Electronic Journal of Probability

Vol. 10 (2005), Paper no. 8, pages 250-302.

Journal URL
http://www.math.washington.edu/~ejpecp/

Return Probabilities of a Simple Random Walk on Percolation Clusters

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Abstract

We bound the probability that a continuous time simple random walk on the infinite percolation cluster on $\mathbb{Z}^d$ returns to the origin at time t. We use this result to show that in dimensions 5 and higher the uniform spanning forest on infinite percolation clusters supported on graphs with infinitely many connected components a.s.

Keywords : percolation, simple random walk, return times, spanning forests, heat kernel.

AMS 2000 Subject Classification : Primary: 60J45; Secondary: 60J60, 60K35.

Submitted to EJP on September 18, 2003. Final version accepted on January 21, 2005.

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1 Introduction

Consider Bernoulli bond percolation on $\mathbb{Z}^d$ with parameter p. Recall that this is the independent process on $\mathbb{Z}^d$ which retains an edge with probability p and deletes an edge with probability $1-p$. For all $d > 1$, there exists a critical parameter $0 < p_c < 1$ that depends on d, such that if $p > p_c$, then a.s. there is a unique connected component with infinitely many edges [1]. This component is called the infinite cluster. The phase where $p > p_c$ is called supercritical Bernoulli percolation. Supercritical Bernoulli percolation can be viewed as a random perturbation of the original graph. For more on this, see [15]. It is then natural to ask what properties persist through this random perturbation. In particular, we focus on the properties of a simple random walk on the infinite cluster.

Rayleigh's monotonicity principle says that if a simple random walk on a graph G is recurrent then a simple random walk on any subgraph of G is recurrent. This implies that a simple random walk on the infinite percolation cluster in $\mathbb{Z}^2$ is recurrent. Grimmett, Kesten, and Zhang proved that in $\mathbb{Z}^d$, $d \geq 3$ a simple random walk on the infinite percolation cluster, $\mathcal{C}_\infty(\mathbb{Z}^d, p)$, is a.s. transient [16]. Häggström and Mossel used a method of Benjamini, Pemantle, and Peres to show that a large class of subgraphs of $\mathbb{Z}^d$ have transient percolation clusters [19] [8]. Although on subgraphs transient percolation clusters corresponds well with transience, the methods in these papers tell us nothing about the probability that a simple random walk on the percolation cluster returns to the origin at time t. They also do not answer the question of whether two independent simple random walks on the percolation cluster will intersect infinitely often a.s.

The probability that a simple random walk on $\mathbb{Z}^d$ returns to the origin at time t decays on the order of $Ct^{-d/2}$. The method of Varopoulos and Carne proves that the probability that a simple random walk returns to the origin at time t is greater than or equal to $Ct^{-d/2}/(\log t)$ [28] [10]. In this paper we get a bound in the other direction. We show that this probability is bounded above by $C't^{-d/2}(\log t)^{6d+14}$.

After this paper was written Mathieu and Remy [24] and then Barlow [3] obtained more refined results on the return probabilities of simple random walk on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$. We mention the beautiful results of Barlow here. For a fixed graph ω and two vertices v and w in the same connected component of ω let $p_t^\omega(v, w)$ be the probability that continuous time simple random walk on ω which is started at v at time 0 is at w at time t.

Theorem 1.1 *[3] There exists constants $c_i = c_i(p,d)$ and a function $S : \{0,1\}^{Edges(\mathbb{Z}^d)} \to \mathbb{R}$ which is finite a.s. with the following property. For all $w \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ if $0, w \in \mathcal{C}_\infty(\mathbb{Z}^d, p)$ $t \geq S(\omega)$, and $t \geq |w| = \sum_d |w_i|$ then*

$$c_1 t^{-d/2} e^{-c_2|w|^2/t} < p_t^\omega(0,w) < c_3 t^{-d/2} e^{-c_4|w|^2/t}.$$

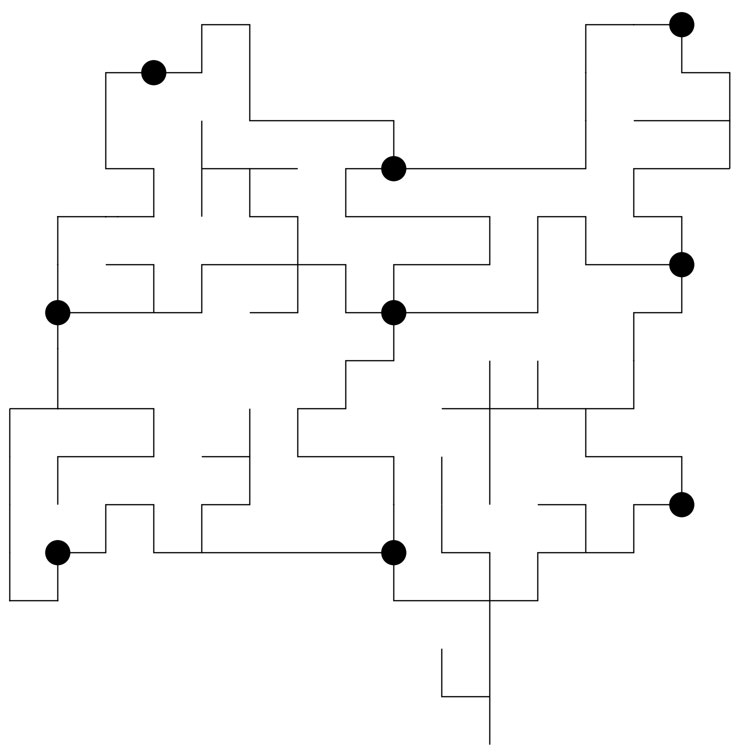
Our result allows us to show that another property of the simple random walk persists after supercritical Bernoulli percolation. Namely, in $\mathbb{Z}^d$, $d \geq 5$, two simple random walks intersect finitely often a.s. [23]. In Section 9 we use the bound on the return probability to show that this property also holds on the infinite percolation cluster in $\mathbb{Z}^d$, $d \geq 5$. Together with results of [7], [6] and [21], this in turn shows that for $\mathbb{Z}^d$, $d \geq 5$, the uniform spanning forest (USF) on percolation clusters is supported on infinitely many components. We define the wired, free and uniform spanning forests in Section 9. This corresponds with a result of Pemantle that the USF in $\mathbb{Z}^d, d \leq 4$, is supported on trees, while in $\mathbb{Z}^d, d \geq 5$ the USF is supported on graphs that have infinitely many connected components [25]. The lower bound proves that two independent simple random walks on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$, $d \leq 4$, have infinite expected number of intersections. If this could be extended to show that two simple random walks have infinite intersections almost surely then this would prove that the USF on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$, $d \leq 4$, is supported on a tree a.s.

2 Outline

The outline of the proof is as follows. Most of this paper is devoted to showing that with high probability there exist subgraphs of $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ where we can calculate the return probabilities. We use a result of Benjamini and Schramm (Theorem 3.1) to show that we can bound the average return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ by the return probabilities on these subgraphs.

Now we will introduce some notation to help us describe the subgraphs that we find embedded in $\mathcal{C}_\infty(\mathbb{Z}^d, p)$. For any positive integers j and d we write $[-j, j]^d$ for the subgraph of $\mathbb{Z}^d$ that includes all vertices $(v_1, ..., v_d)$ such that $-j \leq v_i \leq j$ for all i. The edges of this graph are those that have both their endpoints in this set of vertices. For any graph G and any positive integer n we write nG for the graph that is generated by replacing each edge of G with n edges in series. We say a **distinguished vertex** of nG is a vertex that corresponds with one in G. For any graph G we write $V(G)$ for the vertices of G.

In particular we will show that for some $\alpha > 0$ with high probability there exists a embedding of $n[-2^{\alpha n}, 2^{\alpha n}]^d$ in $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ "centered around the origin". It is easy to bound the return probabilities of a simple random walk on an embedding of $n[-2^{\alpha n}, 2^{\alpha n}]^d$. However we can not use these subgraphs with Theorem 3.1 to bound the return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$. In order to use Theorem 3.1 we will extend the embedding of $n[-2^{\alpha n}, 2^{\alpha n}]^d$ to get a graph that spans the percolation cluster in some finite ball. This subgraph will be the embedding with "bushes" attached to various vertices in the grid. The following is a picture of an embedding of $9[-1, 1]^2$ with bushes.



The bushes ensure that the subgraph spans the finite section of the percolation cluster. They also do not increase the return probabilities significantly. With bounds for the return probabilities on these graphs we can invoke Theorem 3.1 and obtain bounds for the return probabilities on the percolation cluster.

The remainder of the paper is organized as follows. In Section 3 we present the result which allows us to estimate the return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ by studying return probabilities on subgraphs. Then we turn our attention to constructing the subgraphs of $\mathcal{C}_\infty(\mathbb{Z}^d, p)$. In order to show that embeddings of $n[-2^{\alpha n}, 2^{\alpha n}]^d$ exist with high density we need to do two main things, to identify potential distinguished vertices and to show that the distinguished vertices can be connected by disjoint paths of the appropriate length.

We do the second task first. In particular we show that there exists an odd integer γ such that given that x and y are connected, the conditional probability that there exists a path P that connects x and y such that $|P| = \gamma|x - y|$ is approaching 1 as $|x - y|$ approaches ∞. This is done in sections 4 and 5.

Then in Section 6 we show that there are lots of candidates for distinguished vertices of an embedding of $n[-2^{\alpha n}, 2^{\alpha n}]^d$. Also in this section we combine this with the results of the previous section to show that the probability that there exists a embedding of $n[-2^{\alpha n}, 2^{\alpha n}]^d$ centered near the origin inside $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ is increasing to 1 exponentially in n.

Then in Section 7 we show how to extend this grid to a spanning subgraph. Then we bound the return probabilities on the spanning subgraphs. Unfortunately it is rather cumbersome to do so. Our method is as follows. First we place a measure on such spanning subgraphs. Then in Section 8 we prove that with high probability the spanning subgraph has low return probabilities. Finally we use Theorem 3.1 to bound the return probabilities on the percolation cluster.

We conclude the paper with Section 9, where we show that the bounds on the return probabilities are sufficient to prove that the USF on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$, $d \geq 5$ is supported on graphs with infinitely many connected components.

3 Monotonicity of average return probability

This section is due to Itai Benjamini and Oded Schramm. Let F be a finite graph where each edge (i, j) has a conductance $a_{i,j} \geq 0$. (If the conductances are not specified then they are assumed to be one across every edge of F.) Let Z be the nearest neighbor process on $V(F)$ where the rate of moving from i to j is $a_{i,j}$. Define

$$p_t^F(v,w) = P(Z_t = w \mid Z_0 = v).$$

Theorem 3.1 *Fix any $t > 0$. $\sum_v p_t^F(v,v)$ is monotone non-increasing in the conductances.*

Proof: Fix some time $t > 0$, and let $z = \sum_v p_t^F(v,v)$. Observe that $z = \text{trace}\,(exp(tA))$ where $A = (a_{i,j})$ and $a_{i,j}$ is the conductance of the edge $\{i,j\}$ if $i \neq j$ and $a_{i,i} = -\sum_{k\neq i} a_{i,k}$.

Now let $tr = $ trace. It follows from $tr(AB) = tr(BA)$ and the linearity of tr that

$$dz = dtr(exp(tA)) = tr(exp(tA)dA).$$

Without loss of generality we may take dA to be of the form $a_{11} = a_{22} = -1, a_{12} = a_{21} = 1$ (that is, we increase the conductance of the edge $\{1,2\}$). Let $B := exp(tA)$. Then $tr(exp(tA)dA) = b_{12} + b_{21} - b_{11} - b_{22}$. Let $C = exp((t/2)A)$. Then $B = C * C$, and

$$b_{11} + b_{22} - b_{12} - b_{21} = c_1c_1 + c_2c_2 - 2c_1c_2 = |c_1 - c_2|^2,$$

where c_1 is first row of C and the first column of C and c_2 is the second row of C and the second column of C. Consequently, $dz < 0$. □

Corollary 3.1 *Let μ be a shift invariant measure on spanning subgraphs of $\mathbb{Z}^d$, then for any $v \in \mathbb{Z}^d$*

$$\int p_t^X(v,v)d\mu(X) \geq p_t^{\mathbb{Z}^d}(v,v) = \Theta(t^{-d/2}).$$

Note that Barlow and Perkins [4], constructed a subgraph S of $\mathbb{Z}^d$, such that there exists a vertex $v \in \mathbb{Z}^d$, a constant $C > 0$ and a sequence t_n such that $t_n \to \infty$ and

$$p_t^S(v,v) > \frac{Ct_n^{-d/2}}{\log t_n},$$

for all n.

4 Approximate Pathlengths in the Cluster

In this section we show that if two points x and y are connected in the infinite cluster and far apart, then with high probability there exists a path between them in the cluster of approximately any length in a certain interval. Namely, for any length $\tilde{q} \in [1.5, 36]$, there is a path of length $\tilde{q}\rho|x-y| \pm \epsilon\rho|x-y|$. These paths are a first approximation for the sides of the grid we construct in Section 6. In Section 5 we construct paths of length exactly $5\rho|x-y|$. These will be the sides of the grid. First we introduce some notation. Let $\omega \in \{0,1\}^{\mathbb{Z}^d}$ be a realization of percolation. An open edge e is one such that $\omega(e) = 1$. An open **path** P is a connected set of open edges with no cycles such that two vertices have degree 1. These are the **endpoints** of P. An open path P **connects** x and y if x and y are endpoints of edges in P. We also say P is a path from x to y. We write $x \sim y$ if there exists an open path from x to y. We write $x \sim \infty$ if x is part of the unique infinite cluster. If $x \sim y$ then we let $D(x,y)$ be the length of the shortest open path from x to y. A path P is in a set of vertices V if the endpoints of every edge in P are in V. We write $V(G)$ for the vertices of a graph. We say that G' **spans** G if $V(G) \subset V(G')$.

Throughout the next two sections we prove that there exists an odd integer ρ, such that if $x \sim y$ and $|x-y|$ is large, then with high probability there exists a path P which connects x and y such that $|P| = 5\rho|x-y|$. We show that conditioned on $x \sim y$, the probability that there does not exist such a path decreases exponentially in $|x-y|$.

We use the taxicab metric on $\mathbb{Z}^d$, $|(x_1,...,x_d),(y_1,...,y_d)| = |x_1-y_1| + ... + |x_d-y_d|$. Define $L(u,v)$ to be the elements in $\mathbb{Z}^d$ which are within $d/2$ of the line segment joining u and v. Let B_k be all $(x_1,...,x_d) \in \mathbb{Z}^d$ such that $|x_1| + ... + |x_d| \leq k$ and $B_k(x) = x + B_k$. Also let $B_\mu(x,y) = B_{\mu|x-y|}(x) \cup B_{\mu|x-y|}(y)$. All constants A_i and C_i may depend on d and p. There are a number of places in the following sections where in order to be precise in choosing our parameters, it is necessary to use the greatest integer function. However in the sake of readability we have omitted the greatest integer function from some of our definitions. It is easy to check that this causes no essential difficulties.

We will show that for any $\tilde{q} \in (1.5\rho|x-y|, 36\rho|x-y|)$ there exists a path P from x to y, lying in a certain region called the wedge between x and y, such that $|P|$ is approximately $\tilde{q}$. Then in the next section we will use an inductive argument to show that there exists a path of exactly length $5\rho|x-y|$.

There are three main tools that we will use to get a path of approximately the right length. The first is the following result of Antal and Pisztora.

Theorem 4.1 *[2] Let $p > p_c(\mathbf{Z}^d)$. Then there exists $\rho = \rho(p,d) \in [1,\infty)$, and constant C_1 such that for all y*

$$\mathbf{P}(D(0,y) > \rho|y| \mid 0 \sim y) < e^{-C_1|y|}.$$

The second is the method of Grimmett, Kesten, and Zhang [16]. This method was used to show the existence of a tree in $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ with certain branching properties. This tree was used to show that for $d \geq 3$, a simple random walk on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ is transient a.s. The method works just as well for $\mathbb{Z}^2$ [20]. This method requires the following standard facts about the infinite percolation cluster.

Lemma 4.1 *Given $p > p_c(\mathbf{Z}^d)$ there exists C_2 so that $\mathbf{P}(B_k \not\sim \infty) < e^{-C_2 k}$.*

Proof: For $d \geq 3$ a proof of this can be found in [17]. For $d = 2$ this follows from the work of Seymour and Welsh, and Russo [27] [26]. □

The third is a combination two theorems, one of Chayes, Chayes and Newman, and the other of Barsky, Grimmett and Newman, which gives the following result.

Theorem 4.2 *[11], [5] Given $p > p_c(\mathbf{Z}^d)$ there exists C_3 so that*

$$\mathbf{P}(0 \not\sim \infty \mid 0 \sim \partial B_k) < e^{-C_3 k}.$$

Applying the last theorem we get the following lemma.

Lemma 4.2 *Let S be a connected set with diam$(S) = n$. Then there exists C_4 such that*

$$\mathbf{P}(S \not\sim \infty) \leq e^{-C_4 n}.$$

Proof: This argument was given to us by Y. Zhang. For $d = 2$ the result follows easily from the arguments in [15] on pages 194 and 195. Let $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$. Then there exists k such that for any fixed i, $p > p_c(\{(x_1, ..., x_d) \mid 0 \leq x_i \leq k\})$. There also exists a $C > 0$ such that for any $w = (w_1, ...w_d)$, any i and any j such that $j \leq w_i \leq j + k$,

$$\mathbf{P}(\exists \text{ an } \infty \text{ path } P,\ w \in P,\ P \subset \{(x_1, ..., x_d) \mid j \le x_i \le j+k\}) > C.$$

Since $\mathrm{diam}(S) = n$ there exists i and $y, y' \in S$ such that $y'_i - y_i \ge \frac{n}{d}$. Let $\alpha = \lfloor \frac{n}{dk} \rfloor$. Then there exists $w^1, ...w^\alpha \in S$ such that $(w^j)_i \in [y_i + (j-1)k, y_i + jk)$. Then

$$\begin{aligned}
\mathbf{P}(S \not\sim \infty) &\le \mathbf{P}(w^1, ...w^\alpha \not\sim \infty) \\
&\le \prod_{j=1}^{\alpha} \mathbf{P}(\nexists \text{ an } \infty \text{ path } P_j,\ w^j \in P, \\
&\qquad P_j \subset \{(x_1, ..., x_d) \mid x_i \in [y_i + (j-1)k, y_i + jk)\}) \\
&\le (1-C)^\alpha \\
&\le (e^{-C'/dk})^n \\
&\le e^{-C_4 n}
\end{aligned}$$

□

Now we will use a method similar to the one introduced by Grimmett, Kesten, and Zhang to show that infinite percolation clusters in $\mathbb{Z}^d, d \ge 3$ are a.s. transient. We will use this method to show that for any two points $x, y \in \mathbb{Z}^d$ the probability that there exists a reasonably short and direct path from near x to near y is going to 1 exponentially in $|x - y|$.

Define $E_r(z)$ to be the event that there exists $w \in B_r(z)$

1. such that $w \sim \infty$ and
2. any point $w' \in (B_{2r}(z))^C$ such that $w' \sim w$ satisfies $D(w, w') \le \rho |w - w'|$.

Remember that the constant ρ was defined in Theorem 4.1 Notice that $E_r(z)$ is not an increasing event. This makes some of the future arguments more technical since the FKG inequality will not apply.

Lemma 4.3 *There exists $A_4, C_5 > 0$ such that for all r*

$$\mathbf{P}(E_r(z)) > 1 - A_4 e^{-C_5 r}.$$

Proof: By Lemma 4.1 the probability that line 1 in the definition of $E_r(z)$ does not hold is less than $e^{-C_2 r}$. Theorem 4.1 implies that

$$\mathbf{P}\left(\exists\ z_2 \sim w \text{ such that } |z_2 - w| = j\ ,\ D(z_2, w) \geq \rho|z_2 - w|\right) < 2dj^{d-1}e^{-C_1 j}.$$

Thus

$$\begin{aligned} \mathbf{P}(E_r(z)) &> 1 - e^{-C_2 r} - (2r+1)^d \sum_{j \geq 2r} 2dj^{d-1}e^{-C_1 j} \\ &> 1 - A_4 e^{-C_5 r} \end{aligned}$$

for an appropriate choice of A_4 and C_5. □

Define $F_r(z)$ to be the event that

1. $z \sim \infty$ and
2. any point $w' \in B_{2r}(z)^C$ such that $z \sim w'$ satisfies $D(z, w') \leq \rho|z - w|$.

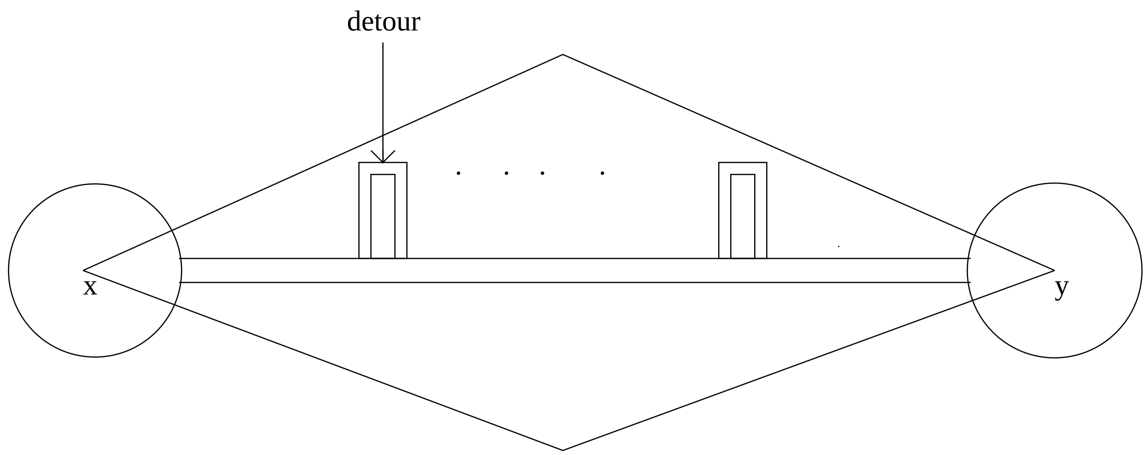
Let $K = 1000$ and $\epsilon = 1/2K^2$. Remember that $\rho > 1$ is an odd integer from Theorem 4.1. We define sets in $\mathbb{Z}^d$, called wedges, around x and y in which the various constructed paths from x to y will lie. The wedges allow enough room to construct paths of many different lengths, while ensuring that when the paths are concatenated, they do not intersect except near x or y. Define $\angle xz, xy$ to be the angle between the line segments xz and xy. Define

$$W(x,y) = \{z \in \mathbb{Z}^d |\ z \in B_{.1/K^2}(x,y) \text{ or } (\angle xz, xy \leq 15^\circ \text{ and } \angle yz, yx \leq 15^\circ)\}$$

and

$$W'(x,y) = \{z \in \mathbb{Z}^d |\ z \in B_{.1/K^2}(x,y) \text{ or } (\angle xz, xy \leq 30^\circ \text{ and } \angle yz, yx \leq 30^\circ)\}. \quad (1)$$

The following is a picture of the wedge W between x and y.



the wedge W between x and y

Definition 4.4 *A pair of points x and y in $\mathbb{Z}^d$ is (ϵ, ρ, ω)* **good***, written $x \simeq y$, if there exists an open path P_0 from x to y such that for all $\tilde{q} \in [1.5, 36]$, there exists open path $P \subset \omega$ such that*

1. $|P| \in [(\tilde{q} - \epsilon)\rho|x - y|, (\tilde{q} + \epsilon)\rho|x - y|]$,
2. $P \subset W(x, y)$,
3. $P|_{B_{1/4}(x,y)} = P_0|_{B_{1/4}(x,y)}$.

Now we show that most pairs of vertices that are connected and far apart are good pairs.

Lemma 4.5 *There exists $A_5, C_6 > 0$ such that*

$$\mathbf{P}(x \simeq y \mid x \sim y) > 1 - A_5 e^{-C_6|x-y|}.$$

Proof: The idea is as follows. By Lemma 4.1, if $x \sim y$, then it is likely that there is a path in the percolation cluster whose length is bounded above by $\rho|x - y|$. We show that it is possible to pick a path, P_0 inside the wedge from x to y with length less than $1.5\rho|x - y|$. This is done as follows.

We lay down a series of evenly spaced points, called markers, on the line segment between x and y. These markers will be labelled $z^1_{i,0}$. These markers lie in a straight line from x to y and are evenly spaced. (In the picture above all of the markers $z^1_{i,0}$ will lie in the thin strip between z and y. The path P_0 will pass close to all of these markers and is also located in the strip.) In particular we will show that with high probability there exists a point in a small ball around each marker such that for any two adjacent markers w and w', there is a path between the points corresponding to w and w' whose length is bounded by $\rho|w - w'|$. This follows from Lemma 4.3. We construct P_0 by concatenating the paths between the points in the balls around the markers on the line segment between x and y and removing any loops. This constructs us a path between x and y that is not too long and lies fairly close to the line segment between x and y.

Moreover we need to show it is possible to pick a different path so that the length falls within any specified range. To do this, we create a disjoint collection of possible detours. Each detour is a path that intersects P_0 on both ends. Thus for each subset of detours we are able to construct a path from x to y by either including the portion of P_0 or the detour. By varying the number of detours we add we can control the length of the path from x to y.

We create the detours in a similar manner to the way we create P_0. First we lay down a series of markers outlining the detours. All of the detours have (roughly) the shape of three sides of a rectangle. (The piece of P_0 which is bypassed makes up the fourth side of the rectangle.) For the jth detour these markers will be labelled $z^2_{i,j}$, $z^3_{i,j}$ and $z^4_{i,j}$. The labels 2,3 and 4 indicate on which side of the rectangle the markers lie. (In the picture the markers $z^2_{i,1}$, $z^3_{i,1}$ and $z^4_{i,1}$ lie on the left hand side, top, and right hand side, respectively, of the object labelled detour.) The index i indicates where along the side the marker lies. The jth detour will roughly follow the points $z^2_{i,j}$ then the points $z^3_{i,j}$ and $z^4_{i,j}$ before returning to P_0.

We will choose our parameters so that if we add all of the detours to P_0 then the resulting path is long enough ($\geq 36\rho|x-y|$). On the other hand adding each detour increases the length of the path by a small amount. Hence it is possible to add the right number of detours so that the length of the path falls within the specified range.

Let $m = 4\rho$ be the number of markers on the short side of each rectangle and $M = 300\rho^2$ be the number of markers on the long side of each rectangle. Define $\# = 60\rho/\epsilon$ to be the total number of possible detours. Let $n = 1000\rho^2/\epsilon$ be the number of markers on the line segment between x and y, $\eta = |x-y|/n$ be the size of the increments between the markers and $v = (y - x)/|y - x|$. Finally let v' be a unit vector perpendicular to v.

We define the markers equally spaced throughout the region in which the path is permitted. Let

$$\begin{array}{lll} z^1_{i,0} = x + i\eta v & 0 \le i \le n, & \\ z^2_{i,j} = x + (n/4 + 2mj)\eta v + i\eta v' & 0 \le i \le M, & 0 \le j \le \#, \\ z^3_{i,j} = x + (n/4 + 2mj + i)\eta v + M\eta v & 0 \le i \le m, & 0 \le j \le \#, \\ z^4_{i,j} = x + (n/4 + (2m+1)j)\eta v + i\eta v' & 0 \le i \le M, & 0 \le j \le \#. \end{array}$$

We will show that if $E_{\eta/4}(z^l_{i,j})$ holds for all i, j and l then x and y are a good pair. First we show that if the events $E_{\eta/4}(z^1_{i,0})$ hold for all i then there is a path P_0 from $B_{\eta/4}(x)$ to $B_{\eta/4}(y)$ inside $L(x, y) + B_{\rho\eta}$ that has $|P_0| < 1.5\rho|x - y|$. To see this, let $w_i \in B_{\eta/4}(z^1_{i,0})$ be the point in the definition of $E_{\eta/4}(z^1_{i,0})$. Then

$$\begin{aligned} |P_0| &= \sum_i D(w_i, w_{i-1}) \\ &\le \sum_i \rho|w_i - w_{i-1}| \\ &\le \sum_i 2\rho(\eta/4) + \sum_i \rho|z^1_{i,0} - z^1_{i-1,0}| \\ &\le \rho/2|x - y| + \rho|x - y|. \end{aligned}$$

Furthermore P_0 lies inside $L(x, y) + B_{\rho\eta}$. This is because the maximum that P_0 can travel away from the line between x and y is bounded by

$$\max_i \frac{D(w_i, w_{i-1})}{2} + \eta/4 \le \rho\eta.$$

If $F_{\eta/4}(x)$ and $F_{\eta/4}(y)$ also hold then there is a path P_0 from x to y inside $W(x, y)$ such that $|P_0| < 1.5\rho|x - y|$.

Now we show that if in addition the events $E_{\eta/4}(z^l_{i,j})$ hold for all i, j, l then there are paths from near x to near y inside $L(x, y) + B_{(\rho+M)\eta}$ of many different lengths. Suppose in addition to the events $E_{\eta/4}(z^1_{i,0})$, $F_{\eta/4}(x)$, and $F_{\eta/4}(y)$ holding that for a fixed j the events $E_{\eta/4}(z^2_{i,j})$, $E_{\eta/4}(z^3_{i,j})$, and $E_{\eta/4}(z^4_{i,j})$ hold for all i. Then there is a path P^j such that $2\eta M < |P^j| < 3\rho\eta M$, $P^j \cap P_0 = \emptyset$ and the only vertices shared by P^j and P_0 are the endpoints of P^j. We can form a new path from x to y by cutting out a piece of P_0

and attaching P^j. We say this path takes detour j. By this construction, the possible detours occur in the middle half of the line segment between x and y. This is because $2m\# \leq 2(4\rho)(60\rho/\epsilon) < n/2$. Furthermore, the maximum height of the detour marker points is $M\eta \leq \epsilon|x-y| < \frac{|x-y|}{4}\tan 15$. This implies that

$$P^j \subset B_{\rho\eta} + (L(z^2_{0,j}, z^2_{M\eta,j}) \cup L(z^3_{0,j}, z^3_{m,j}) \cup L(z^4_{0,j}, z^4_{M\eta,j})) \subset W(x,y).$$

These containment conditions imply that the P^j are all disjoint.

Now suppose $E_{\eta/4}(z^l_{i,j})$ holds for all i, j, l and $F_{\eta/4}(x)$ and $F_{\eta/4}(y)$ also hold. Then for any $i \in [0, ..., \#]$ we can form a path that takes exactly i detours. Since $2\eta M < |P^t|$ for each t a path that takes i detours has length at least $2\eta i M$. Thus there is a path that takes $\#$ detours which has length at least

$$2\eta M\# \geq \frac{2|x-y|\epsilon(300\rho^2)(60\rho)}{1000\rho^2\epsilon} \geq 36\rho|x-y|.$$

Adding one extra detour adds at most length

$$4\rho\eta M \leq \frac{4\rho|x-y|\epsilon 300\rho^2}{1000\rho^2} \leq 2\epsilon\rho|x-y|.$$

to the length of a path. Thus if $E_{\eta/4}(z^l_{i,j})$ holds for all i, j, l and $F_{\eta/4}(x)$ and $F_{\eta/4}(y)$ also hold then x and y are a good pair.

Now we calculate the probability that all of this happens. Notice that $n + 3\#M$ is the total number of markers. Also notice that $\mathbf{P}(x \sim y) \geq \mathbf{P}(x \sim \infty)^2$.

$$\begin{aligned}
\mathbf{P}\left(x \not\approx y \mid x \sim y\right) &< \mathbf{P}\left(\cup_{i,j,l}(E_{\eta/4}(z^l_{i,j}))^c \mid x \sim y\right) + 2\mathbf{P}\left((F_{\eta/4}(x))^c \mid x \sim y\right) \\
&< (n+3\#M)\mathbf{P}\left((E_{\eta/4}(0))^c\right)/\mathbf{P}(x \sim y) + 2\mathbf{P}(x \not\sim \infty \mid x \sim y) + \\
&\quad 2\mathbf{P}(x \sim \infty, (F_{\eta/4}(x))^c \mid x \sim y) \\
&< (n+3\#M)A_4 e^{-C_5\eta/4}/\mathbf{P}(x \sim \infty)^2 + 2e^{-C_3|x-y|} \\
&\quad +2A_4 e^{-C_5\eta/4}/P(x \sim \infty)^2 \\
&< (n+3\#M+2)A_4/\mathbf{P}(x \sim \infty)^2 e^{-C_5\eta/4} + 2e^{-C_3|x-y|} \\
&< (n+3\#M+2)A_4/\mathbf{P}(x \sim \infty)^2 e^{-C_5|x-y|/4n} + 2e^{-C_3|x-y|} \\
&< A_5 e^{-C_6|x-y|}.
\end{aligned}$$

Thus the lemma is true for some appropriately chosen A_5 and C_6. □

Corollary 4.6 *There exists A_6 and C_7 so that*

$$\mathbf{P}(\exists y \in B_r(x)^C \textit{such that } x \sim y,\ x \not\approx y) < A_6 e^{-C_7 r}$$

and

$$\mathbf{P}(\exists y, z \in B_r(x) \textit{such that } y \sim z,\ y \not\approx z, |y - z| > r/2K^2) < A_6 e^{-C_7 r}.$$

5 Exact Pathlengths in the Cluster

This section is devoted to showing that with high probability we can find a path in the percolation cluster from x to y of length exactly $5\rho|x-y|$. We do this so that the sides of the constructed grid in Section 6 are all of the same length. Now we give an overview of this section.

First we define a pair of points (x, y) to be very good if there exists many paths of specific lengths in the percolation cluster connecting x and y (Definition 5.1). The main result of this section is Lemma 5.8 which shows that conditioned on x and y being connected, with high probability, (x, y) is a very good pair. The proof is by induction. We need to establish the existence of many paths between x and y because the existence of 1 path of a certain length does not yield a strong enough induction hypothesis.

The main idea of the induction in Lemma 5.2 is the following. At the nth stage we construct an approximate pathlength path between x and y (we can do this because of the results in Section 4). The path passes through points x' and y' which are close enough to each other to apply the induction hypothesis. The relative distances of x, y, x' and y' are important and are laid out in Lemma 5.2. Since (x', y') is a very good pair, we can modify the path between x' and y' so that we know the exact length of the entire path. We choose the distances between x and y and x' and y' so that if (x, y) is a good pair and (x', y') is a very good pair then (x, y) is also a very good pair. The main idea for the induction step in Lemma 5.8 is encompassed in Lemma 5.2, which we outline in the following paragraph.

In Lemma 5.2 we want to construct a path P from x to y which is exactly some linear multiple of $|x-y|$. We assume that we have points x' and y', with paths P_1 between x and x' and P_2 between y and y' such that $|P_1| + |P_2|$ is approximately the desired linear multiple of $|x-y|$. We further assume that we can find a path P_3 from x' to y' that is exactly the difference between the linear multiple of $|x-y|$ and $|P_1|+|P_2|$. We want to construct $|P|$ by simple concatenation. The problem is that P may have loops. Hence we replace P_1, P_2 and P_3 with paths P_1', P_2' and P_3' which yields a path with no loops when concatenated and such that the new path is the desired length.

After we prove the main part of the induction step in Lemma 5.2 we give a Corollary 5.3 that is more technical but is actually the version we need. It weakens the hypothesis on x' and y' to ensure that with high probability we can fulfill the hypotheses by possible replacing x' and y' with nearby points x'' and y'' lying on the path between x' and y'.

The three lemmas after Corollary 5.3 are devoted to proving the base case of the induction. We now give an idea for the structure of this. In Definition 5.4 we define a very good n-ball around x to be one which contains a very good pair (y,z) such that $|y-z|=K^n$. In Lemma 5.5 we show that if there exists very good balls of large diameter around x and $x\sim y$, then the probability that (x,y) is a very good pair does not decay exponentially to 0 in $|x-y|$. Lemma 5.6 shows that with high probability there exists very good balls around a generic point x in the cluster. In Lemma 5.7 we show that the probability that (x,y) is a very good pair, given that the points are connected and a fixed distance apart is bounded away from 0. This lemma is the base case of the induction. Once we prove Corollary 5.3 and Lemma 5.7, we are ready to prove the main lemma, Lemma 5.8. Remember that ρ is an odd integer from Lemma 4.1.

Definition 5.1 *A pair x and y in $\mathbb{Z}^d$ is said to be (ρ,ω)* **very good**, *written $x\cong y$, if there exists a path P_0 from x to y satisfying the following property. For all $\tilde{q}\in(3\rho|x-y|,10\rho|x-y|)$ such that $\tilde{q}+|x-y|=0 \bmod 2$ there exists $P\subset\omega$ such that*

1. $|P|=\tilde{q}$,
2. $P\subset W'(x,y)$, *and*
3. $P|_{B_{1/16}(x,y)}=P_0|_{B_{1/16}(x,y)}$.

The proof that most pairs that are connected are very good is inductive. The main idea in the inductive step is contained in this next lemma. The three lemmas after that will allow us to get the induction started.

Lemma 5.2 *Suppose $x', y' \in W(x,y)$ such that $\frac{3}{8}|x-y| < |x-x'|, |y-y'| < \frac{5}{8}|x-y|$. Suppose $\frac{1}{K^2}|x-y| < |x'-y'| < \frac{1}{20}|x-x'|$ and similarly for $|y-y'|$. If $x \simeq x'$, $y \simeq y'$ and $x' \cong y'$, then $x \cong y$.*

Proof: The basic idea is that given $\tilde{q}$ we find paths P_1, P_2, and P_3 as follows. The definition of a good pair will allow us to choose P_1 so that P_1 connects x and x', $P_1 \in W(x,x')$, and $|P_1|$ a little less than $.5\tilde{q}$. It also allows us to choose P_2 so that P_2 connects y and y', $P_2 \in W(y,y')$, and $|P_2|$ a little less than $.5\tilde{q}$. Then since $x' \cong y'$ we can choose P_3 so that P_3 connects x' and y', $P_3 \in W(x',y')$, and $|P_3| = \tilde{q} - |P_1| - |P_2|$. Then form a path P by concatenating P_1, P_3 and P_2 we get a path from x to y of length $\tilde{q}$.

The problem is that P may have loops. These can only occur in

$$B_{|x-x'|/10K^2}(x') \cup B_{|x'-y'|/10K^2}(x') \subset B_{|x'-y'|/16}(x')$$

or

$$B_{|y-y'|/10K^2}(y') \cup B_{|x'-y'|/10K^2}(y') \subset B_{|x'-y'|/16}(y').$$

Let $\tilde{q}_1$ be the length of the loops near x' and $\tilde{q}_2$ be the length of the loops near y'. Notice that $\tilde{q}_1 + \tilde{q}_2 < 2\rho|x-y|$. Now modify P_1 to find P_1' with

$$|P_1'| \in (.5(\tilde{q}+\tilde{q}_1+\tilde{q}_2) - 1.6\rho|x'-y'| - \epsilon\rho|x-y|, .5(\tilde{q}+\tilde{q}_1+\tilde{q}_2) - 1.6\rho|x'-y'| + \epsilon\rho|x-y|).$$

This is possible because

$$\begin{aligned} .5(\tilde{q}+\tilde{q}_1+\tilde{q}_2) - 1.6\rho|x'-y'| - \epsilon\rho|x-y| &> (1.5-\epsilon)\rho|x-y| - .1\rho|x-x'| \\ &> 2.4\rho|x-x'| - 1.6\epsilon\rho|x-x'| - .1\rho|x-x'| \\ &> 1.5\rho|x-x'| \end{aligned}$$

and

$$.5(\tilde{q}+\tilde{q}_1+\tilde{q}_2) - 1.6|x'-y'| + \epsilon\rho|x-y| < 6\rho|x-y| + \epsilon\rho|x-y| < 36\rho|x-x'|.$$

Modify P_2 to find P_2' with $|P_2|$ in the same interval. Choose P_3 so that it connects x' and y', $P_3 \in W(x',y')$. Finally find $|P_3'| = \tilde{q} + \tilde{q}_1 + \tilde{q}_2 - |P_1'| - |P_2'|$. This can be done because

$$\begin{aligned} \tilde{q} + \tilde{q}_1 + \tilde{q}_2 - |P_1'| - |P_2'| &\in (3.2|x'-y'| - 2\epsilon\rho|x-y|, 3.2|x'-y'| + 2\epsilon\rho|x-y|) \\ &\subset (3\rho|x'-y'|, 10\rho|x'-y'|). \end{aligned}$$

Since both ends of P_1, P_2, and P_3 are the same as the ends of P_1', P_2', and P_3' this modification does not change the concatenation in $B_{1/10}(x, y)$ or change the loops. Thus concatenating P_1', P_3', and P_2' and then removing the loops near x' and y' gives a path P from x to y of length $\tilde{q}$. Finally $P \subset W(x, x') \cup W'(x', y') \cup W(y, y') \subset W'(x, y)$. $\square$

The same argument gives us the following.

Corollary 5.3 *Suppose $x', y' \in W(x, y)$ such that $\frac{3}{8}|x-y| < |x-x'|, |y-y'| < \frac{5}{8}|x-y|$, $x' \cong y'$ and P_0 is a path between them that satisfies the definition of very good. Also suppose $\frac{1}{K^2}|x-y| < |x'-y'| < \frac{1}{20}|x-x'|$ and similarly for $|y-y'|$. If there exists $x'' \in V(P_0) \cap B_{.01|x'-y'|}(x')$ and $y'' \in V(P_0) \cap B_{.01|x'-y'|}(y')$ such that $x \simeq x''$ and $y \simeq y''$ then $x \cong y$.*

The main result of this section is that $\mathbf{P}(x \cong y \mid x \sim y)$ is converging to 1 exponentially in $|x-y|$. Before we show this, we show in the next two lemmas that it is not converging to 0 exponentially in $|x-y|$. This step is necessary to prove the base case of the induction.

Definition 5.4 *$B_{2K^n}(x)$ is* **very good** *if there exists $y, z \in B_{2K^n}(x)$ such that $y, z \sim \infty$, $y \cong z$ and $|y-z| = K^n$.*

Lemma 5.5 *If there exists an C_8 and a A_7 such that*

$$\mathbf{P}(B_{2K^{n+1}}(x) \text{ is very good} \mid B_{2K^n}(x) \text{ is very good}) > 1 - C_8 e^{-A_7 K^n},$$

then there exist an N and A_{10} such that if $|x-y| = K^n$ for $n \geq N$ then

$$\mathbf{P}(x \cong y \mid x \sim y) > A_{10}|x-y|^{-2d}.$$

Proof: Notice that

$$\sum_{|w-w'|=K^n, w, w' \in B_{2K^n}} \mathbf{P}(w \cong w') \geq \mathbf{P}(B_{2K^n} \text{ is very good }).$$

Since $\mathbf{P}(w \cong w')$ is the same for each of these pairs, this implies that $\mathbf{P}(w \cong w') \geq \mathbf{P}(B_{2K^n} \text{ is very good })/(2K^{nd})^2$. Since

$$\mathbf{P}(B_{2K^{n+1}}(x) \text{ is very good} \mid B_{2K^n}(x) \text{ is very good}) > 1 - C_8 e^{-A_7 K^n},$$

there exist N and C so that

$$\mathbf{P}(B_{2K^n}(x) \text{ is very good for all } n > N) > C.$$

By the previous remark, this implies there exists N such that if $|x-y| = K^n$ for some $n > N$

$$\begin{aligned}
\mathbf{P}(x \cong y \mid x \sim y) &\geq \mathbf{P}(x \cong y) \\
&\geq \mathbf{P}(B_{2K^n} \text{ is very good})/(2K^{nd})^2 \\
&\geq A_{10}|x-y|^{-2d}.
\end{aligned}$$

□

Lemma 5.6 *There exist A_{10} such that if $|x-y| = K^n$ for some n then*

$$\mathbf{P}(x \cong y \mid x \sim y) > A_{10}|x-y|^{-2d}.$$

Proof: To prove this we show that there are C_8 and A_7 such that

$$\mathbf{P}(B_{2K^{n+1}}(x) \text{ is very good} \mid B_{2K^n}(x) \text{ is very good}) > 1 - C_8 e^{-A_7 K^n}.$$

This implies the result by the previous lemma.

The idea of the proof is as follows. It suffices to find a pair of points $y, z \in B_{2K^{n+1}}(x)$ such that $|y-z| = K^{n+1}$ and $y \cong z$. Since $B_{2K^n}(x)$ is very good, we first find points y' and z' in $B_{2K^n}(x)$ such that $|y'-z'| = K^n$ and $y' \cong z'$. Now we need to find points y and z that are the right distance from y', z' and each other, and such that $y' \simeq y$, $z' \simeq z$. This would allow us to apply Lemma 5.2.

Technical considerations force us to add another step to the process. We replace y' and z' with points y'' and z'' near y' and z' respectively, which lie on a path between y' and z' and such that $y \simeq y''$ and $z \simeq z''$. Once we choose y, we need to choose z satisfying the hypotheses of Lemma 5.2 and such that $|y-z| = K^{n+1}$. Thus we choose z lying on $\partial B_{K^{n+1}}(y)$ and so that it is roughly colinear with y, y' and z'. In order to do this, we find z_1 and z_2 near this area, but on either side of $B_{K^{n+1}}(y)$ such that $z_1 \simeq z_2$. The path joining them will intersect $B_{K^{n+1}}(y)$. Let the intersection point be z. Once we find z, we now choose y'' and z'' to satisfy the hypotheses in Lemma 5.2.

If $B_{2K^n}(x)$ is very good then there exist $y', z' \in B_{2K^n}(x)$ such that $y', z' \sim \infty$, $y' \cong z'$ and $|y' - z'| = K^n$. Let P be a path connecting y' and z'. Let v be a unit vector pointing from y' to z'. If there exist y, y'', z, z'' such that

1. $y \in B_{K^n}(y' - (K^{n+1}/2)v)$,
2. $z \in B_{6K^n}(y + (K^{n+1})v)$,
3. $|y - z| = K^{n+1}$,
4. $y'' \in V(P) \cap B_{.01|y'-z'|}(y')$,
5. $z'' \in V(P) \cap B_{.01|y'-z'|}(z')$,
6. $y \simeq y''$, and
7. $z \simeq z''$.

then Corollary 5.3 shows that $y \cong z$. Since $y, z \sim \infty$ we have that $B_{2K^{n+1}}(x)$ is very good. Thus by the FKG inequality

$$\begin{aligned}
&\mathbf{P}(B_{2K^{n+1}}(x) \text{ is not very good} \mid B_{2K^n}(x) \text{ is very good}) \\
\leq\ & \mathbf{P}(\nexists y, y'' \text{ such that } y \in B_{K^n}(y' - (K^{n+1}/2)v),\ y'' \in V(P) \cap B_{.01|y'-z'|}(y'),\ y \simeq y'', \\
&\qquad \text{and } y, y'' \sim \infty \mid B_{2K^n}(x) \text{ is very good}) + \\
& \mathbf{P}(\nexists z, z'' \text{ such that } z \in B_{6K^n}(y + (K^{n+1})v),\ z'' \in V(P) \cap B_{.01|y'-z'|}(z'),\ z \simeq z'' \\
&\qquad |z - y| = K^{n+1} \text{ and } z, z'' \sim \infty \mid B_{2K^n}(x) \text{ is very good}) \\
\leq\ & \mathbf{P}(\nexists y, y'' \text{ such that } y \in B_{K^n}(y' - (K^{n+1}/2)v),\ y'' \in V(P) \cap B_{.01|y'-z'|}(y'),\ y \simeq y'', \\
&\qquad \text{and } y, y'' \sim \infty) + \\
& \mathbf{P}(\nexists z, z'' \text{ such that } z \in B_{6K^n}(y + (K^{n+1})v),\ z'' \in V(P) \cap B_{.01|y'-z'|}(z'),\ z \simeq z'' \\
&\qquad |z - y| = K^{n+1} \text{ and } z, z'' \sim \infty)
\end{aligned}$$

Suppose there exists $y \in B_{K^n}(y' - (K^{n+1}/2)v)$ and such that $y \sim \infty$. If there exists $y'' \in V(P) \cap B_{.01|y'-z'|}(y')$ such that $y'' \sim \infty$ then $y \sim y''$. If there do not exist $a, b \in B_{2K^{n+1}}(x)$ such that $a \sim b$, $a \not\simeq b$ and $|a - b| > K^{n-1}$ then $y \simeq y''$.

Additionally suppose there exist $z_1 \in B_{K^n/\rho}(y+(K^{n+1}-2K^n/\rho)v)$ and $z_2 \in B_{K^n/\rho}(y+(K^{n+1}+2K^n/\rho)v)$ such that $z_1, z_2 \sim \infty$. Then $z_1 \sim z_2$ and $z_1 \simeq z_2$. Thus there is a z such that $|y-z| = K^{n+1}$, $z \sim \infty$ and $|z-(y+(K^{n+1})v)| < 6K^n$. If there exists $z'' \in V(P) \cap B_{.01|y'-z'|}(z')$ such that $z'' \sim \infty$ then $z \sim z''$ and $z \simeq z''$. Thus if all this holds then there exists y, y'', z and z''.

Now we bound the probability that this happens.

$$\begin{aligned}
&\mathbf{P}(B_{2K^{n+1}}(x) \text{ is not very good} \mid B_{2K^n}(x) \text{ is very good}) \\
&\quad < 3\mathbf{P}(B_{K^n/\rho} \not\sim \infty) + 2\mathbf{P}(V(P) \cap B_{|y'-z'|/100}(y') \sim \infty) + \\
&\qquad \mathbf{P}(\exists a, b \in B_{2K^{n+1}}(x) \text{ such that } a \sim b,\ a \not\simeq b \text{ and } |a-b| > K^{n-1}) \\
&\quad < 3e^{-C_2K^n/\rho} + 2e^{-C_4K^{n/2}/100} + e^{-C_7K^{n-1}/\rho} \\
&\quad < C_8 e^{-A_7K^{n/2}}.
\end{aligned}$$

□

The base case of the induction is proved in the next lemma.

Lemma 5.7 *For any d there exists an $N = N(d)$ and $C_9(N)$ such that for all $n \geq N$ and all $0 \leq \alpha \leq 1$*

$$e^{-C_3K^{n+1}} + \frac{2e^{-C_7K^{n+1}}}{\mathbf{P}(x \sim \infty)^2} + (e^{-C_2K^n} + e^{-C_9K^n} + e^{-C_3K^n})^{.0016K^{2+\alpha}} < e^{-C_9K^{n+1+\alpha}}.$$

Furthermore for all x and y such that $|x-y| = K^N$

$$\mathbf{P}(x \not\simeq y \mid x \sim y) < e^{-C_9K^N}.$$

Proof: Fix d. By Lemma 5.6 the last equation is satisfied if $e^{-C_9K^N} = 1 - C/K^{Nd}$. This defines C_9 in terms of N. Notice as $N \to \infty$, $C_9 \to 0$. By choosing N sufficiently large, we can force C_9 to be much smaller than the other constants in the first equation. This implies that the terms with C_9 in them are the dominant terms of the inequality. Since $e^{-.0016C_9K^{n+2+\alpha}} = e^{-1.6C_9K^{n+1+\alpha}} < e^{-C_9K^{n+1+\alpha}}$ the inequality is true for N sufficiently large. □

Now we are ready to prove the main lemma of this section. The first half of the proof in this lemma is constructing a list of events which ensure that the conditions in

Corollary 5.3 are satisfied. The second half of the proof establishes the corresponding estimates on the probabilities that each of these events occurring.

Lemma 5.8 *There exist A_8 and C_9 such that*

$$\mathbf{P}(x \cong y \mid x \sim y) > 1 - A_8 e^{-C_9|x-y|}.$$

Proof: The proof is by induction. We will show that this is true with $A_8 = 1$ if $|x-y|$ is sufficiently large. This implies the result. The previous lemma establishes the base case, which is $n = N$. Given that the induction hypothesis is true for K^n we will show that it is true for all x, y such that $K^{n+1} \leq |x-y| \leq K^{n+2}$. This implies the result for all $|x-y| \geq K^{N+1}$. The idea for the proof is as follows. We start with two points x and y that are connected and $|x-y|$ is approximately K^{n+1}. We lay down about K disjoint copies of B_{2K^n} in the middle of $W'(x,y)$. Using the induction hypothesis, we show that with high probability at least one of these balls contains a pair satisfying the hypotheses of Lemma 5.2. By applying this lemma to the pairs x, x', x', y', and y', y, we obtain that with high probability $x \cong y$.

We claim that if there exist x' and y' satisfying the following conditions then x and y are a very good pair.

1. $x', y' \in W(x,y)$
2. $\frac{5}{8}|x-y| > |x-x'| > \frac{1}{4}|x-y|$,
3. $\frac{5}{8}|x-y| > |y-y'| > \frac{1}{4}|x-y|$,
4. $|x'-y'| = K^n$,
5. $x' \cong y'$,
6. $x', y' \sim \infty$,
7. $x, y \sim \infty$,
8. all w such that $|x-w| > \frac{1}{4}|x-y|$ either $w \not\sim x$ or $x \simeq w$, and
9. all w such that $|y-w| > \frac{1}{4}|x-y|$ either $w \not\sim y$ or $y \simeq w$.

Conditions 6 and 7 imply that x, y, x' and y' are all connected. Combining this with conditions 2 and 8 shows that $x \simeq x'$. Similarly conditions 3 and 9 show that $y \simeq y'$. Thus we have satisfied the hypothesises of Lemma 5.2 and x and y are a very good pair. First we bound the conditional probabilities given $x \sim y$ that conditions 1 - 6 are not satisfied. Then we bound the conditional probabilities that the final three conditions are not satisfied .

To make the first bound let v be the unit vector parallel to $x-y$ and v' be a unit vector perpendicular to $x-y$. Let $\alpha = \log_K(|x-y|/K^{n+1})$. Define $x_{i,j} = (x+y)/2 + 5K^n iv + 5K^n jv'$ for $|i|, |j| \leq .04K^{1+\alpha/2}$. This implies that $x_{i,j} \in W(x,y)$, $\frac{5}{8}|x-y| > |x_{i,j} - x| > .25|x-y|$, and $\frac{5}{8}|x-y| > |x_{i,j} - y| > .25|x-y|$ for all i, j and $|x_{i,j} - x_{i',j'}| > 4K^n$ for all $i, j \neq i', j'$. Now the events that there exist x' and y' in $B_{2K^n}(x_{i,j})$ satisfying conditions 1 - 5 are independent.

$$
\begin{aligned}
&\mathbf{P}(\nexists x', y' \text{ satisfying conditions 1 through 6} \mid x \sim y) \\
&\quad \leq \mathbf{P}(\nexists x', y' \text{ satisfying conditions 1 - 6}) && (2)\\
&\quad \leq \prod_{i,j} \mathbf{P}(\nexists x', y' \in B_{2K^N}(x_{i,j}) \text{ satisfying conditions 1 - 6}) && (3)\\
&\quad \leq \inf_{i,j} (\mathbf{P}(\nexists x', y' \in B_{2K^N}(x_{i,j}) \text{ satisfying conditions 1 - 6}))^{.0016K^{2+\alpha}} && (4)\\
&\quad \leq [\inf_{i,j} (\mathbf{P}(\nexists x', y' \in B_{2K^N}(x_{i,j}) \text{ such that } |x-y| = K^n \text{ and } x' \sim y') + && (5)\\
&\qquad \mathbf{P}(x' \not\simeq y' \mid x' \sim y') + \mathbf{P}(x' \not\sim \infty \mid x' \sim y'))]^{.0016K^{2+\alpha}} && (6)\\
&\quad \leq [\mathbf{P}(B_{K^N}(x_{i,j}) \not\sim \infty) + e^{-C_9 K^n} + e^{-C_3 K^n}]^{.0016K^{2+\alpha}} && (7)\\
&\quad \leq [e^{-C_2 K^n} + e^{-C_9 K^n} + e^{-C_3 K^n}]^{.0016K^{2+\alpha}} && (8)
\end{aligned}
$$

Line 2 follows from the FKG inequality. Line 4 follows from Lemma 4.1, the induction hypothesis, and Lemma 4.2.

$$\mathbf{P}(\text{condition 7 does not hold} \mid x \sim y) = \mathbf{P}(x, y \not\sim \infty \mid x \sim y) \leq e^{-C_3 K^{n+1}}$$

by Lemma 4.2. Corollary 4.6 tells us that

$$
\begin{aligned}
\mathbf{P}(\text{conditions 8 or 9 do not hold} \mid x \sim y) &\leq 2\mathbf{P}(\text{condition 8 does not hold})/\mathbf{P}(x \sim y) \\
&\leq \frac{2e^{-C_7 K^{n+1}}(A_6 e^{-C_7 K^n})}{\mathbf{P}(x \sim \infty)^2}.
\end{aligned}
$$

Putting this all together gives

$$
\begin{aligned}
&\mathbf{P}(\text{lines 1 - 9 hold} \mid x \sim y) \\
&> \;\; 1 - (e^{-C_2 K^n} + e^{-C_9 K^n} + e^{-C_3 K^n})^{.0016 K^{2+\alpha}} - e^{-C_3 K^{n+1}} - \frac{2e^{-C_7 K^{n+1}}(A_6 e^{-C_7 K^n})}{\mathbf{P}(x \sim \infty)^2} \\
&> \;\; 1 - e^{-C_9 K^{n+1+\alpha}} \qquad (9) \\
&> \;\; 1 - e^{-C_9 |x-y|}. \qquad (10)
\end{aligned}
$$

Line 9 is by Lemma 5.7 and line 10 is by the choice of α. □

6 Building the grid

In this section we first show that with high probability there exists corner points, called n hubs, in the percolation cluster. Each corner point has $2d$ paths emanating from it. Then we show that we can glue these points together with paths of exactly the right length to form a large finite grid centered near 0. This is possible because with high probability, the points at the end of adjacent paths of two respective n hubs form a very good pair. Remember we write $[-j, j]^d$ for the subgraph of $\mathbb{Z}^d$ that includes all vertices $(v_1, ..., v_d)$ such that $-j \le v_i \le j$ for all i. We will show that with high probability there exists a copy of $n[-2^{\alpha n}, 2^{\alpha n}]^d$ in ω with center near the origin. Let $u_1, ..., u_{2d}$ be the unit vectors in each of the $2d$ directions in $\mathbb{Z}^d$

Definition 6.1 *Given ω and ρ. A point $x \in \mathbb{Z}^d$ is an* **n hub** *if there exists paths $P_1, ..., P_{2d}$ such that*

1. *P_i connects x to $\partial B_n(x)$,*
2. *$V(P_i) \cap V(P_j) = x$ if $i \neq j$,*
3. *$|P_i| \le 2\rho n$, and*
4. *$V(P_i) \setminus B_{n/2}(x) \subset \{z \mid \angle u_i, xz \le 20\}$.*

Lemma 6.2 *There exists $A_9 > 0$ and J*

$$\mathbf{P}(x \text{ is a } K^j \text{ hub for all } j > J\ ) > A_9.$$

Proof: We prove the lemma by showing that there exists $A, C > 0$ so that

$$\mathbf{P}(x \text{ is a } K^{j+1} \text{ hub } \mid x \text{ is a } K^j \text{ hub and } x \sim \infty) \geq 1 - Ae^{-CK^{j/2}}.$$

(The true rate of decay is exponential in K^j. This will be implicit in a later lemma.)

Suppose x is a K^j hub and $x \sim \infty$. If there exist a disjoint collection of paths P_i' such that

1. $P_i' \subset \{z \mid \angle u_i, xz \leq 30\} \setminus B_{K^j/2}(x)$,
2. $V(P_i') \cap V(P_i) \neq \emptyset$,
3. $|P_i'| \leq 1.6\rho K^{j+1}$,
4. $V(P_i') \setminus B_{K^{j+1}/2}(x) \subset \{z \mid \angle u_i, xz \leq 20\}$, and
5. $V(P_i') \cap \partial B_{K^{j+1}}(x) \neq \emptyset$

then x is a K^{j+1} hub. Conditions 2 and 5 above and condition 1 from the definition of an n hub show that a subset of $P_i \cup P_i'$ satisfies condition 1 in the definition of an K^{j+1} hub. Condition 1 above and condition 2 in the definition of n hub imply condition 2 in the definition of $n+1$ hub. Condition 3 above and condition 3 from the definition of an n hub show it satisfies condition 3 in the definition. Condition 4 above shows it satisfies condition 4 in the definition.

If for a given x and ω there are many possible choices of the P_i, then we adopt some procedure for making a canonical choice of the P_i. If there exists $y_i \in B_{K^j}(x + K^{j+1}u_i) \setminus B_{K^{j+1}}(x)$ and $z_i \in P_i \setminus B_{2K^j/3}(x)$ such that $y_i, z_i \sim \infty$ and $y_i \simeq z_i$ then there exists an appropriate P_i'. Conditions 1,2, and 4 are satisfied because $y_i \simeq z_i$. Condition 3 is satisfied because $y_i \simeq z_i$ and $|y_i - z_i| < (1 + 1/K)K^{j+1}$. Condition 5 is satisfied because $y_i \in (B_{K^{j+1}}(x))^c$, $z_i \in P_i$, and $y_i \simeq z_i$.

$\mathbf{P}(\nexists P_i' \text{ satisfying conditions 2 and 5} \mid x \text{ is a } K^j \text{ hub and } x \sim \infty)$

$$
\begin{aligned}
< \;& \mathbf{P}(\nexists y_i \in B_{K^j}(x+K^{j+1}u_i) \setminus B_{K^{j+1}}(x), z_i \in P_i \setminus B_{2K^j/3}(x) \mid x \text{ is a } K^j \text{ hub and } x \sim \infty) \\
< \;& \mathbf{P}(\nexists y_i \in B_{K^j}(x+K^{j+1}u_i) \setminus B_{K^{j+1}}(x), z_i \in P_i \setminus B_{2K^j/3}(x)) \\
< \;& \mathbf{P}((B_{K^j}(x+K^{j+1}u_i) \setminus B_{K^{j+1}}(x)) \not\sim \infty) + \mathbf{P}(P_i \setminus B_{2K^j/3}(x) \not\sim \infty) + \\
& \mathbf{P}(\exists y, z \in B_{2K^{j+1}}(x) \text{ with } y \sim z, |y-z| \geq K^{j-2}/2, y \not\approx z) \\
< \;& e^{-C_2 K^j/2} + e^{-C_4 2K^j/3} + A_6 e^{-C_7 K^j} \qquad (11) \\
< \;& A e^{-CK^{j/2}}.
\end{aligned}
$$

Line 11 is true by Lemmas 4.1 and 4.2 and Corollary 4.6. This completes the proof of the lemma. □

Lemma 6.3 *There exists* $C_{11} > 0$ *such that*

$$\mathbf{P}(\exists x \in B_n \textit{ such that } x \textit{ is an } n \textit{ hub}) \geq 1 - e^{-C_{11}n}.$$

Proof: The proof is by induction. For the base case we will choose a J and C_{11} such that for all $n \leq K^{J+1}$,

$$\mathbf{P}(\exists x \in B_n \text{ such that } x \text{ is an } n \text{ hub}) \geq 1 - e^{-C_{11}n}.$$

This is possible by the previous lemma, which states that there exists A_9 and J such that

$$\mathbf{P}(x \text{ is a } K^j \text{ hub for all } j > J) > A_9.$$

The ergodic theorem implies that for any j as $J \to \infty$

$$\mathbf{P}(\exists x \in B_{K^J} \text{ such that } x \text{ is a } K^j \text{ hub for all } j > J) \to 1.$$

Choose J so that for all $j > J$

$$\mathbf{P}(\nexists x \in B_{K^J} \text{ such that } x \text{ is a } K^j \text{ hub for all } j > J)^K < .5. \qquad (12)$$

Choose C_{11} so that

1. for all $n \leq K^{J+1}$, $\mathbf{P}(\exists \text{ an } n \text{ hub in } B_n) \geq 1 - e^{-C_{11}n}$, and
2. $e^{-C_3 K^j}/\mathbf{P}(0 \text{ is a } K^j \text{ hub}) + 2de^{-.5C_2 K^j} + A_6 e^{-C_7 K^j} \leq .5e^{-C_{11}K^{j+2}}$

for all $j > J$.

The first condition ensures that the base case of the induction is true. The second can be satisfied because by line 12

$$\mathbf{P}(0 \text{ is a } K^j \text{ hub}) > .5(K^j)^d.$$

Suppose the induction hypothesis has been proven for K^j. We will now show that it is true for all $K^{j+1} \le n \le K^{j+2}$. If there is an $h \in B_n$ which is a K^j hub, $h \sim \infty$, $B_{K^j}(h + u_i n) \setminus B_n(h) \sim \infty$ for all i, and there are no $y, z \in B_{2n}$ with $y \sim z$, $y \not\approx z$, and $|y - z| > n/K^2$ then h is an n hub. This follows from the argument in the previous lemma.

Let $\beta = \log_K(n/K^j)$. Notice that $n = K^{j+\beta}$, and $1 \le \beta \le 2$. Find $x_1, ..., x_{2K^\beta}$ so that $B_{K^j}(x_i) \subset B_{n/3}$ for each i and $B_{K^j}(x_i) \cap B_{K^j}(x_l) = \emptyset$ for all $i \neq l$. This can be done like in Lemma 5.8.

The events that there exists a K^j hub in $B_{K^j}(x_i)$ are independent. So

$$\begin{aligned}
\mathbf{P}(\nexists h \in B_n \text{ such that } h \text{ is a } K^j \text{ hub}) &\le \mathbf{P}(\nexists h \in B_{K^j} \text{ such that } h \text{ is a } K^j \text{ hub})^{2K^\beta} \\
&\le \mathbf{P}(\nexists h \in B_{K^j} \text{ such that } h \text{ is a } K^j \text{ hub})^{K^\beta}(e^{-C_{11}K^j})^{K^\beta} \\
&\le .5(e^{-C_{11}K^{j+\beta}}).
\end{aligned}$$

If there are multiple choices of h such that h is a K^j hub then we employ some procedure for picking a canonical one. By Lemma 4.2 we have that

$$\mathbf{P}(h \not\sim \infty \mid h \text{ is a } K^j \text{ hub}) \le e^{-C_3 K^j} / \mathbf{P}(0 \text{ is a } K^j \text{ hub}).$$

Thus

$$\begin{aligned}
&\mathbf{P}(\nexists h \in B_n \text{ such that } h \text{ is an } n \text{ hub}) \\
&\quad < \mathbf{P}(\nexists h \in B_n \text{ such that } h \text{ is an } K^j \text{ hub}) + \mathbf{P}(h \not\sim \infty \mid h \text{ is an } K^j \text{ hub}) + \\
&\qquad 2d\mathbf{P}((B_{K^j}(h + u_i n) \setminus B_n(h)) \not\sim \infty) + \\
&\qquad \mathbf{P}(\exists y, z \in B_{2n} \text{ such that } y \sim z, |y - z| > K^j \text{ and } y \not\approx z) \\
&\quad < .5(e^{-C_{11}K^{j+\beta}}) + e^{-C_3 K^j} / \mathbf{P}(0 \text{ is a } K^j \text{ hub}) + 2de^{-.5C_2 K^j} + A_6 e^{-C_7 K^j} \\
&\quad < .5(e^{-C_{11}n}) + .5(e^{-C_{11}K^{j+2}}) \\
&\quad < e^{-C_{11}n}.
\end{aligned}$$

□

For an $\alpha = \alpha(d)$ which will be fixed later define the graph

$$S(\alpha, d, n) = [-2^{\alpha n}, 2^{\alpha n}]^d.$$

Then we define the n **grid** to be $nS(\alpha, d, n)$. Remember that nG is the graph derived from G by replacing each edge of G with n edges in series. Also remember that a distinguished vertex of $nS(\alpha, d, n)$ is a vertex that corresponds with one in $S(\alpha, d, n)$. In this case, they are the corner points of the n grid.

We say that **a copy of the** n **grid** is any graph which is isomorphic to the n grid. We say there exists **a copy of the** n **grid around 0 in** ω if there exists an embedding of any copy of the n grid in ω with the distinguished vertex corresponding to $(v_1, ..., v_d)$ in $B_{n/100\rho}((n/5\rho)(v_1, ..., v_d))$.

Lemma 6.4 *If*

1. *for each* $v = (v_1, ..., v_d) \in S(\alpha, d, n)$ *there exists an* $x_v \in B_{n/100\rho}((n/5\rho)v)$ *which is an* $n/100\rho$ *hub and* $x_v \sim \infty$ *and*

2. *each pair* $z, z' \in nS(\alpha, d, n)$ *such that* $|z - z'| > n/6\rho$ *and* $z \sim z'$ *is a very good pair*

then there is a copy of the n *grid around 0 in* ω*.*

Proof: Given a v pick the distinguished vertex corresponding to v to be an $n/100\rho$ hub in $B_{n/100\rho}((n/5\rho)v)$. The fact that distinguished vertices are $n/100\rho$ hubs gives us the start and end of the path between any two neighboring distinguished vertices. The ends of these two paths form a very good pair. Thus we can connect them with a path of the right length so that when we remove loops the distance between any two distinguished vertices is n. It is easy to check that these paths are disjoint. □

Lemma 6.5 *There exist* $\alpha, C_{12} > 0$ *and* A_{11} *such that for all* n

$$\mathbf{P}(\exists \text{ an } n \text{ grid around 0 in } \omega) > 1 - A_{11}e^{-C_{12}n}.$$

Proof: By Lemma 6.3 the probability that there are not the desired $n/100\rho$ hubs is less than $(2^{\alpha n+1}+1)^d(e^{-C_{11}n/100\rho})$. The probability that they are not all connected to the infinite cluster is less than $(2^{\alpha n+1}+1)^d e^{-C_3 n/100\rho}/P(x\sim\infty)^2$. As n hubs are separated by at least $n/6\rho$, Lemma 5.8 implies that the probability that the desired pairs are not very good is less than $(2n2^{\alpha n})^{2d}(A_8 e^{-C_9 n/6\rho})$. Thus the constants can be chosen. □

This fixes α for the rest of the paper.

7 Constructing the spanning subgraph

Now we want to show how to extend the n grid to a subgraph that spans $\mathcal{C}_\infty(\mathbb{Z}^d,p)$ in a finite region. The subgraph that we will construct (which we call a good n graph) has the property that if we remove the n grid then we will be left with a collection of trees. For a random walk on such a graph this ensures that whenever we leave the n grid we will return to it in exactly the same spot. This allows us to keep control on the return probabilities.

A connected graph Q is a **good n graph around 0** if

1. there is a subgraph G of Q which is a copy of the n grid around 0
2. each v which is connected to G is connected by a path (in Q) of length at most $4nd$, and
3. for no v does there exist disjoint paths in Q that connect v to G.

Notice that the third condition implies that any walk in Q which leaves the n grid at vertex x must return to the n grid at vertex x. Given a vertex v in the n grid we define the **bush attached to** v as follows. It contains any vertex v' in Q such that all paths from the n grid to v' go through v. It also contains all vertices which are connected to v by a single edge.

Any walk in Q generates a unique walk in the n grid. Thus it generates a unique nearest neighbor walk in $\mathbb{Z}^d$. This property will be used heavily in the next section.

Define $C(x)$ to be the open cluster containing x. Define

$$C_n(x)=C(x)\cap[-(n/5\rho)2^{\alpha n},(n/5\rho)2^{\alpha n}]^d.$$

Lemma 7.1 *If*

1. $0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)$,
2. *for each* $v = (v_1, ..., v_d) \in S(\alpha, d, n)$ *there exists an* $x_v \in B_{n/100\rho}((n/5\rho)v)$ *which is an* n *hub and* $x_v \sim \infty$ *and*
3. *each pair* $z, z' \in nS(\alpha, d, n)$ *such that* $|z - z'| > n/6\rho$ *and* $z \sim z'$ *is a very good pair.*

then there is a good n *graph around 0 that spans* $C_n(0)$.

Proof: By Lemma 6.4 the last two conditions imply that there is a copy of the n grid around 0. We will inductively show that we can attach each vertex of $C_n(0)$ to the n grid in a way that is consistent with the definition of a good graph. Suppose x is a vertex in $C_n(0)$ but not in the n grid. Then by the third condition there is a path from x to the n grid of length at most nd. This path can be chosen to intersect the n grid in only one vertex. The union of this path and the n grid satisfies the conditions of a good graph.

Now we want to show that if we have an extension E of the n grid that satisfies the conditions of a good graph then we can extend it to include at least one more vertex so that the new extension E' still satisfies the conditions of a good graph. (Note that E' will be an extension of the n grid, but is not necessarily an extension of E.) Let x be a vertex in $C_n(0)$ but which is not in E. By the second property there exists a path, P, connecting x to the n grid with length at most nd. It is clear that $x \cup V(E) \subset V(P \cup E)$. The problem is $P \cup E$ might not satisfy the conditions of a good graph. We will now pare down $P \cup E$ to get a new graph E' which spans $E \cup P$ and also satisfies the conditions of a good graph. Starting at the n grid move on P towards x. Stop at the first vertex v in $V(E)$. From v there may be a vertex $v' \in V(E)$ such that there are two disjoint paths from v to v'. If there are then remove the edge touching v in the longer of the two paths. If they are both the same length then remove either one. Now start at v and move along P toward x and repeat this procedure until you get to x. Call the resulting graph E'.

Now we claim that E' is a good n graph. Suppose there is a vertex y with two disjoint paths to the n grid. Then at least one edge of one of the paths is in P and one edge is in E. Thus there is a vertex in $V(E) \cap V(P)$ connected to one edge in $E' \cap E$ and one edge in $P \cap E'$. But whenever this situation arose we removed one of the two edges. Thus there is no such vertex. Also by the way we chose to remove edges the distance from any vertex in $V(E \cup P)$ to the n grid does not increase. Thus the distance from any point in V$(E \cup P)$ is at most nd. Thus the resulting graph is a good graph. Thus we can construct successive extensions until we get a good n graph that spans $C_n(0)$. □

Lemma 7.2

$$\mathbf{P}(\exists \textit{ a good } n \textit{ graph around 0 that spans } C_n(0) \mid 0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)) > 1 - A_{11}e^{-C_{12}n}.$$

Proof: The proof is the same as the proof of Lemma 6.5. □

We have shown that good n graphs exist with high probability. We would like to be able to bound the return probabilities on any good n graph. Unfortunately we don't know how to do this. So we will place a measure on good n graphs and then bound the return probabilities on a typical good n graph. This will be done in Section 8.

For a good n graph Q and $e = (v_1, v_2) \in E(S(\alpha, d, n))$ we define $Q(e) = Q((v_1, v_2))$ to be the portion of Q between the distinguished vertices corresponding to v_1 and v_2, including v_1 and v_2. Our choice of the measure μ_n will be done in such a way that the random variables $Q((v_1, v_2))$ are C_{13} **dependent**. By this we mean that if $v_1 \dots v_{2n} \in S(\alpha, d, n)$ are such that

1. $|v_{2i-1} - v_{2j-1}| > C_{13}$ for all $1 \le i < j \le n$ and
2. $|v_{2i-1} - v_{2i}| = 1$ for all $1 \le i \le n$

then $\{Q((v_{2i-1}, v_{2i}))\}_{i=1}^n$ are independent.

Lemma 7.3 *There exists C_{13} and measures μ_n on subgraphs of $\mathbb{Z}^d$ such that*

1. *$\mu_n(Q$ is a good n graph around 0 that spans $C_n(0) | 0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)) > 1 - A_{11}e^{-C_{12}n}$ and*
2. *the random variables $Q((v_1, v_2))$ are C_{13} dependent.*

Also we can couple μ_n with $\mathbf{P}$ so that the graph from μ_n is a subgraph of ω.

Proof: We will construct a sequence of distributions on subgraphs of ω. At each stage we pick a graph from the previous distribution. Then we extend the graph in a collection of disjoint regions. If at any stage we cannot complete the proscribed extension in a given region then we define the output of the process in this region to be the null set. We refer to this portion of the graph where we could not extend it as **stunted**. In future stages, if some portion of the graph in the region that we are trying to extend the subgraph in is stunted then we do not extend the graph in this region.

For each $v \in S(\alpha, d, n)$ consider all the

$$v' \in B_{n/100\rho}((n/5\rho)v)$$

which are $n/100\rho$ hubs. We construct a distribution so that all of these have the same probability of being the distinguished vertex corresponding to v. We also do this independently for all $v \in S(\alpha, d, n)$.

For a given choice of the distinguished vertex, x_v, consider the set of all possible sets of $2d$ paths which satisfy the definition of x_v being an $n/100\rho$ hub. Form a new distribution by assigning each of these sets of paths equal mass. Do this independently for all v. Let $H(v)$ be the n hub and paths corresponding to v. We have defined our measure such that the distribution of $H(v)$ depends only on ω in

$$B_{n/50\rho}(n/5\rho)v.$$

Also recall the definition of W' from line 1. Fix a choice of $H(v)$ for all $v \in S(\alpha, d, n)$. For each pair $v_1, v_2 \in S(\alpha, d, n)$ such that $|v_1 - v_2| = 1$ consider all paths in ω that

1. start at x_{v_1} and end at x_{v_2},
2. are contained in $\omega \cap W'(x_{v_1}, x_{v_2})$,
3. have length n, and
4. agree with $H(v_1)$ in $B_{n/200\rho}(x_{v_1})$ and with $H(v_2)$ in $B_{n/200\rho}(x_{v_2})$.

Form the next distribution by giving all of these paths equal mass. We form a distribution on n grids by doing this independently for all such pairs in $S(\alpha, d, n)$. Thus for any

$v_1, v_2 \in S(\alpha, d, n)$ the distribution of the path connecting x_{v_1} and x_{v_2} depends on ω in the region.

$$B_{n/50\rho}(x_{v_1}) \cup B_{n/50\rho}(x_{v_2}) \cup W'(x_{v_1}, x_{v_2}).$$

Fix a choice, H, of the n grid. For each $v \in S(\alpha, d, n)$ consider the set of all subsets S of ω such that:

1. $S \subset R = B_{n/49\rho}(x_v)$.
2. S extends $H|_R$.
3. Every vertex $y \in B_{n/50\rho}(x_v)$ which is connected to H in $\omega|_R$ is connected to H in S.
4. Every vertex $y \in V(S)$ is connected to H by a unique path in S.
5. That path has length $\leq nd$.

Form the next distribution by giving all of these graphs equal mass. We form a distribution by choosing an extension of H independently for all $v \in S(\alpha, d, n)$. The distribution of $Q(v_1, v_2)$ is determined by ω in the region

$$B_{n/4\rho}(x_{v_1}) \cup B_{n/4\rho}(x_{v_2}).$$

Fix H', a choice of the graph constructed at the previous stage. For each pair adjacent pair $v_1, v_2 \in S(\alpha, d, n)$ consider all possible connected subgraphs, S', of ω such that:

1. $S' \subset R' = \big(W'(x_{v_1}, x_{v_2}) \setminus (B_{n/50\rho}(x_{v_1}) \cup B_{n/50\rho}(x_{v_2}))\big) + B_{n/1000\rho}$.
2. S' extends $H'|_{R'}$.
3. Every vertex $y \in W'(x_{v_1}, x_{v_2})$ which is connected to H' in $\omega|_{R'}$ is connected to H' in S'.
4. Every vertex $y \in V(H' \cup S')$ is connected to H' by a unique path.
5. That path has length $\leq 2nd$.

Form the next distribution by giving all of these graphs equal mass. We form a distribution by choosing an extension of H' independently for all adjacent pairs $v_1, v_2 \in S(\alpha, d, n)$. For each adjacent pair $v_1, v_2 \in S(\alpha, d, n)$ the distribution of $Q(v_1, v_2)$ is determined by ω in the region

$$B_{n/4\rho}(x_{v_1}) \cup B_{n/4\rho}(x_{v_2}).$$

For any $v_1, \ldots, v_j$ of $S(\alpha, d, n)$ and choice of the n grid we define

$$\begin{aligned}\mathrm{CH}(v_1, \ldots, v_j) \;&=\; \{z \in \mathbb{Z}^d : \; z \text{ is within distance 1 of the} \\ &\qquad\qquad \text{convex hull of } x_{v_1}, \ldots, x_{v_j}\},\end{aligned}$$

Also define

$$W'(v_1, \ldots, v_{2^{d-1}}) \;=\; \cup_{i,i' \le 2^{d-1}, |v_i - v_{i'}|=1} W'(x_{v_i}, x_{v_{i'}})$$

We say that $v_1, ..., v_{2^{d-1}}$ form a hyperface of $S(\alpha, d, n)$ if $|v_i - v_j|_l = 0$ or 1 for all i, j and l, and there exists l such that $(v_i - v_j)_l = 0$ for all i and j. We say that $v_1, ..., v_{2^d}$ form a hypercube of $S(\alpha, d, n)$ if $|v_i - v_j|_l = 0$ or 1 for all i, j and l.

Given the graph, H'', constructed at the previous stage, and $v_1, \ldots v_{2^{d-1}}$ consider all possible connected subgraphs S'' of ω such that:

1. $S'' \subset R'' = \left(\cup_{z,z' \in \mathrm{CH}(v_1, \ldots, v_{2^{d-1}})} W'(z, z')\right) \setminus W'(v_1, \ldots, v_{2^{d-1}})$.
2. S'' extends $H''|_{R''}$.
3. Every vertex $y \in \mathrm{CH}(v_1, \ldots, v_{2^{d-1}})$ which is connected to H'' in $\omega|_{R''}$ is connected to H'' in S''.
4. Every vertex $y \in V(H'' \cup S'')$ is connected to H'' by a unique path.
5. That path has length $\le 3nd$.

Form the next distribution by giving all of these graphs equal mass. We form a distribution by choosing an extension of H'' independently for all hyperfaces. For each adjacent pair $v_1, v_2 \in S(\alpha, d, n)$ the distribution of $Q(v_1, v_2)$ is determined by ω the region

$$B_{dn/4\rho}(x_{v_1}) \cup B_{dn/4\rho}(x_{v_2})$$

Given the graph H''' constructed at the previous stage, and a hypercube $v_1, \ldots, v_{2^d}$, consider all possible connected subgraphs S''' of ω such that:

1. $S''' \subset R''' = CH(v_1, \ldots, v_{2^d})$.
2. S''' extends $H'''|_{R'''}$.
3. Every vertex $y \in \mathcal{C}_\infty(\mathbb{Z}^d, p) \cap CH(v_1, \ldots, v_{2^d})$ which is connected to H''' in $\omega|_{R'''}$ is connected to H''' in $H''' \cup S'''$.
4. Every vertex $y \in V(S'')$ is connected to H''' by a unique path.
5. That path has length $\leq 4nd$.

We define the measure μ_n by giving all of these graphs equal measure. We do this independently for all hypercubes. For each adjacent pair $v_1, v_2 \in S(\alpha, d, n)$ the distribution of $Q(v_1, v_2)$ is determined by ω in the region

$$B_{dn/2\rho}(x_{v_1}) \cup B_{dn/2\rho}(x_{v_2}).$$

Thus we can pick C_{13} such that, conditioned on the event that Q is a good n graph around 0 that spans $C_n(0)$, the random variables $Q(v_1, v_2)$ are C_{13} dependent.

If the appropriate n hubs exist and every pair in the region which are connected are very good then the proof of Lemma 6.5 shows that Q is good n graph around 0 that spans $C_n(0)$. Thus

$$\mu_n(Q \text{ is good } n \text{ graph around 0 that spans } C_n(0)) > 1 - A_{11}e^{-C_{12}n}.$$

The last condition in the lemma follows easily from the construction. □

Corollary 7.4 *If $e_1, \ldots, e_k$ are separated by at least C_{13} then the random variables $Q(e_1), \ldots, Q(e_k)$ are mutually independent conditioned on the event that none of the random variables $Q(e_1), \ldots, Q(e_k)$ are stunted.*

Proof: The distributions of $Q(e_1), \ldots, Q(e_k)$ are independent as they depend on ω in disjoint regions. The event that none of $Q(e_1), \ldots, Q(e_k)$ are stunted means that a certain event did not occur in any of those regions. Thus $Q(e_1), \ldots, Q(e_k)$ are conditionally independent. □

8 Bounding the return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$

First we bound the average of the return probabilities of a continuous time random walk over the measure we placed on good n graphs. Then we apply Theorem 3.1 to get bounds for the return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$.

Given any graph the continuous time random walk is defined as follows. Let Z^v be a continuous time Markov chain on the vertices, with $Z^v(0) = v$ and jumping rate one across each edge. The times of the jumps are determined by a family of Poisson processes with mean 1. To bound the return probabilities on a good n graph Q, we consider the projection of a walk on Q to a walk on the distinguished vertices of the n grid and then the projection onto $\mathbb{Z}^d$. A walk, P, is the sequence vertices visited by Z^v and P_t is the sequence of vertices visited up to time t. In the case that Q is a good n graph we define **proj($\mathbf{P}_t$)** to be the projection of the walk P to a nearest neighbor walk on $\mathbb{Z}^d$. First we eliminate every element of the sequence P_t which is not a distinguished vertex. Then we replace consecutive occurrences of the same distinguished vertex with one occurrence of that vertex. Finally we obtain the sequence $\text{proj}(P_t)$ by mapping this sequence of distinguished vertices into $\mathbb{Z}^d$ in the canonical way.

The idea of this section is as follows. Our main goal is to bound the return probabilities on good n graphs. We let $|\text{proj}(P_t)|$ represent the number of steps in $\text{proj}(P_t)$. If all walks started at v had projections that have the same length $L = |\text{proj}(P_t)|$ then, by using that the walk projected to $\mathbb{Z}^d$ is simple random walk, we could bound the return probability by $p_t^Q(v, v) \leq CL^{-d/2}$. Clearly this is not the case. However, on the n graphs which we call **great**, there is a large collection of walks such that $|\text{proj}(P_t)|$ falls in a small interval. In Lemma 8.1 we show that most good n graphs are great. Then in Lemma 8.2 we show that this is good enough so that it changes our bound on the return probabilities by only a logarithmic factor. Then, in Theorem 8.1, we apply Theorem 3.1 to bound the return probabilities on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$.

Given a good n graph Q and $v_1, v_2 \in S(\alpha, d, n)$ with $|v_1 - v_2| = 1$ define

$$T_Q(v_1, v_2) \quad = \quad \text{the expected amount of time that simple random walk started at } Q(v_1) \text{ takes before reaching } Q(v_2) \text{ conditioned on } Q(v_2) \text{ being the first distinguished}$$

vertex other than $Q(v_1)$ to be visited.

Also define

$$E = \mathbf{E}_{\mu_n}(T_Q(v_1, v_2)).$$

We let $N(v_1) = \{v : |v - v_1| = 1\}$. Then we can bound E by the cover time of

$$\cup_{w \in N(v_1)} Q((v_1, w))$$

as follows. We couple the distribution of simple random walk paths on $\cup_{w \in N(v_1)} Q((v_1, w))$ with the distribution of simple random walk paths on the same graph conditioned on the event that the first distinguished vertex (other than v_1) to be hit by the walk is v_2. This can be done such that the coupling is supported on pairs such that the path from v_1 to v_2 in the conditioned distribution is a subsequence of the path in the unconditioned distribution. As

$$| \cup_{w \in N(v_1)} Q((v_1, w))| \leq (8nd)^{d+1}$$

the result of [14] implies that the covering time of $\cup_{w \in N(v_1)} Q((v_1, w))$ is at most $(8nd)^{3d+3}$. This implies

$$E \leq (8nd)^{3d+3}. \tag{13}$$

Given any value of t choose n to be the largest integer such that $n \leq \log t$. This implies that there exists C_{14} such that $E \leq C_{14}(\log t)^{3d+3}$.

Fix a good graph Q and $v \in V(Q)$. We now define d_1 and d_2 to be the distinguished vertices closest to v. More precisely if v is not a distinguished vertex we let d_1 and d_2 be the distinguished vertices such that $v \in Q((d_1, d_2))$. If v is a distinguished vertex we choose d_1 and d_2 so that $Q(d_1) = Q(d_2) = v$.

Given a good graph Q, $z \in \{S(\alpha, d, n)\}^{(2^{\alpha n/2})}$ and a vertex of Q which we call v, consider the set of walks on Q starting at v that project onto z. We project the distribution of simple random walk on Q started at v to obtain a distribution $\mathbf{P}_{Q,z}$ on sequences $P_t \in \{S(\alpha, d, n)\}^{(2^{\alpha n/2})}$.

We define the set $Z = Z_{Q,v}$ to be the set of all $z \in \{S(\alpha, d, n)\}^{(2^{\alpha n/2})}$ such that

1. $z_1 = d_1$ or $z_1 = d_2$,

2. $|z_{i+1} - z_i| = 1$ for all i and

3. $\mathbf{P}_{Q,z}\left(|\text{proj}(P_t)| \notin (\frac{t}{E} - C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}, \frac{t}{E} + C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8})\right) < t^{-.51d}$,

where C_{15} will be defined later. Let $\tilde{V} \subset V(Q)$ be the set of all v such that

$$|Z^C| < t^{-.51d}(2d)^{2^{\alpha n/2}}.$$

We say a graph Q is **great for** t if it is a good n graph and

$$|\tilde{V}| > |V(Q)|(1 - t^{-.51d}).$$

The rest of this section is organized as follows. Our goal is to prove the following theorem.

Theorem 8.1 *There exists $C_{22} > 0$ such that*

$$\mathbf{E}(p_t^\omega(0,0) \mid 0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)) < C_{22} t^{-d/2}(\log t)^{6d+14}.$$

There are two main intermediate steps to proving this. First in Section 8.1 we prove

Lemma 8.1 *There exists C_{19} such that for any t*

$$\mu_n(Q \text{ is great for } t) \geq 1 - C_{19} t^{-.51d}.$$

Then in Section 8.2 we prove

Lemma 8.2 *There exists C_{21} such that for any n graph Q which is great for t*

$$\frac{1}{|V(Q)|}\sum_{v \in V(Q)} p_t^Q(v,v) \leq C_{21} t^{-d/2}(\log t)^{6d+14}.$$

Finally in Section 8.3 we combine Lemmas 8.1 and 8.2 to prove Theorem 8.1.

8.1 Probability of great graphs

In Section 7 we placed a measure on subgraphs of $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ and showed that most of them were good. In this subsection we will show that for all large t most of the subgraphs are great for t.

Lemma 8.3 *For any*

$$y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$$

consider the direct product of the measure on good graphs and the uniform distribution on the set of $z \in \{S(\alpha, d, n)\}^{(2^{\alpha n/2})}$ *such that* $z_1 = y$ *and* $|z_{i+1} - z_i| = 1$ *for all* i*. There exists* C_{16} *and* C_{17} *such that for any* $y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$ *the probability of pairs* Q, z *such that* Q *is not good or*

$$\sum_{1}^{\lfloor t/E \rfloor} T_Q(z_i, z_{i+1}) \notin (t - \frac{1}{2} C_{16} \sqrt{t} (\log t)^{3d+8}, t + \frac{1}{2} C_{16} \sqrt{t} (\log t)^{3d+8}) \qquad (14)$$

is less than $C_{17}(t/E)^{-\pi d}$*.*

Proof: For each directed edge

$$e = (v_1, v_2) \in E(S(\alpha, d, n))$$

we define an equivalence class by $e \sim f = (w_1, w_2)$ if $\exists k \in \mathbb{Z}^d$ such that

$$w_1 = v_1 + C_{13} k \text{ and } w_2 = v_2 + C_{13} k,$$

Label the equivalence classes $L_1, L_2, \ldots, L_{2dC_{13}{}^d}$. When we condition of the event that Q is a good n graph $T_Q(e_1), \ldots, T_Q(e_k)$ are mutually independent

By Lemma 7.3 the probability that $Q(e_1), \ldots, Q(e_k)$ are not stunted is at most $A_{11} e^{-C_{12} n}$. By Corollary 7.4 for any set $e_1, \ldots, e_k \in L_j$, with $e_i = e_l$ only if $i = l$, the random variables $T_Q(e_1), \ldots, T_Q(e_k)$ are mutually independent conditioned on the event that $Q(e_1), \ldots, Q(e_k)$ are not stunted.

As in line 13, for any distinguished vertices w and w' and Q

$$T_Q(w, w') < (8nd)^{3d+3}.$$

If the site in $S(\alpha, d, n)$ which z visits most often in the first (t/E) steps was visited at most $(d+1)(\log(t/E))^2$ times then

$$\sum_{i:(z_i, z_{i+1}) \in L_k} T_Q(z_i, z_{i+1}) = \sum_{(z, z') \in L_k} T_Q(z, z') |\{i : (z_i, z_{i+1}) = (z, z')\}|$$

is the sum of independent random variables which are bounded by

$$(d+1)(\log(t/E))^2 (8nd)^{3d+3} < C(\log t)^{3d+5}.$$

Their variance is bounded by $C^2(\log t)^{6d+10}$. Applying Bennett's Inequality (see [29] page 851) we see that if we let $H_k = |\{i : (z_i, z_{i+1}) \in L_k\}|$ then

$$\begin{aligned}
\mathbf{P}\left(\left|\sum_{i:(z_i,z_{i+1})\in L_k} T_Q(z_i, z_{i+1}) - EH_k\right| > 8Cd(\log t)^{3d+8}\sqrt{H_k}\right) &\leq e^{-(\frac{1}{4})(8d^2\log t)^2} \\
&\leq e^{-2d^2\log t} \\
&\leq t^{-2d^2}.
\end{aligned}$$

Thus there exists constants so that

$$\mathbf{P}\left(\sum_k \left|\sum_{i:(z_i,z_{i+1})\in L_k} T_Q(z_i, z_{i+1}) - EH_k\right| > C'\sqrt{t}(\log t)^{3d+8}\right) =$$

$$\mathbf{P}\left(\left|\sum_{i=1}^{\lfloor t/E\rfloor} T_Q(z_i, z_{i+1}) - t\right| > C'\sqrt{t}(\log t)^{3d+8}\right) \leq C''t^{-2d^2}.$$

Then

$$\sum_1^{\lfloor t/E\rfloor} T_Q(z_i, z_{i+1}) \in (t - \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8}, t + \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8}).$$

happens with probability at least $1 - C''t^{-2d^2}$.

By [13] the probability of z for which one vertex is visited too often (more than $(d+1)(\log(t/E))^2$ times) is less than $(t/E)^{-\pi d}$. Thus the probability of z and Q such that Q is not good or Q and z don't satisfy line 14 is less than $C_{17}(t/E)^{-\pi d}$. □

Lemma 8.4 *If*

$$y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$$

then for any good graph Q and z the conditional probability that

$$\left|\sum_1^{\lfloor t/E\rfloor} T_Q(z_i, z_{i+1}) - \inf\{k : |proj(P_k)| = t/E\}\right| > \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8}$$

given Q and z is less than t^{-2d^2}.

Proof: The sequence (indexed by i)

$$r_{Q,z,i}(P) = \inf\{k : |\text{proj}(P_k)| = i+1\} - \inf\{k : |\text{proj}(P_k)| = i\}$$

is independent.

Fix Q, z and i we can bound $\mathbf{P}(r_{Q,z,i}(P) > x)$ by looking at the time it takes for simple random walk on $\cup_{w' \in N(w)} Q((w, w'))$ started at w to hit $N(w)$ (for some appropriate choice of w). As for any Q and w the covering time of $\cup_{w' \in N(w)} Q((w, w'))$ is at most $(8nd)^{3d+3}$ for any Q, z and i and $x > 0$

$$\mathbf{P}(r_{Q,z,i}(P) > x) < 2\left(2^{-x/(8nd)^{3d+3}}\right).$$

This implies

$$\mathbf{E}\left((r_{Q,z,i}(P))^l\right) < 4(16nd)^{3d+3}(l!)(16nd)^{3d+3})^{l-2}/2.$$

Applying Bernstein's inequality (see [29] page 855) we get

$$\mathbf{P}_{Q,v}\left(\left|\sum_1^{\lfloor t/E \rfloor} T_Q(z_i, z_{i+1}) - \inf\{k : |\text{proj}(P_k)| = t\}\right| > \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8}\right) \tag{15}$$

is less than

$$exp\left\{-\frac{\left(C_{16}(\log t)^{3d+8}\right)^2}{8(16d\log t)^{3d+3}}\right\} \leq e^{-(\log t)^2} \leq t^{-2d^2} \tag{16}$$

for all t sufficiently large. □

Lemma 8.5 *If*

$$y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$$

then, with probability at least $1 - C_{17}(t/E)^{-\pi d}$, Q and z are such that the conditional probability that

$$\inf\{k : |proj(P_k)| = t/E\} \in (t - C_{16}\sqrt{t}(\log t)^{3d+8}, t + C_{16}\sqrt{t}(\log t)^{3d+8}) \tag{17}$$

is greater than $1 - t^{-2d^2}$.

Proof: If

$$\inf\{k : |\text{proj}(P_k)| = t/E\} \notin (t - C_{16}\sqrt{t}(\log t)^{3d+8}, t + C_{16}\sqrt{t}(\log t)^{3d+8})$$

then one of the following must have happened

1. $\sum_1^{\lfloor t/E \rfloor} T_Q(z_i, z_{i+1}) \notin (t - \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8}, t + \frac{1}{2}C_{16}\sqrt{t}(\log t)^{3d+8})$ or

2. $\left|\sum_1^{\lfloor t/E \rfloor} T_Q(z_i, z_{i+1}) - \inf\{k : |\text{proj}(P_k)| = t/E\}\right| > \frac{1}{2} C_{16}\sqrt{t}(\log t)^{3d+8}$.

By Lemma 8.3 the fraction of Q and z for which condition 1 is not satisfied is less than

$$C_{17}(t/E)^{-\pi d}.$$

By Lemma 8.4 for all good graphs Q and z the conditional probability that condition 2 is not satisfied is greater than $1 - t^{-2d^2}$. □

Lemma 8.6 *If*

$$y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$$

then with probability at least

$$1 - C_{17}(t/E)^{-\pi d/2}$$

Q and z are such that the conditional probability that

$$\inf\{k : |proj(P_k)| = \lfloor \frac{t}{E} + \frac{1}{2} C_{15} \left(\frac{t}{E}\right)^{\frac{1}{2}} (\log t)^{3d+8} \rfloor\} - \inf\{k : |proj(P_k)| = \lfloor \frac{t}{E} \rfloor\} \geq C_{16}\sqrt{t}(\log t)^{3d+8}, \tag{18}$$

and

$$\inf\{k : |proj(P_k)| = \lfloor \frac{t}{E} \rfloor\} - \inf\{k : |proj(P_k)| = \lfloor \frac{t}{E} - \frac{1}{2} C_{15} \left(\frac{t}{E}\right)^{\frac{1}{2}} (\log t)^{3d+8} \rfloor\} \geq C_{16}\sqrt{t}(\log t)^{3d+8} \tag{19}$$

is at least $1 - t^{-d^2}$.

Proof: We choose C_{15} so that $C_{15}(t/E)^{1/2}(\log t)^{3d+8} > 3C_{16}\sqrt{t}(\log t)^{3d+8}$. The proof of this lemma is the same as the proof of the previous three except that t/E is replaced by $\frac{1}{2} C_{15}(t/E)^{1/2}(\log t)^{3d+8}$. □

Let $A_{Q,y,z}$ be the event that either line 17, 18, or 19 is not satisfied for y, Q, and z.

Combining Lemmas 8.5 and 8.6 we get

Lemma 8.7 *If*

$$y \in [-2^{\alpha n} + 2^{\alpha n/2}, 2^{\alpha n} - 2^{\alpha n/2}]^d$$

then

$$\mathbf{P}(A_{Q,y,z}) < 3C_{17}(t/E)^{-\pi d/2}$$

Lemma 8.8 *If*

1. $v \in Q((y_1, y_2))$,
2. $z_1 = y_1$, *and*
3. $z \notin Z_{Q,v}$

then $A_{Q,y_1,z}$ *occurs.*

Proof: If, for a given Q, y_1 and z lines 17, 18, and 19 are all satisfied then the conditional probability that

$$|\mathrm{proj}(P_t)| - \inf\{k : |\mathrm{proj}(P_k)| = 1\}$$

is in the interval

$$(\frac{t}{E} - \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}, \frac{t}{E} + \frac{1}{2}C_{15}(\frac{t}{E})^{\frac{1}{2}}(\log t)^{3d+8})$$

is at least $1 - 3t^{-2d^2}$.

We see this as follows. Line 17 says that (with high probability) the time it took to go from the first distinguished vertex to the t/Eth distinguished vertex is at most $t + C_{16}\sqrt{t}(\log t)^{3d+8}$. Line 18 says that (with high probability) at least $C_{16}\sqrt{t}(\log t)^{3d+8}$ time passed between when we hit the $\frac{t}{E} - \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}$ distinguished vertex and the t/Eth distinguished vertex. Combining these two says that (with high probability) at time t we have hit at least $\frac{t}{E} - \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}$ distinguished vertices. Similarly combining Lines 17 and 19 show that (with high probability) at time t we have hit at most $\frac{t}{E} + \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}$ distinguished vertices. Thus if they all are satisfied then (with high probability) at time t we have hit between $\frac{t}{E} - \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}$ and $\frac{t}{E} + \frac{1}{2}C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}$ distinguished vertices.

As the probability that

$$\inf\{k : |\mathrm{proj}(P_k)| = 1\} > \frac{1}{2}C_{15}(\frac{t}{E})^{\frac{1}{2}}(\log t)^{3d+8})$$

is decreasing exponentially in $\sqrt{t}$, the conditional probability that $|\mathrm{proj}(P_t)|$ is in the interval

$$\left(\frac{t}{E} - C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}, \frac{t}{E} + C_{15}\left(\frac{t}{E}\right)^{\frac{1}{2}}(\log t)^{3d+8}\right)$$

is at least $1 - 4t^{-2d^2} > 1 - t^{-.51d}$, for t sufficiently large. □

Lemma 8.9 *There exists C_{18} such that the expected value (averaging over μ_n) of*

$$\frac{1}{|v(Q)|}\sum_{v\in V(Q)}|\{z\notin Z_{Q,v}\}| \tag{20}$$

is less than

$$C_{18}(2d)^{2^{\alpha n/2}}(t/E)^{-\pi d/2}(\log t)^{d+1}.$$

Proof: As there are at most $(8nd)^{d+1}$ vertices attached to the n grid between any two distinguished vertices

$$\begin{aligned}\sum_{v\in V(Q)}|\{z\notin Z_{Q,v}\}| &\le 2d(8nd)^{d+1}\sum_{y\in S(\alpha,d,n)}\mathbf{P}(A_{Q,y,z})\\ &\le 2d(8nd)^{d+1}\sum_{y\notin[-2^{\alpha n}+2^{\alpha n/2},2^{\alpha n}-2^{\alpha n/2}]}\mathbf{P}(A_{Q,y,z})+\\ &\quad 2d(8nd)^{d+1}\sum_{y\in[-2^{\alpha n}+2^{\alpha n/2},2^{\alpha n}-2^{\alpha n/2}]}\mathbf{P}(A_{Q,y,z}).\end{aligned}$$

The previous lemmas imply that the expected value of the right hand side of the last inequality is at most

$$C(2d)^{2^{\alpha n/2}}|S(\alpha,d,n)|(t/E)^{-\pi d/2}.$$

Thus the expected value of line 20 is at most

$$C_{18}(2d)^{2^{\alpha n/2}}(t/E)^{-\pi d/2}(\log t)^{d+1}.$$

□

Proof of Lemma 8.1: If Q is not great for t then the number of $v\in V(Q)$ with $v\notin\tilde{V}$ is at least $(t^{-.51d})|V(Q)|$. Thus line 20 is at least

$$(t^{-.51d})(t^{-.51d})(2d)^{2^{\alpha n/2}}.$$

By Lemma 8.9 the expected value of line 20 is at most

$$C_{18}(2d)^{2^{\alpha n/2}}(t/E)^{-\pi d/2}(\log t)^{d+1}.$$

This implies that

$$\begin{aligned}&\mathbf{P}(\text{the } n \text{ graph centered around 0 is great for } t)\\ &\quad\ge 1-\left(C_{18}(2d)^{2^{\alpha n/2}}(t/E)^{-\pi d/2}(\log t)^{d+1}\right)/\left((t^{-.51d})(t^{-.51d})(2d)^{2^{\alpha n/2}}\right)\\ &\quad\ge 1-C_{19}t^{-.51d}.\end{aligned}$$

□

8.2 Return probabilities on great graphs

In this subsection we show that for any t we can bound the average return probability at time t on a graph that is great for t.

Lemma 8.10 *There exists C_{20} such that for any good graph Q, vertex v and path $z \in Z$, and any $i \in (t/E - C_{15}\sqrt{t/E}(\log t)^{3d+8}, t/E + C_{15}\sqrt{t/E}(\log t)^{3d+8})$*

$$\mathbf{P}_{Q,v}(|proj(P(t))| = i) \leq C_{20}(t)^{-1/2}(\log t)^{3d+6}.$$

Proof: Fix $i \in (t/E - C_{15}\sqrt{t/E}(\log t)^{3d+8}, t/E + C_{15}\sqrt{t/E}(\log t)^{3d+8})$. If

$$|\text{proj}(P(t))| = i \tag{21}$$

then either

$$(\inf\{k : \ |\text{proj}(P(k))| = i+1\}) - (\inf\{k : \ |\text{proj}(P(k))| = i\}) \geq (8nd)^{3d+4} \tag{22}$$

or

$$m = \inf\{k : \ |\text{proj}(P(k))| = i\} \in (t - (8nd)^{3d+4}, t). \tag{23}$$

For any Q and distinguished vertex w, the covering time of $\cup_{w' \in N(w)} Q((w, w'))$ is at most $(8nd)^{3d+3}$. Thus the conditional probability of line 22 given Q and z is at most

$$2^{-(8nd)^{3d+4}/(8nd)^{3d+3}} \leq 2^{-8nd} \leq t^{-2d}.$$

To bound the probability of line 23 we condition on any $z' \in V(Q)^{\mathbb{N}}$ which is consistent with z. Let j be such that the ith distinguish vertex of z' is z'_j. The distribution of m given z' is the distribution of the jth occurrence of a Poisson process with parameter one. Call that distribution $D_j(m)$. Since

$$\sup_m D_j(m) \leq Cj^{-\frac{1}{2}} \leq Ci^{-1/2} \leq C\left(t/E - C_{15}\sqrt{t/E}(\log t)^{3d+8}\right)^{-1/2} \leq C'(t/E)^{-1/2}$$

the probability of line 23 conditioned on z' is less than $C'(t/E)^{-1/2}(8nd)^{3d+4}$. As this bound holds for all z' consistent with z, the probability of line 23 conditioned on z is less than

$$C'(t/E)^{-1/2}(\log t)^{3d+4} \leq C_{20}(t)^{-1/2}(\log t)^{3d+6}.$$

□

Proof of Lemma 8.2: Let $t_1 = t/E - C_{15}\sqrt{t/E}(\log t)^{3d+8}$ and $t_2 = t/E + C_{15}\sqrt{t/E}(\log t)^{3d+8}$. We have a set $\tilde{V}$ such that

$$|\tilde{V}| > (1 - t^{-.51d})|V(Q)|.$$

For a fixed $v \in \tilde{V}$ and any $w \in S(\alpha, d, n)$

$$\begin{aligned}
&\mathbf{P}(\bar{P}_t = w \mid P_0 = v) \\
&\quad = \sum_i \sum_{z:z_i = w} \mathbf{P}_{Q,v}(|\mathrm{proj}(P_t)| = i) \cdot P(z) \\
&\quad \le \sum_{i \in (t_1,t_2)} \sum_{z \in Z, z_i = w} \mathbf{P}_{Q,v}(|\mathrm{proj}(P_t)| = i) \cdot P(z) \\
&\qquad + \sum_{i \notin (t_1,t_2)} \sum_{z \in Z : z_i = w} \mathbf{P}_{Q,v}(|\mathrm{proj}(P_t)| = i) \cdot P(z) \\
&\qquad + \sum_i \sum_{z \notin Z} \mathbf{P}_{Q,v}(|\mathrm{proj}(P_t)| = i) \cdot P(z) \\
&\quad \le |\{i, z : i \in (t_1, t_2), z_i = w\}| \cdot \max_{i,Q}(\mathbf{P}_{Q,v}(|\mathrm{proj}(P_t)| = i)P(z)) + 2t^{-.51d} \\
&\quad \le \left(2(2d)^{(2^{\alpha n/2})} t_1^{-d/2} 2C_{15}\sqrt{t/E}(\log t)^{3d+8}\right) \left(C_{20}(\log t)^{3d+6} t^{-1/2}\right) \left((2d)^{-(2^{\alpha n/2})}\right) \\
&\qquad + 2t^{-.51d} \qquad\qquad (24) \\
&\quad \le C(\log t)^{6d+14} t^{-d/2}.
\end{aligned}$$

Line 24 follows from Lemma 8.10.

Thus for any $v \in \tilde{V}$

$$p_t^Q(v, v) \le 2\sup(\bar{P}_t = w \mid P_0 = v) \le 2C(\log t)^{6d+14} t^{-d/2}.$$

Thus

$$\begin{aligned}
\frac{1}{|V(Q)|} \sum_{v \in V(Q)} p_t^{\mathcal{C}_\infty(\mathbb{Z}^d,p)}(v, v) &\le \frac{1}{|V(Q)|} \sum_{v \in \tilde{V}} p_t^{\mathcal{C}_\infty(\mathbb{Z}^d,p)}(v, v) \\
&\quad + \frac{1}{|V(Q)|} \sum_{v \notin \tilde{V}} p_t^{\mathcal{C}_\infty(\mathbb{Z}^d,p)}(v, v) \\
&\le 2Ct^{-d/2}(\log t)^{6d+14} + t^{-.51d} \\
&\le C_{21} t^{-d/2}(\log t)^{6d+14}.
\end{aligned}$$

□

8.3 Return probabilities on the percolation cluster

Now we are ready to prove our main theorem.

Proof of Theorem 8.1: The idea for the proof is as follows. We intersect $\mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m$ with evenly spaced boxes that cover most of this region, but are separated by a sufficient distance when we construct good n graphs in adjacent boxes, they will not intersect. By dividing the cluster this way, it is possible to analyze the behavior of the random walk on each piece separately. It is possible to find an n graph which is great for t in most of these boxes. Then we are able to apply Theorem 3.1 to each piece and average.

Let $\theta = \theta(p, d)$ be the density of $\mathcal{C}_\infty(\mathbb{Z}^d, p)$. For any vertex v let $Q_n(v) \subset \omega$ be the n graph centered around v if it is great for t. Define

$$U_m = \left(\cup_{v \in (10n + 2n2^{\alpha n}/5\rho)\mathbb{Z}^d} Q_n(v)\right) \cap B_m.$$

Notice that U_m is the disjoint union of n graphs which are great for t so we have bounds for $p_t^{U_m}(v, v)$ from Lemma 8.2

By the ergodic theorem,

$$\theta \mathbf{E}(p_t^\omega(0,0) \mid 0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)) = \lim_m \frac{1}{|B_m|} \sum_{v \in \mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m} p_t^{\mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m}(v, v). \tag{25}$$

We apply Theorem 3.1 to the graph $\tilde{U}_m$, which is defined to have the same vertices as $\mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m$ and the same edges as U_m.

$$\begin{aligned}
\lim_m \frac{1}{|B_m|} \sum_{v \in \mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m} p_t^{\mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m}(v, v) &\leq \lim_m \frac{1}{|B_m|} \sum_{v \in \mathcal{C}_\infty(\mathbb{Z}^d, p) \cap B_m} p_t^{\tilde{U}_m}(v, v) \\
&\leq \lim_m \frac{1}{|B_m|} \left(\sum_{v \in U_m} p_t^{U_m}(v, v) + \sum_{v \in (\tilde{U}_m) \setminus U_m} 1 \right) \\
&\leq C_{21} t^{-d/2} (\log t)^{6d+14} + C_{19} t^{-.51d} \\
&\quad + C dn / 2^{\alpha n} \qquad (26) \\
&\leq C_{22} t^{-d/2} (\log t)^{6d+14}.
\end{aligned}$$

Line 26 follows from Lemmas 8.1 and 8.2. □

9 The Uniform Spanning Forest on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$

In this section we show that two independent simple random walks on $\mathcal{C}_\infty(\mathbb{Z}^d, p), d \geq 5$, have finite expected number of intersections. We then define the wired, free and uniform spanning forests on a graph G. We will combine the expectation result with some results about spanning forests to show that the USF on $\mathcal{C}_\infty(\mathbb{Z}^d, p), d \geq 5$, is supported on graphs with infinitely many connected components.

Choose ω such that $0 \in \mathcal{C}_\infty(\mathbb{Z}^d, p)$. Let $\{Z(t)\}_{t\geq 0}$ and $\{Z'(t)\}_{t\geq 0}$ be two independent copies of simple random walks on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ with $Z(0) = Z'(0) = 0$. Let $Z(t)$ represent the vertex that the simple random walk occupies at time t. Given a vertex a in the percolation cluster let $\{X_a(t)\}_{t\geq 0}$ be a simple random walk in the percolation cluster started at a ($X_a(0) = a$). Let $I(Z, Z', \mathcal{C}_\infty(\mathbb{Z}^d, p)) =$ the number of intersections of Z and Z'.

Theorem 9.1 *For almost every ω the number of intersection of two independent random walks is finite a.s. In other words, with probability one*

$$I(Z, Z', \mathcal{C}_\infty(\mathbb{Z}^d, p)) < \infty.$$

Proof: The following argument was indicated to us by Russell Lyons. We only need to show that the expected value of $I(Z, Z', \mathcal{C}_\infty(\mathbb{Z}^d, p))$ is finite.

$$\begin{aligned}
\mathbf{E}(I(Z, Z', \mathcal{C}_\infty)) &\leq \mathbf{E}\left(\sum_k \sum_j \sum_{a\in\mathbb{Z}^d} \mathbf{P}(Z(k) = a \text{ and } Z'(j) = a)\right) && (27)\\
&\leq \mathbf{E}\left(\sum_k \sum_j \sum_{a\in\mathbb{Z}^d} \mathbf{P}(Z(k) = a)\mathbf{P}(Z'(j) = a)\right) && (28)\\
&\leq C\mathbf{E}\left(\sum_k \sum_j \sum_{a\in\mathbb{Z}^d} \mathbf{P}(Z(k) = a)\mathbf{P}(X_a(j) = 0)\right) && (29)\\
&\leq C\mathbf{E}\left(\sum_k \sum_j \sum_{a\in\mathbb{Z}^d} \mathbf{P}(Z(k) = a \text{ and } Z(k+j) = 0)\right) && (30)\\
&\leq C\mathbf{E}\left(\sum_k \sum_j \mathbf{P}(Z(k+j) = 0)\right) && (31)
\end{aligned}$$

$$\leq \; C\mathbf{E}\left(\sum_n (n+1)\mathbf{P}(Z(n)=0)\right) \tag{32}$$

$$\leq \; C' \sum_n n^{1-d/2} C''(\log n)^{6d+14} \tag{33}$$

$$< \; \infty. \tag{34}$$

Line 27 is $\leq$ because you might double count intersections. Line 28 is because of the independence of Z and Z'. Line 29 is true for the following reason. For each path from 0 to a in the percolation cluster there is a corresponding path from a to 0 (just go in reverse). These two paths have probabilities that differ by a constant. The probability of a path from 0 to a is 1 over the product of the degrees of the vertices at the beginning of each segment. The probability of the corresponding path from a to 0 is the product of the degrees of the vertices at the end of each segment. Since these sets of vertices only differ by one element, these two probabilities differ by a factor of the degree of a / degree of 0. Line 31 is because of the independence of X and Z and the independence of the increments in simple random walk. Line 33 follows from Theorem 8.1 and because the sum of the return times for discrete time simple random walk is bounded by a constant times the sum of the return probabilities for continuous time simple random walk. □

We now define spanning forests. Given a finite graph $G = (V, E)$, a spanning tree of that graph is a connected subgraph which contains no cycles and includes all the vertices. Define a probability measure on spanning trees by assigning each spanning tree equal measure and normalizing. Given an infinite graph G, write G as the increasing union of finite graphs G_n. Pemantle shows in [25] that the weak limit of the sequence of finite measures arising from each G_n exists. Furthermore it is independent of the exhaustion. This limit is called the free spanning forest (FSF). Likewise, we define the wired spanning forest (WSF), except now we impose wired boundary conditions. Namely, for each G_n, wire all the edges on the boundary of G_n to a single vertex outside of G_n. Define a sequence of measures on the new finite graphs. Again, this sequence has a weak limit that is independent of exhaustion [25],[18]. This limit is the wired spanning forest. In the case when the wired and free spanning forests coincide, the measure is called the uniform spanning forest. If $G = \mathbb{Z}^d$, then the wired and free spanning forests coincide [25]. In [7], Benjamini, Lyons and Schramm show that the wired and free spanning forests coincide on percolation clusters in $\mathbb{Z}^d$.

In order to show that the USF is supported on infinitely many components, we use some theorems about the wired spanning forest. Let $\alpha(w_1, ...w_k)$ be the probability that independent random walks started at $w_1, ..., w_k$ have no pairwise intersections.

Theorem 9.2 *[6] Let G be a connected network. The number of trees of the WSF is a.s.*

$$\sup\{k \; : \; \exists w_1, ...w_k, \; \alpha(w_1, ..., w_k) > 0\}.$$

Moreover, if the WSF on a graph has finitely many components, then the dimension of the bounded harmonic functions on the graph is equal to the number of components of the WSF.

In particular, the number of trees in a configuration of the WSF is a.s. constant. This theorem, along with Theorem 9.1, implies that the WSF is supported on at least 2 components a.s. We can rule out finitely many components because of the following fact.

Theorem 9.3 *[7],[21] Percolation clusters in $\mathbb{Z}^d$ admit no nonconstant bounded harmonic functions.*

Putting these results together gives the following theorem.

Theorem 9.4 *The USF on $\mathcal{C}_\infty(\mathbb{Z}^d, p)$ $d \geq 5$ is supported on graphs with infinitely many connected components.*

Acknowledgement The authors would like to thank Russell Lyons, Itai Benjamini, and Oded Schramm for suggesting this problem. We would also like to thank Lyons, Benjamini, and Schramm, Yuval Peres, Peter Hoff, Jon Wellner, and Elchanan Mossel for providing references and helpful advice. Finally we wish to thank the referee for helpful comments and pointing out an error in a previous version of this paper.

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