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On the least singular value of random symmetric matrices

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Abstract

Let F_n be an n by n symmetric matrix whose entries are bounded by n^γ for some $\gamma > 0$. Consider a randomly perturbed matrix $M_n = F_n + X_n$, where X_n is a *random symmetric matrix* whose upper diagonal entries $x_{ij}, 1 \le i \le j$, are iid copies of a random variable ξ. Under a very general assumption on ξ, we show that for any $B > 0$ there exists $A > 0$ such that $\mathbf{P}(\sigma_n(M_n) \le n^{-A}) \le n^{-B}$.

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1 Introduction

Let F_n be an n by n matrix whose entries are bounded by $n^{O(1)}$. Consider a randomly perturbed matrix $M_n = F_n + X_n$, where X_n is a *random matrix* whose entries are iid copies of a random variable. It has been shown, under a very general assumption on ξ, that the singular value of M_n cannot be too small.

Theorem 1.1. *[21, Theorem 2.1] Assume that $M_n = F_n + X_n$, where the entries of F_n are bounded by n^γ, and the entries of X_n are iid copies of a random variable of zero mean and unit variance. Then for any $B > 0$, there exists $A > 0$ such that*

$$\mathbf{P}(\sigma_n(M_n) \le n^{-A}) \le n^{-B}.$$

Here $\sigma_n(M_n)$ is the smallest singular value of M_n, defined as

$$\sigma_n(M_n) := \inf_{\|x\|=1} \|M_n x\|.$$

The dependence among the parameters in Theorem 1.1 was made explicitly in [24]. Under the stronger assumption that ξ has sub-Gaussian distribution, Rudelson and Vershynin [16] obtained an almost best possible estimate on the tail bound of $\sigma_n(M_n)$. For more results regarding this random matrix ensemble we refer the reader to [16, 21, 24].

One important application of Theorem 1.1 is a polynomial bound for the condition number of M_n.

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Corollary 1.2. *[21, Corollary 2.10] With the same assumption as in Theorem 1.1, for any $B > 0$, there exists $A > 0$ such that*

$$\mathbf{P}(\sigma_1(M_n)/\sigma_n(M_n) \le n^A) \ge 1 - n^{-B}.$$

The condition number $\kappa(M) = \sigma_1(M)/\sigma_n(M)$ of a matrix M plays a crucial role in numerical linear algebra. The above corollary implies that if one perturbs a fixed matrix F of small spectral norm by a (very general) random matrix X_n, the condition number of the resulting matrix will be relatively small with high probability. This fact has some nice applications in theoretical computer science. (See for instance [17, 18] for further discussions on these applications).

Another popular model of random matrices is that of *random symmetric matrices*; this is one of the simplest models that has non-trivial correlations between the matrix entries. A significant new difficulty in the study of the singularity of X_n (or of M_n in general) is that the symmetry ensures that $\det(X_n)$ is a quadratic function of each row, as opposed to the regular random ensembles in which $\det(X_n)$ is a linear function of each row.

A recent result of Costello, Tao and Vu [2] shows that if the upper diagonal entries x_{ij} of X_n are iid Bernoulli random variables, then X_n is non-singular with probability $1 - n^{-1/8+o(1)}$. In [12], the current author improved the bound to any polynomial order.

The goal of this note is to study the smallest singular value of randomly perturbed matrices M_n, under a general assumption on ξ.

Condition 1.3 (Anti-concentration)**.** *Assume that ξ has zero mean, unit variance, and there exist positive constants c_1 and c_s such that*

$$\mathbf{P}(c_1 \le |\xi - \xi'|) \ge c_2,$$

where ξ' is an independent copy of ξ.

Theorem 1.4 (Main theorem)**.** *Assume that the upper diagonal entries x_{ij} of X_n are iid copies of a random variable ξ satisfying Condition 1.3. Assume also that the entries f_{ij} of the symmetric matrix F_n satisfy $|f_{ij}| \le n^\gamma$ for some $\gamma > 0$. Then for any $B > 0$, there exists $A > 0$ depending on c_1, c_2, γ and B such that for all sufficiently large n,*

$$\mathbf{P}(\sigma_n(M_n) \le n^{-A}) \le n^{-B}.$$

Our result immediately implies a polynomial bound for the condition number of M_n as follows.

Corollary 1.5. *With the same assumptions as of Theorem 1.4, for any $B > 0$, there exists $A > 0$ depending in c_1, c_2, γ and B such that for all sufficiently large n,*

$$\mathbf{P}(\kappa(M_n) \ge n^A) \le n^{-B}.$$

As another application, we provide a relatively fine lower bound for the determinant of random symmetric matrices. This result refines an important case of [25, Theorem 34] obtained by Tao and Vu.

Corollary 1.6. *Assume that the upper diagonal entries x_{ij} of X_n are iid copies of a random variable ξ of zero mean, unit variance, and there is a constant $C > 0$ such that $\mathbf{P}(|\xi| \le C) = 1$. Assume furthermore that the entries f_{ij} of the symmetric matrix F_n also satisfy $|f_{ij}| \le C$. Then for any positive constant B there exists a positive constant D depending on B and C such that the following holds with probability $1 - O(n^{-B})$,*

$$|\det(M_n)| \ge \exp(-Dn^{1/3}\log n)\mathbf{E}(|\det(M_n)|),$$

and

$$\det(M_n)^2 \ge \exp(-Dn^{1/3}\log n)\mathbf{E}(\det(M_n)^2).$$

This corollary complements previously known results on the concentration of the determinant of non-symmetric random matrices (cf. [1, 3, 7, 19]).

Remark. When a preliminary version of this paper was submitted to the arxiv, Vershynin also published a similar result with stronger bounds (see [27]). However, our result is different from Vershynin's in three ways. Firstly, our Condition 1.3 on ξ is weaker, as we do not require it to have bounded fourth-moment. Secondly, our bound for the least singular value works for perturbed matrices of the form $M_n = F_n + X_n$ with $\|F_n\| = n^{O(1)}$. Lastly, the techniques we use are very different. Our proof relies on an almost complete inverse-type result concerning the concentration of quadratic forms, which is of interest of its own.

Notation. For a matrix M we use the notations $\mathbf{r}_i(M)$ and $\mathbf{c}_j(M)$ to denote its i-th row vector and its j-th column vector respectively; we use the notation $(M)_{ij}$ to denote its ij entry.

We use η to denote random Bernoulli variables (thus η takes values ± 1 with probability 1/2).

Here and later, asymptotic notations such as $O, \Omega, \Theta, \omega$, and so for, are used under the assumption that $n \to \infty$. A notation such as $O_C(.)$ emphasizes that the hidden constant in O depends on C. If $a = \Omega(b)$, we write $b \ll a$ or $a \gg b$. If $a = \Omega(b)$ and $b = \Omega(a)$, we write $a \asymp b$.

2 The approach to prove Theorem 1.4

For the sake of simplicity, we will prove our result under the following condition.

Condition 2.1. *With probability one,*

$$|x_{ij}| \le n^{B+1},$$

for all i, j.

In fact, because ξ has unit variance, we have

$$\mathbf{P}(|x_{ij}| \ge n^{B+1}) = O(n^{-2B-2}).$$

Thus, we can assume that $|x_{ij}| \le n^{B+1}$ at the cost of an additional negligible term $o(n^{-B})$ in probability.

We next assume that $\sigma_n(M_n) \le n^{-A}$. Thus

$$M_n\mathbf{x} = \mathbf{y},$$

for some $\|\mathbf{x}\| = 1$ and $\|\mathbf{y}\| \le n^{-A}$. There are two cases to consider.

Case 1. $\det(M_n) = 0$. This is the case to consider when ξ has discrete distribution.

We first show that it is enough to consider the case of M_n having rank $n-1$, thanks to the following result.

Lemma 2.2. *For any* $1 \le k \le n-2$*, we have*

$$\mathbf{P}(\mathrm{rank}(M_n) = k \le n-2) \le O_{c_1}(1)\mathbf{P}(\mathrm{rank}(M_{2n-k-1}) = 2n-k-2).$$

We deduce Lemma 2.2 from a useful observation by Odlyzko, whose simple proof is presented in Appendix A.

Lemma 2.3 (Odlyzko's lemma,[15]). *Let H be a linear subspace in $\mathbf{R}^n$ of dimension at most $k \le n$. Then*

$$\mathbf{P}(\mathbf{u} \in H) \le (\sqrt{1-c_3})^{n-k},$$

where $\mathbf{u} = (f_1 + x_1, \dots, f_n + x_n)$, f_i are fixed and x_i are iid copies of ξ.

Proof. (of Lemma 2.2) View M_{n+1} as the matrix obtained by adding the first row and first column to M_n. Let H be the vector space of dimension k spanned by the row vectors of M_n. Then the probability that the subvector formed by the last n components of the first row of M_{n+1} does not belong to H, by Lemma 2.3, is at least $1-(\sqrt{1-c_3})^{n-k}$.

Observe that if this is the case then the last n columns of M_{n+1} span a vector space of dimension $k+1$. Additionally, by symmetry, as the subvector formed by the last n components of the first column of M_{n+1} does not belong to H, adding the first column will increase the rank of M_{n+1} to $k+2$.

Hence,

$$\mathbf{P}(\operatorname{rank}(M_{n+1}) = k+2 | \operatorname{rank}(M_n) = k) \ge 1 - (\sqrt{1-c_3})^{n-k}.$$

In general, for $1 \le t \le n-k$ we have

$$\mathbf{P}(\operatorname{rank}(M_{n+t}) = k+2t | \operatorname{rank}(M_{n+t-1}) = k+2(t-1)) \ge 1 - (\sqrt{1-c_3})^{n-t-k+1}.$$

Because the rows (and columns) added to M_{n+t-1} at each step (to create M_{n+t}) are independent, we have

$$\begin{aligned} &\mathbf{P}(\operatorname{rank}(M_{2n-k-1}) = 2n-k-2 | \operatorname{rank}(M_n) = k) \ge \\ &\ge \prod_{t=1}^{n-k-1} \mathbf{P}\big(\operatorname{rank}(M_{n+t}) = k+2t | \operatorname{rank}(M_{n+t-1}) = k+2(t-1)\big) \\ &\ge (1-(\sqrt{1-c_3})^{n-k})(1-(\sqrt{1-c_3})^{n-k-1}) \cdots (1-(\sqrt{1-c_3})) = \Omega_{c_3}(1). \end{aligned}$$

□

Next we show that in the case of M_n having rank $n-1$, it suffices to assume that $\operatorname{rank}(M_{n-1}) \ge n-2$, thanks to the following simple observation.

Lemma 2.4. *Assume that M_n has rank $n-1$. Then there exists $1 \le i \le n$ such that the removal of the i-th row and the i-column of M_n results in a matrix M_{n-1} of rank at least $n-2$.*

Proof. (of Lemma 2.4)Without loss of generality, assume that the last $n-1$ rows of M_n span a subspace of dimension $n-1$. Then the matrix obtained from M_n by removing the first row and the first column has rank at least $n-2$. □

Without loss of generality, we assume that the matrix M_{n-1} obtained from M_n by removing its first row and first column has rank at least $n-2$. We next express $\det(M_n)$ as a quadratic function of its first row $(m_{11}, \dots, m_{1n})$ as follows.

$$\det(M_n) = c_{11}(M_n)m_{11} + \sum_{2 \le i,j \le n} c_{ij}(M_{n-1})m_{1i}m_{1j}$$

where $c_{11}(M_n)$ is the first cofactor of M_n, while $c_{ij}(M_{n-1})$ are the corresponding cofactors of the matrix M_{n-1}.

It is crucial to note that, since M_{n-1} has rank at least $n-2$, at least one of the cofactors $c_{ij}(M_{n-1})$ is nonzero. Set $c := (\sum_{2\le i,j\le n} c_{ij}(M_{n-1})^2)^{1/2}$ and $a_{ij} := c_{ij}(M_{n-1})/c$.

Roughly speaking, our approach consists of two main steps.

- *Step 1.* Assume that

$$\mathbf{P}_{x_{11},\dots,x_{1n}}((c_{11}(M_n)/c)m_{11} + \sum_{2\le i,j\le n} a_{ij}m_{1i}m_{1j} = 0|M_{n-1}) \ge n^{-B},$$

  Then there is a strong additive structure among the cofactors $c_{ij}(M_{n-1})$ of M_{n-1}.

- *Step 2.* The probability, with respect to M_{n-1}, that there is a strong additive structure among the $c_{ij}(M_{n-1})$ is negligible.

Here we use the subscript $\mathbf{P}_{x_{11},\dots,x_{1n}}$ to emphasize that the probability under consideration is taken with respect to the random variables $x_{11},\dots,x_{1n}$.

We will execute Step 1 by proving Theorem 2.6 below (as a special case). Step 2 will be carried out by proving Theorem 2.7.

Case 2. $\det(M_n)\neq 0$. Let $C(M_n) = (c_{ij}(M_n))$, $1\le i,j\le n$, be the matrix of the cofactors of M_n. We have

$$C(M_n)\mathbf{y} = \det(M_n)\cdot \mathbf{x}.$$

Thus

$$\|C(M_n)\mathbf{y}\| = |\det(M_n)|.$$

By paying a factor of n in probability, without loss of generality we can assume that

$$|c_{11}(M_n)y_1 + \dots c_{1n}(M_n)y_n| \ge |\det(M_n)|/n^{1/2}.$$

Note that $\|\mathbf{y}\|\le n^{-A}$, thus

$$\sum_{j=1}^{n} |c_{1j}(M_n)|^2 \ge n^{2A-1}\det(M_n)^2. \tag{2.1}$$

For $j\ge 2$, we write

$$c_{1j}(M_n) = \sum_{i=2}^{n} m_{i1}c_{ij}(M_{n-1}),$$

where M_{n-1} is the matrix obtained from M_n by removing its first row and first column, and $c_{ij}(M_{n-1})$ are the corresponding cofactors of M_{n-1}.

Hence, by the Cauchy-Schwarz inequality, by Condition 2.1, and by the bounds $f_{ij}\le n^{\gamma}$ for the entries of F_n, we have

$$\begin{aligned} c_{1j}(M_n)^2 &\le \textstyle\sum_{i=2}^n m_{i1}^2 \sum_{i=2}^n c_{ij}^2(M_{n-1}) \\ &\le n^{2B+2\gamma+3}\textstyle\sum_{i=2}^n c_{ij}^2(M_{n-1}). \end{aligned} \tag{2.2}$$

Similarly, for $j=1$ we write

$$c_{11}(M_n) = \sum_{i=2}^{n} m_{i2} c_{i2}(M_{n-1}).$$

Thus,

$$c_{11}(M_n)^2 \le n^{2B+2\gamma+3} \sum_{i=2}^{n} c_{i2}^2(M_{n-1}). \tag{2.3}$$

It follows from (2.1),(2.2) and (2.3) that

$$\sum_{2 \le i,j \le n} c_{ij}(M_{n-1})^2 \ge n^{2A-2B-2\gamma-4} \det(M_n)^2.$$

Hence, for proving Theorem 1.4, it suffices to justify the following result.

Theorem 2.5. *For any $B > 0$, there exists $A > 0$ such that*

$$\mathbf{P}\big((\sum_{2 \le i,j \le n} c_{ij}(M_{n-1})^2)^{1/2} \ge n^A |\det(M_n)| \big) \le n^{-B}.$$

To prove Theorem 2.5, we again express $\det(M_n)$ as a quadratic form of its first row.

$$\det(M_n) = c_{11}(M_n) m_{11} + \sum_{2 \le i,j \le n} c_{ij}(M_{n-1}) m_{1i} m_{j1}.$$

In other words,

$$\det(M_n)/c = m_{11} c_{11}/c + \sum_{2 \le i,j \le n} a_{ij} m_{1i} m_{1j},$$

where $c := (\sum_{2 \le i,j \le n} c_{ij}(M_{n-1})^2)^{1/2}$ and $a_{ij} := c_{ij}(M_{n-1})/c$.
Roughly speaking, our approach in this case also consists of two main steps.

- *Step 1.* Assume that

  $$\mathbf{P}_{x_{11},\dots,x_{1n}}(|(c_{11}(M_n)/c) m_{11} + \sum_{2 \le i,j \le n} a_{ij} m_{1i} m_{1j}| \le n^{-A} | M_{n-1}) \ge n^{-B}.$$

  Then there is a strong additive structure among the cofactors c_{ij}.

- *Step 2.* The probability, with respect to M_{n-1}, that there is a strong additive structure among the c_{ij} is negligible.

We now state our main supporting lemmas.

Theorem 2.6 (Step 1)**.** *Let $0 < \epsilon < 1$ be given constant. Assume that*

$$\sup_a \mathbf{P}_{x_2,\dots,x_n}(|\sum_{2 \le i,j \le n} a_{ij}(x_i + f_i)(x_j + f_j) - a| \le n^{-A}) \ge n^{-B}$$

for some sufficiently large integer A, where M_{n-1} is the matrix obtained from M_n by removing its first row and first column, $a_{ij} = c_{ij}(M_{n-1})/c$, x_i are iid copies of ξ, and f_i are arbitrary fixed numbers. Then, there exists a vector $\mathbf{u} = (u_1, \dots, u_{n-1})$ satisfying the following properties.

- $\|\mathbf{u}\| \asymp 1$ *and* $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1}) \rangle| \le n^{-A/2 + O_{B,\epsilon}(1)}$ *for* $n - O_{B,\epsilon}(1)$ *rows of* M_{n-1}.

- *There exists a generalized arithmetic progression Q of rank $O_{B,\epsilon}(1)$ and size $n^{O_{B,\epsilon}(1)}$ that contains at least $n-2n^{\epsilon}$ components u_i.*

- *All the components u_i, and all the generators of the generalized arithmetic progression are rational numbers of the form p/q, where $|p|,|q| \le n^{A/2+O_{B,\epsilon}(1)}$.*

We refer the reader to Section 3 for a definition of generalized arithmetic progression.

In the second step of the approach, we show that the probability for M_{n-1} having the above properties is negligible.

Theorem 2.7 (Step 2)**.** *With respect to M_{n-1}, the probability that there exists a vector* $\mathbf{u}$ *as in Theorem 2.6 is* $\exp(-\Omega(n))$.

The rest of the paper is organized as follows. After a short discussion of the main lemmas, we prove Theorem 2.6 in Section 4 and conclude Theorem 2.7 in Section 5. The proof of Corollary 1.6 will be presented in Section 6.

3 The Lemmas

A classical result of Erdős [6] and Littlewood-Offord [11] asserts that if a_i are real numbers of magnitude $|a_i| \ge 1$, then the probability that the random sum $\sum_{i=1}^n a_i x_i$ concentrates on an interval of length one is of order $O(n^{-1/2})$, where x_i are iid copies of a Bernoulli random variable. This remarkable inequality has generated an impressive way of research, particularly from the early 1960s to the late 1980s. We refer the reader to [9, 10] and the references therein.

Motivated by inverse theorems from additive combinatorics (see [26, Chapter 5]), Tao and Vu brought a new view to the problem: find the underlying reason as to why the concentration probability of $\sum_{i=1}^n a_i x_i$ on a short interval is large.

Typical examples of a_i that have large concentration probability are *generalized arithmetic progressions* (GAPs).

A set Q is a *GAP of rank* r if it can be expressed as in the form

$$Q = \{g_0 + k_1 g_1 + \dots + k_r g_r | k_i \in \mathbf{Z}, K_i \le k_i \le K_i' \text{ for all } 1 \le i \le r\}$$

for some $\{g_0,\dots,g_r\},\{K_1,\dots,K_r\},\{K_1',\dots,K_r'\}$.

It is convenient to think of Q as the image of an integer box $B := \{(k_1,\dots,k_r) \in \mathbf{Z}^r | K_i \le k_i \le K_i'\}$ under the linear map

$$\Phi : (k_1,\dots,k_r) \mapsto g_0 + k_1 g_1 + \dots + k_r g_r.$$

The numbers g_i are the *generators* of P, the numbers K_i' and K_i are the *dimensions* of P, and $\mathrm{Vol}(Q) := |B|$ is the *size* of B. We say that Q is *proper* if this map is one to one, or equivalently if $|Q| = \mathrm{Vol}(Q)$. For non-proper GAPs, we of course have $|Q| < \mathrm{Vol}(Q)$. If $-K_i = K_i'$ for all $i \ge 1$ and $g_0 = 0$, we say that Q is *symmetric*.

A closer look at the definition of GAPs reveals that if a_i are very *close* to the elements of a GAP of rank $O(1)$ and size $n^{O(1)}$, then the probability that $\sum_{i=1}^n a_i x_i$ concentrates on a short interval is of order $n^{-O(1)}$, where x_i are iid copies of a Bernoulli random variable.

It was shown by Tao and Vu [22, 21, 24], in an implicit way, that these are essentially the only examples that have high concentration probability. An explicit and optimal version has been given in a recent paper by the current author and Vu.

We say that a is δ-close to a set Q if there exists $q \in Q$ such that $|a-q| \le \delta$.

Theorem 3.1 (Inverse Littlewood-Offord theorem for linear forms, [14])**.** *Let $0 < \epsilon < 1$ and $B > 0$. Let $\beta > 0$ be an arbitrary real number that may depend on n. Suppose that $\sum_{i=1}^n a_i^2 = 1$, and*

$$\sup_a \mathbf{P}_{\mathbf{x}}(|\sum_{i=1}^n a_i(x_i + f_i) - a| \leq \beta) = \rho \geq n^{-B},$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and x_i are iid copies of a random variable ξ satisfying Condition 1.3. Then, for any number n' between n^ϵ and n, there exists a proper symmetric GAP $Q = \{\sum_{i=1}^r k_i g_i : k_i \in \mathbf{Z}, |k_i| \leq L_i\}$ such that

- *At least $n - n'$ elements of a_i are β-close to Q.*
- *Q has small rank, $r = O_{B,\epsilon}(1)$, and small cardinality*

$$|Q| \leq \max\left(O_{B,\epsilon}(\frac{\rho^{-1}}{\sqrt{n'}}), 1\right).$$

- *There is a non-zero integer $p = O_{B,\epsilon}(\sqrt{n'})$ such that all steps g_i of Q have the form $g_i = \beta \frac{p_i}{p}$, with $p_i \in \mathbf{Z}$ and $p_i = O_{B,\epsilon}(\beta^{-1}\sqrt{n'})$.*

In this and all subsequent theorems, the hidden constants could also depend on c_1, c_2, c_3 of Condition 1.3. We could have written $O_{c_1,c_2,c_3}(.)$ everywhere, but these notations are somewhat cumbersome, and this dependence is not our focus, so we omit them.

Theorem 3.1 was proven in [14] with $c_1 = 1, c_2 = 2$ and $c_3 = 1/2$, but the proof there automatically extends to any constants $0 < c_1 < c_2$ and $0 < c_3$.

To prove Theorem 2.6, we need a similar inverse-type result for the quadratic form $\sum_i a_{ij}(x_i + f_i)(x_j + f_j)$. We will invoke the following theorem from [13].

Theorem 3.2 (Inverse Littlewood-Offord theorem for quadratic forms, [13])**.** *Let $0 < \epsilon < 1$ and $B > 0$. Let $\beta > 0$ be an arbitrary real number that may depend on n. Assume that $a_{ij} = a_{ji}$, where $\sum_{i,j} a_{ij}^2 = 1$, and*

$$\sup_a \mathbf{P}_{\mathbf{x}}(|\sum_{i,j \leq n} a_{ij}(x_i + f_i)(x_j + f_j) - a| \leq \beta) = \rho \geq n^{-B}.$$

Then, there exist an integer $k \neq 0, |k| = n^{O_{B,\epsilon}(1)}$, a set of $r = O(1)$ rows $\mathbf{r}_{i1}, \dots, \mathbf{r}_{ir}$ of $A_n = (a_{ij})$, and set I of size at least $n - 2n^\epsilon$ such that for each $i \in I$, there exist integers $k_{ii_1}, \dots, k_{ii_r}$, all bounded by $n^{O_{B,\epsilon}(1)}$, such that the following holds.

$$\mathbf{P}_{\mathbf{z}}(|\langle \mathbf{z}, k\mathbf{r}_i(A_n) + \sum_{j=1}^r k_{ii_j}\mathbf{r}_{i_j}(A_n)\rangle| \leq \beta n^{O_{B,\epsilon}(1)}) \geq n^{-O_{B,\epsilon}(1)}, \tag{3.1}$$

where $\mathbf{z} = (z_1, \dots, z_n)$ and z_i are iid copies of $\eta^{(1/2)}(\xi - \xi')$, where $\eta^{(1/2)}$ is a Bernoulli random variable of parameter $1/2$ independent of ξ and ξ'.

4 proof of Theorem 2.6

We first apply Theorem 3.2 to a_{ij} to obtain

$$\mathbf{P}_{\mathbf{z}}(|\langle \mathbf{z}, k\mathbf{r}_i(A_n) + \sum_j k_{ii_j}\mathbf{r}_{i_j}(A_n)\rangle| \leq n^{-A+O_{B,\epsilon}(1)}) \geq n^{-O_{B,\epsilon}(1)}.$$

For short, we denote by $\mathbf{r}'_i$ the vector $k\mathbf{r}_i(A_n) + \sum_j k_{ii_j}\mathbf{r}_{i_j}(A_n)$. Thus, for any $i \in I$,

$$\mathbf{P}_{\mathbf{z}}(|\langle \mathbf{z}, \mathbf{r}_i' \rangle| \le n^{-A+O_{B,\epsilon}(1)}) \ge n^{-O_{B,\epsilon}(1)}. \tag{4.1}$$

Ideally, our next step is to apply Theorem 3.1 to the $\mathbf{r}_i'$. However, the application is meaningful only when $\|\mathbf{r}_i'\|$ is relatively large. Investigating the degenerate case is our next goal.

Set

$$K = n^{-A/2}.$$

We consider two cases.

Case 1.(*degenerate case*) $\|\mathbf{r}_i'\| \le K$ for all $i \in I$. Hence, with $I_0 := \{i_1, \dots, i_r\}$

$$\|k\mathbf{r}_i(A_n) + \sum_{j \in I_0} k_{ij}\mathbf{r}_j(A_n)\| = \|\mathbf{r}_i'\| \le K. \tag{4.2}$$

Next, because $\sum_j \|\mathbf{c}_j(A_n)\|^2 = 1$, there exists an index j_0 such that $\|\mathbf{c}_{j_0}(A_n)\| \ge n^{-1/2}$. Consider this column vector.

It follows from (4.2) that for any $i \in I$,

$$|k\mathbf{c}_{j_0}(i) + \sum_{j \in I_0} k_{ij}\mathbf{c}_{j_0}(j)| \le K.$$

The above inequality means that the components $\mathbf{c}_{j_0}(i)$ of $\mathbf{c}_{j_0}(A_n)$ belong to a GAP generated by $\mathbf{c}_{j_0}(j)/k, j \in I_0$, up to an error K. This suggests us the following approximation.

For each $j \notin I$, we approximate $\mathbf{c}_{j_0}(j)$ by a number v_j of the form $(1/\lfloor 2K^{-1} \rfloor) \cdot \mathbf{Z}$ such that $|v_j - \mathbf{c}_{j_0}(j)| \le K$. We next set

$$v_i := \sum_{j \in I_0} k_{ij}v_j/k$$

for any $i \in I$.

Thus, v_i belongs to a GAP of rank $O_{B,\epsilon}(1)$ and size $n^{O_{B,\epsilon}(1)}$ for all $i \in I$.

With $\mathbf{v} = (v_1, \dots, v_{n-1})$, we have

$$\|\mathbf{v} - \mathbf{c}_{j_0}(A_n)\| \le Kn^{O_{B,\epsilon}(1)}.$$

Furthermore, by Condition 2.1, and because $\langle \mathbf{c}_{j_0}(A_n), \mathbf{r}_i(M_{n-1}) \rangle = 0$ for $i \ne j_0$, we infer that

$$|\langle \mathbf{v}, \mathbf{r}_i(M_{n-1}) \rangle| \le Kn^{O_{B,\epsilon}(1)}.$$

Note that $\|\mathbf{v}\| \gg n^{-1/2}$. Set $\mathbf{u} := \lfloor 1/\|\mathbf{v}\| \rfloor \cdot \mathbf{v}$, we then obtain

- $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1}) \rangle| \le n^{-A/2+O_{B,\epsilon}(1)}$ for $n-2$ rows of M_{n-1}.
- There exists a GAP of rank $O_{B,\epsilon}(1)$ and size $n^{O_{B,\epsilon}(1)}$ that contains at least $n - 2n^{\epsilon}$ components u_i.
- All the components u_i, and all the generators of the GAP are rational numbers of the form p/q, where $|p|, |q| \le n^{A/2+O_{B,\epsilon}(1)}$.

Case 2.(*non-degenerate case*). There exists $i_0 \in I$ such that $\|\mathbf{r}'_{i_0}\| \geq K$. Because $\mathbf{r}'_{i_0} = k\mathbf{r}_{i_0}(A_n) + \sum_{j \in I_0} k_{i_0 j}\mathbf{r}_j(A_n)$, $\mathbf{r}'_{i_0}$ is orthogonal to $n - |I_0| - 1 = n - O_{B,\epsilon}(1)$ column vectors of M_{n-1}. Consequently, because M_{n-1} is symmetric, $\mathbf{r}'_{i_0}$ is orthogonal to $n - O_{B,\epsilon}(1)$ row vectors of M_{n-1}.

Set

$$\mathbf{v} := \mathbf{r}'_{i_0}/\|\mathbf{r}'_{i_0}\|.$$

Hence, $\langle \mathbf{v}, \mathbf{r}_i(M_{n-1})\rangle = 0$ for at least $n - O_{B,\epsilon}(1)$ row vectors of M_{n-1}.

Also, it follows from (4.1) that

$$\mathbf{P}_{\mathbf{z}}(|\langle \mathbf{z}, \mathbf{v}\rangle| \leq n^{-A/2+O_{B,\epsilon}(1)}) \geq n^{-O_{B,\epsilon}(1)}. \tag{4.3}$$

Next, because the z_i satisfy Condition 1.3, Theorem 3.1 applying to (4.3) implies that $\mathbf{v}$ can be approximated by a vector $\mathbf{u}$ as follows.

- $|u_i - v_i| \leq n^{-A/2+O_{B,\epsilon}(1)}$ for all i.

- There exists a GAP of rank $O_{B,\epsilon}(1)$ and size $n^{O_{B,\epsilon}(1)}$ that contains at least $n - n^{\epsilon}$ components u_i.

- All the components u_i, and all the generators of the GAP are rational numbers of the form p/q, where $|p|, |q| \leq n^{A/2+O_{B,\epsilon}(1)}$.

Note that, by the approximation above, we have $\|\mathbf{u}\| \asymp 1$ and $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \leq n^{-A/2+O_{B,\epsilon}(1)}$ for at least $n - O_{B,\epsilon}(1)$ row vectors of M_{n-1}.

5 Proof of Theorem 2.7

We first bound the number N of vectors $\mathbf{u}$ satisfying the conclusion of Theorem 2.7.

Because each GAP is determined by its generators and dimensions, the number of Qs is bounded by $(n^{A+O_{B,\epsilon}(1)})^{O_{B,\epsilon}(1)}(n^{O_{B,\epsilon}(1)})^{O_{B,\epsilon}(1)} = n^{O_{A,B,\epsilon}(1)}$.

Next, for a given Q of rank $O_{B,\epsilon}(1)$ and size $n^{O_{B,\epsilon}(1)}$ obtained from Theorem 2.6, there are at most $n^{n-2n^{\epsilon}}|Q|^{n-2n^{\epsilon}} = n^{O_{B,\epsilon}(n)}$ ways to choose the $n - 2n^{\epsilon}$ components u_i that Q contains.

The remaining components belong to the set $\{p/q, |p|, |q| \leq n^{A/2+O_{B,\epsilon}(1)}\}$, so there are at most $(n^{A+O_{B,\epsilon}(1)})^{2n^{\epsilon}} = n^{O_{A,B,\epsilon}(n^{\epsilon})}$ ways to choose them.

Hence, we obtain the key bound

$$N \leq n^{O_{A,B,\epsilon}(1)} n^{O_{B,\epsilon}(n)} n^{O_{A,B,\epsilon}(n^{\epsilon})} = n^{O_{B,\epsilon}(n)}. \tag{5.1}$$

Set $\beta_0 := n^{-A/2+O_{B,\epsilon}(1)}$, the bound obtained from the conclusion of Theorem 2.6. For a vector $\mathbf{u}$, we define $\mathbf{P}_{\beta_0}(\mathbf{u})$ as follows

$$\mathbf{P}_{\beta_0}(\mathbf{u}) := \mathbf{P}(|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \leq \beta_0 \text{ for } n - O_{B,\epsilon}(1) \text{ rows of } M_{n-1}).$$

From (5.1), for our task of proving Theorem 2.7, it would be ideal if we can show that the probability $\mathbf{P}_{\beta_0}(\mathbf{u})$ is smaller than $\exp(-\Omega(n))/N$ for each $\mathbf{u}$.

Roughly speaking, our strategy is to classify $\mathbf{u}$ into two classes: one contains of $\mathbf{u}$ of very small $\mathbf{P}_{\beta_0}(\mathbf{u})$, and thus their contribution is negligible; the other contains of $\mathbf{u}$ of relatively large $\mathbf{P}_{\beta_0}(\mathbf{u})$. To deal with those $\mathbf{u}$ of the second type, we will not control $\sum \mathbf{P}_{\beta_0}(\mathbf{u})$ directly but pass to a class of new vectors $\mathbf{u}'$ that are also almost orthogonal to many rows of M_{n-1}, while the probability $\sum \mathbf{P}_{\beta_0}(\mathbf{u}')$ is relatively smaller than $\sum \mathbf{P}_{\beta_0}(\mathbf{u})$. More details follow.

5.1 Technical reductions and key observations

By paying a factor of $n^{O_{B,\epsilon}(1)}$ in probability and without loss of generality we may assume that $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \le \beta_0$ for the first $n - O_{B,\epsilon}(1)$ rows of M_{n-1}. Also, by paying another factor of n^{n^ϵ} in probability, we may assume that the first n_0 components u_i of $\mathbf{u}$ belong to a GAP Q, and $u_{n_0} \ge 1/2\sqrt{n-1}$, where $n_0 := n - 2n^\epsilon$. We refer to remaining u_i as exceptional components. Note that these extra factors do not affect our final bound $\exp(-\Omega(n))$.

For given $\beta > 0$ and $i \le n_0$, we define

$$\rho_\beta^{(i)}(\mathbf{u}) := \sup_a \mathbf{P}_{x_i,\dots,x_{n_0}}(|x_i u_i + \dots + x_{n_0} u_{n_0} - a| \le \beta),$$

where $x_i, \dots, x_{n_0}$ are iid copies of ξ.

A crucial observation is that, by exposing the rows of M_{n-1} one by one, and due to symmetry, the probability $\mathbf{P}_\beta(\mathbf{u})$ that $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \le \beta$ for all $i \le n - O_{B,\epsilon}(1)$ can be bounded by

$$\begin{aligned}
\mathbf{P}_\beta(\mathbf{u}) &\le \prod_{1\le i\le n-O_{B,\epsilon}(1)} \sup_a \mathbf{P}_{x_i,\dots,x_{n-1}}(|x_i u_i + \dots + x_{n-1}u_{n-1} - a| \le \beta) \\
&\le \prod_{1\le i\le n_0} \sup_a \mathbf{P}_{x_i,\dots,x_{n_0}}(|x_i u_i + \dots + x_{n_0}u_{n_0} - a| \le \beta) \\
&= \prod_{1\le i\le n_0} \rho_\beta^{(i)}(\mathbf{u}). \qquad (5.2)
\end{aligned}$$

Also, because of Condition 1.3 and $u_{n_0} \ge 1/2\sqrt{n-1}$, for any $\beta < c_1/2\sqrt{n-1}$ we have

$$\begin{aligned}
\rho_\beta^{(k)}(\mathbf{u}) &\le \sup_a \mathbf{P}_{x_{n_0}}(|x_{n_0}u_{n_0} - a| \le \beta) \\
&\le 1 - c_3, \qquad (5.3)
\end{aligned}$$

and thus,

$$\mathbf{P}_\beta(\mathbf{u}) \le (1-c_3)^{n_0} = (1-c_3)^{(1-o(1))n}.$$

Next, let C be a sufficiently large constant depending on B and ϵ. We classify $\mathbf{u}$ into two classes $\mathcal{B}$ and $\mathcal{B}'$, depending on whether $\mathbf{P}_{\beta_0}(\mathbf{u}) \ge n^{-Cn}$ or not.

Because of (5.1), and as C is large enough,

$$\sum_{\mathbf{u}\in\mathcal{B}'} \mathbf{P}_{\beta_0}(\mathbf{u}) \le n^{O_{B,\epsilon}(n)}/n^{Cn} \le n^{-n/2}. \qquad (5.4)$$

For the rest of the section, we focus on $\mathbf{u} \in \mathcal{B}$.

5.2 Approximation for degenerate vectors

Let $\mathcal{B}_1$ be the collection of $\mathbf{u} \in \mathcal{B}$ satisfying the following property: for any $n' = n^{1-\epsilon}$ components $u_{i_1}, \dots, u_{i_{n'}}$ among the $u_1, \dots, u_{n_0}$, we have

$$\sup_a \mathbf{P}_{x_{i_1},\dots,x_{i_{n'}}}(|u_{i_1}x_{i_1} + \dots + u_{i_{n'}}x_{i_{n'}} - a| \le n^{-B-4}) \ge (n')^{-1/2+o(1)}. \qquad (5.5)$$

For consision we set $\beta = n^{-B-4}$. It follows from Theorem 3.1 that, among any $u_{i_1}, \dots, u_{i_{n'}}$, there are, say, at least $n'/2 + 1$ components that belong to an interval of length 2β. This is because our GAP Q now has only one element as in the size estimate the upper bound $O(\rho^{-1}/\sqrt{n'/2})$ is now $o(1)$. (One may also deduce this fact from the original Littlewood-Offord theorem.)

A simple argument then implies that there is an interval of length 2β that contains all but $n'-1$ components u_i. (To prove this, arrange the components in increasing order, then all but perhaps the first $n'/2$ and the last $n'/2$ components will belong to an interval of length 2β).

Thus there exists a vector $\mathbf{u}' \in (2\beta)\cdot\mathbf{Z}$ satisfying the following conditions.

- $|u_i - u'_i| \leq 2\beta$ for all i.
- $u'_i = u$ for at least $n_0 - n'$ indices i.

Because of the approximation and of Condition 2.1 that $|x_{ij}| \leq n^{B+1}$, whenever $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \leq \beta_0$, we have

$$|\langle \mathbf{u}', \mathbf{r}_i(M_{n-1})\rangle| \leq n^{B+2}(2\beta) + \beta_0 := \beta'.$$

It is clear, from the bound on β and β_0, that $\beta' \leq c_1/2\sqrt{n-1}$, and thus by (5.3),

$$\mathbf{P}_{\beta'}(\mathbf{u}') \leq (1-c_3)^{(1-o(1))n}.$$

Now we bound the number of $\mathbf{u}'$ obtained from the approximation. First, there are $O(n^{n-n_0+n'}) = O(n^{2n^{1-\epsilon}})$ ways to choose those u'_i that take the same value u, and there are just $O(\beta^{-1})$ ways to choose u. The remaining components belong to the set $(2\beta)^{-1}\cdot\mathbf{Z}$, and thus there are at most $O((\beta^{-1})^{n-n_0+n'}) = O(n^{O_{A,B,\epsilon}(n^{1-\epsilon})})$ ways to choose them.

Hence we obtain the total bound

$$\begin{aligned}\sum_{\mathbf{u}\in\mathcal{B}_1} \mathbf{P}_{\beta_0}(\mathbf{u}) \leq \sum_{\mathbf{u}'} \mathbf{P}_{\beta'}(\mathbf{u}') &\leq O(n^{2n^{1-\epsilon}})O(n^{O_{A,B,\epsilon}(n^{1-\epsilon})})(1-c_3)^{(1-o(1))n} \\ &\leq (1-c_3)^{(1-o(1))n}.\end{aligned}$$

5.3 Approximation for non-degenerate vectors

Assume that $\mathbf{u} \in \mathcal{B}_2 := \mathcal{B}\backslash\mathcal{B}_1$. By exposing the rows of M_{n-1} accordingly, and by paying an extra factor $\binom{n_0}{n'} = O(n^{n^{1-\epsilon}})$ in probability, we may assume that the components $u_{n_0-n'+1}, \dots, u_{n_0}$ satisfy the property

$$\begin{aligned}\sup_a \mathbf{P}_{x_{n_0-n'+1},\dots,x_{n_0}}(|u_{n_0-n'+1}x_{n_0-n'+1} + \cdots + u_{n_0}x_{n_0} - a| \leq n^{-B-4}) &\leq (n')^{-1/2+o(1)} \\ &\leq n^{-1/2+\epsilon/2+o(1)}. \qquad (5.6)\end{aligned}$$

Next, define the following sequence $\beta_k, k \geq 0$. $\beta_0 = n^{-A/2+O_{B,\epsilon}(1)}$ is the bound obtained from the conclusion of Theorem 2.6, and

$$\beta_{k+1} := (2n^{B+2}+1)\beta_k.$$

Recall from (5.2) that

$$\mathbf{P}_{\beta_k}(\mathbf{u}) \leq \prod_{1\leq i\leq n_0-n'} \rho^{(i)}_{\beta_k}(\mathbf{u}) =: \pi_{\beta_k}(\mathbf{u}).$$

Roughly speaking, the reason we truncated the product here is that whenever $i \leq n_0 - n^{1-\epsilon}$, and β_k is small enough, the terms $\rho^{(i)}_{\beta_k}(\mathbf{u})$ are smaller than $(n')^{-1/2+o(1)}$, owing to (5.6). This fact will allow us to gain some significant factors when applying Theorem 3.1.

Note that $\pi_{\beta_k}(\mathbf{u})$ increases with k, and recall that $\pi_{\beta_0}(\mathbf{u}) \geq n^{-Cn}$. Thus, by the pigeonhole principle, there exists $k_0 := k_0(\mathbf{u}) \leq C\epsilon^{-1}$ such that

$$\pi_{\beta_{k_0+1}}(\mathbf{u}) \leq n^{\epsilon n}\pi_{\beta_{k_0}}(\mathbf{u}). \tag{5.7}$$

It is crucial to note that, since A was chosen to be sufficiently large compared to $O_{B,\epsilon}(1)$ and C, we have

$$\beta_{k_0+1} \leq n^{-B-4}.$$

Having mentioned the upper bound of $\rho^{(i)}_{\beta_i}(\mathbf{u})$, we now turn to its lower bound. Because of Condition 2.1 and $u_i \leq 1$ for all i, the following trivial bound holds for any $\beta \geq \beta_0$ and $i \leq n_0 - n'$ by pigeonhole principle,

$$\rho^{(i)}_{\beta}(\mathbf{u}) \geq \beta n^{-B-2} \geq \beta_0 n^{-B-2} = n^{-A/2+O_{B,\epsilon}(1)}.$$

We next divide the interval $I = [n^{-A/2+O_{B,\epsilon}(1)}, n^{-1/2+\epsilon/2+o(1)}]$ into $K = (A/2 + O_{B,\epsilon}(1))\epsilon^{-1}$ sub-intervals $I_k = [n^{-A/2+O_{B,\epsilon}(1)+k\epsilon}, n^{-A/2+O_{B,\epsilon}(1)+(k+1)\epsilon}]$. For short, we denote by ρ_k the left endpoint of each I_k. Thus $\rho_k = n^{-A/2+O_{B,\epsilon}(1)+k\epsilon}$.

With all the necessary settings above, we now classify $\mathbf{u}$ basing on the distributions of the $\rho^{(i)}_{\beta_{k_0}}(\mathbf{u}), 1 \leq i \leq n_0 - n^{1-\epsilon}$.

For each $0 \leq k_0 \leq C\epsilon^{-1}$ and each tuple $(m_0, \dots, m_K)$ satisfying $m_0 + \cdots + m_K = n_0 - n^{1-\epsilon}$, we let $\mathcal{B}^{(m_0,\dots,m_K)}_{k_0}$ denote the collection of those $\mathbf{u}$ from $\mathcal{B}_2$ that satisfy the following conditions.

- $k_0(\mathbf{u}) = k_0$.

- There are exactly m_k terms of the sequence $(\rho^{(i)}_{\beta_{k_0}}(\mathbf{u}))$ belonging to the interval I_k. In other words, if $m_0 + \cdots + m_{k-1} + 1 \leq i \leq m_0 + \cdots + m_k$ then $\rho^{(i)}_{\beta_{k_0}}(\mathbf{u}) \in I_k$.

Now we will use Theorem 3.1 to approximate $\mathbf{u} \in \mathcal{B}^{(m_0,\dots,m_K)}_{k_0}$ as follows.

- *First step.* Consider each index i in the range $1 \leq i \leq m_0$. Because $\rho^{(1)}_{\beta_{k_0}} \in I_0$, we apply Theorem 3.1 to approximate u_i by u_i' such that $|u_i - u_i'| \leq \beta_{k_0}$ and the u_i' belong to a GAP Q_0 of rank $O_{B,\epsilon}(1)$ and size $O(\rho_0^{-1}/n^{1/2-\epsilon})$ for all but $n^{1-2\epsilon}$ indices i. Furthermore, all u_i' have the form $\beta_{k_0} \cdot p/q$, where $|p|, |q| = O(n\beta_{k_0}^{-1}) = O(n^{A/2+O_{B,\epsilon}(1)})$.

- *k-th step,* $1 \leq k \leq K$. We focus on i from the range $n_0 + \cdots + n_{k-1} + 1 \leq i \leq n_0 + \cdots + n_k$. Because $\rho^{(n_0+\cdots+n_{k-1}+1)}_{\beta_{k_0}} \in I_k$, we apply Theorem 3.1 to approximate u_i by u_i' such that $|u_i - u_i'| \leq \beta_{k_0}$ and u_i belongs to a GAP Q_k of rank $O_{B,\epsilon}(1)$ and size $O(\rho_k^{-1}/n^{1/2-\epsilon})$ for all but $n^{1-2\epsilon}$ indices i. Furthermore, all u_i' have the form $\beta_{k_0} \cdot p/q$, where $|p|, |q| = O(n\beta_{k_0}^{-1}) = O(n^{A/2+O_{B,\epsilon}(1)})$.

- For the remaining components u_i, we just simply approximate them by the closest point in $\beta_{i_0} \cdot \mathbf{Z}$.

We have thus provided an approximation of $\mathbf{u}$ by $\mathbf{u}'$ satisfying the following properties.

1. $|u_i - u'_i| \le \beta_{k_0}$ for all i.

2. $u'_i \in Q_k$ for all but $n^{1-2\epsilon}$ indices i in the range $m_0+\cdots+m_{k-1}+1 \le i \le m_0+\cdots+m_k$.

3. All the u'_i, including the generators of Q_k, belong to the set $\beta_{k_0}\cdot\{p/q, |p|, |q| \le n^{A/2+O_{B,\epsilon}(1)}\}$.

4. Q_k has rank $O_{B,\epsilon}(1)$ and size $|Q_k| = O(\rho_k^{-1}/n^{1/2-\epsilon})$.

Let ${\mathcal{B}'}_{k_0}^{(m_1,\dots,m_K)}$ be the collection of all $\mathbf{u}'$ obtained from $\mathbf{u} \in \mathcal{B}_{k_0}^{(m_1,\dots,m_K)}$ as above. Observe that, as $|\langle \mathbf{u}, \mathbf{r}_i(M_{n-1})\rangle| \le \beta_{k_0}$ for all $i \le n - O_{B,\epsilon}(1)$, we have

$$|\langle \mathbf{u}', \mathbf{r}_i(M_{n-1})\rangle| \le (n^{B+2}+1)\beta_{k_0}. \tag{5.8}$$

Hence, in order to justify Theorem 2.7 in the case $\mathbf{u} \in \mathcal{B}_2$, it suffices to show that the probability that (5.8) holds for all $i \le n - O_{B,\epsilon}(1)$, for some $\mathbf{u}' \in {\mathcal{B}'}_{k_0}^{(m_1,\dots,m_K)}$, is small.

Consider a $\mathbf{u}' \in {\mathcal{B}'}_{k_0}^{(m_1,\dots,m_K)}$ and the probability $\mathbf{P}_{(n^{B+2}+1)\beta_{k_0}}(\mathbf{u}')$ that (5.8) holds for all $i \le n - O_{B,\epsilon}(1)$. We have

$$\begin{aligned}
\mathbf{P}_{(n^{B+2}+1)\beta_{k_0}}(\mathbf{u}') &\le \prod_{1\le i\le n_0-n^{1-\epsilon}} \sup_a \mathbf{P}_{x_i,\dots,x_{n_0}}(|u'_i x_i + \cdots + u'_{n-1}x_{n_0} - a| \le (n^{B+2}+1)\beta_{k_0}) \\
&\le \prod_{1\le i\le n_0-n^{1-\epsilon}} \sup_a \mathbf{P}_{x_i,\dots,x_{n_0}}(|u_i x_i + \cdots + u_{n-1}x_{n_0} - a| \le (2n^{B+2}+1)\beta_{k_0}) \\
&= \pi_{\beta_{k_0+1}}(\mathbf{u}) \le n^{\epsilon n}\pi_{\beta_{k_0}}(\mathbf{u}),
\end{aligned}$$

where in the last inequality we used (5.7).
We recall from the definition of $\mathcal{B}_{k_0}^{(m_1,\dots,m_K)}$ that

$$\begin{aligned}
\pi_{\beta_{k_0}}(\mathbf{u}) \le \prod_{k=1}^K \rho_{k+1}^{m_k} &= n^{\epsilon(m_1+\cdots+m_k)}\prod_{k=1}^K \rho_k^{m_k} \\
&\le n^{\epsilon n}\prod_{k=1}^K \rho_k^{m_k}.
\end{aligned}$$

Hence,

$$\mathbf{P}_{(n^{B+2}+1)\beta_{k_0}}(\mathbf{u}') \le n^{2\epsilon n}\prod_{k=1}^K \rho_k^{m_k}. \tag{5.9}$$

In the next step we bound the size of ${\mathcal{B}'}_{k_0}^{(m_1,\dots,m_K)}$.

Because each Q_k is determined by its $O_{B,\epsilon}(1)$ generators from the set $\beta_{k_0}\cdot\{p/q, |p|, |q| \le n^{A/2+O_{B,\epsilon}(1)}\}$, and its dimensions from the integers bounded by $n^{O_{B,\epsilon}(1)}$, there are $n^{O_{A,B,\epsilon}(1)}$ ways to choose each Q_k. So the total number of ways to choose $Q_1,\dots,Q_K$ is bounded by

$$(n^{O_{A,B,\epsilon}(1)})^K = n^{O_{A,B,\epsilon}(1)}.$$

Next, after locating Q_k, the number N_1 of ways to choose u'_i from each Q_k is

$$\begin{aligned}N_1 &\le \prod_{k=1}^K \binom{m_k}{n^{1-2\epsilon}} |Q_k|^{m_k - n^{1-2\epsilon}}\\ &\le 2^{m_1+\cdots+m_K} \prod_{k=1}^K |Q_k|^{m_k}\\ &\le (O(1))^n \prod_{k=1}^K \rho_k^{-m_k}/n^{(1/2-\epsilon)(m_1+\cdots+m_k)}\\ &\le \prod_{k=1}^K \rho_k^{-m_k}/n^{(1/2-\epsilon-o(1))n},\end{aligned}$$

where we used the bound $|Q_k| = O(\rho_k^{-1}/n^{1/2-\epsilon})$.

The remaining components u'_i can take any value from the set $\beta_{k_0} \cdot \{p/q, |p|, |q| \le n^{A/2+O_{B,\epsilon}(1)}\}$, so the number N_2 of ways to choose them is bounded by

$$N_2 \le (n^{A+O_{B,\epsilon}(1)})^{2n^\epsilon + Kn^{1-2\epsilon}} = n^{O_{A,B,\epsilon}(n^{1-2\epsilon})}.$$

Putting the bound for N_1 and N_2 together, we obtain a bound N for $|\mathcal{B}'^{(m_1,\dots,m_K)}_{k_0}|$,

$$N \le \prod_{k=1}^K \rho_k^{-m_k}/n^{(1/2-\epsilon-o(1))n}. \tag{5.10}$$

It follows from (5.9) and (5.10) that

$$\sum_{\mathbf{u}' \in \mathcal{B}'^{(m_1,\dots,m_K)}_{k_0}} \mathbf{P}_{(n^{B+2}+1)\beta_{k_0}}(\mathbf{u}') \le n^{2\epsilon n} \prod_{k=1}^K \rho_k^{m_k} \prod_{k=1}^K \rho_k^{-m_k}/n^{(1/2-\epsilon-o(1))n} \le n^{-(1/2-3\epsilon-o(1))n}. \tag{5.11}$$

Summing over the choices of k_0 and $(m_1,\dots,m_K)$ we obtain the bound

$$\sum_{k_0,m_1,\dots,m_K} \sum_{\mathbf{u}' \in \mathcal{B}'^{(m_1,\dots,m_K)}_{k_0}} \mathbf{P}_{(n^{B+2}+1)\beta_{k_0}}(\mathbf{u}') \le n^{-(1/2-3\epsilon-o(1))n},$$

completing the proof of Theorem 2.7.

6 Proof of Corollary 1.6

Assume that the upper diagonal entries of M_n satisfy the conditions of Corollary 1.6. We denote by $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ the real eigenvalues of M_n.

Our first ingredient is the following special form of the spectral concentration result of Guionnet and Zeitouni.

Lemma 6.1. *[8, Theorem 1.1] Assume that f is a convex Lipschitz function. Then for any $\delta \ge \delta_0 := 16C\sqrt{\pi}|f|_L/n$,*

$$\mathbf{P}\left(|\sum_{i=1}^n f(\lambda_i) - \mathbf{E}(\sum_{i=1}^n f(\lambda_i))| \ge \delta n\right) \le 4\exp(-\frac{n^2(\delta-\delta_0)^2}{16C^2|f|_L^2}).$$

Following [3] and [7], we will apply the above theorem to the cut-off functions $f_\epsilon^+(x) := \log(\max(\epsilon, x))$ and $f_\epsilon^-(x) = \log(\max(\epsilon, -x))$, for some $\epsilon > 0$ to be determined. The main reason we have to truncate the $\log$ function is because it is not Lipschitz. Note that f^+ and f^{-1} both have Lipschitz constant ϵ^{-1}. Although they are not convex, it is easy to write them as difference of convex functions of Lipschitz constant $O(\epsilon^{-1})$, and so Lemma 6.1 applies. Thus the following estimates hold for $\delta \gg (\epsilon n)^{-1}$

$$\mathbf{P}\left(|\sum_{\lambda_i \in S_\epsilon^+} \log \lambda_i - \mathbf{E}(\sum_{\lambda_i \in S_\epsilon^+} \log \lambda_i)| \geq \delta n\right) \leq \exp(-\Theta(n^2\delta^2\epsilon^2))$$

and

$$\mathbf{P}\left(|\sum_{\lambda_i \in S_\epsilon^-} \log |\lambda_i| - \mathbf{E}(\sum_{\lambda_i \in S_\epsilon^-} \log |\lambda_i|)| \geq \delta n\right) \leq \exp(-\Theta(n^2\delta^2\epsilon^2)),$$

where $S_\epsilon^+ := \{\lambda_i, \lambda_i \geq \epsilon\}$ and $S_\epsilon^- := \{\lambda_i, \lambda_i \leq -\epsilon\}$.
Hence,

$$\mathbf{P}\left(|\sum_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} \log |\lambda_i| - \mathbf{E}(\sum_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} \log |\lambda_i|)| \geq 2\delta n\right) \leq \exp(-\Theta(n^2\delta^2\epsilon^2)). \tag{6.1}$$

Roughly speaking, (6.1) implies that $\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} |\lambda_i|$ is well concentrated around its mean. It thus remains to control the factor $R := \prod_{|\lambda_i| \leq \epsilon} |\lambda_i|$. We will bound R away from zero, relying on Theorem 1.4 and Lemma 6.2 below.

Lemma 6.2. *[25, Proposition 66], [5, Theorem 5.1] Assume that M_n is a random symmetric matrix of entries satisfying the conditions of Corollary 1.6. Then for all $I \subset \mathbf{R}$ with $|I| \geq K^2 \log^2 n / n^{1/2}$, one has*

$$N_I \ll n^{1/2}|I|$$

with probability $1 - \exp(-\omega(\log n))$, where N_I is the number of λ_i belonging to I.

We refer the readers to [4] for a survey of recent results on the distribution of the eigenvalues of M_n.

By Lemma 6.2, we have $|\{i, |\lambda_i| \leq \epsilon\}| \ll n^{1/2}\epsilon$. Also, Theorem 1.4 implies that $\min_i\{|\lambda_i|\} \geq n^{-A}$ with probability $1 - O(n^{-B})$. Thus

$$R = \prod_{|\lambda_i| \leq \epsilon} |\lambda_i| \geq (\min_i\{|\lambda_i|\})^{n^{1/2}\epsilon} = n^{-O(n^{1/2}\epsilon)}. \tag{6.2}$$

Our next goal is the following result.

Proposition 6.3. *With probability $1 - n^{-\omega(1)}$ we have*

$$\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} |\lambda_i| = \exp(-O(\epsilon^{-1}\log n + \epsilon^{-2}))\mathbf{E}(\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} |\lambda_i|) - \exp(\frac{2\log n}{\epsilon}) \tag{6.3}$$

and

$$\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} \lambda_i^2 = \exp(-O(\epsilon^{-1}\log n + \epsilon^{-2}))\mathbf{E}(\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} \lambda_i^2) - \exp(\frac{2\log n}{\epsilon}). \tag{6.4}$$

Let us complete the proof of the first half of Corollary 1.6 assuming Proposition 6.3. The second half follows by the same reasoning.

Firstly, because $\prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}|\lambda_i| \geq \prod_{i=1}^n |\lambda_i|/\epsilon^{n-|S_\epsilon^-\cup S_\epsilon^+|} \geq \prod_{i=1}^n |\lambda_i| = |\det(M_n)|$, it follows from Proposition 6.3 that with probability $1-n^{-\omega(1)}$,

$$\prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}|\lambda_i| = \exp(-O(\epsilon^{-1}\log n+\epsilon^{-2}))\mathbf{E}(|\det(M_n)|)-\exp(\frac{2\log n}{\epsilon}). \tag{6.5}$$

Secondly, by (6.2), the following holds with probability $1-O(n^{-B})$

$$|\det(M_n)| = \prod_{\lambda_i\notin S_\epsilon^-\cup S_\epsilon^+}|\lambda_i| \prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}|\lambda_i| \geq n^{-O(n^{1/2}\epsilon)}\prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}|\lambda_i|.$$

Combining with (6.5), we have

$$|\det(M_n)| = \exp(-O(\epsilon^{-1}\log n+\epsilon^{-2}+\epsilon n^{1/2}\log n))\mathbf{E}(|\det(M_n|)-n^{-O(n^{1/2}\epsilon)}\exp(\frac{2\log n}{\epsilon}).$$

By choosing $\epsilon = n^{-1/6}$, we obtain the conclusion of Corollary 1.6, noting that $\mathbf{E}(|\det(M_n)|)\gg \exp(n)$.

It remains to prove Proposition 6.3.

Proof. (of Proposition 6.3) Set

$$U := \sum_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}\log|\lambda_i| - \mathbf{E}(\sum_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}\log|\lambda_i|).$$

By (6.1) we have

$$\mathbf{P}(|U|\geq 2\delta n) \leq \exp(-\Theta(n^2\delta^2\epsilon^2)), \tag{6.6}$$

for $\delta \gg (n\epsilon)^{-1}$.

Also, note that $\mathbf{E}(U)=0$. Thus, by Jensen inequality and by (6.6),

$$\begin{aligned} 1\leq \mathbf{E}(\exp(U)) &\leq \mathbf{E}(\exp(|U|)) \\ &\leq 1+\int_0^\infty \exp(t)\mathbf{P}(|U|\geq t)dt \\ &\leq 1+\int_0^{\log n/\epsilon}\exp(t)dt + \int_{\log n/\epsilon}^\infty \exp(t)\exp(-\Theta(t^2\epsilon^2))dt \\ &= \exp(O(\epsilon^{-1}\log n+\epsilon^{-2})). \end{aligned} \tag{6.7}$$

Observe that

$$\mathbf{E}(\exp(U)) = \mathbf{E}(\prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon+}|\lambda_i|)/\exp(\mathbf{E}(\sum_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}\log|\lambda_i|)).$$

It thus follows from (6.7) that

$$\exp(\mathbf{E}(\sum_{\lambda_i\in S_\epsilon^-\cup S_\epsilon^+}\log|\lambda_i|)) = \exp(-O(\epsilon^{-1}\log n+\epsilon^{-2}))\mathbf{E}(\prod_{\lambda_i\in S_\epsilon^-\cup S_\epsilon+}|\lambda_i|).$$

This relation, together with (6.6), imply that with probability $1-n^{-\omega(1)}$,

$$\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} |\lambda_i| = \exp(-O(\epsilon^{-1}\log n + \epsilon^{-2}))\mathbf{E}(\prod_{\lambda_i \in S_\epsilon^- \cup S_\epsilon^+} |\lambda_i|) - \exp(\frac{2\log n}{\epsilon}).$$

The second half of Proposition 6.3 follows from the identical calculation applied to $\exp(2U)$.

□

A Proof of Lemma 2.3

Assume that $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbf{R}^n$ are independent vectors that span H. Also, without loss of generality, we assume that the subvectors $(v_{11}, \dots, v_{1k}), \dots, (v_{k1}, \dots, v_{kk})$ generate a full space of dimension k.

Consider a random vector $\mathbf{u} = (f_1 + x_1, \dots, f_n + x_n)$, where $x_1, \dots, x_n$ are iid copies of ξ. If $\mathbf{u} \in H$, then there exist $\alpha_1, \dots, \alpha_k$ such that

$$\mathbf{u} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i.$$

Note that $\alpha_1, \dots, \alpha_k$ are uniquely determined once the first k components of $\mathbf{u}$ are exposed. Thus we have

$$\mathbf{P}(\mathbf{u} \in H) \le \prod_{k+1 \le j} \mathbf{P}_{x_j}(x_j + f_j = \sum_{i=1}^{k} \alpha_i v_{ij}) \le (\sqrt{1-c_3})^{n-k},$$

where in the last estimate we use the fact (which follows from Condition 1.3) that $\sup_a \mathbf{P}(\xi = a) \le \sqrt{1-c_3}$.

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