Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 74, 1–21.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-2287

Multiparameter processes with stationary increments: Spectral representation and integration

Andreas Basse-O'Connor* Svend-Erik Graversen* Jan Pedersen*

Abstract

In this article, a class of multiparameter processes with wide-sense stationary increments is studied. The content is as follows. (1) The spectral representation is derived; in particular, necessary and sufficient conditions for a measure to be a spectral measure is given. The relations to a commonly used class of processes, studied e.g. by Yaglom, is discussed. (2) Some classes of deterministic integrands, here referred to as predomains, are studied in detail. These predomains consist of functions or, more generally, distributions. Necessary and sufficient conditions for completeness of the predomains are given. (3) In a framework covering the classical Walsh-Dalang theory of a temporal-spatial process which is white in time and colored in space, a class of predictable integrands is considered. Necessary and sufficient conditions for completeness of the class are given, and this property is linked to a certain martingale representation property.

Keywords: Multiparameter processes; stationary increments; spectral representation; integration.
AMS MSC 2010: 60G51; 60G12; 60H05.
Submitted to EJP on November 16, 2011, final version accepted on August 29, 2012.

1 Introduction

Let $d \geq 1$ be an integer which is fixed throughout. In this article we consider a class of real-valued processes $X = \{X_u : u \in \mathbb{R}^d\}$ indexed by $\mathbb{R}^d$ with *wide-sense stationary increments*. We refer to Section 2 for the precise definition so for now it suffices to say that this class is large and contains e.g. the d-parameter fractional Brownian sheet and a well-known example from the theory of stochastic partial differential equations cf. e.g. Dalang [2], p. 5–6; see also Example 2.6. The main purpose is to study different kinds of integrals with respect to such processes, focusing in particular on completeness of various sets of integrands.

In Section 3 we discuss classes of deterministic integrands, referred to as *predomains*. Predomains are not necessarily sets of functions but the corresponding integral takes values in the set of square-integrable random variables. On predomains we use the metric induced by the L^2-distance between corresponding integrals. If completeness is present, a predomain is referred to as a *domain*. In the one-dimensional case

*Aarhus University, Denmark. Email: basse,matseg,jan@imf.au.dk

$d = 1$ several predomains have been studied for processes with stationary increments. A key reference in the case of fractional Brownian motion is Taqqu and Pipiras [7] where various (pre)domains consisting of functions are analyzed. These authors show that many natural predomains studied in the literature are in fact not complete and hence not domains. To remedy this, Jolis [5] introduced a larger predomain consisting of distributions in the case of a continuous processes with stationary increments. In particular she showed that this will often lead to a domain. In Section 3 we follow [5] and study predomains containing functions as well as distributions. Generalizing results of [5, 7], necessary and sufficient conditions on the spectral measure for a predomain to be a domain are given. Moreover, we show that the integral of an integrand φ belonging to any of the predomains considered is given by

$$\int_{\mathbb{R}^d} \varphi(u)\, X(du) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z)\, Z(dz), \tag{1.1}$$

where $\mathcal{F}$ denotes the Fourier transform and Z the random spectral measure of X.

As is obvious from (1.1) the integral is closely linked to the spectral representation of X. Therefore we study the spectral representation of X in detail in Section 2. Moreover, a comparison to the class of processes studied e.g. by Yaglom [13] is given.

Finally, in Section 4 we add a temporal component and thus consider Gaussian processes $X = \{X_u : u = (t,x) \in \mathbb{R}^{1+d}\}$ where $t \in \mathbb{R}$ is time and $x \in \mathbb{R}^d$ a spatial component. We assume that X is *white in time and colored in space*. A martingale integral with respect to X is constructed akin to the classical papers by Walsh [12] and Dalang [2] although it should be noticed that in the present situation, unlike these papers, X does in general not induce a martingale measure. For example, when $d = 1$, X could be fractional in space with Hurst exponent H in $(0,1)$ in which case X only induces a martingale measure as in [2] when $H > 1/2$. We show that the integral of a predictable integrand $\varphi_t(x)$ with respect to X is

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)\, X(d(t,x)) = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi_t(z)\, dZ_t(x),$$

where $\mathcal{F}$ denotes the Fourier transform in the space variable, and for fixed t, $Z_t(\cdot)$ is the random spectral measure of $X((0,t] \times \cdot)$ in the space variable. Necessary and sufficient conditions for completeness for a class of integrands are given and in particular this property is linked to a martingale representation property with respect to X.

Definitions and notation: For any measure μ, $L^2_{\mathbb{C}}(\mu)$ denotes the set of complex-valued μ-square integrable functions and $L^2_{\mathbb{R}}(\mu)$ the subset hereof taking values in $\mathbb{R}$. Likewise, for any $A \subseteq L^2_{\mathbb{C}}(\mu)$, $\overline{\mathrm{sp}}_{\mathbb{C}} A$ is the closed complex linear span and $\overline{\mathrm{sp}}_{\mathbb{R}} A$ the corresponding closed real linear span of A. Observe that $\overline{\mathrm{sp}}_{\mathbb{R}} A$ coincides with the real-valued elements in $\overline{\mathrm{sp}}_{\mathbb{C}} A$ if all elements in A are real-valued. According to usual notation the space of tempered distributions, that is the dual of the Schwartz space $\mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$ consisting of complex-valued C^∞–functions on $\mathbb{R}^d$ of rapid decrease, is denoted $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$. The subspace of $\mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$ consisting of real-valued functions is denoted $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$, and likewise $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ is the set of elements Ψ in $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ such that $\Psi(\phi) \in \mathbb{R}$ for all $\varphi \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$. Similarly, $\mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ (resp. $\mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)$) denotes the set of complex-valued (resp. real-valued) C^∞-functions on $\mathbb{R}^d$ of compact support. The class of non-negative elements in $\mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)$ is denoted by $\mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)_+$. For the general theory of distributions and especially tempered distributions we refer to Schwartz [9].

Let λ_d denote Lebesgue measure on $\mathbb{R}^d$. The Fourier transform $\mathcal{F}$ maps $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ onto $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ and with the usual identification of a locally integrable function with its corresponding tempered distribution when it exists, we have for $f \in L^1_{\mathbb{C}}(\lambda_d)$ that

$$\mathcal{F}f(z) = \int_{\mathbb{R}^d} e^{i\langle z,\cdot\rangle} f(\cdot)\, d\lambda_d = \int_{\mathbb{R}^d} e^{i\langle z,u\rangle} f(u)\, du, \quad \text{for } z \in \mathbb{R}^d.$$

Here, $\langle \cdot, \cdot \rangle$ is the canonical inner product on $\mathbb{R}^d$ with corresponding norm $\|\cdot\|$. The notation differs from the one used e.g. in [9] where, for $f \in L^1_{\mathbb{C}}(\lambda_d)$, $\mathcal{F}f(-2\pi\cdot)$ is used as the Fourier transform of f. But apart from a constant $(2\pi)^d$ appearing in Parseval's identity and the explicit form of the inverse $\mathcal{F}^{-1}$, all results from the general theory of distributions remain valid with the definition given above. When $d = 1$ we also use the notation $\mathcal{F}_1$ instead of $\mathcal{F}$.

All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is fixed throughout. Equality in distribution is denoted $\stackrel{\mathscr{D}}{=}$. Finally, $\mathcal{B}_b(\mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}^d$.

2 Spectral representation

In Definition 2.4 the class of processes with wide-sense stationary increments is defined and the spectral representation is given in Theorem 2.7. This representation is stated in terms of the following class of random measures.

Definition 2.1. *Let F be a symmetric Borel measure on $\mathbb{R}^d$ finite on compacts. A set function $Z\colon \mathcal{B}_b(\mathbb{R}^d) \to L^2_{\mathbb{C}}(\mathbb{P})$ is said to be an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure F if*

(1) $Z(A \cup B) = Z(A) + Z(B)$ $\mathbb{P}$-a.s. whenever $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ are disjoint;

(2) $Z(A) = \overline{Z(-A)}$ $\mathbb{P}$-a.s. for $A \in \mathcal{B}_b(\mathbb{R}^d)$;

(3) $\mathbb{E}[Z(A)\overline{Z(B)}] = F(A \cap B)$ for $A, B \in \mathcal{B}_b(\mathbb{R}^d)$;

(4) $\mathbb{E}[Z(A)] = 0$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$.

Remark 2.2. *Let Z be a random measure as above. From (1) and (3) it follows that $Z(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty Z(A_n)$ in $L^2_{\mathbb{C}}(\mathbb{P})$ for any disjoint sequence $(A_n)_{n\geq 1}$ in $\mathcal{B}_b(\mathbb{R}^d)$ satisfying $\cup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbb{R}^d)$.*

Decompose Z as $Z(A) = Z_1(A) + iZ_2(A)$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$; that is, Z_1 is the real part of Z, Z_2 the imaginary part, and $Z_1(A), Z_2(A) \in L^2_{\mathbb{R}}(\mathbb{P})$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$. For $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$\mathbb{E}[Z_1(A)Z_2(B)] = 0, \tag{2.1}$$
$$\mathbb{E}[Z_1(A)Z_1(B)] = \tfrac{1}{2}\big[F(A\cap B) + F(A \cap (-B))\big], \tag{2.2}$$
$$\mathbb{E}[Z_2(A)Z_2(B)] = \tfrac{1}{2}\big[F(A\cap B) - F(A \cap (-B)\big]. \tag{2.3}$$

To see this, notice that by (2) in Definition 2.1,

$$Z_1(A) = \tfrac{1}{2}\big[Z(A) + Z(-A)\big] \quad \textit{and} \quad Z_1(B) = \tfrac{1}{2}\big[\overline{Z(B)} + \overline{Z(-B)}\big],$$
$$Z_2(A) = \tfrac{1}{2i}\big[Z(A) - Z(-A)\big] \quad \textit{and} \quad Z_2(B) = \tfrac{-1}{2i}\big[\overline{Z(B)} - \overline{Z(-B)}\big].$$

Hence, (2.1)–(2.3) *follow by symmetry of the measure F and Definition 2.1(3).*

Let Z be an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure F. As usual, integration with respect to Z can be defined starting with simple functions and extending to $L^2_{\mathbb{C}}(F)$ using the isometry condition Definition 2.1(3). Thus, the integral $\varphi \mapsto \int \varphi\, dZ$ maps $L^2_{\mathbb{C}}(F)$ linearly isometrically onto a closed subset of $L^2_{\mathbb{C}}(\mathbb{P})$ consisting of zero-mean random variables, and satisfies, for $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $\varphi, \psi \in L^2_{\mathbb{C}}(F)$,

$$\int \mathbf{1}_A\, dZ = Z(A), \quad \text{and} \quad \mathbb{E}\Big[\int \varphi\, dZ \overline{\int \psi\, dZ}\Big] = \int \varphi\overline{\psi}\, dF.$$

Denoting by $\mathcal{R}_{\mathbb{C}}(Z)$ the set of integrals $\int \varphi\, dZ$, $\varphi \in L^2_{\mathbb{C}}(F)$, $\mathcal{R}_{\mathbb{R}}(Z)$ refers to the real-valued elements in $\mathcal{R}_{\mathbb{C}}(Z)$. With $\tilde{L}^2_{\mathbb{C}}(F)$ denoting the set of functions in $L^2_{\mathbb{C}}(F)$ satisfying $\varphi(x) = \overline{\varphi(-x)}$ for all $x \in \mathbb{R}^d$ we have

$$\mathcal{R}_{\mathbb{R}}(Z) = \Big\{ \int \varphi\, dZ : \varphi \in \tilde{L}^2_{\mathbb{C}}(F) \Big\}.$$

Indeed, the inclusion "$\supseteq$" follows from Definition 2.1(2) and "$\subseteq$" from the fact that for all $\varphi \in L^2_{\mathbb{C}}(F)$, $\frac{1}{2}(\varphi + \overline{\varphi(-\cdot)})$ is in $\tilde{L}^2_{\mathbb{C}}(F)$ with integral equal to the real part of $\int \varphi\, dZ$.

For $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $\mathbb{R}^d$ write $u \le v$ if $u_j \le v_j$ for all j, and $u < v$ if $u_j < v_j$ for all j. Let $(u, v] = \{y \in \mathbb{R}^d : u < y \le v\}$. Consider a family $H = \{H_u \colon u \in \mathbb{R}^d\}$ with $H_u \in \mathbb{C}$. For $u \le v$ in $\mathbb{R}^d$ define *the increment of* H *over* $(u, v]$, $H((u, v])$, as

$$H((u,v]) = \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_d)\in\{0,1\}^d} (-1)^{\epsilon_\cdot} H_{(c_1(\epsilon_1),\ldots,c_d(\epsilon_d))}, \tag{2.4}$$

where $\epsilon_\cdot = \epsilon_1 + \cdots + \epsilon_d$, $c_j(0) = v_j$ and $c_j(1) = u_j$. That is, $H((u,v]) = H_v - H_u$ if $d = 1$ and

$$H((u,v]) = H_{(v_1,v_2)} + H_{(u_1,u_2)} - H_{(u_1,v_2)} - H_{(v_1,u_1)} \quad \text{if } d = 2.$$

Notice that $H((u,v]) = 0$ if $u \le v$ and $u \not< v$. Later we shall occasionally write $\triangle^h H(u)$ for $H((u, u+h])$ for $u \in \mathbb{R}^d$ and any $h \in \mathbb{R}^d_+$.

Remark 2.3. *The set function* H *defined in* (2.4) *on the semi-ring* $\mathcal{R} := \{(u,v] : u \le v \text{ in } \mathbb{R}^d\}$ *is finitely additive. Conversely, let* μ *be a finitely additive set function on* $\mathcal{R}$ *and set*

$$H_u = (-1)^{n_u} \mu((u \wedge 0, u \vee 0]), \qquad \textit{for } u \in \mathbb{R}^d, \tag{2.5}$$

where $u \wedge 0 = (\min(u_1, 0), \ldots, \min(u_d, 0))$, $u \vee 0 = (\max(u_1, 0), \ldots, \max(u_d, 0))$ *and* n_u *is the number of coordinates in* u *that are strictly less than* 0. *Then* $\mu((u,v]) = H((u,v])$ *for all* $u \le v$ *in* $\mathbb{R}^d$.

To show the first claim it is enough to show that for all $u \le v$ *in* $\mathbb{R}^d$, $k = 1, \ldots, d$ *and* $r \in (u_k, v_k)$ *we have*

$$\begin{aligned} H((u,v]) &= H((u_1,v_1] \times \cdots \times (u_{k-1},v_{k-1}] \times (u_k, r] \times (u_{k+1}, v_{k+1}] \times \cdots \times (u_d, v_d]) \\ &\quad + H((u_1,v_1] \times \cdots \times (u_{k-1},v_{k-1}] \times (r, v_k] \times (u_{k+1}, v_{k+1}] \times \cdots \times (u_d, v_d]). \end{aligned}$$

This follows, however, directly from definition (2.4). *To show the last claim let* μ *be a finitely additive set function on* $\mathcal{R}$ *and* $\{H_u : u \in \mathbb{R}^d\}$ *be given by* (2.5). *The function* H *vanishes on the axes, i.e.* $H_u = 0$ *for all* $u = (u_1, \ldots, u_d)$ *satisfying* $u_j = 0$ *for some* $j = 1, \ldots, d$. *Hence, by definition of the increments* $H((0 \wedge u, 0 \vee u])$, $u \in \mathbb{R}^d$, *there is at most one non-zero term in the sum* (2.4), *namely when* $c(\epsilon) = u$, *and in this case* $\epsilon_\cdot = n_u$. *That is,*

$$H((0 \wedge u, 0 \vee u]) = (-1)^{n_u} H_u = \mu((0 \wedge u, 0 \vee u]).$$

Since any half-open interval $(u,v]$ *in* $\mathbb{R}^d$ *can be expressed in terms of a finite number of intervals of the form* $(w \wedge 0, w \vee 0]$, $w \in \mathbb{R}^d$, *using elementary set operations it follows by finite additivity of* H *and* μ *that* $H((u,v]) = \mu((u,v])$.

Definition 2.4. *A real-valued process* $X = \{X_u : u \in \mathbb{R}^d\}$ *is said to have* wide-sense stationary increments *if* $X((u,v]) \in L^2_{\mathbb{R}}(\mathbb{P})$ *for all* $u \le v$ *in* $\mathbb{R}^d$ *with* $\mathbb{E}[X((u,v])] = 0$ *and*

$$\mathbb{E}\big[X\big((u_1+h, v_1+h]\big)X\big((u_2+h, v_2+h]\big)\big] = \mathbb{E}\big[X\big((u_1,v_1]\big)X\big((u_2,v_2]\big)\big] \tag{2.6}$$

for all $h \in \mathbb{R}^d$ *and* $u_1 \le v_1, u_2 \le v_2$ *in* $\mathbb{R}^d$.

It is enough that (2.6) holds for all $h \in \mathbb{R}^d_+$ for X to have wide-sense stationary increments. To see this assume that (2.6) holds for all $h \in \mathbb{R}^d_+$ and let $h \in \mathbb{R}^d$ be given. Choose $\tilde{h} \in \mathbb{R}^d$ such that $\tilde{h} \le 0$ and $\tilde{h} \le h$. An application of (2.6) with h replaced by $h - \tilde{h} \in \mathbb{R}^d_+$ and $-\tilde{h} \in \mathbb{R}^d_+$ yields

$$\begin{aligned}&\mathbb{E}\big[X\big((u_1+h, v_1+h]\big)X\big((u_2+h, v_2+h]\big)\big]\\&\quad= \mathbb{E}\big[X\big((u_1+\tilde{h}, v_1+\tilde{h}]\big)X\big((u_2+\tilde{h}, v_2+\tilde{h}]\big)\big] = \mathbb{E}\big[X\big((u_1, v_1]\big)X\big((u_2, v_2]\big)\big]\end{aligned}$$

which shows that (2.6) holds for general $h \in \mathbb{R}^d$.

Remark 2.5. *In this article we let increments be defined as in* (2.4)*. However, when* $d \ge 2$ *an alternative way of defining an increment of* $H = \{H_u : u \in \mathbb{R}^d\}$ *could be as* $H_v - H_u$ *for* $u \le v$*, and this leads to the very different kind of wide-sense stationary increments studied e.g. by Yaglom [13]. In this context, notice that in contrast to the set function* $(u, v] \mapsto H((u, v])$ *we have in the case* $d \ge 2$ *that the set function* $(u, v] \mapsto H_v - H_u$ *is only finitely additive when* H *is constant.*

Let us compare Yaglom's definition to the one given above. A real-valued process $X = \{X_u : u \in \mathbb{R}^d\}$ *for which* $X_v - X_u \in L^2_{\mathbb{R}}(\mathbb{P})$ *and* $\mathbb{E}[X_v - X_u] = 0$ *for all* $u \le v$ *in* $\mathbb{R}^d$ *is said to have wide-sense stationary increments in Yaglom's sense if*

$$\begin{aligned}&\mathbb{E}[(X_{v_1+h} - X_{u_1+h})(X_{v_2+h} - X_{u_2+h})]\\&= \mathbb{E}[(X_{v_1} - X_{u_1})(X_{v_2} - X_{u_2})], \quad \text{for all } h \in \mathbb{R}^d \text{ and } u_1 \le v_1, u_2 \le v_2 \text{ in } \mathbb{R}^d.\end{aligned}$$

It is easily seen that this implies that X *has wide-sense stationary increments in the sense of Definition 2.4. But conversely there are many processes with wide-sense stationary increments that do not have wide-sense stationary increments in Yaglom's sense. One such example is the Brownian sheet, where increments over disjoint intervals are independent and* $X((u, v]) \overset{\mathscr{D}}{=} N(0, \lambda_d((u, v]))$ *for* $u \le v$*, in the case* $d \ge 2$*. See also Example 2.6 for another example. However, when* $d = 1$ *the two definitions coincide.*

The term "wide-sense" refers to a property of the covariance function. In [1] (resp. in [13], Definition 8.1.2) a process is said to have strict-sense stationary increments, where increments are defined as $X_v - X_u$*, if the finite dimensional distributions of the increments are invariant under translations (resp. under translations and rotations). In the following we only consider "wide-sense" stationary increments.*

Assume that X *has wide-sense stationary increments in Yaglom's sense. Yaglom [13], Remark 3, p. 295, shows that, up to addition of a random variable not depending on* u*,* X_u *is given by*

$$X_u = \int_{\mathbb{R}^d} (e^{i\langle z,u\rangle} - 1)\, \tilde{Z}(dz) + \langle V, u\rangle, \quad \text{for } u \in \mathbb{R}^d, \tag{2.7}$$

where $\tilde{Z} = \{\tilde{Z}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ *is an* $L^2_{\mathbb{C}}(\mathbb{P})$*-valued random measure with control measure* $\tilde{F}$ *satisfying*

$$\int_{\mathbb{R}^d} (\|z\|^2 \wedge 1)\, \tilde{F}(dz) < \infty$$

and V *is a random vector in* $\mathbb{R}^d$*. After a few calculations it follows that when* $d \ge 2$*,*

$$X((u, v]) = \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(z)\, Z(dz), \quad \text{for } u < v, \tag{2.8}$$

where $Z(dz) = i^d z_1 \cdots z_d \tilde{Z}(dz)$*. That is, the control measure* F *of* Z *is* $F(dz) = \prod_{j=1}^d z_j^2 \tilde{F}(dz)$ *which satisfies*

$$\int_{\mathbb{R}^d} \frac{1 \wedge \|z\|^2}{\prod_{j=1}^d z_j^2}\, F(dz) < \infty.$$

Example 2.6. *In some situations one can define $X(A)$ not only for $A = (u, v]$ but also for arbitrary bounded Borel sets in $\mathbb{R}^d$. In this case, if the mapping $X : \mathcal{B}_b(\mathbb{R}^d) \to L^2_{\mathbb{R}}(\mathbb{P})$ is σ-additive one can in fact define $X(\varphi)$ for a large class of Borel functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, including all bounded Borel functions with compact support.*

An important example of this appears in the theory of stochastic partial differential equations and is presented in Dalang [2], p. 5–6. Let $X = \{X(\varphi) : \varphi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)\}$ be a centered Gaussian process with covariance function

$$\mathbb{E}[X(\varphi)X(\psi)] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y)g(x-y)\,dx\,dy, \tag{2.9}$$

where $g : \mathbb{R}^d \to [0, \infty]$ is a locally integrable function such that $g = \mathcal{F}\mu$ in $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ for a tempered measure μ. By approximating with a sequence in $\mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)$ in the norm

$$\|\phi\|_g = \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\phi(x)\phi(y)|g(x-y)\,dx\,dy\Big)^{1/2} \tag{2.10}$$

one can define $X(\varphi)$ for any bounded Borel function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support by $L^2_{\mathbb{R}}(\mathbb{P})$-continuity. Putting $X(A) = X(\mathbf{1}_A)$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$ it follows easily that the mapping $(A \in \mathcal{B}_b(\mathbb{R}^d)) \mapsto X(A) \in L^2_{\mathbb{R}}(\mathbb{P})$ is σ-additive. Finally, if we let $\{X_u : u \in \mathbb{R}^d\}$ be defined as $X_u = (-1)^{n_u} X((0 \wedge u, 0 \vee u])$ (cf. Remark 2.3) then the increment over any interval $(u, v]$ with $u < v$ in $\mathbb{R}^d$ is precisely $X((u, v])$. The process $\{X_u : u \in \mathbb{R}^d\}$ has wide-sense stationary increments since

$$\begin{aligned}\mathbb{E}[X(A+h)X(B+h)] &= \int_{A+h}\int_{B+h} g(x-y)\,dx\,dy \\ &= \int_A\int_B g(x-y)\,dx\,dy = \mathbb{E}[X(A)X(B)]\end{aligned}$$

for any $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ and any $h \in \mathbb{R}^d$.

In the next result we give the spectral representation of processes with wide-sense stationary increments. In this case it is natural to look for a representation as in (2.8) rather than (2.7). Recall that for $u, v \in \mathbb{R}^d$ with $u < v$,

$$\mathcal{F}\mathbf{1}_{(u,v]}(z) = \prod_{j=1}^d \Big(\frac{e^{iv_j z_j} - e^{iu_j z_j}}{iz_j}\Big), \quad \text{for } z = (z_1, \dots, z_d) \in \mathbb{R}^d, \tag{2.11}$$

where the right-hand side should be understood by continuity if $z_j = 0$ for some j, i.e. the j'th factor equals $v_j - u_j$ for $z_j = 0$.

Theorem 2.7. *Let $X = \{X_u : u \in \mathbb{R}^d\}$ be a real-valued process. Then X has wide-sense stationary increments and the mapping $(u \in \mathbb{R}^d_+) \mapsto X((0, u])$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$ if and only if there is a symmetric measure F on $\mathbb{R}^d$ satisfying*

$$\int_{\mathbb{R}^d}\prod_{j=1}^d \frac{1}{1+z_j^2}\,F(dz) < \infty, \quad (\text{where } z = (z_1, \dots, z_d)), \tag{2.12}$$

and an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure Z with control measure F such that

$$X((u, v]) = \int \mathcal{F}\mathbf{1}_{(u,v]}\,dZ, \quad \text{for } u < v. \tag{2.13}$$

If this is the case then for $u_1 < v_1$ and $u_2 < v_2$,

$$\mathbb{E}[X((u_1, v_1])X((u_2, v_2])] = \int \mathcal{F}\mathbf{1}_{(u_1,v_1]}\overline{\mathcal{F}\mathbf{1}_{(u_2,v_2]}}\,dF. \tag{2.14}$$

The measures F and Z are uniquely determined by X. In addition, $\mathcal{R}_{\mathbb{C}}(Z) = \overline{\mathrm{sp}}_{\mathbb{C}}\{X((u, v]) : u \le v\}$ and $\mathcal{R}_{\mathbb{R}}(Z) = \overline{\mathrm{sp}}_{\mathbb{R}}\{X((u, v]) : u \le v\}$.

The measure F above is called the *spectral measure of* X and Z is the *random spectral measure of* X. The last statement in Theorem 2.7 shows that Z is Gaussian if X is Gaussian.

If X vanishes on the axes then by Remark 2.3 $X_u = (-1)^{n_u} X((0\wedge u, 0\vee u])$ for $u \in \mathbb{R}^d$. In particular, in this case (2.13) implies that

$$X_u = \int \mathcal{F}\mathbf{1}_{(0,u]}\, dZ, \quad \text{for } 0 < u \text{ in } \mathbb{R}^d.$$

Proof. The "if" part: Let F satisfy (2.12) and Z be an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure F. By (2.11) $\mathcal{F}\mathbf{1}_{(u,v]} \in \widetilde{L}^2_{\mathbb{C}}(F)$ thus making the right-hand side of (2.12) well-defined for all $u < v$ in $\mathbb{R}^d$. Assume that the increments of X are given by (2.13) and notice that we have (2.14) as well by definition of the integral with respect to Z. Hence, since for arbitrary $h \in \mathbb{R}^d$ and $u_i \le v_i$ in $\mathbb{R}^d$ for $i = 1, 2$,

$$\mathcal{F}\mathbf{1}_{(u_1+h,v_1+h]}\overline{\mathcal{F}\mathbf{1}_{(u_2+h,v_2+h]}} = \mathcal{F}\mathbf{1}_{(u_1,v_1]}\overline{\mathcal{F}\mathbf{1}_{(u_2,v_2]}},$$

it follows from (2.14) that (2.6) holds, that is, X has wide-sense stationary increments. From (2.11) and (2.14) it follows that $(u \in \mathbb{R}^d_+) \mapsto X((0,u])$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$.

The "only if" part: In the case $d = 1$ the result goes back to [11] and [6]; see also Itô [4], Theorem 6.1. In the general case we follow Itô's approach closely. More specifically, we first define three processes $X(\cdot)$, $X^{(1)}(\cdot)$ and $X_1(\cdot)$ as well as Z and F. Then we establish the fundamental formula (2.20) below and finally we prove (2.12)–(2.13).

Assume that X has wide-sense stationary increments and the mapping $(u \in \mathbb{R}^d_+) \mapsto X((0,u])$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$. Define $\{X(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ as

$$X(\varphi) = \int_{\mathbb{R}^d} X_u \varphi(u)\, du, \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

where the integral is constructed in the $L^2_{\mathbb{C}}(\mathbb{P})$-sense using that $u \mapsto X_u\varphi(u)$ is $L^2_{\mathbb{C}}(\mathbb{P})$-continuous with compact support. Clearly, $\{X(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ constitutes a random distribution in the sense of Itô [4] or Yaglom [13].

Denote by D the differential operator $\partial^d/\partial u_1 \cdots \partial u_d$ and define $\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ according to

$$X^{(1)}(\varphi) = (-1)^d \int_{\mathbb{R}^d} X_u D\varphi(u)\, du, \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

Since, with $e = (1, \dots, 1) \in \mathbb{R}^d$ denoting the vector of ones,

$$D\varphi(u) = \lim_{\epsilon\downarrow 0} \varphi((u-\epsilon e, u])/\epsilon^d = \lim_{\epsilon\downarrow 0} \triangle^{\epsilon e}\varphi(u-\epsilon e)/\epsilon^d, \quad \text{for } u \in \mathbb{R}^d \text{ and } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

we get, using the assumptions and linear change of variables together with the formula

$$\int_{\mathbb{R}^d} f(u)\triangle^h g(u)\, du = (-1)^d \int_{\mathbb{R}^d} \triangle^h f(u-h) g(u)\, du, \quad h \in \mathbb{R}^d_+, \tag{2.15}$$

that for $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$

$$(-1)^d \int_{\mathbb{R}^d} X_u D\varphi(u)\, du = \lim_{\epsilon\downarrow 0} \epsilon^{-d} \int_{\mathbb{R}^d} X((u, u+\epsilon e])\varphi(u)\, du, \quad \text{in } L^2_{\mathbb{C}}(\mathbb{P}). \tag{2.16}$$

A key point is that $X^{(1)}$ is stationary in the sense that

$$\mathbb{E}[\tau_h X^{(1)}(\varphi)\, \overline{\tau_h X^{(1)}(\psi)}] = \mathbb{E}[X^{(1)}(\varphi)\, \overline{X^{(1)}(\psi)}], \quad \text{for } h \in \mathbb{R}^d,\ \varphi, \psi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

where

$$\tau_h X^{(1)}(\varphi) = X^{(1)}(\varphi(\cdot - h)), \quad \text{for } h \in \mathbb{R}^d,\ \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

To see this, let $h \in \mathbb{R}^d$ and $\varphi, \psi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$. Using (2.16) it follows that

$$\begin{aligned}
&\mathbb{E}[\tau_h X^{(1)}(\varphi)\overline{\tau_h X^{(1)}(\psi)}] \\
&= \lim_{\epsilon\downarrow 0} \epsilon^{-2d}\mathbb{E}[\int_{\mathbb{R}^d} X((u, u+\epsilon e])\varphi(u-h)\, du \int_{\mathbb{R}^d} X((v, v+\epsilon e])\overline{\psi(v-h)}\, dv] \\
&= \lim_{\epsilon\downarrow 0} \epsilon^{-2d}\mathbb{E}[\int_{\mathbb{R}^d} X((u+h, u+h+\epsilon e])\varphi(u)\, du \int_{\mathbb{R}^d} X((v+h, v+h+\epsilon e])\overline{\psi(v)}\, dv] \\
&= \lim_{\epsilon\downarrow 0} \epsilon^{-2d} \int_{\mathbb{R}^{2d}} \mathbb{E}[X((u+h, u+h+\epsilon e])X((v+h, v+h+\epsilon e])]\varphi(u)\,\overline{\psi(v)}\, dudv \\
&= \lim_{\epsilon\downarrow 0} \epsilon^{-2d} \int_{\mathbb{R}^{2d}} \mathbb{E}[X((u, u+\epsilon e])X((v, v+\epsilon e])]\varphi(u)\,\overline{\psi(v)}\, dudv \\
&= \mathbb{E}[X^{(1)}(\varphi)\,\overline{X^{(1)}(\psi)}].
\end{aligned}$$

Applying [13], Theorem 3, there exists an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure Z with symmetric control measure F satisfying

$$\int_{\mathbb{R}^d} \frac{1}{(1+\|z\|^2)^q}\, F(dz) < \infty, \quad \text{for some } q \geq 1, \tag{2.17}$$

such that

$$X^{(1)}(\varphi) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z)\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

Let $\mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$ denote the set of $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ of the form

$$\varphi(z) = \prod_{j=1}^d g_j(z_j) \tag{2.18}$$

where $g_j \in \mathscr{D}_{\mathbb{C}}(\mathbb{R})$. Notice that for such a φ, $\mathcal{F}\varphi(z) = \prod_{j=1}^d (\mathcal{F}_1 g_j)(z_j)$. Following Itô [4], set

$$X_1(\varphi) = \int_{\mathbb{R}^d} G\varphi(z)\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d), \tag{2.19}$$

where

$$G\varphi(z) = \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{e^{iu_j z_j} - \mathbf{1}_{\{|z_j|\leq 1\}}}{iz_j}\, \varphi(u)\, du.$$

Observe that for φ as in (2.18) $G\varphi(z) = \prod_{j=1}^d G_1 g_j(z_j)$ where

$$G_1 g_j(z_j) = \int_{\mathbb{R}} \frac{e^{iu_j z_j} - \mathbf{1}_{\{|z_j|\leq 1\}}}{iz_j}\, g_j(u_j)\, du_j.$$

Since $G_1 g_j(z_j)$ is bounded and

$$G_1 g_j(z_j) = \frac{\mathcal{F}_1 g_j(z_j)}{iz_j}, \quad \text{for } |z_j| > 1\ ,$$

and thus tends to zero faster than any polynomial it follows, with q given in (2.17), that

$$\sup_{z\in\mathbb{R}^d} |G\varphi(z))|^2(1+\|z\|^2)^q < \infty.$$

Hence, by (2.17), $G\varphi$ belongs to $L^2_{\mathbb{C}}(F)$ making (2.19) well-defined. Maintaining the definition of the differential operator D from above and using integration by parts we get $G(D\varphi) = (-1)^d \mathcal{F}\varphi$ for $\varphi \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$, implying that

$$X(D\varphi) = (-1)^d X^{(1)}(\varphi) = X_1(D\varphi), \quad \text{for } \varphi \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d),$$

or equivalently,

$$X_1(\varphi) = X(\varphi), \quad \text{for } \varphi \in \mathscr{D}^p_0(\mathbb{R}^d),$$

where $\mathscr{D}^p_0(\mathbb{R}^d)$ is the subspace of $\mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$ consisting of φ on the form (2.18) where, for $j = 1, \ldots, d$, $\int_{\mathbb{R}} g_j(z_j)\, dz_j = 0$.

For $\varphi \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d_+$ we have $\triangle^h \varphi \in \mathscr{D}^p_0(\mathbb{R}^d)$ and thus

$$X(\triangle^h \varphi) = \int_{\mathbb{R}^d} G(\triangle^h \varphi)(z)\, Z(dz)$$

implying, since

$$G(\triangle^h \varphi)(z) = \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{e^{-ih_j z_j} - 1}{iz_j},$$

that

$$X(\triangle^h \varphi) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{e^{-ih_j z_j} - 1}{iz_j}\, Z(dz). \tag{2.20}$$

Splitting $\mathbb{R}^d$ into disjoint sets according to the coordinates being numerically greater than or less than 2π we see, using that F is finite on bounded sets, that equation (2.12) is equivalent to that

$$\int_{C_I} \prod_{j\in I} \frac{1}{z_j^2}\, F(dz) < \infty \tag{2.21}$$

for each non-empty $I \subseteq \{1, \ldots, d\}$, where

$$C_I = \{z \in \mathbb{R}^d : |z_j| > 2\pi \text{ for } j \in I \text{ and } |z_j| \le 2\pi \text{ for } j \notin I\}.$$

To show (2.21) for a given I we argue as follows. By (2.20) we have for $\varphi \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d_+$ that

$$\begin{aligned}\|X(\triangle^h \varphi)\|^2_{L^2_{\mathbb{C}}(\mathbb{P})} &= \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \prod_{j=1}^d \Big|\frac{1 - e^{-ih_j z_j}}{iz_j}\Big|^2 F(dz) \\ &\ge \int_{C_I} \prod_{j=1}^d |1 - e^{-ih_j z_j}|^2\, |\mathcal{F}\varphi(z)|^2 \frac{F(dz)}{\prod_{j=1}^d z_j^2}.\end{aligned}$$

In particular this holds for every $\varphi_n \in \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d)$, $n \ge 1$, of the form $\varphi_n(z) = \prod_{j=1}^d g_n(z_j)$ for $z \in \mathbb{R}^d$, where $g_n \in \mathscr{D}_{\mathbb{R}}(\mathbb{R})_+$ satisfies

$$g_n(x) = 0, \quad \text{for } |x| \ge 1/n, \text{ and } \int_{\mathbb{R}} g_n(x)\, dx = 1.$$

In this case $|\mathcal{F}_1 g_n(z_j)| \ge 1/2$ for all z_j satisfying $|z_j| \le 2\pi n/16$; see [4], p. 221. Hence for arbitrary n and $h \in \mathbb{R}^d_+$,

$$\|X(\triangle^h \varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} \ge (1/2)^{2d} \int_{C_I} \prod_{j=1}^d \frac{|1 - e^{-ih_j z_j}|^2\, \mathbf{1}_{\{|z_j| \le 2\pi n/16\}}}{z_j^2} F(dz). \tag{2.22}$$

Following [4] integrate both sides with respect to dh over the cube $[0,1]^d$. Using the definition of C_I and the product structure the integral of the integrand on the right-hand side of (2.22) equals for each $z \in C_I$ and $n \geq 16$

$$\prod_{j\in I} \frac{1}{z_j^2} \int_0^1 |1-e^{-ih_j z_j}|^2\, dh_j \mathbf{1}_{\{2\pi<|z_j|\leq 2\pi n/16\}} \prod_{j\notin I} \frac{1}{z_j^2} \int_0^1 |1-e^{-ih_j z_j}|^2\, dh_j \mathbf{1}_{\{|z_j|\leq 2\pi\}}. \quad (2.23)$$

Now, there is a constant $b > 0$ such that for all $|z_j| \leq 2\pi$ and $0 \leq h_j \leq 1/2$

$$\frac{|1-e^{-ih_j z_j}|^2}{|z_j|^2} \geq b h_j^2,$$

and, as shown on p. 221 in [4], there exists a constant $c > 0$ such that

$$\int_0^1 |1-e^{-ih_j z_j}|^2\, dh_j \geq c$$

for all $2\pi < |z_j|$. Hence, using that we get a smaller value by integrating over $[0,1/2]$ instead of over $[0,1]$ for $j \notin I$, it follows that the expression in (2.23) is greater than or equal to

$$\prod_{j\in I} \frac{c}{|z_j|^2} \mathbf{1}_{\{2\pi<|z_j|\leq 2\pi n/16\}} \prod_{j\notin I} \frac{b}{24} \mathbf{1}_{\{|z_j|\leq 2\pi\}}.$$

Inserting into (2.22) and applying monotone convergence (2.21) follows if

$$\sup_{n\geq 1,\, h\in[0,1]^d} \|X(\triangle^h \varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} < \infty.$$

But using Jensen's inequality we have, for all $n \geq 1$ and $h \in [0,1]^d$,

$$\begin{aligned}\|X(\triangle^h \varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} &= \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} X_u \triangle^h \varphi_n(u)\, du \Big)^2 \Big] = \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} (-1)^d \triangle^h X_{u-h} \varphi_n(u)\, du \Big)^2 \Big] \\ &\leq \int_{\mathbb{R}^d} \mathbb{E}[(\triangle^h X_{u-h})^2] \varphi_n(u)\, du \leq \sup_{u\in\mathbb{R}^d,\, h\in[0,1]^d} \mathbb{E}[(\triangle^h X_{u-h})^2]\end{aligned}$$

which is finite due to the $L^2_{\mathbb{R}}(\mathbb{P})$-continuity and the stationary increments.

Let $h = v - u$. From (2.20) and (2.15) it follows that

$$\int_{\mathbb{R}^d} \triangle^h X_{x-h} \varphi_n(x)\, dx = (-1)^d \int_{\mathbb{R}^d} \mathcal{F}\varphi_n(z) \prod_{j=1}^d \frac{e^{-ih_j z_j}-1}{iz_j}\, Z(dz), \quad (2.24)$$

for all $n \geq 1$, where $(\varphi_n)_{n\geq 1} \subseteq \mathscr{D}^p_{\mathbb{C}}(\mathbb{R}^d) \cap \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)_+$ is a sequence satisfying

$$\int_{\mathbb{R}^d} \varphi_n(x)\, dx = 1 \text{ for } n \geq 1 \quad \text{and} \quad \varphi_n(x)\, dx \to \delta_v \text{ weakly.}$$

As n tends to infinity both sides of (2.24) converge in $L^2_{\mathbb{R}}(\mathbb{P})$ due to the continuity assumption on X and the integrability property (2.12) of F, giving the identity

$$\begin{aligned}X((u,v]) = \triangle^{v-u} X_u &= \int_{\mathbb{R}^d} e^{i\langle z,v\rangle} \prod_{j=1}^d \frac{1-e^{-i(v_j-u_j)z_j}}{iz_j}\, Z(dz) \\ &= \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{e^{iv_j z_j}-e^{iu_j z_j}}{iz_j}\, Z(dz) = \int_{\mathbb{R}^d} \mathcal{F}1_{(u,v]}(z)\, Z(dz)\end{aligned}$$

which is (2.13).

To prove the last part notice that X and $X^{(1)}$ are in one-to-one correspondence, that $\overline{\mathrm{sp}}_{\mathbb{C}}\{X((u,v]) : u \leq v\} = \overline{\mathrm{sp}}_{\mathbb{C}}\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$, and that there is a similar result with subscript $\mathbb{C}$ replaced by $\mathbb{R}$. By construction (see [13] p. 281), Z is uniquely determined; moreover we have $\mathcal{R}_{\mathbb{C}}(Z) = \overline{\mathrm{sp}}_{\mathbb{C}}\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ as well as the corresponding result with subscript $\mathbb{C}$ replaced by $\mathbb{R}$. This concludes the proof. $\square$

In connection with the integrability condition (2.12) on the spectral measure F we have the following inequalities for all $z = (z_1, \ldots, z_d)$ in $\mathbb{R}^d$:

$$\Big(\frac{1}{1+\|z\|^2}\Big)^d \leq \prod_{j=1}^d \frac{1}{1+z_j^2} \leq \frac{1}{1+\|z\|^2}. \tag{2.25}$$

This should compared with the integrability condition satisfied by general tempered measures cf. Lemma 3.4 below. The inequalities (2.25) can be shown as follows:

$$\begin{aligned} 1+\|z\|^2 = 1+\sum_{j=1}^d z_j^2 &\leq \sum_{\epsilon_1,\ldots,\epsilon_d \in \{0,1\}} \prod_{j=1}^d z_j^{2\epsilon_j} = \prod_{j=1}^d (1+z_j^2) \\ &= \sum_{\epsilon_1,\ldots,\epsilon_d \in \{0,1\}} \prod_{j=1}^d z_j^{2\epsilon_j} \leq \sum_{\epsilon_1,\ldots,\epsilon_d \in \{0,1\}} \prod_{j=1}^d \|z\|^{2\epsilon_j} = \big(1+\|z\|^2\big)^d. \end{aligned}$$

The following corollary gives necessary and sufficient conditions for a Gaussian process with stationary increments to be of the form described in Example 2.6.

Corollary 2.8. *Let $X = \{X_u : u \in \mathbb{R}^d\}$ be a centered Gaussian process with wide-sense stationary increments, spectral measure F and random spectral measure Z. If $\mathcal{F}F$ is a positive locally integrable function then the process*

$$X(\phi) = \int \mathcal{F}\phi \, dZ, \qquad \phi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d), \tag{2.26}$$

is of the form (2.9) *with $\mu = F$ and $g = \mathcal{F}\mu$ in $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$.*

Conversely, let $\{X(\phi) : \phi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)\}$ be of the form (2.9) *and $\{X_u : u \in \mathbb{R}^d\}$ be the corresponding process with wide-sense stationary increments constructed in Example 2.6. Then $\{X_u : u \in \mathbb{R}^d\}$ has spectral measure $F = \mu$, and therefore*

$$\int \prod_{j=1}^d \frac{1}{1+z_j^2} \, \mu(dz) < \infty. \tag{2.27}$$

Proof. Assume that $g = \mathcal{F}F$ is a positive locally integrable function and let $\{X(\phi) : \phi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)\}$ be given by (2.26). By elementary properties of Fourier transforms and convolutions

$$\mathbb{E}[X(\phi)X(\psi)] = \int \mathcal{F}\phi\overline{\mathcal{F}\psi} \, dF = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\psi(y)g(x-y)\,dx\,dy$$

for $\phi, \psi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)$, which shows the first part.

Conversely, consider the process $\{X_u : u \in \mathbb{R}^d\}$ with wide-sense stationary increments constructed in Example 2.6 and let us show that $F = \mu$. Again by elementary properties of Fourier transforms and convolutions

$$\mathbb{E}[X(\phi)X(\psi)] = \int \mathcal{F}\phi \, \overline{\mathcal{F}\psi} \, d\mu, \qquad \text{for } \phi, \psi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d). \tag{2.28}$$

Let $\mathscr{E}$ be the set of simple functions on $\mathbb{R}^d$ defined in (3.1) below. By (2.14) and linearity we have that

$$\mathbb{E}[X(\phi)X(\psi)] = \int \mathcal{F}\phi\,\overline{\mathcal{F}\psi}\,dF, \qquad \text{for } \phi,\psi \in \mathcal{E}. \tag{2.29}$$

For $(\phi_j)_{j=1}^d \subseteq \mathscr{D}_{\mathbb{R}}(\mathbb{R})$ set $\phi(z) = \prod_{j=1}^d \phi_j(z_j)$ and

$$\phi_n(z) = \prod_{j=1}^d \phi_{n,j}(z_j) \quad \text{where} \quad \phi_{n,j} = \sum_{k=-\infty}^{\infty} \phi_j(\tfrac{k-1}{n})\mathbf{1}_{((k-1)/n,k/n]}.$$

Notice that $(\phi_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ and $\phi \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)$. By continuity of ϕ we have that $\phi_n \to \phi$ in $\|\cdot\|_g$ (see (2.10)) and hence $X(\phi_n) \to X(\phi)$ in $L^2_{\mathbb{R}}(\mathbb{P})$. According to (2.28)–(2.29) we have

$$\begin{aligned}\int |\mathcal{F}\phi|^2\,d\mu &= \mathbb{E}[X(\phi)^2] = \lim_{n\to\infty} \mathbb{E}[X(\phi_n)^2] \\ &= \lim_{n\to\infty} \int |\mathcal{F}\phi_n|^2\,dF = \int |\mathcal{F}\phi|^2\,dF,\end{aligned}$$

where in the last equality we have used that $\mathcal{F}\phi_n \to \mathcal{F}\phi$ in $L^2_{\mathbb{C}}(F)$, cf. Lemma 3.6 below. This proves that $\mu = F$. Finally, (2.27) follows from Theorem 2.7. □

Example 2.9. *Consider the case $d = 1$ and let $X = \{X_u : u \in \mathbb{R}\}$ be a fractional Brownian motion with Hurst exponent $H \in (0,1)$. Then X has absolutely continuous spectral measure F with density $f\colon x \mapsto C|x|^{1-2H}$ for some $C > 0$. By Corollary 2.8, X is of the form* (2.9) *if and only if $\mathcal{F}F = \mathcal{F}f$ is a positive locally integrable function. For $H \in (\frac{1}{2},1)$, $\mathcal{F}f = (x \mapsto K|x|^{2H-2})$ for some constant $K > 0$ and hence X is of the form* (2.9)*. On the other hand, X is not of the form* (2.9) *when $H \in (0,\frac{1}{2}]$ because $\mathcal{F}f$ is not a locally integrable function. Indeed, to show the last claim let $r_\alpha\colon x \mapsto C|x|^\alpha$ for all $\alpha > -1$. For the moment assume that $\mathcal{F}r_\alpha$ is a locally integrable function. By the scaling property of r_α we have for all $u \in \mathbb{R}$ that $\mathcal{F}r_\alpha(x) = |u|^{\alpha+1}(\mathcal{F}r_\alpha)(ux)$ for λ_1-a.e. x, which implies that $\mathcal{F}r_\alpha(x) = K'|x|^{-1-\alpha}$ for λ_1-a.e. x and some constant $K' \in \mathbb{R}\setminus\{0\}$. This shows that $\mathcal{F}r_\alpha$ is not a locally integrable function when $\alpha \geq 0$. In particular, $\mathcal{F}f$ is not a locally integrable function when $H \in (0,\frac{1}{2}]$.*

3 Deterministic integrands

Let $X = \{X_u : u \in \mathbb{R}^d\}$ be a real-valued process with wide-sense stationary increments having spectral measure F satisfying (2.12) and random spectral measure Z. Assume furthermore that F is absolutely continuous with respect to λ_d with density f. In the following we study classes of deterministic integrands with respect to X.

Let $\mathscr{E}$ be the set of simple functions on $\mathbb{R}^d$ of the form

$$\varphi = \sum_{j=1}^n \alpha_j \mathbf{1}_{(u_j,v_j]} \tag{3.1}$$

where $\{\alpha_j\} \subseteq \mathbb{R}$ and $\{u_j\},\{v_j\} \subseteq \mathbb{R}^d$ satisfy $u_j \leq v_j$ for all j. For $\varphi \in \mathcal{E}$ represented as in (3.1) define the *simple integral* as

$$\int \varphi\,dX := \sum_{j=1}^n \alpha_j X((u_j,v_j]), \tag{3.2}$$

and equip $\mathscr{E}$ with the norm $\|\varphi\|_{\mathscr{E}} := \|\int \varphi\,dX\|_{L^2_{\mathbb{R}}(\mathbb{P})}$. Notice that the integral (3.2) is well-defined by finite additivity of the mapping $(u,v] \mapsto X((u,v])$, that is, $\int \varphi\,dX$ does not depend on the representation (3.1). By Theorem 2.7,

$$\int_{\mathbb{R}^d} \varphi(u)\, dX_u = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z)\, dZ_z \text{ and } \|\varphi\|_{\mathscr{E}}^2 = \int_{\mathbb{R}^d} |\mathcal{F}\varphi|^2\, dF \text{ for } \varphi \in \mathscr{E}. \tag{3.3}$$

Definition 3.1. *A pseudo normed linear space $(\Lambda, \|\cdot\|_\Lambda)$ containing $\mathscr{E}$ as a dense subspace and satisfying $\|\varphi\|_{\mathscr{E}} = \|\varphi\|_\Lambda$ for $\varphi \in \mathscr{E}$ is called a predomain for X. A domain is a complete predomain. Given a predomain Λ, there is a unique continuous linear mapping $\int \cdot\, dX : \Lambda \to L^2_{\mathbb{R}}(\mathbb{P})$, extending the simple integral* (3.2)*. This mapping is called the integral with respect to X.*

Notice that Λ is not assumed to be a function space. By definition, a domain is a completion of $\mathscr{E}$ and thus uniquely determined up to an isometric isomorphism. Below we give concrete examples of predomains and domains.

Remark 3.2. *Using the completeness of $L^2_{\mathbb{R}}(\mathbb{P})$ we see that a predomain Λ is a domain if and only if*

$$\Big\{ \int \varphi\, dX : \varphi \in \Lambda \Big\} = \overline{\mathrm{sp}}_{\mathbb{R}} \big\{ X((u,v]) : u, v \in \mathbb{R}^d, u \le v \big\}. \tag{3.4}$$

This emphasizes why domains are more attractable than predomains since for the latter we only have "$\subseteq$" in (3.4)*.*

For ease of reading we formulate two lemmas. The first generalizes Lemma 3.1 of [5] to $d \ge 2$. For the second see [9], Chapter VII, Théorème VII.

Lemma 3.3. *Let $\varphi \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ be given such that $\mathcal{F}\varphi$ is a function. Then $\varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ if and only if $\mathcal{F}\varphi(-x) = \overline{\mathcal{F}\varphi(x)}$ for λ_d-a.e. x.*

Lemma 3.4. *Let μ be a signed Borel measure on $\mathbb{R}^d$. Then μ is a tempered measure, that is $\mu \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ if*

$$\int_{\mathbb{R}^d} (1 + \|u\|^2)^{-k}\, |\mu|(du) < \infty$$

for some positive integer $k \ge 1$. This condition is also necessary if μ is a positive measure. In particular, a real-valued Borel function h is a tempered distribution if, and in case h is non-negative only if,

$$\int_{\mathbb{R}^d} \frac{|h(u)|}{(1 + \|u\|^2)^k}\, du < \infty$$

for some positive integer $k \ge 1$.

In view of (3.3) it is natural to look for predomains consisting of objects for which a Fourier transform can be defined, that is spaces of distributions. Let φ be a tempered distribution. If the Fourier transform $\mathcal{F}\varphi$ is a function, then this function is determined up to Lebesgue null sets and hence by absolute continuity of F the following spaces are well-defined:

$$\Lambda_{\mathrm{dist}} = \Big\{ \varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d) : \mathcal{F}\varphi \text{ is a function such that } \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2\, F(dz) < \infty \Big\},$$
$$\Lambda_{\mathrm{func}} = \Big\{ \varphi \in L^2_{\mathbb{R}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2\, F(dz) < \infty \Big\}.$$

Moreover, let Λ_{dist} and Λ_{func} be equipped with the pseudo norms

$$\|\varphi\|^2_{\Lambda_{\mathrm{func}}} = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2\, F(dz), \qquad \|\varphi\|^2_{\Lambda_{\mathrm{dist}}} = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2\, F(dz).$$

Notice that $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d) \subseteq \Lambda_{\mathrm{func}} \subseteq \Lambda_{\mathrm{dist}}$.

Theorem 3.5. *(1) Λ_{dist} is a predomain for X and the integral on Λ_{dist} is given by*

$$\int \varphi \, dX = \int \mathcal{F}\varphi \, dZ, \quad \varphi \in \Lambda_{\mathrm{dist}}. \tag{3.5}$$

(2) Λ_{dist} is a domain for X if and only if

$$\forall g \in L^2_{\mathbb{R}}(F) \; \exists k \in \mathbb{N} : \int_{\{f>0\}} \frac{|g(u)|}{(1+\|u\|^2)^k} \, du < \infty. \tag{3.6}$$

In particular, Λ_{dist} is a domain for X if there exists $k \in \mathbb{N}$ such that

$$\int_{\{f>0\}} \frac{1}{f(u)(1+\|u\|^2)^k} \, du < \infty. \tag{3.7}$$

(3) Λ_{func} is a predomain, and it is a domain if and only if

$$L^2_{\mathbb{R}}(F) \subseteq L^2_{\mathbb{R}}(\mathbf{1}_{\{f(u)>0\}} \, du). \tag{3.8}$$

By Lemma 3.6 we further have that Λ_{dist} is complete if and only if $\mathcal{F}(\Lambda_{\mathrm{dist}}) = \tilde{L}^2_{\mathbb{C}}(F)$.

Proof. (1): Lemma 3.6 below implies that $\mathscr{E}$ is dense in Λ_{dist} showing together with (3.3) that Λ_{dist} is a predomain for X. The continuous linear mapping $\varphi \mapsto \int \varphi \, dX$ from Λ_{dist} to $L^2_{\mathbb{C}}(\mathbb{P})$ defined by (3.5) extends the simple integral by (3.3) and is hence the corresponding integral since $L^2_{\mathbb{R}}(\mathbb{P})$ is a closed subspace of $L^2_{\mathbb{C}}(\mathbb{P})$.

(2): Assume that for all $g \in L^2_{\mathbb{R}}(F)$, (3.6) holds for some k and let us show that Λ_{dist} is a domain for X. Let $\{\varphi_n\}$ be a Cauchy sequence in Λ_{dist}. By completeness of $L^2_{\mathbb{C}}(F)$ there exists $g \in L^2_{\mathbb{C}}(F)$ with $\overline{g} = g(-\cdot)$ such that $\mathcal{F}\varphi_n \to g$ in $L^2_{\mathbb{C}}(F)$. Since we may assume that $g = 0$ on $\{f = 0\}$, (3.6) and Lemma 3.4 show that $g \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$. Hence, using Lemma 3.3, $\varphi := \mathcal{F}^{-1}g$ is in Λ_{dist} and $\varphi_n \to \varphi$ in Λ_{dist} which shows that Λ_{dist} is complete.

Conversely, assume that Λ_{dist} is complete. For contradiction consider an $h \in L^2_{\mathbb{R}}(F)$ which does not satisfy (3.6) with g replaced by h. Without loss of generality we may assume that $h \geq 0$ and $h = 0$ on $\{f = 0\}$. By Lemma 3.4, $h \notin \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$. Let $h_1 = \frac{1}{2}(h + h(-\cdot))$ and $h_2 = \frac{1}{2}(h - h(-\cdot))$ be the even and odd parts of h and set $g = h_1 + ih_2$. By linearity, $g \in L^2_{\mathbb{C}}(F)$ and if $g \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ then $h_1, h_2 \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ which implies that $h = h_1 + h_2 \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$. Thus $g \in L^2_{\mathbb{C}}(F) \setminus \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ and by construction $\overline{g} = g(-\cdot)$. Since F is a tempered measure, $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$ is dense in $L^2_{\mathbb{R}}(F)$ and therefore there exist sequences $\{g_{e,n}\}$ and $\{g_{o,n}\}$ in $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$ consisting of even and odd functions approximating h_1 and h_2 in $L^2_{\mathbb{R}}(F)$. Setting $g_n = g_{e,n} + ig_{o,n}$ for $n \geq 1$ we have $\{g_n\} \subseteq \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d) \subseteq \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ satisfying $\overline{g}_n = g_n(-\cdot)$ and $g_n \to g$ in $L^2_{\mathbb{C}}(F)$. Thus $\varphi_n := \mathcal{F}^{-1}g_n$ is a Cauchy sequence in Λ_{dist} which does not converge.

The last statement in (2) follows since for any measurable function $g \colon \mathbb{R}^d \to \mathbb{R}$, we have by the Cauchy-Schwarz inequality that

$$\int_{\{f>0\}} \frac{|g(u)|}{(1+\|u\|^2)^k} \, du \leq \Big(\int_{\{f>0\}} |g(u)|^2 f(u) \, du \Big)^{1/2} \Big(\int_{\{f>0\}} \frac{1}{f(u)} \frac{1}{(1+\|u\|^2)^{2k}} \, du \Big)^{1/2}.$$

(3): Assume (3.8) and let $\{\varphi_n\}$ be Cauchy in Λ_{func}. As in the proof of (2) there is a $g \in L^2_{\mathbb{C}}(F)$ with $\overline{g} = g(-\cdot)$ and satisfying $g = 0$ on $\{f = 0\}$ such that $\mathcal{F}\varphi_n \to g$ in $L^2_{\mathbb{C}}(F)$. Since $g \in L^2_{\mathbb{C}}(\mathbb{R}^d)$ we have by Lemma 3.3 that $\varphi := \mathcal{F}^{-1}g$ is in Λ_{func} and $\varphi_n \to \varphi$ in Λ_{func}, showing that the latter space is complete.

Conversely assume that Λ_{func} is complete. As in the proof of (2), if (3.8) is not satisfied there is a function $g : \mathbb{R}^d \to \mathbb{C}$ satisfying $g = 0$ on $\{f = 0\}$ and $\overline{g} = g(-\cdot)$ such that $g \in L^2_{\mathbb{C}}(F) \setminus L^2_{\mathbb{C}}(\mathbf{1}_{\{f(u)>0\}} du)$. Again as in (2) we can construct a sequence $\{g_n\}$ in $L^2_{\mathbb{C}}(\mathbb{R}^d) \cap L^2_{\mathbb{C}}(F)$ satisfying $\overline{g_n} = g_n(-\cdot)$ such that $g_n \to g$ in $L^2_{\mathbb{C}}(F)$. Then $\varphi_n := \mathcal{F}^{-1}g_n$ is a non-converging Cauchy sequence in Λ_{func}. □

Lemma 3.6. *For $(\phi_j)_{j=1}^d \subseteq \mathscr{D}_\mathbb{R}(\mathbb{R})$ let $\phi(z) = \prod_{j=1}^d \phi_j(z_j)$ and*

$$\phi_n(z) = \prod_{j=1}^d \phi_{n,j}(z_j) \quad \textit{where} \quad \phi_{n,j} = \sum_{k=-\infty}^{\infty} \phi_j(\tfrac{k-1}{n})\mathbf{1}_{((k-1)/n,k/n]}.$$

Then $\mathcal{F}\phi_n \to \mathcal{F}\phi$ in $L^2_\mathbb{C}(F)$. In particular, $\mathcal{F}(\mathscr{E})$ is a dense subspace of $\tilde{L}^2_\mathbb{C}(F)$.

Proof. For simplicity assume that $\operatorname{supp}(\phi_j) \subseteq (0,1)$ for all j. For all n, j and $t \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned}(\mathcal{F}_1\phi_{n,j})(t) &= \int_\mathbb{R} e^{ist}\phi_{n,j}(s)\,ds = \sum_{k=1}^{n+1} \phi_j(\tfrac{k-1}{n})\Big(\frac{e^{i(k/n)t} - e^{i((k-1)/n)t}}{it}\Big) \\ &= \frac{-1}{it}\sum_{k=1}^{n}(\phi_j(\tfrac{k}{n}) - \phi_j(\tfrac{k-1}{n}))e^{i(k/n)t}.\end{aligned}$$

Hence, denoting by K_j the total variation of ϕ_j on $[0,1]$,

$$|(\mathcal{F}_1\phi_{n,j})(t)| \leq \frac{K_j}{|t|}.$$

Furthermore, for all j,

$$\|\mathcal{F}_1\phi_{n,j} - \mathcal{F}_1\phi_j\|_\infty \leq \|\phi_{n,j} - \phi_j\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } n \to \infty \tag{3.9}$$

implying that $C_j := \sup_{n\in\mathbb{N}}\|\mathcal{F}_1\phi_{n,j}\|_\infty < \infty$. Thus $\mathcal{F}\phi_n(z) = \prod_{j=1}^d \mathcal{F}_1\phi_{n,j}(z_j) \to \mathcal{F}g(z)$ pointwise by (3.9), and

$$|(\mathcal{F}\phi_n)(z)| = \prod_{j=1}^d |\mathcal{F}_1\phi_{n,j}(z_j)| \leq \prod_{j=1}^d \Big(\mathbf{1}_{\{|z_j|\leq 1\}}C_j + \mathbf{1}_{\{|z_j|>1\}}\frac{K_j}{|z_j|}\Big)$$

which by dominated convergence implies that $\mathcal{F}\phi_n \to \mathcal{F}\phi$ in $L^2_\mathbb{C}(F)$.

Let G be the real linear span of functions ϕ of the form $\phi(u) = \prod_{j=1}^d \phi_j(u_j)$ where $\{\phi_j\} \subseteq \mathscr{D}_\mathbb{R}(\mathbb{R})$. Since $(\mathcal{F}\phi)(z) = \prod_{j=1}^d (\mathcal{F}\phi_j)(z_j)$ it follows by arguments as in [4], Theorem 4.1, that $\mathcal{F}(G)$ is dense in $\tilde{L}^2_\mathbb{C}(F)$. To show that $\mathcal{F}(\mathscr{E})$ is dense in $\tilde{L}^2_\mathbb{C}(F)$ it is hence enough to show that for all $\phi \in G$ there exists a sequence $\{\phi_n\} \subseteq \mathscr{E}$ such that $\mathcal{F}\phi_n \to \mathcal{F}\phi$ in $L^2_\mathbb{C}(F)$ which, however, follows by the above. □

Remark 3.7. *Theorem 3.5(2)–(3) underlines that it is easier for Λ_{dist} than for Λ_{func} to be a domain. As an illustration, consider the fractional Brownian sheet, which corresponds to X being Gaussian and $f(u) = \prod_{j=1}^d c_{H_j}|u_j|^{1-2H_j}$, where $H_1, \dots, H_d \in (0,1)$ and $c_{H_j} > 0$ are constants, see e.g. [10]. Since f satisfies* (3.7)*, Theorem 3.5 shows that Λ_{dist} is complete. Moreover, by Theorem 3.5 it follows that Λ_{func} is complete if and only if $H_1 = \dots = H_d = \frac{1}{2}$, that is, X is a Brownian sheet. In the case $d = 1$, where X is a fractional Brownian motion, a quite long proof of the non-completeness of Λ_{func} can be founded in Taqqu and Pipiras [7], Theorem 3.1, and the completeness of Λ_{dist} is shown by Jolis [5], Proposition 4.1.*

Remark 3.8. *Λ_{dist} is not always a domain. For instance, if $d = 1$ and $f(u) = e^{-u^2}$ then $g(u) = e^u$ belongs to $L^2(F)$ but $\int_\mathbb{R}|g(u)|(1+u^2)^{-k}\,du = \infty$ for all $k \in \mathbb{N}$. Hence, by Theorem 3.5(2) Λ_{dist} is not a domain.*

4 Stochastic integrands for processes white in time and colored in space

In the following we add a temporal component; that is, we consider processes indexed by $\mathbb{R}^{1+d}$ rather than $\mathbb{R}^d$. A generic element $u \in \mathbb{R}^{1+d}$ will be decomposed as $u = (t, x)$ where $t \in \mathbb{R}$ is time and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ a space variable. Intervals in $\mathbb{R}^{1+d}$ will be written either as $(u, v]$ or $(s, t] \times (x, y]$ where $(s, t]$ is an interval in $\mathbb{R}$ and $(x, y]$ is an interval in $\mathbb{R}^d$. Functions on $\mathbb{R}^{1+d}$ will often be denoted by $\varphi_t(x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, and $\mathcal{F}\varphi_t(z)$ denotes the Fourier transform in the space variable for fixed t.

Let F denote a symmetric measure on $\mathbb{R}^d$ with density f with respect to λ_d satisfying $f(x) = f(-x)$ for all $x \in \mathbb{R}^d$. Assume throughout that F satisfies condition (2.12). The measure $\lambda_1 \times F$ on $\mathbb{R}^{1+d}$ then satisfies (2.12) as well. Consider an $L^2_{\mathbb{R}}(\mathbb{P})$-continuous *Gaussian* process $X = \{X_u : u \in \mathbb{R}^{1+d}\}$ with wide-sense stationary increments and spectral measure $\lambda_1 \times F$. By Parseval's identity and (2.14) we have, for $s_i < t_i$ (in $\mathbb{R}$) and $x_i < y_i$ (in $\mathbb{R}^d$), $i = 1, 2$,

$$\begin{aligned} &\mathbb{E}[X((s_1, t_1] \times (x_1, y_1])X((s_2, t_2] \times (x_2, y_2])] \\ &\quad = 2\pi \int_{\mathbb{R}} \mathbf{1}_{(s_1,t_1]}\mathbf{1}_{(s_2,t_2]}\, d\lambda_1 \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(x_1,y_1]}\overline{\mathcal{F}\mathbf{1}_{(x_2,y_2]}}\, dF. \end{aligned} \tag{4.1}$$

Thus, there are independent increments in time, and the correlation in space is determined by F. That is, X is *white in time but colored in space*.

From now on we consider only time points in $\mathbb{R}_+$. Notice that in general $X((0, t] \times \cdot)$ may not extend to an $L^2_{\mathbb{R}}(\mathbb{P})$-valued measure defined on $\mathcal{B}_b(\mathbb{R}^d)$ as in [2]. As an example, if $d = 1$ and X is fractional in space we have $f(x) = C|x|^{1-2H}$ for $x \in \mathbb{R}$ (where $H \in (0, 1)$) for some constant $C > 0$. In this case X extends to a measure in space as in [2] if and only if $H > 1/2$ cf. Example 2.9. We shall see that one can nevertheless define a martingale integral with respect to X; moreover, we show that the set of integrands forms a complete space if and only if Λ_{dist} is complete.

For $t \geq 0$ let $Z_t = \{Z_t(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ denote the random spectral measure of $X((0, t] \times \cdot)$. Notice that by (4.1) the latter has spectral measure $2\pi t F$.

Define the filtration $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$ as

$$\mathcal{G}_t := \sigma\{X((0, s] \times (u, v]) : s \leq t, u \leq v\} \vee \mathcal{N}, \quad t \geq 0,$$

where $\mathcal{N}$ denotes the set of $\mathbb{P}$-null sets. A standard argument based on the stationary independent increments in X shows that $\mathcal{G}$ is right-continuous. For fixed u and v the process $\{X((0, t] \times (u, v]) : t \geq 0\}$ is a (non-standard) Brownian motion with respect to $\mathcal{G}$. By the last property in Theorem 2.7 it follows that

$$\mathcal{G}_t = \sigma\{Z_s(A) : s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\} \vee \mathcal{N}.$$

Let us first describe the basic properties of $Z = \{Z_t(A) : t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$. For $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ let

$$c^1(A, B) = \pi[F(A \cap B) + F(A \cap (-B))] \text{ and } c^2(A, B) = \pi[F(A \cap B) - F(A \cap (-B))].$$

Decompose $Z_t(A)$ into the real and imaginary parts as $Z_t(A) = Z^1_t(A) + iZ^2_t(A)$.

Proposition 4.1. *(1) The process Z is Gaussian and has stationary independent increments in the sense that $Z_t - Z_s = \{Z_t(A) - Z_s(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is independent of $\mathcal{G}_s$ and $Z_t - Z_s \overset{\mathscr{D}}{=} Z_{t-s}$ for all $0 \leq s < t$.*

(2) Let $A, B \in \mathcal{B}_b(\mathbb{R}^d)$. The processes $\{Z^1_t(A) : t \geq 0\}$ and $\{Z^2_t(B) : t \geq 0\}$ are independent Brownian motions with respect to $\mathcal{G}$. Moreover, $\{(Z^j_t(A), Z^j_t(B)) : t \geq 0\}$ is a bivariate Brownian motion with respect to $\mathcal{G}$ and $\mathbb{E}[Z^j_t(A)Z^j_t(B)] = c^j(A, B)t$ for $j = 1, 2$.

Proof. For $0 \le s < t$ let $Z_{s,t} = \{Z_{s,t}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ denote the random spectral measure of $X((s,t] \times \cdot)$. The processes Z_s and $Z_{s,t}$ are independent by independence of $X((0,s] \times \cdot)$ and $X((s,t] \times \cdot)$ and the last part of Theorem 2.7. Thus, $Z_s + Z_{s,t} = \{Z_s(A) + Z_{s,t}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure. Since, in addition,

$$\begin{aligned} X((0,t] \times (u,v]) &= X((0,s] \times (u,v]) + X((s,t] \times (u,v]) \\ &= \int \mathcal{F}\mathbf{1}_{(u,v]}\, d(Z_s + Z_{s,t}) \text{ for } u \le v \text{ in } \mathbb{R}^d, \end{aligned}$$

it follows by uniqueness of the random spectral measure that $Z_t - Z_s = Z_{s,t}$. Hence $Z_t - Z_s$ is Gaussian and independent of $\mathcal{G}_s$. Since Z has independent Gaussian increments it follows that Z is Gaussian. Finally, since $Z_t - Z_s$ and Z_{t-s} are Gaussian and both have control measure $2\pi(t-s)F$ they have the same law by (2.1)–(2.3). This concludes the proof of (1).

By (1), $\{(Z^1_t(A), Z^2_t(B)) : t \ge 0\}$ and $\{(Z^j_t(A), Z^j_t(B)) : t \ge 0\}$ ($j = 1,2$) are bivariate Brownian motions with respect to $\mathcal{G}$. Hence, since $Z^1_t(A)$ and $Z^2_t(B)$ are independent by (2.1), the first part of (2) follows. Similarly, (2.2) and (2.3) show that $\mathbb{E}[Z^j_t(A)Z^j_t(B)] = c^j(A,B)t$. ☐

It follows from Proposition 4.1 that the process $Z = \{Z_t(A) : t \ge 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is an orthogonal (and hence worthy) martingale measure in the sense of [12]. (In fact, the only difference compared to [12] is that we use complex martingales rather than real ones.) More precisely, we have the following:

(a) For $A \in \mathcal{B}_b(\mathbb{R}^d)$ $\{Z_t(A) : t \ge 0\}$ is a complex-valued continuous martingale with respect to $(\mathcal{G}_t)_{t \ge 0}$ with $Z_0(A) = 0$.

(b) For fixed $t \ge 0$ the mapping $A \mapsto Z_t(A)$ is σ-additive from $\mathcal{B}_b(\mathbb{R}^d)$ to $L^2_{\mathbb{C}}(\mathbb{P})$.

(c) For all $A, B \in \mathcal{B}_b(\mathbb{R}^d)$, $\langle Z_\cdot(A), Z_\cdot(B)\rangle_t = 2\pi t F(A \cap B)$.

Here, for two complex continuous square-integrable martingales M and N which are 0 at 0, $\langle M, N\rangle$ is the continuous complex process of bounded variation characterized by being 0 at 0 and $M\overline{N} - \langle M, N\rangle$ being a martingale.

Notice that (a) is immediate from Proposition 4.1(2) and (b) is simply the σ-additivity of the random spectral measure mentioned in Remark 2.2. Finally, using Proposition 4.1(2), it follows that $\langle Z^1_\cdot(A), Z^2_\cdot(B)\rangle_t = 0$ for all $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ and hence

$$\begin{aligned} \langle Z_\cdot(A), Z_\cdot(B)\rangle_t &= \langle Z^1_\cdot(A), Z^1_\cdot(B)\rangle_t + \langle Z^2_\cdot(A), Z^2_\cdot(B)\rangle_t \\ &= [c^1(A,B) + c^2(A,B)]t = 2\pi F(A \cap B)t, \end{aligned}$$

which is (c).

Denote by $\mathscr{P}$ the predictable σ-field on $\mathbb{R}_+ \times \Omega$. Set $\tilde{\mathscr{P}} := \mathscr{P} \times \mathcal{B}(\mathbb{R}^d)$ and

$$\begin{aligned} L^2_{\mathbb{C}}(Z) := \Big\{ \varphi : \varphi &\text{ is a } \tilde{\mathscr{P}}\text{-measurable mapping from } \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \text{ to } \mathbb{C} \\ &\text{satisfying } \mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\varphi_t(x)|^2 \, dt\, F(dx) \Big] < \infty \Big\}. \end{aligned}$$

This is clearly a complete space when equipped with the norm

$$\mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\varphi_t(x)|^2 \, dt\, F(dx) \Big]^{\frac{1}{2}}, \quad \varphi \in L^2_{\mathbb{C}}(Z).$$

Thus, also $\tilde{L}^2_{\mathbb{C}}(Z)$, the set of φ's in $L^2_{\mathbb{C}}(Z)$ satisfying $\varphi_t(x) = \overline{\varphi_t(-x)}$ for all (t,x) $\mathbb{P}$-a.s., is complete.

Since Z is a worthy martingale measure it induces a stochastic integral cf. [12], that is a unique continuous linear mapping

$$(\varphi \in L^2_{\mathbb{C}}(Z)) \mapsto \int \varphi \, dZ \in L^2_{\mathbb{C}}(\mathbb{P})$$

determined by

$$\int \varphi \, dZ = c(Z_{s_2}((u,v]) - Z_{s_1}((u,v]))\mathbf{1}_G$$

if

$$\varphi_t(\omega, x) = c\mathbf{1}_G(\omega)\mathbf{1}_{(s_1,s_2]}(t)\mathbf{1}_{(u,v]}(x) \tag{4.2}$$

for some $c \in \mathbb{R}$, $s_1 < s_2$ (in $\mathbb{R}$), $G \in \mathcal{G}_{s_1}$ and $u \leq v$ (in $\mathbb{R}^d$), and

$$\Big\| \int \varphi \, dZ \Big\|^2_{L^2_{\mathbb{C}}(\mathbb{P})} = \mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^d} |\varphi_t(x)|^2 \, dt \, F(dx) \Big] \quad \text{for } \varphi \in L^2_{\mathbb{C}}(Z).$$

The real-valued integral processes correspond to integrands in $\tilde{L}^2_{\mathbb{C}}(Z)$. For $\varphi \in L^2_{\mathbb{C}}(Z)$ the process $M^\varphi_t := \int_0^t \int_{\mathbb{R}^d} \varphi \, dZ$, $t \geq 0$, is by construction a complex square-integrable continuous martingale up to infinity which is 0 at 0; moreover,

$$\langle M^\varphi, M^\psi \rangle_t = \int_0^t \int_{\mathbb{R}^d} \varphi_s(y)\overline{\psi_s(y)} \, ds F(dy), \quad \text{for } \varphi, \psi \in L^2_{\mathbb{C}}(Z).$$

To define an integral with respect to X introduce the set

$$\begin{aligned} \Lambda_X = \Big\{ \varphi \colon \mathbb{R}_+ \times \Omega \to \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d) : \varphi \text{ is predictable, } \mathcal{F}\varphi_t(\omega) \text{ is a function} \\ \text{for all } (\omega, t) \text{, and } \mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\mathcal{F}\varphi_t(x)|^2 \, dt \, F(dx) \Big] < \infty \Big\}. \end{aligned}$$

On $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ we use the cylindrical σ-algebra $\sigma(\Psi \mapsto \Psi(\psi) : \psi \in \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d))$, that is, $\varphi \colon \mathbb{R}_+ \times \Omega \to \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ is predictable if and only if

$$((t, \omega) \in \mathbb{R}_+ \times \Omega) \mapsto \varphi_t(\omega)(\psi) \in \mathbb{C}$$

is predictable for all $\psi \in \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$. Furthermore, the following lemma shows that $\mathcal{F}\varphi_t(x)$ can be chosen bimeasurable making Λ_X well-defined.

Lemma 4.2. *Let $\varphi \colon \mathbb{R}_+ \times \Omega \to \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ be predictable such that $\mathcal{F}\varphi_t(\omega)$ is a function for all (ω, t). Then there exists a $\tilde{\mathscr{P}}$-measurable mapping $\Phi \colon \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{C}$ such that for all (t, ω), $\Phi(t, \omega, \cdot) = \mathcal{F}\varphi_t(\omega)(\cdot)$ λ_d-a.e.*

Proof. Since $\mathcal{F}$ maps $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ continuously into $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$

$$\Phi_\psi \colon \mathbb{R}_+ \times \Omega \to \mathbb{C}, \quad (t, \omega) \mapsto \int_{\mathbb{R}^d} (\mathcal{F}\varphi_t(\omega))(x)\psi(x) \, dx$$

is predictable, that is $\mathscr{P}$-measurable for all $\psi \in \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$. Hence by a Monotone Class Lemma (cf. e.g. II.3 in [8]) argument, Φ_ψ is predictable for all bounded measurable functions $\psi \colon \mathbb{R}^d \to \mathbb{C}$ with compact support. In particular, for all compact sets $K \subseteq \mathbb{R}^d$, the mapping $\mathbb{R}_+ \times \Omega \to L^1_{\mathbb{C}}(K) \colon (t, \omega) \mapsto \mathcal{F}\varphi_t(\omega)_{|K}$ is weakly measurable and hence (strongly) measurable by Pettis' theorem since $L^1_{\mathbb{C}}(K)$ is a separable Banach space. By applying [3], Exc. 1.75, there exists a $\tilde{\mathscr{P}}$-measurable mapping $\Phi_K \colon \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{C}$ such that for all (t, ω), $\Phi_K(t, \omega, \cdot) = \mathcal{F}\varphi_t(\omega)(\cdot)$ $\lambda_{d|K}$-a.e., which shows the existence of Φ since K was arbitrary and $\mathbb{R}^d$ is a countable union of compact sets. $\square$

Notice that Λ_X is Dalang's space $\overline{\mathcal{P}}$ considered in [2], page 9, with a few modifications: We consider the time interval $[0,\infty)$ rather than $[0,T]$ and as mentioned above our X does in general not induce a martingale measure. For $\varphi \in \Lambda_X$ define

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX := \int_0^\infty \int_{\mathbb{R}^d} (\mathcal{F}\varphi_t)(x)\, dZ.$$

By the above lemma, the integral is well-defined and maps Λ_X into $L^2_{\mathbb{R}}(\mathbb{P})$. On Λ_X define the norm $\|\cdot\|_{\Lambda_X}$ as

$$\|\varphi\|^2_{\Lambda_X} := \Big\| \int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX \Big\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} = \mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^d} |(\mathcal{F}\varphi_t)(x)|^2\, dt\, F(dx)\Big].$$

The integral with respect to X just defined extends the simple integral since if φ is given by (4.2) then, by definition,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX &= c\mathbf{1}_G\Big(\int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(x)\, Z_{s_2}(dx) - \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(x)\, Z_{s_1}(dx)\Big) \\ &= c\mathbf{1}_G\big(X((0,s_2]\times(u,v]) - X((0,s_1]\times(u,v])\big) = c\mathbf{1}_G X((s_1,s_2]\times(u,v]),\end{aligned}$$

where the second equality is due to (2.13). Moreover, if $\psi : \mathbb{R}_+ \to \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ is a deterministic measurable mapping, then the integral $\int_0^\infty \int_{\mathbb{R}^d} \psi\, dX$ exists if and only if $\psi \in \Lambda_X$, that is, $\mathcal{F}\psi_t$ is a function satisfying $\int_{\mathbb{R}^{1+d}} |\mathcal{F}\psi_t(x)|^2 dt F(dx) < \infty$. Thus, in view of the the first part of Theorem 4.3 below, this improves Theorem 3 in [2].

Theorem 4.3. *The real linear span of processes given by* (4.2) *is dense in* Λ_X. *Moreover, the following three statements are equivalent:*

(a) f *satisfies* (3.6)*,*

(b) Λ_X *equipped with the norm* $\|\cdot\|_{\Lambda_X}$ *is complete,*

(c) for every $\mathcal{G}_\infty$*-measurable random variable* $V \in L^2_{\mathbb{R}}(\mathbb{P})$ *there is a* $\varphi \in \Lambda_X$ *such that*

$$V = \mathbb{E}[V] + \int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX.$$

Proof. The first part: Using Lemma 3.3 and Lemma 4.2 it suffices to show that whenever $\varphi \in L^2_{\mathbb{C}}(Z)$ is of the form $\varphi_t(\omega,x) = \mathbf{1}_F(\omega)1_{(s_1,s_2]}(t)\psi(x)$, where $\psi \in \tilde{L}^2_{\mathbb{C}}(F)$ then there is a sequence $(\varphi_n)_{n\geq 1}$ in $L^2_{\mathbb{C}}(Z)$ of the form $\varphi_{n,t}(\omega,x) = \mathbf{1}_F(\omega)1_{(s_1,s_2]}(t)\mathcal{F}\psi_n(x)$ where $\psi_n \in \mathscr{E}$ (see (3.1)) approximating φ in Λ_X. However, this follows since by Lemma 3.6 the ψ_n's can be chosen such that $\mathcal{F}\psi_n$ approximates ψ in $L^2_{\mathbb{C}}(F)$.

(b) implies (a): Suppose that Λ_X is complete. To show that f satisfies (3.6) it is enough to show that Λ_{dist} is complete, cf. Theorem 3.5(2). Let therefore $(\psi^n)_{n\geq 1}$ be a Cauchy sequence in Λ_{dist} and set $\phi^n_t(\omega) = \mathbf{1}_{(0,1]}(t)\psi^n(\cdot) \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$. We note that ϕ^n are deterministic elements in Λ_X with $\mathcal{F}\phi^n_t = \mathbf{1}_{(0,1]}(t)\mathcal{F}\psi^n$ and

$$\|\phi^n - \phi^m\|_{\Lambda_X} = \|\psi^n - \psi^m\|_{\Lambda_{\mathrm{dist}}}, \quad n,m \geq 1. \tag{4.3}$$

Eq. (4.3) shows that $(\phi^n)_{n\geq 1}$ is a Cauchy sequence in Λ_X and hence convergent with a limit point ϕ. That is,

$$\mathbf{1}_{(0,1]}\mathcal{F}\psi^n \xrightarrow[n\to\infty]{} \mathcal{F}\phi \qquad \text{in } L^2_{\mathbb{C}}(\mathbb{P}\times\lambda_1\times F).$$

By Tonelli's Theorem there exist $\omega_0 \in \Omega$ and $s_0 \in (0,1]$ such that with $\psi = \phi_{s_0}(\omega_0)$ we have $\mathbf{1}_{(0,1]}\mathcal{F}\psi = \mathcal{F}\phi$ $(\mathbb{P}\times\lambda_1\times F)$-a.s. By definition of Λ_X, $\psi \in \Lambda_{\mathrm{dist}}$ and $\|\psi^n - \psi\|_{\Lambda_{\mathrm{dist}}} = \|\phi^n - \phi\|_{\Lambda_X} \to 0$, proving the completeness of Λ_{dist}.

(a) implies (c): Assume (3.6). Every $\psi \in \tilde{L}^2_{\mathbb{C}}(Z)$ is given by $\psi_t(\omega, x) = \mathcal{F}\rho_t(\omega, x)$ for some $\rho \in \Lambda_X$. Indeed, by disregarding a null set if necessary we may and do assume that $\psi_t(\omega) \in \tilde{L}^2_{\mathbb{C}}(F)$ for all (t, ω). By Theorem 3.5, $\mathcal{F}(\Lambda_{\text{dist}}) = \tilde{L}^2_{\mathbb{C}}(F)$ so we can use $\rho_t(\omega) := \mathcal{F}^{-1}\psi_t(\omega)$. Hence, it suffices to show that V can be written as

$$V = \mathbb{E}[V] + \int_0^\infty \int_{\mathbb{R}^d} \psi \, dZ \quad \text{for some } \psi \in \tilde{L}^2_{\mathbb{C}}(Z). \tag{4.4}$$

In the following fix $n \geq 1$ and $A_1, \ldots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ satisfying $A_j \cap (-A_j) = \emptyset$ for all j and $(A_1 \cup (-A_1)) \cap \ldots \cap (A_n \cup (-A_n)) = \emptyset$. Define M^j and N^j as

$$M^j_t = Z_t(A_j) \quad \text{and} \quad N^j_t = Z_t(-A_j).$$

Decompose M^j_t in the real and imaginary parts as $M^j_t = M^{1,j}_t + iM^{2,j}_t$. By Proposition 4.1 $M^{1,j}$ and $M^{2,j}$ are independent Brownian motions. Thus, if V^j is any real-valued square integrable random variable measurable with respect to the σ-algebra generated by $(M^{1,j}, M^{2,j})$ there are two $(\mathcal{G}_t)$-predictable processes $\alpha^{1,j}, \alpha^{2,j} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying

$$V^j = \mathbb{E}[V^j] + \int_0^\infty \alpha^{1,j}_t \, dM^{1,j}_t + \int_0^\infty \alpha^{2,j}_t dM^{2,j}_t \quad \mathbb{P}\text{-a.s.}$$

(These two processes are even predictable in the filtration generated by $(M^{1,j}, M^{2,j})$.) Using that, by Definition 2.1(2), $N^j_t = \overline{M^j_t}$, it is readily seen that the right-hand side equals

$$\mathbb{E}[V^j] + \int_0^\infty \beta^j_t \, dM^j_t + \int_0^\infty \overline{\beta^j_t} \, dN^j_t,$$

where $\beta^j_t = (\alpha^{1,j}_t - i\alpha^{2,j}_t)/2$. Thus,

$$V^j = \mathbb{E}[V^j] + \int_0^\infty \int_{\mathbb{R}^d} \varphi^j dZ,$$

where $\varphi^j_t(x) = \mathbf{1}_{A_j}(x)\beta^j_t + \mathbf{1}_{-A_j}(x)\overline{\beta^j_t}$. By the assumptions on the A_j's, the martingales $M^{1,j}, M^{2,j}$, $j = 1, \ldots, n$, are orthogonal, so Itô's formula implies

$$\prod_{j=1}^n V^j = \prod_{j=1}^n \mathbb{E}[V^j] + \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}^d} \Big(\prod_{k: k \neq j} M^{\varphi^k}_s \Big) \varphi^j_s(y) \, Z(ds, dy).$$

This gives (4.4) when $V = \prod_{j=1}^n V^j$ from which the general case follows using the Monotone Class Lemma.

(c) implies (b): Follows from completeness of $L^2_{\mathbb{R}}(\mathbb{P})$. □

References

[1] Aline Bonami and Anne Estrade, *Anisotropic analysis of some Gaussian models*, J. Fourier Anal. Appl. **9** (2003), no. 3, 215–236. MR-1988750

[2] Robert C. Dalang, *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s*, Electron. J. Probab. **4** (1999), no. 6, 29 pp. MR-1684157

[3] J. Hoffmann-Jørgensen, *Probability with a view toward statistics. Vol. I*, Chapman & Hall Probability Series, Chapman & Hall, New York, 1994. MR-1278485

[4] Kiyosi Itô, *Stationary random distributions*, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28** (1954), 209–223. MR-0065060

[5] Maria Jolis, *The Wiener integral with respect to second order processes with stationary increments*, J. Math. Anal. Appl. **366** (2010), no. 2, 607–620. MR-2600506

[6] A. Kolmogorov, *Kurven im Hilbertschen Raum, die gegenüber einer einparametrigen Gruppe von Bewegungen invariant sind*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **26** (1940), 6–9. MR-0003440

[7] Vladas Pipiras and Murad S. Taqqu, *Integration questions related to fractional Brownian motion*, Probab. Theory Related Fields **118** (2000), no. 2, 251–291. MR-1790083

[8] L. C. G. Rogers and David Williams, *Diffusions, Markov Processes, and Martingales. Vol. 1*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000, Foundations, Reprint of the second (1994) edition. MR-1796539

[9] Laurent Schwartz, *Théorie des distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée, Hermann, Paris, 1966. MR-0209834

[10] György Terdik and Wojbor A. Woyczyński, *Notes on fractional Ornstein-Uhlenbeck random sheets*, Publ. Math. Debrecen **66** (2005), no. 1-2, 153–181. MR-2128690

[11] J. von Neumann and I. J. Schoenberg, *Fourier integrals and metric geometry*, Trans. Amer. Math. Soc. **50** (1941), 226–251. MR-0004644

[12] John B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439. MR-876085

[13] A. M. Yaglom, *Certain types of random fields in n-dimensional space similar to stationary stochastic processes*, Teor. Veroyatnost. i Primenen **2** (1957), 292–338. MR-0094844

Acknowledgments. We thank the referee for an extremely constructive and detailed report. In particular, the original version of Theorem 2.7 was wrong as discovered by the referee. In addition, the referee suggested to add Example 2.6 and provided a very detailed list of comments and suggestions that has resulted in a much improved article.