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Bounds for the annealed return probability on large finite percolation graphs*

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Abstract

Bounds for the expected return probability of the delayed random walk on finite clusters of an invariant percolation on transitive unimodular graphs are derived. They are particularly suited for the case of critical Bernoulli percolation and the associated heavy-tailed cluster size distributions. The upper bound relies on the fact that cartesian products of finite graphs with cycles of a certain minimal size are Hamiltonian. For critical Bernoulli bond percolation on the homogeneous tree this bound is sharp. The asymptotic type of the expected return probability for large times t in this case is of order $t^{-3/4}$.

Keywords: Random walks; Annealed Return Probability; Critical Invariant Percolation; Anomalous Diffusion; Integrated Density of States; Number of open clusters per vertex.
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1 Introduction

1.1 Context and Results

This paper is about the expected return probability of the delayed random walk on the finite clusters of percolation graphs with heavy-tailed cluster size distributions (such as critical Bernoulli percolation).

The asymptotics of the integrated density of states (IDS) of the graph Laplacian on percolation subgraphs of the Euclidean lattice has recently been studied in the subcritical phase by Kirsch and Müller [19], and the supercritical phase by Müller and Stollmann [24]. The question of the IDS' asymptotics in the critical phase was left open. For the two-dimensional Euclidean lattice, we present upper and lower polynomial bounds (Theorem 2.6) for general invariant percolation. More generally, we find polynomial bounds for the expected return probability on finite critical percolation clusters on any planar transitive unimodular graph (Theorem 2.2). The upper estimates also hold in the non-planar case. For homogeneous trees, this bound proves to be sharp if the asymptotic type of decay of the cluster size' probability density function is known (Theorem

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2.4). Furthermore, improved bounds for the number of open clusters per vertex [13] in terms of the expected return probability are found (Theorem 2.7).

The method from which these bounds are derived are comparison theorems for random walks on finite graphs. For the upper bound, the main idea is the comparison of *all* the eigenvalues of the transition matrices. Taking into account the whole spectrum instead of only the spectral gap leads to an additional polynomially decreasing prefactor in front of the exponentially converging return probability. For the *expected* return probability an additional integration over all finite random clusters is involved. As in critical percolation, the corresponding cluster size distribution is heavy-tailed, i.e. integral moments do not exist [4]. The result is a polynomial decay in time. For this decay the additional prefactor is an essential improvement.

The comparison theorem is obtained from the property of cartesian products of finite graphs with maximum vertex-degree δ and cycles C of size equal to δ to be Hamiltonian [6]. This cycle exists due to Hamiltonicity. In addition to this fact, we will use that the return probability of a continuous time random walk on a finite cartesian product graph factorises into the return probabilities on its factors. Since the return probabilities are known on the cycle, this gives a bound for the return probability on the original graph. For the lower bound, we resort to a result by Boshier [8] about the isoperimetric number of a finite graph (see [23]): This is an upper bound for the isoperimetric number of graphs with bounded genus. For planar graphs, this gives us a bound of the spectral gap from above by Cheeger's inequality.

1.2 Delayed Random walk on finite graphs

We now recall some standard facts from finite random walk theory. We write $\mathbb{N}_0$ for $\{0, 1, 2, 3, ...\}$, and $\mathbb{R}_+ := [0, \infty)$. Since we will assume $|\mathcal{C}_o| < \infty$, we will reserve subscript 'o' for objects defined in connection with *finite* graphs.

Let $G_o = \langle V_o, E_o \rangle$ be a finite simple graph, i.e. the vertex-set V_o has **finite** cardinality and there are no multiple edges in E_o, nor are they directed or have coinciding incident vertices ('loops'). Let δ be the maximal occurring degree, i.e. $\delta := \max\{\ \deg(v) \mid v \in V_o\}$, where $\deg(v) = |\{w \in V_o \mid \{v, w\} \in E_o\}|$.

We define the discrete-time **delayed random walk (DRW)** on G_o to be the nearest neighbour random walk [28] with state space V_o, some initial distribution $\nu \in \mathcal{M}_{+,1}(V_o)$, and transition probabilities $P_{vw} := (P(G_o))_{vw}$ with $v, w \in V_o$, and

$$(P(G_o))_{vw} \;\; = \;\; \begin{cases} 1/\delta & \{v,w\} \in E_o, \\ 1 \; - \; \deg(v)/\delta & v = w, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the transition probabilities of v to w after n steps is given by the element of the matrix-power $(P^n)_{vw}$, for all $v, w \in V_o$.

The continuous-time version of the delayed random walk with coordinate-map X_t is defined as the Markov-process on the right-continuous V_o-valued functions depending on $t \in \mathbb{R}_+$, with some initial distribution $\nu \in \mathcal{M}_{+,1}(V_o)$, and transition probabilities

$$\mathbb{P}[X_t = w | X_0 = v] \;\; = \;\; \left(e^{-t(1-P)}\right)_{vw}, \qquad v, w \in V_o. \tag{1.1}$$

We note that $\left(e^{-t(1-P)}\right)_{vw} = \sum_{n=0}^{\infty} (P^n)_{vw} \frac{t^n}{n!} e^{-t}$, and that $(P^n)_{vw}$ is also the probability of X_t to reach w from v *conditioned* on the event of there having been exactly n jumps up to time t. The number $e^{-t}t^n/n!$ is the probability of that event, which is also characterised by $t \in [t_n, t_{n+1})$, where t_n is the sum of n independent exponentially distributed random variables ('waiting times') with parameter 1. So $\left(e^{-t(1-P)}\right)_{vw} = \sum_{n=0}^{\infty} \mathbb{P}[X_t = w | X_0 = v,\ t \in [t_n, t_{n+1})\,] \mathbb{P}[t \in [t_n, t_{n+1})]$, (see [25]).

Finally, we note that choosing the initial distribution $\nu \in \mathcal{M}_{+,1}(V_o)$ to be the **uniform distribution**, i.e. $X_0 \sim$UNIF(V_o), and $\nu(\{v\}) = 1/|V_o|$ gives the return probability as the value of a normalised trace

$$\mathbb{P}[X_t = X_0] \;=\; \sum_{v \in V_o} \left(e^{-t(\mathbb{I}-P)}\right)_{vv} \frac{1}{|V_o|} \;=\; \frac{1}{|V_o|} \mathrm{Tr}[e^{-t(\mathbb{I}-P)}], \tag{1.2}$$

as $\mathbb{P}[X_t = X_0] = \sum_{v \in V_o} \mathbb{P}[X_t = X_0 | X_0 = v] \mathbb{P}[X_0 = v]$, and $\mathbb{P}[X_0 = v] = \frac{1}{|V_o|}$.

1.3 Invariant percolation on unimodular graphs

We now define the setting for which the results of section 2.1 will be applied (see section 2.2).

Let $G = \langle V, E \rangle$ be an **infinite** simple (see above) graph, which has a transitive, unimodular subgroup Γ of the automorphism group Aut(G). 'Transitive' means vertex-transitive, here, i.e. for all $v, w \in V$, there is an automorphism $\gamma \in \Gamma$, s.t. $w = \gamma(v)$. 'Unimodular' means that the left Haar measure of Γ is the same as the right Haar measure. We call such a graph a **unimodular graph**.

A well-known result for unimodular graphs is the so called **mass-transport-principle** (see [22],[7]). It says that for all Γ-diagonally invariant functions ($f(\gamma(v), \gamma(w)) = f(v, w)$ for all $\gamma \in \Gamma$) it holds that

$$\sum_{w \in V} f(v, w) \;=\; \sum_{w \in V} f(w, v).$$

Let now $(\Omega, \mathcal{F}, \mu)$ be the probability space with $\Omega = 2^E$ the two-valued functions on the edges and $\mathcal{F} = \otimes_E \mathcal{F}_o$ the product $\sigma-$ algebra with $\mathcal{F}_o = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. On $\mathcal{F}$, we consider a probability distribution $\mu : \mathcal{F} \to [0,1]$ with the property of Γ-invariance:

$$\mu(A) \;=\; \mu(\gamma(A)), \qquad \text{for all } A \in \mathcal{F}, \gamma \in \Gamma.$$

In this way, for any fixed $\omega \in \Omega$, we obtain a random subgraph $G'(\omega) \leq G = \langle V, E \rangle$, of the form $G'(\omega) = \langle V, E'(\omega) \rangle$, where

$$E'(\omega) \;=\; \{\ e \in E \mid \omega(e) = 1\ \} \;=\; \omega^{-1}(\{1\}).$$

A subgraph of G in which only edges are removed is called a *partial graph* of G. Therefore, with every $\omega \in \Omega$, we associate the random partial graph

$$G'(\omega) = \langle V, \omega^{-1}(\{1\}) \rangle.$$

We call the pair $\langle G, \mu \rangle$ an **invariant percolation μ on a unimodular graph G**.

We will now fix an arbitrary vertex $o \in V$, the 'root', and look for fixed $\omega \in \Omega$ at the connected component of the graph $G'(\omega)$ which contains o, and call it $\mathcal{C}_o(\omega)$. Since we will assume $|\mathcal{C}_o| < \infty$, we will be interested in invariant percolation measures μ with μ-almost surely finite connected components, i.e.

$$\mu(\{\omega \in \Omega \mid |\mathcal{C}_o(\omega)| < \infty\}) \;=\; 1.$$

Examples: a.) *Bernoulli Percolation on the Euclidean Lattice*: $G = \langle \mathbb{Z}^d, N.N. \rangle$ ('Nearest Neighbours'), and μ is the product measure on Ω: $\mu = \otimes_{e \in E} \pi_e$, where $\pi_e : \mathcal{F}_o \to [0,1]$, and $p = \pi(\omega(e) = 1) \in [0,1]$, for all $e \in E$. It is well-known that for sufficiently small p, the connected components are a.s. finite ('subcritical regime'). Also, in the 'supercritical regime' or the 'critical regime', for which $\mu(|\mathcal{C}_o| = \infty) \;>\; 0$, we may condition on the event $A := \{\omega \in \Omega \mid |\mathcal{C}_o| < \infty\}$. The conditional measure $\mu(\cdot|A) = \mu(\cdot \cap A)/\mu(A)$ is also Γ-invariant. It is a celebrated result that Bernoulli bond-percolation has almost surely finite clusters in the case $d = 2$.

b.) *Bernoulli Percolation on homogeneous trees*. The Bernoulli percolation measure μ on a homogeneous tree of degree δ is invariant under the action of any transitive subgroup of its automorphism group. It is well-known (see [12], Chap. 10.1), that for critical percolation on the binary tree, we have that the $\mathbb{P}_\mu[|\mathcal{C}_o| \geq m] \sim m^{-\frac{1}{2}}$.

Now, we define the delayed random walk on a random partial graph: Given $\omega \in \Omega$, consider the finite subgraph of $G'(\omega)$ induced by $\mathcal{C}_o(\omega)$, i.e. (using a standard notation) consider

$$G_o(\omega) \;\;:=\;\; G'(\omega)|\mathcal{C}_o(\omega).$$

As discussed in Section 1.2 this induces a random finite random walk with random state space $\mathcal{C}_o(\omega)$, random initial distribution $\nu^{(\omega)} \in \mathcal{M}_{+,1}(\mathcal{C}_o(\omega))$, and corresponding random return probabilities

$$P^{(\omega)}_{vw} := \left(P(G_o(\omega)\right)_{vw}, \tag{1.3}$$

which form a $|\mathcal{C}_o(\omega)| \times |\mathcal{C}_o(\omega)|$ matrix, where $|\mathcal{C}_o(\omega)|$ is μ-a.s. finite.

The random continuous-time random walk is formed analogously to the procedure of section 1.2, with $G_o = G_o(\omega)$. Choosing $\nu^{(\omega)} \in \mathcal{M}_{+,1}(\mathcal{C}_o(\omega))$as the initial distribution of the process to be the **uniform distribution** on $\mathcal{C}_o(\omega)$, the random continuous-time return probabilities turn out to be (compare with (1.2))

$$\mathbb{P}[X^{(\omega)}_t = X^{(\omega)}_0] \;\;=\;\; \frac{1}{|\mathcal{C}_o(\omega)|}\mathrm{Tr}[e^{-t(\mathbb{I}-P(\omega))}],$$

where $P(\omega) = ((P^{(\omega)})_{vw})$ with $v, w \in \mathcal{C}_o(\omega)$ is the transition probability matrix (1.3) of the random discrete-time random walk on $\mathcal{C}_o(\omega)$.

We are interested in the asymptotic corrections of the expectation value of the return probabilities

$$P_t(o) \;\;:=\;\; \mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\mathrm{Tr}[e^{-t(\mathbb{I}-P)}]\right]$$

for large values of the time $t > 0$ from its limiting value, which is given by $\mathbb{E}_\mu[1/|\mathcal{C}_o|]$.

2 Results

We first present our estimates for finite graphs in Section 2.1, and apply them in Section 2.2 to bound the expected return probability. Sections 2.3 and 2.4 contain the applications concerning the integrated density of states and the expected number of open clusters per vertex.

2.1 Bounds of the Return Probability on finite graphs

Theorem 2.1. *Let $G_o = \langle V_o, E_o\rangle$ be a simple, finite, connected graph with N vertices and largest degree δ. Let X_t be the delayed random walk on G_o, and β_2 the second-largest eigenvalue of its transition kernel. For $X_0 \sim$ UNIF(V_o), $k \in \{1, ..., N-2\}$ and $t > 0$*

$$i.)\quad \mathbb{P}[X_t = X_0] \;\leq\; \frac{1}{N} + 2\cdot\frac{k}{N}\,e^{-t(1-\beta_2)} + \sqrt{\frac{\pi}{32}}\frac{\delta\sqrt{\delta+2}}{\sqrt{t}}\exp\left(-\frac{32tk^2}{(\delta+2)\delta^2N^2}\right),$$

$$ii.)\quad \mathbb{P}[X_t = X_0] \;\leq\; \frac{1}{N} + 2\cdot\frac{k}{N}\,e^{-t(1-\beta_2)} + \frac{\delta^2(\delta+2)}{16t}\frac{N}{k}\exp\left(-\frac{32tk^2}{(\delta+2)\delta^2N^2}\right).$$

iii.) If G_o is also planar, and $N > 288$, it holds for $t > 0$

$$\mathbb{P}[X_t = X_0] \;\geq\; \frac{1}{N} + \frac{\exp\left(-tK/\sqrt{N}\right)}{N}, \quad \textit{with } K = 12\sqrt{2}\cdot\delta.$$

These bounds allow choosing an optimal value of k if something about the relation between β_2 and N is known. If k in Theorem 2.1, i.) and ii.) is of the order of N, the bound is qualitatively the same as the obvious estimate resulting from using the Poincaré inequality $1-\beta_j \geq \delta/(4N^2)$ for all $j \in \{2, ..., N\}$ (see [26], Chap. 3.2).

2.2 Annealed Return Probability on finite Percolation Subgraphs

Let $P_t := P_t(o) = \mathbb{E}_\mu\mathbb{P}[X_t = o \mid X_0 = o]$ denote the expected return probability to the vertex o of the continuous-time delayed random walk on $\mathcal{C}_o(\omega)$ at time $t \geq 0$.

Theorem 2.2. *For μ being any invariant percolation on a unimodular transitive graph $G = \langle V, E\rangle$, let $A, B, a, b > 0$, with $b \leq 2$ such that for all $m \in \mathbb{N}$*

$$A\,m^{-a} \;\leq\; \mathbb{P}_\mu[|\mathcal{C}_o| \geq m] \;\leq\; B\,m^{-b}. \tag{2.1}$$

i.) Then with $C = 5\,(4b/\delta)^b\,(2\cdot 4^b + \delta(\delta+2)/2)$, for all α with $0 < \alpha < b$ and $t > 0$

$$P_t - \mathbb{E}_\mu\left[\tfrac{1}{|\mathcal{C}_o|}\right] \;\leq\; C\cdot\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha]\,t^{-\frac{1}{2}(1+\alpha)}.$$

ii.) If G is also assumed planar, and K as in Theorem 2.1, then for $t > \sqrt{288}$

$$D\cdot t^{-2a(1+1/b)} \;\leq\; P_t - \mathbb{E}_\mu\left[\tfrac{1}{|\mathcal{C}_o|}\right], \qquad \textit{where } D = e^{-K}\frac{A/2}{1+(2B/A)^{1/b}}.$$

Remarks: The folklore rule about easily obtained lower bounds doesn't apply in this general setting of transitive graphs. The quality of the argument of comparing the graph with the 'host graph' G on which the percolation is defined (see e.g. Lemma 2.2 in [11]) generally gives poor results. If for example $\mathcal{C}_o$ is the finite connected component containing the root of Bernoulli percolation on a homogeneous tree with vertex-degree δ, then the subtree of the homogeneous tree induced by a ball with radius equal to that of $\mathcal{C}_o$ has typically a much *smaller* spectral gap. Thus, it cannot be used for *lower* bounds of the return probability. In the case of amenable graphs, however, this comparison technique is successful (see e.g. [3] for results beyond the Euclidean lattice). Furthermore, as upper bounds on the volume-growth give lower bounds on the return probability (see e.g. [28], Chap. 14.C), the lack of such a bound on the volume-growth under the present assumptions comes at the cost of weaker results in Theorem 2.2, ii.).

Nevertheless, from the following discussion it will be seen that it is for tree-like graphs G, for which the upper bounds perform well. The upper bounds turn out better if few manipulations of the finite graph in form of removals and additions of edges have to be undertaken to retrieve a spanning cycle (we say that the graph is *similar* to the spanning cycle). The proof (see Section 3.2) involves the comparison of the graph with that of a cycle having length comparable to the graph's order (number of vertices). An example of graphs for which this property may be likely to prevail is given by finite subgraphs of the *incipient infinite cluster* of Bernoulli percolation (see [12], Chap. 9.4). It occurs at the critical retention probability p_c under the additional condition of being infinite [17]. It is therefore of interest to compare the expected return probability of the delayed random walk on the incipient infinite cluster with the corresponding quantity on the ordinary connected components of critical percolation to which Theorem 2.2 can be applied, as long as it has clusters at criticality which are almost surely finite ([16], Theorem 2; [7]):

Corollary 2.3. *Consider P_t, the expected return probability of the delayed random walk on finite percolation clusters of critical Bernoulli bond percolation:*

i.) For the 2-dimensional Euclidean lattice, with $\alpha \in (0, 1/5]$ such that $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$, there is $C_2 > 0$ such that for $t \geq 1$

$$C_2^{-1} t^{-(1+\alpha^{-1})} \quad \leq \quad P_t \;-\; \mathbb{E}_\mu[1/|\mathcal{C}_o|] \quad \leq \quad C_2\, \mathbb{E}_\mu[|\mathcal{C}_o|^\alpha]\, t^{-\frac{1}{2}(1+\alpha)}.$$

ii.) For the homogeneous tree of degree δ, there is $\epsilon > 0$, and a constant C_δ depending on ϵ, such that for $t \geq 1$

$$C_\delta(\epsilon)^{-1} t^{-3} \quad \leq \quad P_t \;-\; \mathbb{E}_\mu[1/|\mathcal{C}_o|] \quad \leq \quad C_\delta(\epsilon)\, \mathbb{E}_\mu[|\mathcal{C}_o|^{\frac{1}{2}-2\epsilon}]\, t^{-\frac{3}{4}+\epsilon}. \tag{2.2}$$

Remarks: It is easy to show that given $b > 0$, the condition $\mathbb{P}_\mu[|\mathcal{C}_o| \geq m] \;\leq\; B\, m^{-b}$ for some $B > 0$ implies $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$ for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < b$.

The upper bound for the range of α in Corollary 2.3, i.) is a result by Kesten [18] (see also [12], Table 10.1). The results obtained in Theorem 2.2 are valid for the very general setting of any invariant percolation on a unimodular transitive graph G, and therefore their quality varies strongly depending on the structure of G and the type of percolation measure μ. The $\alpha \leq 1/5$ condition implies that the upper bound Corollary 2.3 i.) for P_t isn't stronger than $\sim t^{-2/3}$, which would distinguish DRW on the finite critical percolation cluster from the incipient infinite cluster if the Alexander-Orbach conjecture would be true. However, it isn't believed that the this conjecture holds for the Euclidean lattice in dimensions $d \leq 6$ ([15], Chap. 7.4.4).

The situation with Corollary 2.3, ii.) is different. It is clear from Lemma 1.6 of [27], that DRW and the simple random walk SRW on any finite subgraph of an infinite graph of polynomial growth have the same decay-type of the expected or quenched return probability, as long as the maximum vertex-degree is uniformly bounded. Kozma and Nachmias (see Theorem 1.2 and 1.3 in [21]) have shown that the volume growth of the incipient infinite cluster in high dimensional Euclidean lattices is almost surely polynomial. The same follows from Lemma 2.2 of Barlow and Kumagai [5] for homogeneous trees. Both of these cases are percolation models on transitive graphs with uniformly bounded vertex-degree. Corollary 2.3 is therefore interesting when compared with the results obtained in [21] and [5] for the asymptotics of the simple random walk on the incipient infinite cluster on trees. It is proved there that the expected return probability is - regardless of the degree δ - of the order of $t^{-2/3}$. (That the so called spectral dimension $-2\lim_n \log \mathbb{P}_o[X_n = o]/\log n$ is equal to $-4/3$ is known as the Alexander-Orbach conjecture [2], proven for homogeneous trees [5], and Euclidean lattices for high dimensions [21].) Since (2.2) represents an upper bound for $P_t - \mathbb{E}_\mu[1/|\mathcal{C}_o|]$ that can be chosen to have an exponent arbitrarily close to $-3/4$, it proves that the expected return probability at criticality on ordinary finite percolation clusters displays a different asymptotic decay towards its limit than on the incipient infinite cluster.

We expect the upper bound Theorem 2.2, i.) to be a good approximation when G is similar to a homogeneous tree and $\mathbb{P}_\mu[|\mathcal{C}_o| = m]$ is polynomially decreasing in m:

Theorem 2.4. *Let G be the homogeneous tree of degree δ and μ an invariant percolation on G obeying assumption (2.1), and $A \leq \mathbb{P}_\mu[|\mathcal{C}_o| = m]m^{a+1}$ for all $m \in \mathbb{N}$. Then there is $c > 0$, such that for all $t > 0$*

$$P_t(o) - \mathbb{E}_\mu[1/|\mathcal{C}_o|] \;\; \geq \;\; c\, t^{-\frac{1}{2}(1+a)}. \tag{2.3}$$

We conclude the discussion of our results by the following tight estimate for independent percolation on the homogeneous tree:

Corollary 2.5. *For critical Bernoulli bond percolation on the homogeneous tree*

$$\lim_{t\to\infty} \frac{\log(\, P_t(o) \; - \; \mathbb{E}_\mu[1/|\mathcal{C}_o|]\,)}{\log t} \;\; = \;\; -\frac{3}{4}.$$

These findings allow to conclude that the observation by Kirsch and Müller [19] of the predominance of path-like clusters also determines the asymptotics of critical percolation in the present case. The difference of (2.3) over subcritical percolation considered in [19] consists of the necessity to include, in addition to the ‘linear’ clusters [see Remark 1.15, iii.) in [19]), the larger class of clusters $\mathcal{C}_o$ which have diameters D comparable to the cluster’s size $|\mathcal{C}_o|$.

The fact that path-like clusters are the dominating structures for the large-time asymptotics in the case of trees is also illustrated by the following ‘heuristic’, but wrong argument: Suppose that for a given realisation $\omega \in \Omega$ at time $t > 0$, the cluster size $|\mathcal{C}_o|$ is larger than $t^{2/3}$. One might guess that up to this time the Markov chain hasn’t equilibrated and this cluster contributes significantly in the averaging over $\mathbb{P}[X_t = X_0] - 1/|\mathcal{C}_o|$. For times larger than $t^{2/3}$ one then assumes that $\mathbb{P}[X_t = X_0] \sim 1/|\mathcal{C}_o|$. Assuming further that up to the time of equilibriation the return probability on the finite cluster typically decays just like on the incipient infinite cluster, namely like $t^{-2/3}$ (see [5], Theorem 1.4), then by using $\mathbb{P}_\mu[|\mathcal{C}_o| = m] \sim m^{-3/2}$, one arrives at the following rough estimate:

$$P_t(o) - \mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\right] \;\sim\; \sum_{m \geq t^{2/3}} \left(t^{-2/3} - \frac{1}{m}\right) m^{-3/2} \;\sim\; t^{-1},$$

where the first $\sim$ (meaning ‘of this order, for large t’) follows from assuming that only unsignificant terms are neglected. This, however, contradicts Corollary 2.5.

The reason for restricting the considered clusters to sizes of at least $t^{2/3}$ in this argument comes from the idea that because the characteristic asymptotic decay of the random walk on the incipient infinite cluster is $t^{-2/3}$ the random walk on smaller clusters will have already reached equilibrium, and all of the significant contributions to $(t^{-2/3} - 1/|\mathcal{C}_o|)_+$ are accounted for. It is therein implicitly assumed, that the typical decay on clusters of smaller size before equilibriation is also $\sim t^{-2/3}$. However, the path-like clusters have a characteristic heat-kernel decay towards $1/|\mathcal{C}_o|$ of order $t^{-1/2}$ instead of $t^{-2/3}$ (see Part ii. and iii. in the proof of Theorem 2.4). And so our result shows that the regime of cluster sizes between $t^{1/2}$ and $t^{3/2}$ plays the dominant part in the averaging for Bernoulli percolation on the homogeneous tree.

The reason why these contributions are not relevant in the case of the infinite incipient cluster follows from the results of Barlow and Kumagai [5]: According to their Lemmata 2.2 and 2.3 the incipient infinite cluster on homogeneous trees has realisations which, if restricted to subtrees with radius n, typically have a size of order n^2 , so that the diameter ($\sim n$) is never a positive fraction of the cluster size. From this it becomes apparent that the main characteristic responsible for the stronger decay of the upper bounds in Theorem 2.2 is the existence of a significant fraction of finite clusters with diameter comparable to their size.

2.3 Integrated density of states for $\mathbb{Z}^2$

Let μ be an invariant bond percolation on the 2-dimensional Euclidean lattice $G = \langle \mathbb{Z}^2, N.N.\rangle$ with a μ-a.s. finite percolation cluster $\mathcal{C}_o$ having a size distribution obeying (2.1). Let $\alpha \in (0,2)$ such that $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$.

Let $N(E)$ be the **integrated density of states** (IDS) of the graph Laplacian $L(\omega)$ belonging to the percolation subgraphs $G'(\omega)$. This means for $\Lambda_N = \{-N+1, ..., N\}^2 \subset \mathbb{Z}^2$ the limit

$$N(E) \quad = \quad \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \#\{\lambda \text{ eigenvalue of } L_{\Lambda_N}(\omega) \le E\}$$

exists, where $L_{\Lambda_N}(\omega)$ is the graph Laplacian of the finite induced subgraph $G'(\omega)|\Lambda_N$ (see e.g. [19], Lemma 1.12).

Theorem 2.6. *There is $C_3 > 0$, s.t. for $E > 0$ sufficiently small the integrated density of states $E \mapsto N(E)$ of the graph Laplacian obeys*

$$C_3^{-1} \frac{E^{1+1/\alpha}}{(\log 1/E)^{1+1/\alpha}} \quad \le \quad N_N(E) \ - \ N_N(0) \quad \le \quad C_3\, E^{\frac{1}{2}(1+\alpha)}.$$

Remarks: This shows that independently of the vertex-degree δ, the type of the asymptotics of these bounds for small values of $E > 0$ is polynomial and only depends on the decay of the cluster size distribution. By comparison with Theorem 1.14 of [19], by which for subcritical percolation on the Euclidean lattice in any dimension ($d = \delta/2$)

$$\exp(-\alpha_-/\sqrt{E}) \ \le \ N(E) \ - \ N(0) \ \le \ \exp(-\alpha_+/\sqrt{E}),$$

for some $\alpha_-, \alpha_+ > 0$, and $E > 0$ sufficiently close to zero, it is seen that observation of the type of asymptotics of the IDS for small energies suffices to decide about whether the finite random cluster of the origin is generated with a critical, or subcritical percolation measure.

2.4 Number of open clusters per vertex

A central theme in percolation theory on the Euclidean lattice $G = \langle \mathbb{Z}^d, N.N.\rangle$ is the so called **number of open clusters per vertex**. Given a finite box $\Lambda_N = \{-N + 1, ..., N\}^d$, and the number $M_N(\omega)$ of connected components of the induced subgraph $G'(\omega)|\Lambda_N$, the μ-a.s. existence of the limit

$$\kappa(p) = \lim_{N\to\infty} \frac{M_N(\omega)}{|\Lambda_N|}$$

and its almost sure independence of $\omega \in \Omega$ has been shown by Grimmett [13]. Its value equals $\kappa(p) \;=\; \mathbb{E}_\mu[1/|\mathcal{C}_o|]$ (see [12], (4.18)). Note that the number $1/\mathcal{C}_o(\omega)$ is the value of the density of the uniform distribution on $\mathcal{C}_o(\omega)$.

In [13] there are upper and lower bounds for $\kappa(p)$ (there it is defined by $\mathbb{E}_\mu[1/|\mathcal{C}_o|] - \mu[|\mathcal{C}_o| = 1]$) in the case of Bernoulli percolation on the Euclidean lattice. They entail expansions which are converging slowly in the regime of the retention probability p being close to the critical value. We present the consequences of our bounds in terms of the expected cluster size $\chi(p) = \mathbb{E}_\mu[|\mathcal{C}_o|]$:

Theorem 2.7. *Let μ be subcritical Bernoulli bond percolation on the d-dimensional Euclidean lattice $G = \langle \mathbb{Z}^d, N.N.\rangle$ with almost surely finite connected components. Let $\chi(p) = \mathbb{E}_\mu[|\mathcal{C}_o|]$. Then, for $t > 0$*

$$P_t \;-\; c\,\frac{\chi(p)}{t} \quad\leq\quad \kappa(p) \quad\leq\quad P_t\,. \tag{2.4}$$

with $c = \min\{\, \frac{1}{2}(d^3 + d^2 + 4),\;\; \frac{20}{d}(4 + d(d+1))\}$.

Remarks: The power of the method for the upper bound (mainly due to Lemma 3.3) becomes visible if one compares Theorem 2.7 with the simple bound obtained by using Poincaré's inequality for λ, together with $\lambda \leq 1 - \beta_j$, for $j \geq 2$: In this case $P_t \;-\; \kappa(p) \;\;\leq\;\; \mathbb{E}_\mu[e^{-t\delta/(4|\mathcal{C}_o|^2)}$ instead of (2.4) which yields for $t > 0$

$$P_t \;-\; \frac{2}{d}\cdot\frac{\mathbb{E}_\mu[|\mathcal{C}_o|^2]}{t} \quad\leq\quad \kappa(p) \quad\leq\quad P_t.$$

The constant in front of the term t^{-1} includes the second moment of the cluster size, while in (2.4) only the first moment appears.

3 Proofs

3.1 Auxiliary results

The proofs rest on the theory of infinite unimodular transitive graphs [7]. 'Unimodularity' of a graph refers to the existence of a vertex-transitive subgroup of the automorphism group of the graph.

Lemma 3.1. *Let G be an infinite unimodular vertex-transitive graph, an μ an invariant percolation measure on G. If $\mathbb{E}_\mu$ refers to the integration of the expected value over all partial graphs $\omega \in \Omega$,*

$$\mathbb{E}_\mu\left[\mathbb{P}_o[X_t \;=\; o]\right] \quad=\quad \mathbb{E}_\mu\left[\mathbb{P}[X_t \;=\; X_0]\right]. \tag{3.1}$$

Proof: (see [27] for a detailed discussion) Let $\mathcal{C}_v$ be the connected component of $H(\omega)$ containing the vertex $v \in V$. Since the Euclidean lattice is a graph with a unimodular group of automorphisms, by the mass-transport-principle [7], [22], the left-hand side of (3.1) equals

$$\sum_{v\in V} \mathbb{E}_\mu \left[\mathbb{P}_o[X_t \;=\; o] \frac{\chi_{\{v\in\mathcal{C}_o\}}}{|\mathcal{C}_o|} \right] = \sum_{v\in V} \mathbb{E}_\mu \left[\mathbb{P}_v[X_t \;=\; v] \frac{\chi_{\{o\in\mathcal{C}_v\}}}{|\mathcal{C}_v|} \right] = \sum_{v\in V} \mathbb{E}_\mu \left[\mathbb{P}_v[X_t \;=\; v] \frac{\chi_{\{v\in\mathcal{C}_o\}}}{|\mathcal{C}_o|} \right]$$

since $v \in \mathcal{C}_o \;\Leftrightarrow\; o \in \mathcal{C}_v$, which equals the right-hand side of(3.1). $\square$

Lemma 3.2. *For $N > 3$ and $k \in \{1, ..., N-2\}$, let $I_t(k,N) := \sum\limits_{j=k+1}^{N-1} e^{-t(1-\cos\pi\frac{j}{N})}$. Then*

$$i.) \qquad I_t(k,N) \quad \leq \quad \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{N}{\sqrt{t}}\, e^{-2t\frac{k^2}{N^2}}, \qquad \textit{and} \tag{3.2}$$

$$ii.) \qquad I_t(k,N) \quad \leq \quad \frac{1}{2}\frac{N^2}{kt}\, e^{-2t\frac{k^2}{N^2}}. \tag{3.3}$$

Proof: From $\cos\pi x \leq 1 - 2x^2$, if $x \in [0,1]$, we obtain by following [26], (Ex. 2.1.1)

$$\begin{aligned} \sum_{j=k+1}^{N-1} e^{-t(1-\cos\pi\frac{j}{N})} \quad &\leq \quad \int_k^\infty e^{-2t\frac{x^2}{N^2}}\,dx \;=\; \frac{N}{\sqrt{2t}} \int_{\sqrt{2t}k/N}^\infty e^{-y^2}\,dy \\ &\leq \quad \frac{N}{\sqrt{2t}} e^{-2t\frac{k^2}{N^2}} \int_0^\infty e^{-y^2 - 2\sqrt{2t}\frac{k}{N}y}\,dy \;\leq\; \frac{N}{\sqrt{2t}} e^{-2t\frac{k^2}{N^2}} \int_0^\infty e^{-y^2}\,dy, \end{aligned} \tag{3.4}$$

which proves (3.2). Moreover, we have

$$\int_z^\infty e^{-u^2}\,du \;=\; \frac{1}{2}\int_{z^2}^\infty e^{-y}\frac{dy}{\sqrt{y}} \;=\; \frac{1}{2}\int_0^\infty e^{-(y+z^2)}\frac{dy}{\sqrt{y+z^2}} \;\leq\; \frac{e^{-z^2}}{2z}\int_0^\infty e^{-y}\,dy.$$

Applying this inequality to the right-hand side of (3.4) with $z = \frac{\sqrt{2t}k}{N}$ gives (3.3). $\square$

Lemma 3.3. *Let $\widehat{G} = G_X \Box G_Y$ be the cartesian product of the simple, connected, finite graphs G_X, G_Y. Let $\widehat{X}_t$ be the continuous-time delayed random walk on $\widehat{G}$ with uniform initial distribution on the vertices of $\widehat{G}$. Let X_t and Y_t be the continuous-time delayed random walk on G_X and G_Y, also with uniform initial distribution on the vertex-sets of G_X and G_Y, respectively . Then*

$$\mathbb{P}[\,\widehat{X}_{2t} \;=\; \widehat{X}_0\,] \;=\; \mathbb{P}[\,X_t \;=\; X_0\,] \cdot \mathbb{P}[\,Y_t \;=\; Y_0\,].$$

Proof: Let $N = |V(G_X)|$, and $M = |V(G_Y)|$. Let P_X and P_Y be the transition kernels of X_t and Y_t, respectively. For the delayed random walk on $\widehat{G}$, with equal transition weights across edges of type $\{\langle x,v\rangle, \langle y,v\rangle\}$, and $\{\langle x,v\rangle, \langle x,w\rangle\}$ (where $x, y \in V(G_X)$, and $v, w \in V(G_Y)$), the transition kernel is given by $\frac{1}{2}(P_X \otimes \mathbb{I} \,+\, \mathbb{I} \otimes P_Y)$ (see [28], Chap. 18). Therefore,

$$\begin{aligned} \mathbb{P}[\widehat{X}_{2t} = \widehat{X}_0] \quad &= \quad \frac{1}{N\cdot M}\mathrm{Tr}[e^{-2t(\mathbb{I}-\frac{1}{2}(P_X\otimes\mathbb{I} \,+\, \mathbb{I}\otimes P_Y))}] = \frac{1}{N\cdot M}\mathrm{Tr}[e^{-t(\mathbb{I}-P_X)} \otimes e^{-t(\mathbb{I}-P_Y)}] \\ &= \quad \frac{1}{N}\mathrm{Tr}[e^{-t(\mathbb{I}-P_X)}]\;\frac{1}{M}\mathrm{Tr}[e^{-t(\mathbb{I}-P_Y)}] = \mathbb{P}[X_t = X_0]\;\mathbb{P}[Y_t = Y_0]. \qquad \square \end{aligned}$$

Remark: This auxiliary result can also be derived by using the fact that the sum of two independent Poisson processes is also a Poisson process, however with rate equal to the sum of the two components' rates (see e.g. [25], Theorem 2.4.4).

Lemma 3.4. *Let $\phi : \mathbb{N}_0 \to \mathbb{R}_+$, s.t. $\sum_{k=0}^{\infty} \phi(k) = 1$ with $\Phi(m) := \sum_{k=m}^{\infty} \phi(k)$. Let there exist $A, B, a, b \in \mathbb{R}_+$ such that $\frac{A}{m^a} \leq \Phi(m) \leq \frac{B}{m^b}$ for all $m \in \mathbb{N}$. Then*

$$\sum_{k=m}^{\infty} \frac{1}{k}\,\phi(k) \;\geq\; \frac{C}{m^{a(1+1/b)}}, \qquad \textit{with} \quad C = \frac{(A/2)^{1-1/b}}{B^{1/b}}.$$

Proof:

$$\sum_{k=m}^{\infty} \frac{1}{k}\,\phi(k) \;\geq\; \sum_{k=m}^{L} \frac{1}{k}\,\phi(k) \;\geq\; \frac{1}{L}\left(\Phi(m) \,-\, \Phi(L+1)\right) \;\geq\; \frac{1}{L}\left(\frac{A}{m^a} \,-\, \frac{B}{(L+1)^b}\right).$$

We set $\tilde{L} > 0$ to be the real value L, such that the parentheses on the right-hand side are exactly $\frac{1}{2} \cdot A/m^a$, i.e. $\tilde{L} := \left(\frac{2B}{A}\right)^{1/b} m^{a/b}$. Now, by defining $L_- := \lfloor \tilde{L} \rfloor$ and $L_+ := L_- + 1$, we have as a lower bound for the right-hand side

$$\frac{1}{L_-}\left(\frac{A}{m^a} \,-\, \frac{B}{(L_+)^b}\right) \;\geq\; \frac{1}{\tilde{L}}\left(\frac{A}{m^a} \,-\, \frac{B}{\tilde{L}^b}\right) \;\geq\; \frac{1}{m^{a/b}\,\left(\frac{2B}{A}\right)^{1/b}} \cdot \frac{A}{2\,m^a}. \qquad \square$$

3.2 Proofs of main results

Theorem 2.1; Upper bounds: By the Theorem of [6] (see also the discussion in [9]) the cartesian product $\widehat{G} := G_o \square C_\delta$ is Hamiltonian. Let Y_t be the continuous-time delayed random walk on the cycle C_δ of order δ, with transition kernel P_Y. Since $1/\delta \leq \mathbb{P}[Y_t = Y_0] = (1/\delta)\mathrm{Tr}\exp(-t(\mathbb{I} - P_Y)) \leq 1$, and from Lemma 3.3 it follows

$$\mathbb{P}[\widehat{X}_{2t} = \widehat{X}_0] \leq \mathbb{P}[X_t = X_0] \leq \delta \cdot \mathbb{P}[\widehat{X}_{2t} = \widehat{X}_0],$$

where $\widehat{X}_t$ is the continuous-time delayed random walk on $\widehat{G}$. By Theorem 1 in [14], the eigenvalues of the transition kernel $\widehat{P}$ of $\widehat{X}_t$ can be compared with the eigenvalues of the delayed random walk on $C_{\delta N}$; namely,

$$\widehat{\beta}_j \;\leq\; 1 - \frac{2}{\delta+2}\left(1 - \cos 2\pi\,\frac{j-1}{\delta N}\right), \qquad (j \in \{1, ..., \delta N\}), \tag{3.5}$$

where $1 = \widehat{\beta}_1 > \widehat{\beta}_2 \geq \widehat{\beta}_3 \geq \widehat{\beta}_4 \geq ... \geq \widehat{\beta}_{\delta N}$, and $N = |V(G_o)|$. The factor $2/(\delta+2)$ in front of the parentheses results from the regularisation with loops, characteristic of the delayed random walk on a graph ($\widehat{G}$) with maximal degree $\delta + 2$, where the extra 2 comes from taking the cartesian product with C_δ (see [27]). Note, the eigenvalue of $\widehat{P}$ can also be enumerated differently: $\{\widehat{\beta}_j\}_{j=1}^{\delta N} = \{\widehat{\beta}_{j,l}\}_{j,l=1}^{N,\delta}$, where $\widehat{\beta}_{j,l} = \frac{1}{2}(\beta_j + \cos(2\pi(l-1)/\delta)$, with $j \in \{1, ..., N\}$ and $l \in \{1, ..., \delta\}$. From (1.2), we have $\mathbb{P}[\widehat{X}_{2t} = \widehat{X}_0] = \frac{1}{\delta N}\mathrm{Tr}[e^{-2t(1-\widehat{P})}]$, so

$$\begin{aligned} \delta\,\mathbb{P}[\widehat{X}_{2t} = \widehat{X}_0] &= \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{\delta} e^{-2t(1-\frac{1}{2}(\beta_j + \cos(2\pi(i-1)/\delta)))} \\ &\leq \frac{1}{N}(1 \,+\, 2 \cdot k\,e^{-t(1-\beta_2)}) \,+\, \frac{1}{N}\sum_{j=k+2}^{\delta N - k} e^{-2t(\frac{2}{\delta+2}(1-\cos 2\pi\frac{j-1}{\delta N}))} \\ &\leq \frac{1}{N}(1 \,+\, 2 \cdot k\,e^{-t(1-\beta_2)}) + \frac{2}{N}\sum_{j=k+1}^{\lfloor \delta\frac{N}{2}\rfloor - 1} e^{-\frac{4t}{\delta+2}(1-\cos 2\pi\frac{j}{\delta N})}. \end{aligned} \tag{3.6}$$

The first inequality follows from bounding the first $2k$ eigenvalues of $\widehat{P}$ less than one from above by $\beta_2 = \widehat{\beta}_{2,1}$, and from (3.5), giving that the n-th largest element of $\{\widehat{\beta}_{j,l}\}_{j,l=1}^{N,\delta}$ is less than the n-th largest eigenvalue of DRW on $C_{\delta N}$, which however is only applied to $n > 2k+1$. The second inequality follows from the symmetry of the cosine, and an index-shift, with equality if δN is even. Since $I_t(\cdot,\cdot)$ is monotone in the second argument, the claim follows from applying Lemma 3.2 i.) and ii.) to $I_t(4t/(\delta+2), \delta N/2)$. $\square$

Remark: (Theorem 2.1) For $1-\beta_2$ we have the standard lower bound given by the Poincaré inequality. The delayed random walk has the same spectrum as the simple random walk on the path 'decorated' with loops to yield a regular graph of degree δ [27]. In particular, $1-\beta_2 \geq 1-(1-2/\delta(1-\cos(\pi/N))) \geq 4/(\delta N^2)$, by $\cos \pi x \leq 1-2x^2$ for $x \in [0,1]$. If $k \in \{1, ..., N-2\}$ in (3.6) is chosen such that $\frac{4}{\delta N^2} \leq \frac{32k^2}{(\delta+2)\delta^2 N^2}$, or, equivalently $k^2 \geq \delta(\delta+2)/8$, then the first exponential term $\exp(-t(1-\beta_2))$ has weaker decay than the second. We see this is the case for a number k independent of N. Therefore, provided that N is sufficiently large, even if nothing else is known about β_2, Theorem 2.1 is an improvement over simply using $\beta_2 \geq \beta_j$ for $j \geq 2$ and the Poincaré inequality for $1-\beta_2$, which would be the bound corresponding to $k = N-1$ and the second term in (3.6) vanishing.

Theorem 2.1; Lower bound: For a given finite simple graph $G_o = \langle V_o, E_o \rangle$, let $I(G_o)$ be the isoperimetric number (or 'Cheeger-constant') of G_o, defined by

$$I(G_o) \quad = \quad \min_{A \subset V_o \,:\, |A| \leq \frac{1}{2}|V_o|} \frac{|\partial_{G_o} A|}{|A|} \ ,$$

where $\partial_{G_o} A = \{\{k,l\} \in E_o \mid k \in A, l \notin A\}$ is the *edge-boundary* of A in G_o, and $|A| = \#A$ denotes cardinality of the finite set A.

By a theorem of A.G. Boshier [8] the isoperimetric number I for graphs with genus bounded by g obeys $I \leq 3\,\delta(g+2)/(\sqrt{|V_o|/2} - 3(g+2))$ if $|V_o| > 18(g+2)^2$ (see [23] for a discussion). From this result it holds that if $|V_o| \geq 4 \cdot 72$ and G_o a planar finite graphs (for which $g = 0$!) that

$$I \quad \leq \quad K/\sqrt{|V_o|}, \qquad\qquad \text{with } K \ = \ 12\sqrt{2} \cdot \delta. \tag{3.7}$$

By Cheeger's inequality (see [26], Lemma 3.3.7), the spectral gap $\lambda = \frac{1}{2} \min_{v \neq \text{const}} (v, (1-P)v)/(v,v) \ = \ 1-\beta_2$ for the delayed random walk with transition probability matrix P can be estimated from above,

$$\lambda \ \leq \ I.$$

By (3.7) this implies a lower bound on the return probability of the continuous-time delayed random walk for planar graphs with the uniform distribution as the initial distribution. We have $\mathbb{P}[X_t = X_0] \ - \ 1/|V_o|$ is

$$\mathbb{P}[X_t = X_0] - \tfrac{1}{|V_o|} \ = \ \tfrac{1}{|V_o|} \sum_{j=2}^{|V_o|} e^{-t(1-\beta_j)} \ \geq \ \tfrac{1}{|V_o|} e^{-t\lambda} \ \geq \ \tfrac{1}{|V_o|} e^{-\frac{t\,K}{|V_o|^{1/2}}}. \qquad \square$$

Theorem 2.2; Lower bound: Compare this with [11], Lemma 2.2 and [28]. Let G be transitive, with a unimodular, transitive subgroup of Aut(G), the automorphism group of G. Given $\omega \in \Omega$, for $G'(\omega)$ being the whole percolation subgraph of G, the graph G_o is the connected subgraph of $G'(\omega)$ induced by $\mathcal{C}_o(\omega)$, i.e. $V_o = \mathcal{C}_o(\omega)$. (In what follows, we will drop the dependence on ω, wherever it doesn't cause confusion. For example, we write $\mathcal{C}_o$ instead of $\mathcal{C}_o(\omega)$.)

From Theorem 2.1, iii.), since G_o is almost surely finite, there is a lower bound for the expected return probability of the delayed random walk. Namely, since

$$\mathbb{E}_\mu\left[\mathbb{P}[X_t = X_0]\right] \;-\; \mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\right] \;\geq\; \mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\, e^{-\frac{tK}{\sqrt{|\mathcal{C}_o|}}}\chi_{|\mathcal{C}_o|>288}\right],$$

and due to the assumption $t > \sqrt{288}$, we have

$$\mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\, e^{-\frac{tK}{\sqrt{|\mathcal{C}_o|}}}\chi_{|\mathcal{C}_o|\geq t^2}\right] \;\geq\; \sum_{m\geq t^2}^{\infty}\frac{1}{m}\, e^{-\frac{tK}{\sqrt{m}}}\phi(m) \;\geq\; e^{-K}\sum_{m\geq t^2}^{\infty}\frac{1}{m}\,\phi(m).$$

The lower bound of Theorem 2.2 now follows by Lemma 3.4, with $D = e^{-K}\frac{A/2}{1+(2B/A)^{1/b}}$ and by applying Lemma 3.1 to express $P_t(o)$ by the normalised trace. ☐

Theorem 2.2; Upper bound: By assumption, μ is invariant under a unimodular transitive subgroup of $\mathrm{Aut}(G)$, and by the remark after Corollary 2.3 there are almost surely only finite cluster. In particular μ-a.s. $|\mathcal{C}_o| < \infty$.

Let $N = |\mathcal{C}_o|$. In order to use Theorem 2.1 most effectively, we want to choose $k \in \{1, ..., N-2\}$ as small as possible while keeping the exponents of the same order in t/N^2. We differentiate between two cases: First, we assume $\lfloor N\sqrt{q\lambda}\rfloor + 1 \leq N - 2$, where $q = \delta^2(\delta+2)/32$. Then, we choose k in Theorem 2.1, i.) such that

$$\lambda \;:=\; 1 \;-\; \beta_2 \;\leq\; \frac{32}{\delta^2(\delta+2)}\cdot\frac{k^2}{N^2}.$$

This is accomplished if we set $k = \lfloor N\sqrt{q\lambda}\rfloor + 1$. (Note, $k \leq N-2$.) This choice implies $N\sqrt{q\lambda} < k \leq 1 + N\sqrt{q\lambda}$. Setting $c = \sqrt{\pi q/2}$, it follows

$$\mathbb{P}[X_t \;=\; X_0] \quad\leq\quad \frac{1}{N} \;+\; \left(\left(\frac{2}{N} \;+\; 2\sqrt{q\lambda}\right) \;+\; \frac{c}{\sqrt{t}}\right)e^{-\lambda t}. \tag{3.8}$$

From $e^{-x} \leq y^y/x^y$ and $e^{-x} \leq ((y-1/2)/x)^{y-1/2}$ for $y > \frac{1}{2}$, we get

$$\mathbb{P}[X_t \;=\; X_0] \quad\leq\quad \frac{1}{N} \;+\; \frac{1}{t^y}\left(y^y\left(\frac{2}{N\lambda^y} + \frac{2\sqrt{q}}{\lambda^{y-1/2}}\right) \;+\; c\frac{(y-\frac{1}{2})^{y-1/2}}{\lambda^{y-1/2}}\right).$$

Now using the Poincaré inequality $\lambda \geq \delta/(4N^2)$, we obtain the following estimate:

$$\begin{aligned}\mathbb{P}[X_t = X_0] \;&\leq\; \frac{1}{N} + \frac{1}{t^y}\left(y^y\left(2\frac{16^y N^{2y-1}}{\delta^y} + \frac{\sqrt{q}\,2^{2y}N^{2y-1}}{\delta^{y-1/2}}\right) + c\frac{(y-\frac{1}{2})^{y-1/2}(2N)^{2y-1})}{\delta^{y-1/2}}\right)\\ &\leq\; \frac{1}{N} \;+\; c_\delta\frac{N^{2y-1}}{t^y},\end{aligned} \tag{3.9}$$

with

$$c_\delta \;=\; 2^{2y}\left(\frac{y}{\delta}\right)^y\left(2^{2y+1} \;+\; \frac{\delta\sqrt{\delta(\delta+2)}}{4\sqrt{2}}\left(1 \;+\; \sqrt{2\pi}\right)\right). \tag{3.10}$$

We have used that $b > 0$ and that by (2.1) and the remark after Corollary 2.3 the exponent of N in (3.9) is $\alpha := 2y-1 < b$, so that $1/2 < y < 1/2 + b/2$ and $(y-\frac{1}{2})^{y-\frac{1}{2}} < 2y^{\,y}$. With $\sqrt{\delta(\delta+2)} \leq \delta + 2$ and $(1+\sqrt{2\pi})/(4\sqrt{2}) \leq 1/2$, this leads to the upper bound

$c_\delta \le (4\alpha/\delta)^\alpha (2 \cdot 4^\alpha + \delta(\delta+2)/2)$. Since $b \le 2$, we see that the constant in front of N^{2y-1}/t^y in (3.9) is bounded below by $\frac{1}{2}$, independently of δ.

Now, turning to $\lfloor N\sqrt{q\lambda}\rfloor + 1 \ge N - 2$, which is equivalent to $\lambda \ge (1-3/N)^2/q \ge 1/(16q)$, if $N \ge 4$. If $N < 4$, we have the Poincaré inequality $\lambda \ge \delta/(4N^2) \ge \delta/36$. So, in both cases, the function $t \mapsto \mathbb{P}[X_t = X_0] - 1/N$ is decreasing exponentially fast. Since it is smaller than 1, the overall estimate covering all three possibilities (including the polynomially decreasing one) is given if the constant $c_\delta > 1$ in (3.10) is multiplied by five, yielding $5 \cdot (4b/\delta)^b (2 \cdot 4^b + \delta(\delta+2)/2)$.

Taking the expected value of both sides of the inequality and applying Lemma 3.1 to express $P_t(o)$ by the normalised trace yields the result. □

Corollary 2.3, i.); Upper bound: Since Bernoulli bond percolation on the Euclidean lattice is invariant under the unimodular transitive group of translations of the Euclidean lattice, this is a special case of Theorem 2.2. The result follows from the well-known fact [18], that there exists $\alpha > 0$, s.t. $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$. □

Corollary 2.3, i.); Lower bound: By the power law inequality $\Phi(m) = \mathbb{P}_\mu[|\mathcal{C}_o| \ge m] \ge \frac{1}{2}\, m^{-\frac{1}{2}}$ (see [12], Theorem 11.89), we have $a = \frac{1}{2}$ in (2.1). For any $t_o > 0$ it is now possible to choose C_2 depending on t_o such that $C_2^{-1}(288)^{-(1+1/\alpha)} = 1$. So, by choosing $t_o = 1$ the given estimate follows from Theorem 2.2 for $\alpha < b$ and for all $t \ge 1$. □

Corollary 2.3, ii.); Upper and Lower bound: It is well-known that for the homogeneous tree of finite degree, $a = b = \frac{1}{2}$ (see [1], [20], and [4]). Just as in the previous proof, the constant C_δ can be chosen so large, that the estimate is valid for all $t \ge 1$. □

Theorem 2.4: a.) Let $I_t = \mathbb{E}_\mu\left[\mathbb{P}[X_t = X_0] - \frac{1}{|\mathcal{C}_o|}\right]$, and let $\lambda = 1 - \beta_2$ be the smallest non-zero eigenvalue of $\mathbb{I} - P$, as above. We have, for any $c > 0$, that

$I_t \ge \mathbb{E}_\mu\left[\mathbb{P}[X_t = X_0] - \frac{1}{|\mathcal{C}_o|} \;\middle|\; \lambda \le \frac{c}{|\mathcal{C}_o|^2}\right] \cdot \mathbb{P}\left[\lambda \le c/|\mathcal{C}_o|^2\right]$, and that by (1.2)

$$\mathbb{E}_\mu\left[\mathbb{P}[X_t = X_0] - \frac{1}{|\mathcal{C}_o|} \;\middle|\; \lambda \le \frac{c}{|\mathcal{C}_o|^2}\right] \ge \mathbb{E}_\mu\left[\frac{e^{-t\lambda}}{|\mathcal{C}_o|} \;\middle|\; \lambda \le \frac{c}{|\mathcal{C}_o|^2}\right] = \mathbb{E}_\mu\left[\frac{e^{-\frac{ct}{|\mathcal{C}_o|^2}}}{|\mathcal{C}_o|}\right].$$

b.) Let for $\omega \in \Omega$ the diameter $D(\omega)$ of $\mathcal{C}_o(\omega)$ be defined by $D = \max_{v,w\in\mathcal{C}_o} d(v,w)$, with $d(.,.)$ the graph metric of G_o. Let $\pi = (v_0, v_1, v_2, ..., v_D)$ be a geodesic path in G_o of length D. Consider the function $g : \mathcal{C}_o \to \mathbb{R}$ with $g(v) = \cos(\pi k/D)$ where k is uniquely defined by $d(v_k, v) = \min\{\, d(v_j, v) \mid j \in \{0, ..., D\}\,\}$.

Now, we show that if for some number $\epsilon > 0$ it holds $\epsilon|\mathcal{C}_o| \le D$, then the function g gives an upper estimate of λ in terms of $|\mathcal{C}_o|^{-2}$:

$$\lambda = \min_{f\perp\text{const}} \frac{\sum_{i<j\in\mathcal{C}_o}(f_i - f_j)^2}{\sum_{v\in\mathcal{C}_o}|f(v)|^2} \le \frac{\sum_{v\sim w\in\mathcal{C}_o}(g(v)-g(w))^2}{\sum_{v\in\mathcal{C}_o}|g(v)|^2} \le \frac{\sum_{j=1}^D (g(v_j) - g(v_{j-1}))^2}{\sum_{j=1}^D |g(v_j)|^2},$$

where the second inequality results from neglecting the terms in the denominator not belonging to the geodesic π. By Taylor's Theorem $\cos(\pi j/D) = \cos(\pi(j-1)/D) + (\pi/D)\sin(\pi(j-1)/D) + O(1/D^2)$ as $D \mapsto \infty$, so for some number $c > 0$

$$\lambda \le \frac{\pi^2}{D^2}\frac{\sum_{j=1}^D(\sin(\pi(j-1)/D)^2}{\sum_{j=1}^D|\cos(\pi j/D)|^2}\left(1 + O(\frac{1}{D^2})\right) \le \frac{c}{D^2} \le \frac{c}{\epsilon^2|\mathcal{C}_o|^2}.$$

c.) By Markov's inequality, for $\alpha < b$

$$\mathbb{P}_\mu\left[\frac{|\mathcal{C}_o|}{D} \ge \epsilon^{-1}\right] \le \epsilon^\alpha \mathbb{E}_\mu\left[\frac{|\mathcal{C}_o|^\alpha}{D^\alpha}\right] \le \epsilon^\alpha \mathbb{E}_\mu[|\mathcal{C}_o|^\alpha]$$

of which the right-hand side can be made smaller than one by choosing ϵ sufficiently small. For such an ϵ the probability of the complement is positive, or, in other words, $C := \mathbb{P}_\mu[\epsilon|\mathcal{C}_o| < D] > 0$. So, from b.), $\mathbb{P}[\lambda < c/(\epsilon^2|\mathcal{C}_o|^2)]$ for some $c > 0$ with a probability bounded below by $C > 0$.

d.) Let $\phi(m) = \mathbb{P}_\mu[|\mathcal{C}_o| = m]$, and $t > 0$. Under the assumptions

$$\sum_{m>\sqrt{t}} \frac{\phi(m)}{m} \;\geq\; A \sum_{m>\sqrt{t}} m^{-a-2} \;\geq\; A \int_{\sqrt{t}}^{\infty} x^{-a-2}dx \;=\; \frac{A}{a+1}\frac{1}{(\sqrt{t})^{a+1}}$$

and so, by the foregoing arguments (a., b., c.), I_t is bounded from below by

$$C \cdot \mathbb{E}_\mu\left[\frac{e^{-t/(\epsilon^2|\mathcal{C}_o|^2)}}{|\mathcal{C}_o|}\right] \;\geq\; C \sum_{m>\sqrt{t}} \frac{1}{m} e^{-\frac{t}{\epsilon^2 m^2}}\, \mathbb{P}[|\mathcal{C}_o| = m] \;\geq\; \frac{C\,A\,e^{-1/\epsilon^2}}{(a+1)}\; t^{-\frac{a+1}{2}}. \qquad \square$$

Corollary 2.5: Since by Corollary 2.3, ii.) it holds for all $\epsilon > 0$ that

$$\lim_{t\to\infty} \frac{\log(\,P_t(o)\;-\;\mathbb{E}_\mu[1/|\mathcal{C}_o|]\,)}{\log t} \quad\leq\quad -\frac{3}{4} \;+\; \epsilon,$$

it must be true for $\epsilon = 0$, and the upper bound follows. Furthermore, it is well known [10] that for critical percolation on the homogeneous tree $\mathbb{P}_\mu[|\mathcal{C}_o| = m] \sim m^{-3/2}$. Therefore, the assumptions of Theorem 2.4 are fulfilled where $a = b = 1/2$ (see [12], Chap. 10.1, and [15], Chap. 1.3), which implies the lower bound. $\square$

Theorem 2.6; Upper bound: The integrated density of states $N(E)$ obeys [19], [24], [27]] the relation $\int_0^\infty e^{-tE}dN(E) = \mathbb{E}_\mu[\mathbb{P}_o[X_t = o]]$, such that by Theorem 2.2, i.)

$$e^{-t\epsilon}(N(\epsilon)\;-\;N(0)) \;\leq\; \int_0^\epsilon e^{-tE}dN(E) \;\leq\; P_t\;-\;\kappa \;\leq\; c_4\,\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha]\,t^{-\nu},$$

where $\nu = \frac{1}{2}(1+\alpha)$, with α such that $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$, and $c_4 = \left(8+\sqrt{3\pi}\right)$. Choosing $t = \nu/\epsilon$ and thereby optimising the upper bound for $N(\epsilon)\;-\;N(0)$ leads to the result.

Theorem 2.6; Lower bound: Again, by $\int_0^\infty e^{-tE}dN(E) = \mathbb{E}_\mu[\mathbb{P}_o[X_t = o]]$, Lemma 3.1 and Corollary 2.3, with $\alpha > 0$ s.t. $\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha] < \infty$,

$$\frac{C_2^{-1}}{t^{(1+1/\alpha)}} \leq \int_0^\infty e^{-t\,E}dN(E) \;\leq\; \int_0^\epsilon dN(E) \;+\; e^{-t\epsilon}\int_\epsilon^\infty dN(E) \;\leq\; N(\epsilon) - N(0) + e^{-t\epsilon}.$$

So, $N(\epsilon) - N(0) \geq \frac{1}{2}C_2^{-1}t^{-(1+1/\alpha)} - e^{-t\epsilon}$. Choosing $t = -(\bar{c}/\epsilon)\log\epsilon$ for $\epsilon > 0$ produces the result if, for example, $\bar{c} = 2\cdot(1+1/\alpha)$. Then $C_3 = \max\{C_2^{-1}/(\bar{c}\log\epsilon)^{1+\alpha^{-1}}$, $c_4\;\mathbb{E}_\mu[|\mathcal{C}_o|a]\}$. $\square$

Theorem 2.7: Bernoulli bond percolation on the d-dimensional Euclidean lattice is a percolation invariant under the unimodular translation group of the lattice. The degree is $\delta = 2\cdot d$. Assuming subcritical Bernoulli bond-percolation, we have existence of the first moment of the cluster size. By repeating the argument of the proof of the upper bound in Theorem 2.2 (which lead to (3.8)) with Theorem 2.1 ii.) instead of 2.1 i.) yields for all $t > 0$ with $q = 4/(d^2(d+1))$

$$\mathbb{E}_\mu\mathbb{P}[X_t \;=\; X_0] \quad\leq\quad \mathbb{E}_\mu\left[\frac{1}{|\mathcal{C}_o|}\right] \;+\; \mathbb{E}_\mu\left[\frac{2k}{|\mathcal{C}_o|}e^{-\frac{t}{|\mathcal{C}_o|^2}} \;+\; \frac{2}{q\,t}\frac{|\mathcal{C}_o|}{k}\exp\left(-\frac{q\,t\,k^2}{|\mathcal{C}_o|^2}\right)\right].$$

Now, choosing $k = 1$, and using $\exp(-x) \le 1/x$ for $x > 0$ gives for all $t > 0$

$$\mathbb{E}_\mu \mathbb{P}[X_t = X_0] \quad \le \quad \mathbb{E}_\mu \left[\frac{1}{|\mathcal{C}_o|} \right] + \mathbb{E}_\mu \left[2 \frac{|\mathcal{C}_o|}{t} + \frac{2|\mathcal{C}_o|}{q\,t} \right].$$

Calling $\kappa(p) = \mathbb{E}_\mu[1/|\mathcal{C}_o|]$ (note the difference to [13] regarding the cluster which consist of only one vertex), letting $\chi(p) := \mathbb{E}_\mu[|\mathcal{C}_o|]$ and noting $2 + 2/q = (d^3 + d^2 + 4)/2$, leads to the lower bound after a subsequent application of Lemma 3.1, and a rearrangement of the terms in the inequality.

The other constant $\frac{20}{d}(4 + d(d+1))$ follows from the method used for proving the upper bound of Theorem 2.2, and by using $b = 1$ and setting α in $\mathbb{E}_p[|\mathcal{C}_o|^\alpha] < \infty$ equal to b, which is possible due to the existence of the first moment.

The upper bound follows from observing $P_t - \kappa(p) = \mathbb{E}_\mu[(1/|\mathcal{C}_o|)\cdot \mathrm{Tr}\exp(-t(1-P))] \ge 0$, since $1 - P$ has only non-negative eigenvalues. □

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