Electron. Commun. Probab. **17** (2012), no. 38, 1–11.
DOI: 10.1214/ECP.v17-2282
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Large deviation results for random walks conditioned to stay positive

Ronald A. Doney* Elinor M. Jones†

Abstract

Let $X_1, X_2, ...$ denote independent, identically distributed random variables with common distribution F, and S the corresponding random walk with $\rho := \lim_{n\to\infty} P(S_n > 0)$ and $\tau := \inf\{n \geq 1 : S_n \leq 0\}$. We assume that X is in the domain of attraction of an α-stable law, and that $P(X \in [x, x + \Delta))$ is regularly varying at infinity, for fixed $\Delta > 0$. Under these conditions, we find an estimate for $P(S_n \in [x, x + \Delta)|\tau > n)$, which holds uniformly as $x/c_n \to \infty$, for a specified norming sequence c_n.

This result is of particular interest as it is related to the bivariate ladder height process $((T_n, H_n), n \geq 0)$, where T_r is the rth strict increasing ladder time, and $H_r = S_{T_r}$ the corresponding ladder height. The bivariate renewal mass function $g(n, dx) = \sum_{r=0}^{\infty} P(T_r = n, H_r \in dx)$ can then be written as $g(n, dx) = P(S_n \in dx|\tau > n)P(\tau > n)$, and since the behaviour of $P(\tau > n)$ is known for asymptotically stable random walks, our results can be rephrased as large deviation estimates of $g(n, [x, x + \Delta))$.

Keywords: Limit theorems; Random walks; Stable laws.
AMS MSC 2010: 60G50; 60G52; 60E07.
Submitted to ECP on August 8, 2011, final version accepted on August 1, 2012.

Let $X_1, X_2, ...$ denote independent, identically distributed random variables with common distribution F, and S the corresponding random walk. Suppose that there is a norming sequence c_n and a stable random variable Y such that

$$\frac{S_n}{c_n} \xrightarrow{D} Y \text{ as } n \to \infty.$$

If Y has index α and positivity parameter $\rho := P(Y > 0)$ which satisfy $\alpha \in (0, 1) \cup (1, 2)$ and $\rho \in (0, 1)$ or $\alpha = 1$ or 2 and $\rho = 1/2$ we say that S is **asymptotically stable** and write $S \in D(\alpha, \rho)$. It can be shown that $\{c_n : n \geq 1\}$ is regularly varying with index α^{-1}, so that for some slowly varying function $l(n)$, $c_n = n^{1/\alpha} l(n)$.

The important recent paper by Vatutin and Wachtel [13] has established local limit theorems for the position at time n of an asymptotically stable random walk conditioned to stay positive, in the zones of "normal deviations" and "small deviations". That is, they obtained uniform estimates of $P(S_n \in [x, x + \Delta)|\tau > n)$ as $n \to \infty$, where $\tau = \inf\{n \geq 1 : S_n \leq 0\}$ and $0 < \Delta < \infty$, in the case that $x = O(c_n)$ and the case $x = o(c_n)$ respectively.

These results are of particular interest in the study of fluctuations of random walks since they can be rephrased in terms of the of the bivariate ladder height process

*University of Manchester, UK. E-mail: rad@maths.man.ac.uk
†University of Leicester, UK. E-mail: emj8@le.ac.uk

$((T_n, H_n), n \geq 0)$. Specifically, if we introduce the bivariate renewal mass function by

$$g(n, dx) = \sum_{r=0}^{\infty} P(T_r = n, H_r \in dx),$$

where $T_0 = H_0 = 0$, and for $r \geq 1$, T_r is the rth strict increasing ladder time and $H_r = S_{T_r}$ the corresponding ladder height, we have the duality relation

$$g(n, dx) = P(S_n \in dx, \tau > n) = P(S_n \in dx | \tau > n) P(\tau > n).$$

Since the behaviour of $P(\tau > n)$ is known for asymptotically stable random walks we see that the results in [13] also give uniform estimates for $g(n, dx)$ in the case that $x = O(c_n)$ and $x = o(c_n)$. As many questions about the asymptotic behaviour of random walks involve controlling the dependence between the ladder times and ladder heights, these results will find many applications, and in fact have already been exploited in [6] to find uniform asymptotic estimates for the probability mass functions of first passage times. It is also noteworthy that these results hold for **all** asymptotically stable random walks, including the cases where Y is spectrally positive (i.e. $\alpha(1-\rho) = 1$) or spectrally negative (i.e. $\alpha\rho = 1$).

The main aim of this work is to find an estimate for $P(S_n \in [x, x+\Delta) | \tau > n)$, and hence for $g(n, [x, x+\Delta))$, which holds uniformly in x as $x/c_n \to \infty$, for c_n as defined above. However, since we are in the scenario where the "one large jump" principle is applicable, we cannot hope to get a general result in the spectrally negative case, $\alpha \in (0, 2)$ and $\alpha\rho = 1$,: we know little about the behaviour of the righthand tail $\overline{F}(x) := P(X_1 > x)$, and the corresponding tail estimate does not seem to be known. Also, if $S \in D(2, 1/2)$ and $\overline{F} \in RV(-\alpha)$ with $\alpha > 2$ it would be useful to have similar estimates, but it is clear that we have to choose the sequence c_n in a different way.

From now on, we will let $\bar{F}_\Delta(x) := P(X_1 \in [x, x+\Delta))$ for fixed $\Delta \in (0, \infty]$, so that $\Delta = \infty$ corresponds to $\bar{F}(x) := \bar{F}_\infty(x) = P(X_1 > x)$. In what follows, we will consider two different scenarios;

- Case A: $S \in D(\alpha, \rho)$ with $0 < \alpha < 2$, $\alpha\rho < 1$ and if $\alpha = 1$ $\rho = 1/2$. In this situation we know that $\bar{F}(\cdot) \in RV(-\alpha)$, $1 - F(x) = O(\ \bar{F}(x))$ and we can define the norming sequence to be the restriction to the integers of a continuous, increasing function $c(\cdot) \in RV(\alpha^{-1})$ which satisfies

$$\lim_{x\to\infty} x\bar{F}(c(x)) = 1.$$

- Case B: $EX_1 = 0, EX_1^2 < \infty$,and $\bar{F}(\cdot) \in RV(-\alpha)$ with $2 < \alpha < \infty$. In this situation we write $c(x) := \sqrt{x \log x}$, so that

$$\lim_{x\to\infty} x\bar{F}(c(x)) = 0.$$

Unconditional large deviation results are well known under these assumptions; see [3] for general results. Special cases of this result can be found in Doney [5] when $\Delta = \infty$and $\alpha < 1$, and in [12, 10] when $\Delta = \infty$ and $\alpha \geq 1$. For lattice random walks, local results can also be found in [4]. These results are combined in Proposition 0.1 below.

Proposition 0.1. *Suppose that the assumptions of Case A or Case B hold, and additionally, in the local case ($\Delta < \infty$), $\bar{F}_\Delta(\cdot) \in RV(-(\alpha+1))$. Then uniformly in $n \geq 1$ such that $x/c_n \to \infty$,*

$$P(S_n \in [x, x+\Delta)) \sim nP(X_1 \in [x, x+\Delta)) \text{ as } x \to \infty. \quad (0.1)$$

Although this result applies when n is fixed, to get our conditional results it is clear that we must have $n \to \infty$. In virtue of Proposition 0.1, it is then clear how $P(S_n \in [x, x+\Delta) | \tau > n)$ should behave, and our result in Theorem 0.2 confirms this.

Theorem 0.2. *Suppose that the assumptions of Case A or Case B hold, and additionally, in the local case ($\Delta < \infty$), $\bar{F}_\Delta(\cdot) \in RV(-(\alpha+1))$. Then uniformly in x such that $x/c_n \to \infty$,*

$$P(S_n \in [x, x+\Delta) | \tau > n) \sim \frac{P(S_n \in [x, x+\Delta))}{\rho} \sim \frac{nP(X_1 \in [x, x+\Delta))}{\rho} \text{ as } n \to \infty. \tag{0.2}$$

Remark 0.3. *To be quite specific, (0.2) means that, given any $\varepsilon > 0$ we can find $n(\varepsilon)$ and $\Gamma(\varepsilon) > 0$ such that, whenever $n \geq n(\varepsilon)$ and $x \geq \Gamma(\varepsilon) c_n$,*

$$\left| \frac{\rho P(S_n \in [x, x+\Delta) | \tau > n)}{nP(X_1 \in [x, x+\Delta))} - 1 \right| \leq \varepsilon.$$

From now on we will often use the notation $\overset{u}{\sim}$ to signify this.

Remark 0.4. *The cases where $\Delta = \infty$, or that S is a lattice random walk, are taken from the thesis [9], and have already found applications in part C of Theorem 1 of [6] and Theorem 1 of [14].*

1 Proofs

1.1 Preliminary lemma.

The following lemma provides a recursive relation for the quantity $g(n, dx) = \sum_{k=1}^n P(T_k = n, H_k \in dx)$, which will provide the basis to prove theorem 0.2. Recall that $g(n, dx)$ coincides with $P(S_n \in dx, \tau > n)$. Another recursive relation,

$$ng(n, dy) = \sum_{m=1}^{n-1} \int_{z=0}^{x-} g(m, dz) P(S_{n-m} \in dy - z), \quad y > 0,$$

can be found in [1], and plays a key role in [13]. But it turns out that the following variant is appropriate in our context. (It should be mentioned that the Lévy process version of this has been used in [7].)

Lemma 1.1. *For $n \geq 1$ and $x > 0$ we have the identity*

$$xg(n, dx) = \int_{w=0}^{x-} \sum_{m=1}^{n-1} \frac{x-w}{n-m} g(m, dw) P(S_{n-m} \in dx - w). \tag{1.1}$$

Proof. First we prove

$$xg(n, dx) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} E\{H_j : T_{k+j} = n, H_{k+j} \in dx\} \tag{1.2}$$

To see this put $\phi(\alpha, \beta) = E\{\alpha^{T_1} \beta^{H_1}\}$, so that

$$\begin{aligned} &\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{x=0+}^{\infty} xP(T_k = n, H_k \in dx) \alpha^n \beta^x \\ &= \sum_{k=1}^{\infty} \beta \frac{\partial \phi}{\partial \beta} k \phi(\alpha, \beta)^{k-1} = \frac{\beta \frac{\partial \phi}{\partial \beta}}{(1 - \phi(\alpha, \beta))^2}. \end{aligned}$$

However, writing $(T_{k+j}, H_{k+j}) \stackrel{D}{=} (T_k, H_k) + (\tilde{T}_j, \tilde{H}_j)$ and using the independence property, we see that

$$\sum_{n=1}^{\infty} \int_{x=0+}^{\infty} \alpha^n \beta^x \cdot \text{ RHS of (1.2)} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \cdot \left(\beta j \phi^{j-1} \frac{\partial \phi}{\partial \beta} \right) \phi^k = \frac{\beta \frac{\partial \phi}{\partial \beta}}{(1-\phi)^2}.$$

So (1.2) holds, and we can write the RHS of it as

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \sum_m \int_w (x-w) P(T_j = n-m, H_j \in dx - w) P(T_k = m, H_k \in dw). \tag{1.3}$$

Now, from equation (3) of [1], with $\sigma_x := \min(m : H_m \geq x)$, we have the identity

$$P(T_j = n-m, H_j \in dx - w) = \frac{j}{n-m} P(S_{n-m} \in dx - w, \sigma_x = j),$$

so (1.3) can be written as

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_m \int_w \frac{x-w}{n-m} P(S_{n-m} \in dx - w, \sigma_x = j) P(T_k = m, H_k \in dw) \\
=&\sum_{k=1}^{\infty} \sum_m \int_w \frac{x-w}{n-m} P(S_{n-m} \in dx - w) P(T_k = m, H_k \in dw) \\
=&\sum_m \int_w \frac{x-w}{n-m} P(S_{n-m} \in dx - w) g(m, dw),
\end{aligned}$$

and Lemma 1.1 follows. □

1.2 Proof of theorem 0.2.

The following lemma is required to prove Theorem 0.2.

Lemma 1.2. *Under the assumptions of Theorem 0.2, in both the local case* $\Delta < \infty$ *and the global case* $\Delta = \infty$, *there is a* $\Lambda \in (0, \infty)$ *and* $n^* < \infty$ *such that for* $\bar{F}_\Delta(x) := P(X_1 \in [x, x+\Delta))$,

$$\sup_{x \geq \Lambda c_n, n \geq n^*} \frac{P(S_n \in [x, x+\Delta), \tau > n)}{n \bar{F}_\Delta(x) P(\tau > n)} < \infty. \tag{1.4}$$

Proof. Let $T = \inf\{n : S_n > y'\}$, where $y' = c_n d^{1/2}$ and $d = d(x,n) = x/c_n$, so that $c_n = o(y')$ and $y' = o(x)$, where c_n is defined as before, and write

$$\begin{aligned}
P(S_n \in [x, x+\Delta), \tau > n) =& \sum_{t=1}^{n} \int_{w \in (y', x/2]} P(S_n \in [x, x+\Delta), \tau > n, T = t, S_t \in dw) \\
&+ \sum_{t=1}^{n} P(S_n \in [x, x+\Delta), \tau > n, T = t, S_t > x/2) =: Q_1 + Q_2.
\end{aligned}$$

Note that

$$\begin{aligned}
Q_1 =& \sum_{t=1}^{n} \int_{w \in (y', x/2]} P(\tau > t, T = t, S_t \in dw) P_w(S_{n-t} \in [x, x+\Delta), S_i > 0, \forall i = 1, \cdots n-t) \\
\leq& \sum_{t=1}^{n} \sup_{w \in (y', x/2]} P_w(S_{n-t} \in [x, x+\Delta)) \int_{w \in (y', x/2]} P(\tau > t, T = t, S_t \in dw) \\
\leq& C \sup_{t \leq n; w \in (y', x/2]} P_w(S_{n-t} \in [x, x+\Delta)) \sum_{t=1}^{n} P(\tau > t, T = t).
\end{aligned}$$

To see that for some constant C,

$$P(\tau > n) \geq C \sum_{t=1}^{n} P(\tau > t, T = t), \tag{1.5}$$

note that

$$\begin{aligned} P(\tau > n) \geq & P(S \text{ exits } [0, y'] \text{ before time } n \text{at } y', \tau > n) \\ \geq & \sum_{t=1}^{n} P(T = t, \tau > t) P(\inf_{w \leq n} S_w > -y'). \end{aligned}$$

However, $\lim_{n\to\infty} P(\inf_{w\leq n} S_w > -y') > 0$ uniformly as $d \to \infty$, since, using Pruitt's inequality [11] in the case that $\alpha \in (0,2)$,

$$P(\inf_{w\leq n} S_w \leq -y') \leq P(\max_{w\leq n} |S_w| > y') \leq Cn \left(\bar{F}(y') + F(-y')\right) \to 0,$$

uniformly, for some constant C, and in the case that $\alpha > 2$, an application of Chebyshev's inequality gives the desired result. Therefore, by applying Proposition 0.1,

$$\begin{aligned} Q_1 \leq & \sup_{t\leq n; w\in(y', x/2]} P_w(S_{n-t} \in [x, x+\Delta)) P(\tau > n) \\ \sim & nP(X_1 \in [x/2, x/2+\Delta)) P(\tau > n) \sim Cn\bar{F}_\Delta(x) P(\tau > n). \end{aligned}$$

In addition, letting x_i be a sequence such that $x_0 = x/2$ and $x_i = x_{i-1} + \Delta$ for $i = 1, \ldots$, and for a constant $C > 0$ which may change from line to line,

$$\begin{aligned} Q_2 = & \sum_{t=1}^{n} \sum_i P(S_n \in [x, x+\Delta), \tau > n, T = t, S_t \in [x_i, x_i+\Delta)) \\ \leq & \sum_{t=1}^{n} \sum_i \int_{u=0}^{y'} P(\tau > t-1, S_{t-1} \in du) P(X_t \in [x_i - u, x_i - u + \Delta)) P(S_{n-t} \in [x - x_i - \Delta, x + \Delta - x_i)) \\ \leq & \sup_{u\leq y', i} P(X_1 \in [x_i - u, x_i - u + \Delta)) \sum_{t=1}^{n} \sum_i P(\tau > t-1) P(S_{n-t} \in [x - x_i - \Delta, x + \Delta - x_i)) \\ = & 2 \sup_{u\leq y', i} P(X_1 \in [x_i - u, x_i - u + \Delta)) \sum_{t=1}^{n} P(\tau > t-1) \sum_i P(S_{n-t} \in [x - x_i, x + \Delta - x_i)) \\ \leq & 2\bar{F}_\Delta(x/2 - y) \sum_{t=1}^{n} P(\tau > t-1) \sim C\bar{F}_\Delta(x/2 - y) nP(\tau > n) \sim C\bar{F}_\Delta(x) nP(\tau > n) \end{aligned}$$

and (1.2) follows. □

It turns out that the first part of the following proof is also valid in both the local and global situations.

Proof. (Of Theorem 0.2) Our aim is to show

$$\frac{P(S_n \in [x, x+\Delta), \tau > n)}{P(\tau > n)} \overset{u}{\sim} \frac{n}{\rho} P(X_1 \in [x, x+\Delta)),$$

and we can rewrite the result of Lemma 1.1 as

$$P(S_n \in [x, x+\Delta), \tau > n) = \int_{y=x}^{x+\Delta} \int_{w=0}^{y} \sum_{m=1}^{n-1} \frac{y - w}{n - m} P(S_m \in dw, \tau > m) \frac{P_w(S_{n-m} \in dy)}{y}.$$

Therefore, showing that

$$Q_\Delta := \int_{y=x}^{x+\Delta}\int_{w=0}^{y}\sum_{m=1}^{n-1}\frac{y-w}{n-m}\frac{P(S_m\in dw,\tau>m)}{nP(\tau>n)}\frac{P_w(S_{n-m}\in dy)}{y\bar{F}_\Delta(x)} \overset{u}{\to} \rho^{-1}$$

as $n\to\infty$, is sufficient to show that Theorem 0.2 holds. We write $Q_\Delta = Q_\Delta^{(1)} + Q_\Delta^{(2)}$ with

$$Q_\Delta^{(1)} := \int_{y=x}^{x+\Delta}\int_{w=0}^{x\delta}\sum_{m=1}^{n-1}\frac{y-w}{n-m}\frac{P(S_m\in dw,\tau>m)}{nP(\tau>n)}\frac{P_w(S_{n-m}\in dy)}{y\bar{F}_\Delta(x)}.$$

Firstly, for any $\delta\in(0,1)$, when $w\le\delta x$ note that $(y-w)/y$ must lie in the interval $(1-\delta,1)$, and by applying Proposition 0.1,

$$\sum_{m=1}^{n-1}\int_{y=x}^{x+\Delta}\int_{w=0}^{\delta x}\frac{P(S_m\in dw,\tau>m)}{nP(\tau>n)}\frac{1}{(n-m)}\frac{P_w(S_{n-m}\in dy)}{\bar{F}_\Delta(x)}\sim\sum_{m=1}^{n-1}\int_{w=0}^{\delta x}\frac{P(S_m\in dw,\tau>m)}{nP(\tau>n)}\frac{\bar{F}_\Delta(x-w)}{\bar{F}_\Delta(x)}.$$

Furthermore, note that for sufficiently large x, $\bar{F}_\Delta(x-w)/\bar{F}_\Delta(x)$ is bounded between 1 and $(1-\delta)^{-\alpha-2}$, so that by defining $I_\delta := \sum_{m=1}^{n-1}P(\tau>m, S_m\le x\delta)$,

$$(1-\delta)\frac{I_\delta}{nP(\tau>n)\bar{F}_\Delta(x)} \le Q_\Delta^{(1)} \le (1-\delta)^{-\alpha-2}\frac{I_\delta}{nP(\tau>n)\bar{F}_\Delta(x)}$$

so we need only deal with

$$I_\delta = \sum_{m=1}^{n-1}P(\tau>m) - \sum_{m=1}^{n-1}P(\tau>m, S_m>x\delta).$$

Recalling that $T:=\inf\{t:S_t>y'\}$, where y' is defined as before, we write

$$\begin{aligned}
&P(\tau>m,S_m>x\delta) = \sum_{t=1}^{m}P(T=t,\tau>m,S_m>x\delta)\\
=&\sum_{t=1}^{m}P(T=t,\tau>m,S_m>x\delta,S_t\in[y',x\delta/2]) + \sum_{t=1}^{m}P(T=t,\tau>m,S_m>x\delta,S_t>x\delta/2).
\end{aligned}$$

Note that, using (1.5),

$$\begin{aligned}
\sum_{t=1}^{m}P(T=t,\tau>m,S_m>\delta x,S_t\in[y',x\delta/2]) &\le \sum_{t=1}^{m}\int_{u=y'}^{x\delta/2}P(T=t,\tau>t,S_t\in du)P(S_{m-t}>\delta x-u)\\
&\le \sup_{r\le m}P(S_r>x\delta/2)\sum_{t=1}^{m}P(T=t,\tau>t)P(S_{m-t}>x\delta/2)\\
&\le Cm\overline{F}(x\delta/2)P(\tau>m),
\end{aligned}$$

for some constant C, and

$$\begin{aligned}
&\sum_{t=1}^{m}P(T=t,S_t>x\delta/2,\tau>m,S_m>\delta x)\\
\le&\sum_{t=1}^{m}P(T=t,\tau>t-1,S_t>x\delta/2)\\
\le&\bar{F}(x\delta/2-y')\sum_{t=1}^{m}P(\tau>t-1).
\end{aligned}$$

It then follows easily that, for any fixed δ

$$\sum_{m=1}^{n-1} \frac{P(\tau > m, S_m \leq x\delta)}{nP(\tau > n)} \overset{u}{=} \sum_{m=1}^{n-1} \frac{P(\tau > m)}{nP(\tau > n)} + o(1) \overset{u}{\to} \frac{1}{\rho}.$$

Thus, given any $\varepsilon > 0$ we can find $\delta_\varepsilon > 0, \Delta_\varepsilon$, and n_ε such that

$$\sum_{m=1}^{n-1} \int_{w=0}^{x\delta_\varepsilon} \frac{P(\tau > m, S_m \in dw)}{nP(\tau > n)} \frac{(x-w)}{(n-m)} \frac{P_w(S_{n-m} > x)}{xp(x)} \in (\rho^{-1} - \varepsilon, \rho^{-1} + \varepsilon)$$

whenever $n \geq n_\varepsilon$ and x is sufficiently large.

To show that $Q_\Delta^{(2)} = o(1)$, we consider the cases $\Delta = \infty$ and $\Delta < \infty$ separately.

(i) THE GLOBAL CASE ($\Delta = \infty$)

In this case we have

$$Q_\infty^{(2)} = \sum_{m=1}^{n-1} \int_{y>x} \int_{w=\delta x}^{y} \frac{P(\tau > m, S_m \in dw)}{nP(\tau > n)} \frac{(y-w)}{(n-m)} \frac{P_w(S_{n-m} \in dy)}{y\bar{F}(x)}.$$

For *fixed* m, note that

$$\begin{aligned}
&\int_{y>x} \int_{w=\delta x}^{y} P(\tau > m, S_m \in dw) \frac{(y-w)}{(n-m)} \frac{P_w(S_{n-m} \in dy)}{y} \\
\leq &\int_{w=\delta x}^{\infty} \int_{y>w} \frac{P(\tau > m, S_m \in dw)}{(n-m)} P_w(S_{n-m} \in dy) \\
\leq &\int_{w=\delta x}^{\infty} \frac{P(\tau > m, S_m \in dw)}{(n-m)} \\
\leq &\frac{P(S_m > \delta x)}{(n-m)} \sim \frac{m\bar{F}(\delta x)}{(n-m)} = o\left(\bar{F}(x) nP(\tau > n)\right),
\end{aligned}$$

so that we can discard the terms with $m \leq n^*$, where n^* features in Lemma 1.2. Next we write

$$\begin{aligned}
&\sum_{m=n^*+1}^{n-1} \int_{w=\delta x}^{\infty} \int_{y \geq w} P(\tau > m, S_m \in dw) \frac{(y-w)}{(n-m)} \frac{P_w(S_{n-m} \in dy)}{y} \\
&\quad = \sum_{m=n^*+1}^{n-1} \int_{w=\delta x}^{\infty} \int_{z \geq 0} P(\tau > m, S_m \in dw) \frac{z}{(n-m)} \frac{P(S_{n-m} \in dz)}{(w+z)} := P_\delta^{(1)} + P_\delta^{(2)},
\end{aligned}$$

where $P_\delta^{(1)}$ has $z \in (0, \sqrt{d} c_{n-m})$, and $P_\delta^{(2)}$ has $z \geq \sqrt{d} c_{n-m}$. Now, using the result of Lemma 1.2, and noting that $mP(\tau > m)$ is asymptotically increasing,

$$\begin{aligned}
P_\delta^{(1)} \leq \frac{1}{\delta x} \sum_{m=n^*+1}^{n-1} P(\tau > m, S_m > \delta x) \frac{\sqrt{d} c_{n-m}}{n-m} &\sim \frac{\sqrt{d}}{\delta x} \sum_{m=n^*+1}^{n-1} m\bar{F}(\delta x) P(\tau > m) \frac{c_{n-m}}{n-m} \\
&\leq \frac{\sqrt{d}}{\delta x} nP(\tau > n) \bar{F}(\delta x) \sum_{m=1}^{n} \frac{c_m}{m} \\
&\sim \frac{C\sqrt{d} c_n}{\delta x} nP(\tau > n) \bar{F}(\delta x) = o\left(nP(\tau > n)\bar{F}(\delta x)\right).
\end{aligned}$$

To deal with $P_\delta^{(2)}$, we use the following lemma:

Lemma 1.3. *For any random variable X, and positive constant w,*

$$E\left[\frac{X}{X+w} : X \geq c\right] \leq 2\frac{c\bar{F}(c)}{c+w}$$

Proof.

$$\begin{aligned} E\left[\frac{X}{X+w} : X \geq c\right] = \int_c^\infty \frac{x}{w+x} d(-\bar{F}(x)) =& \frac{c}{c+w}\bar{F}(c) + \int_c^\infty \bar{F}(x)\frac{w}{(w+x)^2}dx \\ \leq& \frac{c}{c+w}\bar{F}(c) + \bar{F}(c)\int_c^\infty \frac{w}{(w+x)^2}dx \\ =& 2\frac{c\bar{F}(c)}{c+w}, \end{aligned}$$

as required. □

Applying this lemma to $P_\delta^{(2)}$, and recalling that $P(S_{n-m} > \sqrt{d}c_{n-m}) \sim (n-m)\bar{F}(\sqrt{d}c_{n-m})$,

$$\begin{aligned} P_\delta^{(2)} \leq& \sum_{m=1}^{n-1}\int_{w=\delta x}^\infty \frac{P(\tau > m, S_m \in dw)}{n-m}\frac{2\sqrt{d}c_{n-m}P(S_{n-m} > \sqrt{d}c_{n-m})}{w+\sqrt{d}c_{n-m}} \\ \sim& \sum_{m=1}^{n-1}\int_{w=\delta x}^\infty P(\tau > m, S_m \in dw)\frac{2\sqrt{d}c_{n-m}\bar{F}(\sqrt{d}c_{n-m})}{w+\sqrt{d}c_{n-m}} \\ \leq& C\frac{\sqrt{d}}{\delta x}\sum_{m=1}^{n-1}P(S_m > \delta x)P(\tau > m)\frac{c_{n-m}}{n-m} = 2\frac{C\sqrt{d}c_n}{\delta x}nP(\tau > n)\bar{F}(x) = o(1)nP(\tau > n)\bar{F}(x), \end{aligned}$$

as required, so that when $\Delta = \infty$,

$$Q_\infty^{(2)} = o\left(nP(\tau > n)\bar{F}(x)\right).$$

(ii) THE LOCAL CASE ($\Delta < \infty$)

Now with $\Delta < \infty$, we prove that

$$Q_\Delta^{(2)} = \sum_{m=1}^{n-1}\int_{w=\delta x}^{x+\Delta}\int_{y=x\vee w}^{x+\Delta} P(\tau > m, S_m \in dw)\frac{(y-w)}{(n-m)}\frac{P(S_{n-m} \in dy - w)}{y} = o\left(nP(\tau > n)\bar{F}_\Delta(x)\right).$$

Note first that

$$\int_{y=x\vee w}^{x+\Delta}\frac{(y-w)P(S_{n-m} \in dy-w)}{y} \leq \frac{1}{x}\int_{z=(x-w)^+}^{\Delta+x-w} zP(S_{n-m} \in dz).$$

Next we see that for any fixed m and suitably large k when $y \geq kc(n-m)$ and n is large enough we have

$$\begin{aligned} &\int_{z=y}^{\Delta+y} zP(S_{n-m} \in dz) \leq (\Delta+y)P(S_{n-m} \in (y, y+\Delta]) \\ \leq& 2(n-m)(\Delta+y)\overline{F}_\Delta(y) \backsim 2(n-m)y\overline{F}_\Delta(y), \end{aligned}$$

and hence,

$$\begin{aligned} &\int_{w=\delta x}^{x+\Delta-kc_{n-m}}\int_{y=x\vee w}^{x+\Delta} P(\tau > m, S_m \in dw)\frac{(y-w)}{(n-m)}\frac{P(S_{n-m} \in dy-w)}{y} \\ \leq& \frac{2}{x}\int_{w=\delta x}^{x+\Delta-kc_{n-m}} P(\tau > m, S_m \in dw)(x-w)\overline{F}_\Delta(x-w) \\ \leq& 2(1-\delta)\overline{F}_\Delta((1-\delta)x)P(S_m > \delta x) = o(\overline{F}_\Delta(x)) = o(\overline{F}_\Delta(x)nP(\tau > n)). \end{aligned}$$

Note that the Gnedenko and Stone local limit theorem implies that there exists a constant $C > 0$ such that

$$P(S_r \in [(u-\Delta)^+, u)) \leq \frac{C\Delta}{c_r},\ r \geq 1. \tag{1.6}$$

Using this gives

$$\begin{aligned}
&\frac{1}{x(n-m)} \int_{x+\Delta-kc_{n-m}}^{x+\Delta} P(\tau > m, S_m \in dw) \int_{z=(x-w)^+}^{\Delta+x-w} zP(S_{n-m} \in dz) \\
\leq &\frac{\Delta + x}{x(n-m)} \int_0^{kc_{n-m}} P(\tau > m, S_m \in x - dw) P(S_{n-m} \in ((x-w)^+, \Delta + x - w) \\
\leq &\frac{(\Delta+x)\Delta}{x(n-m)c_{n-m}} \int_0^{kc_{n-m}} P(\tau > m, S_m \in x - dw) \\
\leq &\frac{(\Delta+x)\Delta P(S_m \in (x - kc_{n-m}, x])}{x(n-m)c_{n-m}} \sim \frac{m\Delta P(S_1 \in (x-kc_{n-m}, x])}{nc_{n-m}} \\
\sim &\frac{m\Delta kc_{n-m}\overline{F}_\Delta(x)}{nc_{n-m}} = \frac{m\Delta k\overline{F}_\Delta(x)}{n} = o(\overline{F}_\Delta(x)nP(\tau > n)).
\end{aligned}$$

So we can discard the terms with $m \leq n^*$, where n^* features in Lemma 1.2. In addition, if we write $\eta = 1 - \delta/2$, then for all large x,

$$\begin{aligned}
&\sum_{m=n^*+1}^{n-1} \int_{w=\delta x}^{x+\Delta} \int_{y=x\vee w}^{x+\Delta} P(\tau > m, S_m \in dw) \frac{(y-w)}{(n-m)} \frac{P(S_{n-m} \in dy - w)}{y} \\
\leq &\frac{1}{x} \sum_{m=n^*+1}^{n-1} \frac{1}{n-m} \int_{w=\delta x}^{x+\Delta} P(\tau > m, S_m \in dw) \int_{z=(x-w)^+}^{x-w+\Delta} zP(S_{n-m} \in dz) \\
= &\frac{1}{x} \sum_{m=n^*+1}^{n-1} \frac{1}{n-m} \int_{u=0}^{(1-\delta)x+\Delta} P(\tau > m, S_m \in x + \Delta - du) \int_{z=(u-\Delta)^+}^{u} zP(S_{n-m} \in dz) \\
\leq &\frac{1}{x} \sum_{m=n^*+1}^{n-1} \frac{1}{n-m} \int_{u=0}^{\eta x} uP(\tau > m, S_m \in x + \Delta - du) P(S_{n-m} \in [(u-\Delta)^+, u)) =: J.
\end{aligned}$$

Showing that J is $o\left(nP(\tau > n)\bar{F}_\Delta(x)\right)$ completes the proof. We write $J = J^{(1)} + J^{(2)}$, such that for suitably large k,

$$J^{(1)} = \frac{1}{x} \sum_{m=n^*+1}^{n-1} \frac{1}{n-m} \int_{u=0}^{kc_{n-m}} P(\tau > m, S_m \in x + \Delta - du) uP(S_{n-m} \in [(u-\Delta)+, u)).$$

Using the bound (1.6) again gives

$$J^{(1)} \leq \sum_{m=n^*+1}^{n-1} \frac{C\Delta}{x(n-m)c_{n-m}} \int_{u=0}^{kc_{n-m}} uP(\tau > m, S_m \in x + \Delta - du).$$

Now let $N = [kc_{n-m}/\Delta]$, then for some constant $C > 0$ which may change from line to line,

$$
\begin{aligned}
\int_{u=0}^{kc_{n-m}} uP(\tau > m, S_m \in x + \Delta - du) \le & \sum_{r=0}^{N} \int_{r\Delta}^{(r+1)\Delta} uP(\tau > m, S_m \in x + \Delta - du) \\
\le & \sum_{r=0}^{N} (r+1)\Delta P(\tau > m, S_m \in [x - r\Delta, x - r\Delta + \Delta]) \\
\le & mP(\tau > m) \sum_{r=0}^{N} (r+1)\Delta \bar{F}_\Delta(x - r\Delta) \\
\sim & mP(\tau > m)\bar{F}_\Delta(x) \sum_{r=0}^{N} (r+1)\Delta \\
\le & CmP(\tau > m)\bar{F}_\Delta(x) c_{n-m}^2. \qquad (1.7)
\end{aligned}
$$

Now,

$$
\begin{aligned}
J^{(1)} \le & \sum_{m > n^*} \frac{C}{x(n-m)c_{n-m}} mP(\tau > m)\bar{F}_\Delta(x) c_{n-m}^2 \\
\sim & \frac{CnP(\tau > n)\bar{F}_\Delta(x)}{x} \sum_{m > n^*} \frac{c_{n-m}}{(n-m)} \\
\sim & \frac{CnP(\tau > n)\bar{F}_\Delta(x)}{x} c_n = o\left(nP(\tau > n)\bar{F}_\Delta(x)\right)
\end{aligned}
$$

as required. Finally, to show that $J^{(2)}$ is also of the correct form, note that for $u \in [kc_{n-m}, \eta x]$, k sufficiently large and all m

$$
P(S_{n-m} \in [(u-\Delta)^+, u]) \le 2(n-m)\bar{F}_\Delta(u - \Delta) \backsim 2(n-m)\bar{F}_\Delta(u).
$$

Now, since $u\bar{F}_\Delta(u) \sim KF(u)$ for some constant K,

$$
\begin{aligned}
J^{(2)} \le & \frac{1}{x} \sum_{m > n^*} \int_{kc_{n-m}}^{\eta x} u\bar{F}_\Delta(u) P(\tau > m, S_m \in x + \Delta - du) \\
\sim & \frac{K}{x} \sum_{m > n^*} \int_{kc_{n-m}}^{\eta x} \bar{F}(u) P(\tau > m, S_m \in x + \Delta - du)
\end{aligned}
$$

and we can use a variation of the argument leading up to (1.7) to show that

$$
\int_{kc_{n-m}}^{\eta x} \bar{F}(u) P(\tau > m, S_m \in x + \Delta - du) \le CnP(\tau > n)\bar{F}_\Delta(x) \int_{kc_{n-m}}^{\eta x} \bar{F}(u) du.
$$

When $\alpha < 2$, c satisfies $\bar{F}(c(x)) \sim 1/x$ for $x \ge 1$ so that

$$
\begin{aligned}
\sum_{m=1}^{n} \int_{y=kc_m}^{x} \bar{F}(y) \asymp & \int_1^n dz \int_{kc(z)}^{x} \bar{F}(y) dy \\
= & \int_{y=kc(1)}^{x} \bar{F}(y) dy \int_1^{c^{-1}(y/k) \wedge n} dz \\
= & \int_{y=kc(1)}^{kc(n)} c^{-1}(y/k)\bar{F}(y) dy + n \int_{kc(n)}^{x} \bar{F}(y) dy \\
\sim & k \int_{w=1}^{n} w\bar{F}(kc(w))c'(w) dw + n \int_{kc(n)}^{x} \bar{F}(y) dy \\
= & c(n) - c(1) + n \int_{kc(n)}^{x} \bar{F}(y) dy. \qquad (1.8)
\end{aligned}
$$

Note that

$$\frac{n}{x}\int_{kc(n)}^{x}\bar{F}(y)dy \leq \frac{n}{x}x\bar{F}(kc_n) \sim \frac{n}{k^{\alpha}}\bar{F}(c_n), \tag{1.9}$$

we can choose k to be as large as we want, giving the desired result. If $\alpha > 2$ we have $x\overline{F}(c(x)) \to 0$, and the same argument applies. $\square$

References

[1] Alili, L., and Doney, R. A.: Wiener-Hopf factorization revisited and some applications. *Stochastics and Stochastics Reports*, **66**, (1997), 87–102. MR-1687803

[2] Bingham, N. H., Goldie, C. G. and Teugels, J.: *Regular Variation*, Cambridge University Press, (1987). MR-0898871

[3] Denisov, D., Dieker, A. B., and Shneer, V.: Large deviations for random walks under subexponentiality: the big-jump domain. *The Annals of Probability* **36**, (2008), 1946–1991. MR-2440928

[4] Doney, R. A.: A large deviation local limit theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, **105**, (1989), 575-577. MR-0985693

[5] Doney, R. A.: One-sided local large deviation and renewal theorems in the case of infinite mean. *Probability Theory and Related Fields*, **107**, (1997), 451–465. MR-1440141

[6] Doney, R. A.: Local behaviour of first passage probabilities. *Probability Theory and Related Fields*. **152**, (2010), 559–588. MR-2892956

[7] Doney, R. A. and Savov, M. S.: The asymptotic behaviour of densities related to the supremum of a stable process. *The Annals of Probability*, **38**(1), (2010), 316–326. MR-2599201

[8] Fuk, D. H. and Nagaev, S. V.: Probabilistic inequalities for sums of independent random variables. *Teor. Veroyatnost. i Primenen.*, **16**, (1971), 660–675. MR-0293695

[9] Jones, E. M.: PhD thesis, The University of Manchester, (2008).

[10] Nagaev, S. V.: On the asymptotic behavior of probabilities of one-sided large deviations. *Teor. Veroyatnost. i Primenen.*, **26**(2), (1981), 369–372. MR-0616627

[11] Pruitt, W. E.: The growth of random walks and Lévy processes. *The Annals of Probability*, **9**(6), (1981), 948–956. MR-0632968

[12] Tchachkuk, S. G.: Limit theorems for sums of independent random variables belonging to the domain of attraction of a stable law. Candidate's dissertation, Tashent (in Russian), (1977).

[13] Vatutin, V. A. and Wachtel, V.: Local probabilities for random walks conditioned to stay positive. *Probability Theory and Related Fields*, **143**, (2009), 177–217. MR-2449127

[14] Wachtel, V.: Local limit theorem for the maximum of asymptotically stable random walks. *Probability Theory and Related Fields*, **152**, (2010), 407–424. MR-2892952

Acknowledgments. We would like to thank an anonymous referee for valuable suggestions which lead to an improved version of this paper.