$

Convergence of the product follows as in [3], Section 5. Thus defined, γ satisfies the functional equation

$$\gamma(z + 1, \beta) = \psi(\beta + \alpha z)\gamma(z, \beta)$$

reminiscent of the well-known equation for Euler's gamma function.

All singularities of the function $\gamma(\cdot\,, \beta)$ are the poles located at roots of (2). Let

$$\mathcal{P}_\beta = \{z :\ \exists\, n \geq 0,\ \psi(\beta + \alpha(n + z)) = 0\}$$

be the set of singular points. Because (2) has no solutions to the right of β^*, the rightmost point of $\mathcal{P}_\beta$ is $z_\beta := (\beta^* - \beta)/\alpha$, where $\gamma(\cdot\,, \beta)$ has a simple pole provided $\beta^* > \beta_a$.

Still assuming β nonsingular we have $\mathcal{P}_\beta \cap \{0, 1, \ldots\} = \emptyset$. Since the poles of $\Gamma(-z)$ are nonnegative integers, the function $\Gamma(-z)\gamma(z, \beta)t^z$ (with t as parameter) also has these poles, with residue $(-1)^n t^n \gamma(n, \beta)/n!$ at $z = n$. Defining $\mathcal{C}$ to be a vertical line between $\Re z_\beta$ and $n_\beta := \min(0, \lceil \Re z_\beta \rceil)$ we obtain by the residue theorem and an estimate of γ

$$\sum_{n=n_\beta}^{\infty} \frac{(-t)^n}{n!}\,\gamma(n, \beta) = \frac{1}{2\pi \mathtt{i}} \int_{\mathcal{C}} \Gamma(-z)\gamma(z, \beta)z^t \, \mathrm{d}z\,. \tag{12}$$

If $\beta^* > \beta_a$ the function $\gamma(\cdot\,, \beta)$ is meromorphic in an open strip containing the line $\Re z = z_\beta$, and the residue at z_β is

$$\mathrm{Res}_{z_\beta}\gamma(z, \beta) = \frac{\psi(\beta)}{\alpha\psi'(\beta^*)}\,\gamma\left(\frac{\beta^* - \beta}{\alpha}\,,\, \alpha + \beta\right)$$

as it follows from the identity

$$\gamma(s,\beta) = \frac{\psi(\beta)}{\psi(\beta+\alpha s)}\,\gamma(s,\alpha+\beta)$$

upon expanding the ratio. Replacing $\mathcal{C}$ by another integration contour $\mathcal{C}'$ located in the half-plane $\Re z < \Re z_\beta$ so that all poles of $\gamma(\cdot\,,\,\beta)$ in this half-plane lie to the left of $\mathcal{C}'$ we obtain the principal-term asymptotics of m.

Theorem 1 *Suppose $\beta^* > \beta_a$, the structural measure σ is nonarithmetic and conditions* (1) *hold, then*

$$m(t,\beta) \sim \Gamma\left(\frac{\beta-\beta^*}{\alpha}\right)\frac{\psi(\beta)}{\alpha\psi'(\beta^*)}\;\gamma\left(\frac{\beta^*-\beta}{\alpha}\,,\,\alpha+\beta\right)\;t^{(\beta^*-\beta)/\alpha}\,, \qquad \text{as} \quad t\to\infty. \tag{13}$$

for $\Re\beta > \beta_a$.

Using the identity $\gamma(-z\,,\,\alpha z+\beta)\,\gamma(z,\beta) = 1$ we can re-write the γ-factor in (13) as

$$\gamma\left(\frac{\beta^*-\beta}{\alpha}\,,\,\alpha+\beta\right) = \frac{1}{\gamma\left((\beta-\beta^*)/\alpha\,,\,\alpha+\beta^*\right)}\,.$$

The restriction of ψ to the real segment $]\beta_a,\infty[$ is plainly a concave increasing function, so the condition $\beta^* > \beta_a$ entails $0 < \psi'(\beta^*) < \infty$. We also remark that in the arithmetic case other poles on the line $\Re z = \beta^*$ would contribute to the coefficient. Further terms of the asymptotic expansion can be obtained by pulling the integration contour through other poles left of β^*, as long as ϕ admits a meromorphic continuation, which can go beyond the convergence abscissa. If β^* is the only point of $\mathcal{P}_\beta$ in a closed strip $\beta^*-\theta \le \Re z \le \beta^*$ then the rest-term in (13) is estimated as $O(t^{(\beta^*-\beta)/\alpha-\epsilon})$ with $\epsilon = \min(1,\theta/\alpha)$.

5 Convergence of the mean measures

We encode the configuration of masses $X(t) = \{X_j(t)\}$ into the random measure

$$\sum_j X_j^{\beta^*}(t)\delta_{t^{1/\alpha}X_j(t)}.$$

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty f(x)\sigma_t^*(\mathrm{d}x) = \mathbb{E}\sum_j f(t^{1/\alpha}X_j(t))X_j^{\beta^*}(t) \tag{14}$$

which is required to hold for all compactly supported continuous functions f. It is easily seen that σ_t^* is a probability measure. Our next goal is to show that the measures σ_t^* converge weakly to a probability measure ρ on $]0,\infty[$.

Because

$$t^{(\beta-\beta^*)/\alpha}m(t,\beta) = \int_0^\infty x^{\beta-\beta^*}\sigma_t^*(\mathrm{d}x)$$

the convergence of $t^{(\beta-\beta^*)/\alpha}m(t,\beta)$ implied by (13) amounts to the convergence of power moments

$$\int_0^\infty x^{\beta-\beta^*}\sigma_t^*(\mathrm{d}x) \to \Gamma\left(\frac{\beta-\beta^*}{\alpha}\right)\frac{\psi(\beta)}{\alpha\psi'(\beta^*)}\ \frac{1}{\gamma\left((\beta-\beta^*)/\alpha\,,\,\alpha+\beta^*\right)}\,.$$

Specialising $\beta=\beta^*+\alpha k$ this simplifies to

$$\int_0^1 x^{k\alpha}\,\sigma_t^*(\mathrm{d}x) \to \frac{(k-1)!}{\alpha\psi'(\beta^*)}\prod_{j=1}^{k-1}\frac{1}{\psi(\beta^*+\alpha j)}$$

in application of (11). It is easy to check and is well known [12, 17] that for $n=0,1,\ldots$ the related moment problem is determinate, whence the following result.

Theorem 2 *Under assumptions of* Theorem 1 *the measures σ_t^* converge weakly, as $t\to\infty$, to a probability measure ρ uniquely determined by its power moments*

$$\int_0^\infty x^{k\alpha}\rho(\mathrm{d}x) = \frac{(k-1)!}{\alpha\psi'(\beta^*)}\prod_{j=1}^{k-1}\frac{1}{\psi(\beta^*+\alpha j)}\,,\qquad k=1,2,\ldots \tag{15}$$

Theorems 1 and 2 imply that

$$t^{(\beta-\beta^*)/\alpha}m(t,\beta) \to \int_0^\infty x^{\beta-\beta^*}\,\rho(\mathrm{d}x) \tag{16}$$

which extends the convergence of expectations in (14) to a wider class of functions f.

6 Self-similar stick-breaking process

A key tool in the conservative case treated in [16, 17, 5, 6, 7] has been the following observation related, somewhat paradoxically, to the simplest dissipative case of a singleton ensemble with reproduction law $\{\eta\}$, where η is a random variable assuming values in $]0,1]$. That is to say, if at some time the particle has mass x then, independently of the history, the particle shrinks with probability rate x^α and the new mass after the shrink becomes $x\eta$ where η follows $\widehat{\sigma}$, and $\widehat{\sigma}$ is the probability law of η.

We recollect briefly a result from [17] (also see [7, 9]). Introduce

$$\widehat{\psi}(\beta) = 1-\int_0^1 x^\beta\,\widehat{\sigma}(\mathrm{d}x)$$

and suppose $\widehat{\psi}(0+)<\infty$. Let L_t be the sole mass at time t, and $\widehat{m}(t,\beta)=\mathbb{E}L_t^\beta$. Assuming $\widehat{\sigma}$ non-arithmetic, Brennan and Durrett [17] proved that for $t\to\infty$

$$t^{1/\alpha}L_t \xrightarrow{d} Y^{1/\alpha} \qquad \text{and} \qquad \widehat{m}(t,\beta)\to\mathbb{E}\,Y^{\beta/\alpha}$$

where Y is a random variable with moments

$$\mathbb{E}\,Y^k = \frac{(k-1)!}{\alpha\widehat{\psi}'(0+)}\prod_{j=1}^{k-1}\frac{1}{\widehat{\psi}(\alpha j)}.$$

The convergence of moments $\widehat{m}(t,\beta)$ was shown for all real β strictly to the right of the convergence abscissa of $\widehat{\psi}$ (which is nonpositive due to the normalisation $\widehat{\sigma}\,]0,1]=1$). They also suggested the explicit representation

$$Y \stackrel{d}{=} \sum_{k=0}^{\infty} \epsilon_k \prod_{j=0}^{k} \eta_j^{\alpha} \tag{17}$$

where all η_j, ϵ_j are independent, ϵ_j is mean one exponential, η_j for $j>0$ are replicas of η, and η_0 follows the law

$$\mathbb{P}(\eta_0 \in \mathrm{d}x) = \frac{\widehat{\sigma}[x,1]\mathrm{d}x}{\widehat{\psi}'(0)x} \qquad x \in\,]0,1].$$

Each product in (17) corresponds to the mass of the particle after k splits, conditionally given the initial mass is η_0, thus formula (17) identifies Y with the well-known exponential functional of a (stationary) compound Poisson process, see e.g. the survey [12].

In the conservative fragmentation case, the above 'stick-breaking' process describes the evolution of a particle tagged by an atom of isotope that was injected at a random uniform location into the progenitor unit mass. The mechanism which determines the line of descent of the tagged particle amounts, at each consecutive split, to a random mass-biased pick from the child particles. Thus, defining $\widehat{\sigma}(\mathrm{d}x) := x\,\sigma(\mathrm{d}x)$ to be the distribution of a mass-biased pick from $\{\xi_j\}$, we obtain the relation $\widehat{m}(t,\beta-1)=m(t,\beta)$ which was observed in [17], p. 112.

In the general nonconservative case choosing $\widehat{\sigma}(\mathrm{d}x) := x^{\beta^*}\sigma(\mathrm{d}x)$, we still get $\widehat{m}(t,\beta-\beta^*) = m(t,\beta)$ and $\widehat{\psi}(z)=\psi(z+\beta^*)$, though no interpretation of these relations akin to the tagged fragment process is known. The measure ρ may be identified with the distribution of $Y^{1/\alpha}$. A consequence of this discussion and the result of Brennan and Durrett is the following corollary.

Corollary 3 *The conclusion of* Theorem 2 *remains valid even if the assumption $\beta^* > \beta_a$ is replaced by the weaker $\psi'(\beta^*+)<\infty$*

The tiny improvement upon Theorem 2 appears in the case where the characteristic function is defined in a closed half-plane and $\beta^*=\beta_a$, i.e. the critical exponent falls exactly on the convergence abscissa. An example of such situation is the structural measure of the form $\sigma(\mathrm{d}x) = c\mathbf{1}_{\{x<1/2\}}x^{-3/2}|\log x|^{-3}\,\mathrm{d}x$ with a suitable c. The method based on contour integration requires in such cases the analytical continuation of ϕ in a domain to the left of β_a.

Alternatively, along the lines in [16, 17, 9], the renewal theory can be applied also in the nonconservative case, to prove first the convergence of measures $\sigma_t^* \to \rho$ and then to justify the asymptotics of mean power-sums using the uniform integrability.

7 A martingale

The total β^*-mass of coexisting particles deserves a special notation $M_t := M(t,\beta^*)$, as this quantity plays an important role. By definition of the critical exponent, each reproduction preserves the mean β^*-mass, hence M_t is a nonnegative martingale. We denote $M_\infty = \lim_{t\to\infty} M_t$ its terminal value. Recall from Section 3 that in the homogeneous case $\alpha=0$, M_t can be viewed as the so-called additive martingale associated to some branching random walk in continuous times. In the case $\alpha > 0$, the same standard argument yields the following convergence result.

Proposition 4 *Under assumptions* (1) *and* (3)*, the martingale* $(M_t, t \geq 0)$ *is bounded in* L^2*. Its terminal value is strictly positive and satisfies the distributional identity*

$$M_\infty \stackrel{d}{=} \sum \xi_j^{\beta^*} M^{(j)} \tag{18}$$

where $M^{(j)}$ *are independent copies of* M_∞*, also independent of* $\{\xi_j\}$*. The identity taken together with conditions* (1), (3) *and* $\mathbb{E}M_\infty = 1$ *characterise* M_∞ *uniquely.*

Proof. Since M_t is a pure-jump martingale we need to check that the sum of its squared jumps is integrable. Observe that when a mass x splits into masses $\{x\xi_j\}$ this induces a jump of M_t of size $x^{\beta^*}\left(\Sigma\xi_j^{\beta^*} - 1\right)$. On the other hand, such event occurs at rate x^α. This entails that the predictable compensator of the sum of squared jumps $\sum_{s \leq t}(M_s - M_{s-})^2$ is equal to

$$c \int_0^t \sum_j X_j^{2\beta^* + \alpha}(s) \mathrm{d}s\,,$$

where

$$c = \mathbb{E}\left(\sum_j \xi_j^{\beta^*} - 1\right)^2$$

is finite (3). We thus get

$$\mathbb{E}\sum_{t \geq 0}(M_s - M_{s-})^2 = c\,\mathbb{E}\int_0^\infty \sum_j X_j^{2\beta^* + \alpha}(t)\,\mathrm{d}t \;= c/\psi(2\beta^*)$$

which is finite and positive because $2\beta^* > \beta^*$ and ψ is strictly increasing. The expectation was computed by isolating a contribution of each particle. Indeed, a generic particle of mass x contributes $x^{2\beta^*}$, because this is the $(2\beta^* + \alpha)$-mass multiplied by the expected lifetime $x^{-\alpha}$, whence the expectation equals the mean total $2\beta^*$-mass of all ever existing particles, which is $1/\psi(2\beta^*)$, as we know from the homogeneous case in Section 3.

The fixed-point equation (18) follows readily as the limit form of (7). The uniqueness part is a consequence of [27] (see remark on p. 200).

To show that $M_\infty > 0$ assume that ξ_j's are in decreasing order. Then by (1) $\xi_1 > 0$ and $\mathbb{P}(\xi_2 > 0) > 0$. From the fixed-point equation we find, using $M_\infty \geq 0$, $\xi_j \geq 0$ and independence, that

$$\mathbb{P}(M_\infty = 0) \leq \mathbb{P}(\xi_1^{\beta^*} M^{(1)} = \xi_2^{\beta^*} M^{(2)} = 0) = \mathbb{P}(M^{(1)} = 0)\,\mathbb{P}(\xi_2^{\beta^*} M^{(2)} = 0)$$

where $\mathbb{P}(M^{(j)} > 0) > 0$ (since the mean is 1) and $\mathbb{P}(\xi_2^{\beta^*} M^{(2)} = 0) = 1 - \mathbb{P}(\xi_2 > 0)\,\mathbb{P}(M^{(2)} > 0) < 1$. This implies readily $\mathbb{P}(M_\infty = 0) = 0$. □

Note that in the conservative case $\Sigma\xi_j = 1$ we have $\beta^* = 1$ and (18) is satisfied for M_∞ a constant. We also mention that if we replace the assumption (3) by the weaker

$$\mathbb{E}\left(\sum_j \xi_j^{\beta^*}\right)^p < \infty$$

for some $1 < p \leq 2$, a similar argument based on the calculation of the expectation of the sum of the p-th powers of jumps then shows that the martingale M_t is bounded in L^p, see e.g. Neveu [26].

8 L^2-convergence

Our principal result improves on the convergence of the mean measures $\sigma_t \to \rho$ and says that the scaled empirical measures induced by $X(t)$ converge in a L^2-sense to the measure $M_\infty \rho$.

Theorem 5 *Assume* (1), (3), *that* $\beta^* > \beta_a$ *and that* σ *is nonarithmetic. Then for any bounded continuous* f

$$L^2\text{–}\lim_{t\to\infty} \sum_j X_j^{\beta^*}(t) f(t^{1/\alpha} X_j(t)) = M_\infty \int_0^\infty f(x)\rho(\mathrm{d}x).$$

Proof. We need to show that

$$\mathbb{E}\left(\sum_{i,j} X_i^{\beta^*}(t) f(t^{1/\alpha} X_i(t)) X_j^{\beta^*}(t) g(t^{1/\alpha} X_j(t))\right) \to \mathbb{E}M_\infty^2 \left(\int_0^\infty f(x)\rho(\mathrm{d}x)\right)\left(\int_0^\infty g(x)\rho(\mathrm{d}x)\right) \tag{19}$$

for positive f and g bounded from above by 1. Indeed, suppose (19) is shown. Denote

$$A_t = \sum_j X_j^{\beta^*}(t) f(t^{1/\alpha} X_j(t)).$$

Take $f = g$ to conclude from (19) that

$$\lim_{t\to\infty} \mathbb{E}A_t^2 = \mathbb{E}M_\infty^2 \left(\int_0^\infty f(x)\rho(\mathrm{d}x)\right)^2.$$

Similarly, setting $g = 1$

$$\lim_{t\to\infty} \mathbb{E}(A_t M_t) = \mathbb{E}M_\infty^2 \int_0^\infty f(x)\rho(\mathrm{d}x).$$

Recalling that $\mathbb{E}M_t^2 \to \mathbb{E}M_\infty^2$ and combining the above we get the desired

$$\lim_{t\to\infty} \mathbb{E}\left(A_t - M_t \int_0^\infty f(x)\rho(\mathrm{d}x)\right)^2 = 0\,.$$

To prove (19) let us replace t by $t+s$ and condition on the configuration of masses $X(s)$. At time $t+s$ two coexisting particles may stem from the same ancestor that lived at time s or from two different ancestors; write $i \sim_s j$ in the first case, and write $i \not\sim_s j$ in the second. The sum in the left-hand side of (19) is split then in two

$$S_1 + S_2 = \mathbb{E}\left(\sum_{i\sim_s j} \cdots | X(s)\right) + \mathbb{E}\left(\sum_{i\not\sim_s j} \cdots | X(s)\right).$$

Using the fundamental self-similarity relation (4) and the Markov nature of the fragmentation process we estimate the first sum as

$$S_1 \le \sum_k X_k^{2\beta^*}(s)\,\mathbb{E}\left(\sum_j X_j^{\beta^*}(t)\right)^2$$

hence by (13) and Proposition 4

$$\mathbb{E}S_1 < \mathrm{const}\, s^{-\beta^*/\alpha} \to 0 \qquad \text{as} \quad s \to \infty$$

uniformly in t.

Dealing with S_2 requires more effort. We use the parallel notation $y_j = X_j(s)$. Write $i \searrow k$ if the mass $X_i(t+s)$ stems from y_k. By independence, the descendants of different particles with masses y_k and y_ℓ evolve independently, thus grouping the masses $X_j(t+s)$ by the ancestors at time s yields

$$S_2 = \sum_{k \ne \ell} \left(\mathbb{E}\sum_{i \searrow k} \cdots\right)\left(\mathbb{E}\sum_{j \searrow \ell} \cdots\right)$$

However, by self-similarity and convergence of the mean measures

$$\mathbb{E}\sum_{i \searrow k} y_k^{\beta^*} X_i^{\beta^*}(t y_k^{\alpha}) f((t+s)^{1/\alpha} y_k X_i(t y_k^{\alpha})) \to y_k^{\beta^*} \int_0^\infty f(x)\rho(\mathrm{d}x)$$

$$\mathbb{E}\sum_{j \searrow \ell} y_\ell^{\beta^*} X_j^{\beta^*}(t y_\ell^{\alpha}) g((t+s)^{1/\alpha} y_\ell X_j(t y_\ell^{\alpha})) \to y_\ell^{\beta^*} \int_0^\infty g(x)\rho(\mathrm{d}x)$$

as $t \to \infty$, therefore by dominated convergence

$$\mathbb{E}S_2 \sim \left(\int_0^\infty f(x)\rho(\mathrm{d}x)\right)\left(\int_0^\infty g(x)\rho(\mathrm{d}x)\right)\mathbb{E}\sum_{k \ne \ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s)$$

as $s \to \infty$. It remains to note that

$$\mathbb{E}\sum_{k \ne \ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s) \sim \mathbb{E}\sum_{k\,,\,\ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s) = \mathbb{E}M_s^2 \to \mathbb{E}M_\infty^2$$

because

$$\mathbb{E}\sum_k X_k^{2\beta^*}(s) = m(t, 2\beta^*) \to 0\,.$$

$\square$

Remarks. This result bears obvious similarities with (6) for the homogeneous case $\alpha = 0$. It is interesting to observe that in the homogeneous case, masses decay exponentially fast and the limiting scaled empirical measure is always exponential (up-to a random factor), whereas for $\alpha > 0$ the decay of masses is polynomial and the limiting scaled empirical measure depends crucially on the structural measure σ (more precisely, σ can be recovered from the limiting scaled empirical measure for $\alpha > 0$, but not for $\alpha = 0$).

The limit measure $M_\infty \rho$ has no atom at 0. If we omit the assumption $\#\{\xi_j\} \neq \emptyset$, the result should be modified properly by allowing a positive probability of extinction.

Finally, it is known that in the binary conservative case the scaled empirical measures converge with probability one [17]. It would be interesting to extend this result to the nonconservative case.

In parallel to (16) there is the following extension of Theorem 5 to power functions.

Corollary 6 *Under assumptions of* Theorem 5

$$L^2 - \lim_{t\to\infty} t^{(\beta-\beta^*)/\alpha} \sum_j X_j^\beta(t) = M_\infty \int_0^\infty x^{\beta-\beta^*} \rho(\mathrm{d}x)$$

for $\Re\beta > \beta_a$.

Proof. Along the same line, the proof is reduced to showing that

$$\sup_{t\geq 0} t^{(\beta^*-\beta)/\alpha} y^{-\beta^*} \mathbb{E} \sum_j \left(X_j^{(y)}(t)\right)^\beta$$

is bounded uniformly in $y \in]0,1[$. And the latter follows by noting that (4) and (16) imply

$$t^{(\beta-\beta^*)/\alpha} \mathbb{E} \sum_j \left(X_j^{(y)}(t)\right)^\beta \to y^{\beta^*} \int_0^\infty x^{\beta-\beta^*} \rho(\mathrm{d}x)\,.$$

$\square$

9 Examples

9.1 Filippov's example revisited

Extending an example in Filippov (see [18], section 8) consider the structural measure

$$\sigma(\mathrm{d}x) = \lambda x^{\theta-1}\,\mathrm{d}x\,, \qquad x \in]0,1]$$

with parameters $\lambda > \min(\theta, 0)$ and *arbitrary* $\theta \in \mathbb{R}$. For $\lambda < \theta + 1$ the fragmentation is mean-value dissipative. We have

$$\phi(\beta) = \frac{\lambda}{\theta+\beta}\,, \qquad \beta^* = \lambda - \theta\,, \qquad \psi(\beta) = \frac{\beta-\beta^*}{\beta+\theta}.$$

The characteristic function is thus meromorphic in $\mathbb{C}$ with a unique simple pole at $\beta_a = -\theta$.

Computing

$$\gamma(n,\beta) = \frac{(A)_n}{(B)_n}$$

we see that this is the ratio of two Pochhammer factorials, with $A = (\beta - \beta^*)/\alpha$ and $B = (\theta+\beta)/\alpha$, thus $m(t,\beta) = {}_1F_1(A;B;-t)$ is Kummer's hypergeometric function. The anatytical extension of γ is

$$\gamma(z,\beta) = \frac{\Gamma(A+z)\Gamma(B)}{\Gamma(B+z)\Gamma(A)}.$$

Computing the moments we obtain

$$\int_0^\infty x^{\alpha k}\,\rho(\mathrm{d}x) = (\lambda/\alpha)_k$$

which identifies ρ as

$$\rho(\mathrm{d}x) = \frac{\alpha}{\Gamma(\lambda/\alpha)}\,x^{\lambda-1}e^{-x^\alpha}\,\mathrm{d}x\,, \qquad x \geq 0\,.$$

Note that the shape parameter θ cancels and does not appear in ρ. It follows that $\sigma_t := x^{-\beta^*}\sigma_t^*$ (the intensity of $\Sigma\,\delta_{t^{1/\alpha}X_j(t)}$) converges to the measure

$$x^{-\beta^*}\rho(\mathrm{d}x) = \frac{\alpha}{\Gamma(\lambda/\alpha)}\,x^{\theta-1}e^{-x^\alpha}\,\mathrm{d}x, \qquad x \geq 0$$

in accord with the case $\lambda = 2$, $\theta = 1$ considered in [17] in connection with the conservative binary fragmentation with ξ_1 uniform and $\xi_2 = 1 - \xi_1$.

It follows that the mean number of particles satisfies

$$m(t,0) \sim \frac{\Gamma(\theta/\alpha)}{\Gamma(\lambda/\alpha)}\,t^{(\lambda-\theta)/\alpha}\,, \qquad t \to \infty\,.$$

which agrees with a special case in [18]. Of course, this formula makes sense only for $\theta > 0$, because our asymptotics for $m(t,\beta)$ hold only for $\Re\beta > \beta_a$, thus for $\theta \leq 0$ the value $\beta = 0$ is not considered.

The case $\lambda = 1$, $\theta = 0$, when $\sigma(\mathrm{d}x) = x^{-1}\mathrm{d}x$ corresponds to the conservative fragmentation generated by the uniform stick-breaking, as in the example in Section 2 (but without removing a piece). It is well known that the distribution of a size-biased pick from $\{\xi_j\}$ is uniform, and this implies that the intensity measure of $\Sigma\delta_{\xi_j}$ is indeed $\sigma(\mathrm{d}x) = x^{-1}\mathrm{d}x$.

9.2 Hypergeometrics

The following is a further generalisation of Filippov's example, and covers the class of dissipative binary fragmentations treated in [3]. Consider a Dirichlet polynomial

$$g(x) = \sum_{j=1}^{p} \lambda_j\,x^{\theta_j - 1} \tag{20}$$

which is non-negative on $]0,1]$ and has real parameters satisfying

$$\sum_{j=1}^{p} \frac{\lambda_j}{\theta_j} > 1\,.$$

Then $\sigma(\mathrm{d}x) = g(x)\mathrm{d}x$ is a measure on $]0,1]$ with rational characteristic function

$$\phi(\beta) = \sum_{j=1}^{p} \frac{\lambda_j}{\theta_j + \beta}$$

and by the assumption the righmost root of $\phi(\beta) = 1$ is positive, denote it also $\beta_1 = \beta^*$ and denote further roots $\beta_2, \ldots, \beta_p$ (the roots are certainly different from the poles of ϕ).

Observe that

$$\psi(\beta) = \prod_{j=1}^{p} \frac{\beta - \beta_j}{\beta + \theta_j}\,,$$

thus assuming $\alpha = 1$ (without loss of generality) we have

$$m(t,\beta) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \prod_{j=1}^{p} \frac{(\beta - \beta_j)_n}{(\beta + \theta_j)_n}$$

where we recognise a generalised hypergeometric function of the type ${}_pF_p$. By (1) we have $m(t,\beta) \sim c(\beta) t^{\beta-\beta^*}$ for $\Re\beta > \beta_a$. Noting that

$$\psi'(\beta^*) = \frac{1}{\beta^* + \theta_1} \prod_{j=2}^{p} \frac{\beta^* - \beta_j}{\beta^* + \theta_j}$$

and manipulating infinite products, the coefficient is evaluated in terms of the gamma function as

$$c(\beta) = \prod_{j=2}^{p} \frac{\Gamma(\beta^* - \beta_j)}{\Gamma(\beta - \beta_j)} \prod_{j=1}^{p} \frac{\Gamma(\beta + \theta_j)}{\Gamma(\beta^* + \theta_j)}\,.$$

This allows to recover the density by Mellin inversion as

$$\frac{\mathrm{d}\rho}{\mathrm{d}x} = \frac{1}{2\pi\mathtt{i}} \frac{\prod_{j=2}^{p} \Gamma(\beta^* - \beta_j)}{\prod_{j=1}^{p} \Gamma(\beta^* + \theta_j)} \int_{-\mathtt{i}\infty}^{\mathtt{i}\infty} \frac{\prod_{j=1}^{p} \Gamma(z + \beta^* + \theta_j)}{\prod_{j=2}^{p} \Gamma(z + \beta^* - \beta_j)}\, x^{-z-1}\, \mathrm{d}z\,, \qquad x \geq 0$$

which is an instance of Meijer's G-function, see [22].

The limit measure is uniquely determined by the integer moments which can be written as

$$\int_0^\infty x^k\, \rho(\mathrm{d}x) = \frac{(k-1)!}{\psi'(\beta^*)} \prod_{j=1}^{p} \frac{(\beta^* + 1 + \theta_j)_{k-1}}{(\beta^* + 1 - \beta_j)_{k-1}}\,,$$

where the derivative may be also computed as

$$\psi'(\beta^*) = \sum_{j=1}^{p} \frac{\lambda_j}{(\beta^* + \theta_j)^2}$$

10 Fragmentations with infinite reproduction measure

We sketch how the preceding results can be extended to a class of self-similar conservative or dissipative fragmentations with infinite reproduction measure. Such processes were introduced in [6] where the reproduction law was called 'dislocation measure'.

Let ν be a measure on the infinite simplex

$$\Delta = \{(s_j) : s_j \geq 0,\, s_j \downarrow 0,\, \Sigma s_j \leq 1\}$$

such that the integral

$$\psi(\beta) := \int_{\Delta} (1 - \Sigma\, s_j^{\beta})\, \nu(\mathrm{d}s) \tag{21}$$

satisfies $1 < \psi(\beta) < \infty$ for some $\beta > 0$. For $\alpha \geq 0$ we can define a fragmentation process $(X(t), t \geq 0)$ with the property that a generic particle of mass x gives birth to a collection of particles of sizes xs_j with $s = (s_j) \in B$ at rate $x^{\alpha}\nu(B)$, where s_j runs over nonzero coordinates of s and $B \subset \Delta$ runs over Borel sets of finite ν-measure.

If ν is a probability measure it can be regarded as a reproduction law by defining (ξ_j) to be a random element of Δ with distribution ν. In this case the structural measure is identified with the superposition of marginal distributions of ν, and the definition (21) agrees with our definition of the characteristic function in Section 2. The case $\nu < \infty$ is easily reduced to the case $\nu(\Delta) = 1$ by the obvious time-change.

In the case $\nu(\Delta) = \infty$ some features of the fragmentation process are different, in particular, each particle produces infinitely many generations within arbitrarily small time period. As a consequence, the life-time of a particle is not a well-defined quantity and we do not have equation like (7). Still, we can define β_a and β^* exactly as in the case of finite measure, and consider β-masses for $\Re\,\beta > \beta_a$. Formula (10) remains valid and can be proved by an argument exploiting approximation of ν by suitable finite measures, or by using the methods developed in [9, 11] (same applies to (5) in the case $\alpha = 0$). For $\alpha > 0$ conclusions of Theorems 2 and 5 remain valid if we assume that

$$\psi'(\beta^*+) < \infty \quad \text{and} \quad \int_{\Delta} \left(\sum_{j=1}^{\infty} s_j^{\beta^*} - 1 \right)^2 \nu(\mathrm{d}s) < \infty\,,$$

and impose a non-arithmeticity condition on ν. The analog of Proposition 4 is shown by arguments similar to those in Theorem 2 of [7], and the analog of distributional equation (18) generalises in the form of the Laplace transform identity

$$\mathcal{L}(\theta) = \int_{\Delta} \prod_{j=1}^{\infty} \mathcal{L}(\theta s_j)\, \nu(\mathrm{d}s) \tag{22}$$

for $\mathcal{L}(\theta) := \mathbb{E}\exp(-\theta M_{\infty})$. The question about the uniqueness of solution to (22) with infinite measure seems to have not been considered before and remains open.

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