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Propagating Lyapunov functions to prove noise–induced stabilization

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Abstract

We investigate an example of noise-induced stabilization in the plane that was also considered in (Gawedzki, Herzog, Wehr 2010) and (Birrell, Herzog, Wehr 2011). We show that despite the deterministic system not being globally stable, the addition of additive noise in the vertical direction leads to a unique invariant probability measure to which the system converges at a uniform, exponential rate. These facts are established primarily through the construction of a Lyapunov function which we generate as the solution to a sequence of Poisson equations. Unlike a number of other works, however, our Lyapunov function is constructed in a systematic way, and we present a meta-algorithm we hope will be applicable to other problems. We conclude by proving positivity properties of the transition density by using Malliavin calculus via some unusually explicit calculations.

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1 Introduction

Stabilization by noise is a mathematically intriguing phenomenon. For instance, in the classic example of the inverted pendulum, the addition of noise opens up a small neighborhood of local stability around a deterministically unstable fixed point [2, 17]; in the striking examples of [28], the addition of noise leads to global stabilization. In general, however, there are few rigorous proofs of this phenomenon for specific systems, and most existing proofs depend upon correctly "guessing" a Lyapunov function and then verifying that it satisfies the requisite properties.

In three recent, interesting works [16, 10, 11], a global Lyapunov function is constructed by patching together functions which are locally Lyapunov in a collection of regions whose union covers all of the possible routes to infinity. These papers are concerned with specific examples in which the stabilization by noise is a property of the global dynamics rather than the local dynamics near a fixed point. As such, they are

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closer in spirit to [28] than to examples such as the inverted pendulum. In these examples, the noise is only important in localized regions of phase space, but its effect is global, in that it changes the global nature of the flow. This nature is hinted at in the patchwork constructions used in [16, 10, 11]. In [11], this structure is the most explicit; there, local asymptotic expansions are used to construct a patchwork of local Lyapunov functions. Still in all three works, the local constructions have mainly the flavor of “guess–and–check" with some information of the presumed overall structure of the transport in phase space.

Here we take a more systematic approach to proving global stabilization by noise; we outline a meta-algorithm which we hope can be used to produce Lyapunov functions in a number of different dynamical systems. Inspired by the examples in [16, 10, 11], we apply our meta-algorithm to a system of stochastic differential equations in the plane whose underlying deterministic dynamics display finite-time blow-up for certain initial conditions, but in which the addition of an arbitrarily small amount of noise leads to an invariant probability measure. We consider essentially the same system as in [16] and one of the examples in [10] with the specific choice of parameters $\alpha_1 = \alpha_2 = 1$ and the change of coordinates induced by $(x, y) \mapsto (-x, -y)$. The choice of parameters is representative of the stable regime we are interested in. We demonstrate the existence of an invariant measure by constructing a Lyapunov function. Our general approach is to build local Lyapunov functions as solutions to associated partial differential equations (PDEs), where the PDEs are defined in regions delineated by different asymptotic behaviors of the flow. While it is related to that in [16, 10], the Lyapunov function we construct might better be called a “super” Lyapunov function, in that it enables us to prove a stronger form of convergence than in [16, 10]. Our exponential convergence results apply equally well to the case of degenerate stochastic forcing while those in [10] only prove exponential convergence in the uniformly elliptic setting. Our analysis also adapts to the specifics of the problem and is likely to produce closer-to-optimal results.

In [14, 23, 15, 27], the scaling limit of a discrete time Markov chain, called the “fluid limit” in the context of these works, is used to build a Lyapunov function. In some ways this is related in spirit to our work in this paper. However, we are explicitly interested in the case where the noise and its fluctuations are fundamentally important to the behavior at infinity. The naive fluid limit model, however, is a singular limit and hence does not capture the behavior of those systems in which noise plays an essential role. We will see in our example that the naive fluid limit—which is the underlying unperturbed dynamical system—is in fact unstable. Our constructions capture the essential stochasticity in the regions where it matters at infinity.

Combining our Lyapunov function with a result on the positivity of transition densities, we prove a strong result on the convergence to equilibrium of our specific dynamical system. Though the positivity result is neither the most general nor the most powerful, it is nevertheless of independent interest, since the proof employs sophisticated ideas from control theory and Malliavin calculus in a very concrete and transparent way. We hope that it will help develop the readers intuition in such matters.

2 Lyapunov Functions

We are interested in the stability of a Markov process (X_t, Y_t) which is the solution to a stochastic differential equation (SDE) on the state space $\mathbb{R}^2$ with generator L. In the deterministic setting, a Lyapunov function is a positive function of the state space which decreases, often exponentially, along trajectories. In the stochastic setting, one requires that the function decrease on average. More precisely, we define a Lyapunov function V on an unbounded set $\mathcal{R} \subset \mathbb{R}^2$ as follows:

Definition 2.1. *A C^2 function $V : \mathcal{R} \to (0, \infty)$ is a* Lyapunov function *on $\mathcal{R}$ if*

1. *$V(x, y) \to \infty$ as $|(x, y)| \to \infty$ with $(x, y) \in \mathcal{R}$,*
2. *there exist constants $m, b > 0$ and $\gamma > 0$ such that for all $(x, y) \in \mathcal{R}$,*

$$(LV)(x, y) \le -mV^{\gamma}(x, y) + b\,.$$

We say that V is a super Lyapunov function *on $\mathcal{R}$ if $\gamma > 1$ and a* standard Lyapunov function *on $\mathcal{R}$ if $\gamma = 1$. We call γ the* Exponent *of the Lyapunov function. If $\mathcal{R}$ is a strict subset of $\mathbb{R}^2$, we say that V is a* local *(super/standard) Lyapunov function; if $\mathcal{R}$ is all of $\mathbb{R}^2$, we say that V is a* global *(super/standard) Lyapunov function.*

We remark that there are several different notions of a Lyapunov function in the literature, but the one above will be used in this particular paper. See for example [22, 26, 28, 29, 30].

Remark 2.2. *Notice that the continuity of V coupled with its growth at infinity implies that the sub-level sets $\{(x, y) \in \mathcal{R} : V(x, y) \le R\}$ are compact for all R. In a certain sense, this is the more fundamental condition, but we will not belabor this point here. See, for example, Proposition 5.1 in [20] for more details.*

It is well-known that the existence of a global Lyapunov function satisfying the properties in Definition 2.1 implies the existence of an invariant measure [22, 28, 26, 20]. If one adds a mild mixing/minorization condition and assumes that the Lyapunov function is a standard Lyapunov function, it is possible to prove exponential convergence to this invariant measure[26, 18]. If the Lyapunov function is weaker ($\gamma < 1$), then the convergence is generally slower [26, 29, 30, 13, 19]. in this case, one might rightly call V a sub Lyapunov function.

For the system considered in this paper, we show the existence of a global super Lyapunov function, along with needed mild mixing/minorization conditions. Together, these imply that the rate of convergence to equilibrium is not only exponential, but also independent of initial condition.

2.1 General Construction Strategy

We now give an outline of our general approach to constructing a Lyapunov function. Since many of the details of the implementation depend on the specific example under consideration, this outline is meant as an overall rubric. On first reading, this section may seem rather heuristic and overly vague. We encourage the reader to take it as motivation at first and then reread this section after Section 8 and Section 9; in these later sections, the following abstract discussion is made concrete.

In our general construction algorithm, we begin by identifying a region in phase space where there is an obvious choice of a Lyapunov function. We refer to this region as the “priming” region; it is often characterized as a subset of phase space in which the deterministic flow is directed toward the origin. We refer to the associated local Lyapunov function in this region as the “priming” Lyapunov function. Next, by contrast, we identify the region in which the deterministic dynamics exhibit instability—for example, blow-up in finite time—and for which noise is essential to the stabilization, at least insofar as the noise ensures that the system leaves this region. Since this region is noise-dominated to some degree, we refer to this region as the “diffusive" region. The construction of a local Lyapunov function in this diffusive region is a key component of our methodology. We then “propagate” the priming Lyapunov function, from the priming region to the diffusive region, through a series of intermediate regions of phase space, which we call “transport" regions, until we have covered all possible routes to infinity. Following this prescription, we obtain a sequence of local Lyapunov functions, which we mollify to obtain a global Lyapunov function. Beyond this general, overarching strategy is a philosophy of determining the relevant scaling at infinity in each of the above regions and systematically producing local Lyapunov function which respect this scaling.

To determine more precisely the boundaries of these regions, we study the scaling of the generator L of the SDE as $|(x, y)| \to \infty$. Each region corresponds to a different dominant balance of the terms in the generator. (See [9, 31] for a discussion of the concept of dominate balances in asymptotic analysis and Section 8 for the details in our setting.) In order to facilitate the mollification of the local Lyapunov functions, we choose the regions so that the intersection between adjoining regions is both nonempty and unbounded. Neglecting all but the terms involved in the associated dominant balance, each region also has a differential operator associated to it which captures the dominant behavior of the generator in the region as $|(x, y)| \to \infty$. Beginning with the region adjacent to the priming region, we propagate the priming Lyapunov function through the adjacent region by solving an associated Poisson equation of the form

$$\begin{cases} (\tilde{L}v)(x, y) = -f(x, y) \\ \quad v(x, y) = g(x, y) \text{ on the boundary} \,. \end{cases}$$

The differential operator $\tilde{L}$ is governed by the dominant balance in the region under consideration. Since it represents a dominate balanced at infinity, it is necessarily an operator which scales homogeneously. Hence if the righthand side and the boundary conditions are chosen to scale homogeneously at infinity in a compatible way the solution v will also scale homogeneously at infinity. The boundary data g for the Poisson equation is given by the dominant behavior/scaling of the priming Lyapunov function on the boundary between the priming and adjacent region. The right-hand side, f, of the Poisson equation is chosen to be a positive definite function which grows unboundedly and satisfies certain scaling properties that we specify in Section 9.3 and that are compatible with the scaling of the region.

We iterate this procedure to construct local Lyapunov functions as solutions to associated Poisson equations in each of the transport regions. Furthermore, we construct a Lyapunov function in the diffusive region by solving a Poisson equation as well, again with the boundary data determined by the dominant behavior of the local Lyapunov function in the adjacent transport region. An advantage of this approach, therefore, is its consistency: the same procedure is used to construct local Lyapunov functions in all but the priming region (where the local Lyapunov function is usually straightforward to deduce).

As we solve the sequence of Poisson equations, we encounter boundaries without boundary data. While *a priori* this could be an issue, we will see that in our model problem, in all but the diffusive region, the deterministic dynamics are dominant. Hence the associated Poisson equations are governed by first-order operators requiring only one boundary condition. This is consistent with the idea of the priming Lyapunov function being propagated through a sequence of transport regions. Again, *a priori* this could lead to an incompatibility between two different boundaries of a given region, particularly if the relevant operator in a region is only first-order and cannot accept generic initial data on two boundaries. However, in our model problem and all of the other problems we have explored, sequences of compatible transport regions are separated from each other by diffusive regions. Since the associated differential operator in the diffusive region is second-order, the associated Poisson equation produces a smooth solution even with all of its boundary data specified.

3 The Model Problem

As our model problem, we consider essentially the same problem as in [16, 10] and which was suggested to us by one of the authors:

$$\begin{aligned} dX_t &= (X_t^2 - Y_t^2)dt + \sqrt{2\sigma_x}\, dW_t^{(1)} \\ dY_t &= 2X_tY_tdt + \sqrt{2\sigma_y}\, dW_t^{(2)} \end{aligned} \tag{3.1}$$

with $\sigma_x \geq 0$ and $\sigma_y \geq 0$. Notice that when

$(\sigma_x, \sigma_y) = (0,0)$, the resulting deterministic equation blows up in finite time if $x_0 > 0$ and $y_0 = 0$. In light of this, it is striking that for any $\sigma_y > 0$, system (3.1) has a unique invariant probability measure π. This was first proven in [16] and is also a consequence of one of our main results, which is given below in Theorem 3.2. In the sequel to [10], the authors prove exponential convergence to equilibrium. The principal difficulty in both of these works was the establishment of a standard Lyapunov function.

Let P_t be the Markov semi-group associated to the process (X_t, Y_t) and defined by

$$(P_t\phi)(x,y) = \mathbb{E}_{(x,y)}[\phi(X_t,Y_t)]\,. \tag{3.2}$$

Define the action of P_t on a probability measure μ by $(\mu P_t)(A) = \int P_t(x,A)\mu(dx)$ for any measurable set A. An invariant probability measure μ is any measure such that $\mu P_t = \mu$ for all t.

We prove the following theorem, which is stronger than the previously cited results on the existence of a standard Lyapunov function.

Theorem 3.1. *There exists a C^2 function $V : \mathbb{R}^2 \to (0,\infty)$ which is a super Lyapunov function for the dynamics given by (3.1). More exactly, for any choice of $\delta \in (0, \frac{2}{5})$ the super Lyapunov function V can be chosen to have an exponent $\frac{5\delta+5}{5\delta+3}$ and satisfy $c|(x,y)|^\delta \leq V(x,y) \leq C|(x,y)|^{\frac{5}{2}\delta+\frac{3}{2}}$ for some positive constants c and C.*

The existence of an invariant measure μ is an easy consequence of this theorem. To determine rates of convergence to the equilibrium measure μ, we introduce the following family of weighted total variation metrics. For $\beta > 0$ and probability measures μ_1 and μ_2, we define

$$\rho_\beta(\mu_1,\mu_2) = \sup_{\|\phi\|_\beta \leq 1} \int \phi(z)(\mu_1 - \mu_2)(dz) \tag{3.3}$$

where

$$\|\phi\|_\beta = \sup_z \frac{|\phi(z)|}{1+\beta V(z)}\ .$$

Notice that ρ_0 is just the standard total-variation norm.

The standard Lyapunov function and supporting estimates developed in [10] essentially show that there exists positive C and η so that

$$\rho_1(\mu_1 P_t, \mu_2 P_t) \leq Ce^{-\eta t}\rho_1(\mu_1,\mu_2)$$

for any probability measures μ_1 and μ_2. Using Theorem 3.1 on the existence of a super Lyapunov function, we establish the following stronger convergence result:

Theorem 3.2. *If $\sigma_y > 0$, then for any $\beta \geq 0$ there exist positive C and η so that for all probability measures μ_1 and μ_2 one has*

$$\rho_\beta(\mu_1 P_t, \mu_2 P_t) \leq Ce^{-\eta t}\|\mu_1 - \mu_2\|_{TV}$$

for all $t \geq 0$, where $||_{TV}$ represents the total variation norm. Here the constant C depends on the choice of β but the constant η does not.

The strength of this result is that the dominating norm on the right-hand side is scale and translation invariant. As we will see, when a super Lyapunov function exists, one can usually prove a stronger result than the standard Harris-type ergodic theorem associated to a standard Lyapunov function.

Remark 3.3. *As already mentioned, the existence of an invariant measure μ follows quickly from Theorem 3.1 or Theorem 3.2. The fact that there is only one invariant measure is immediate from Theorem 3.2.*

One consequence of our estimates is the following information on the unique invariant measure μ. The proof of Theorem 3.4 below is given in Section 11.3.

Theorem 3.4. *As long as $\sigma_y > 0$, then μ has a smooth density with respect to Lebesgue measure which we denote by $m(z)$. If $\sigma_x, \sigma_y > 0$, then $m(z) > 0$ for all $z \in \mathbb{R}^2$. If $\sigma_x = 0$ and $\sigma_y > 0$, then $m(z) = 0$ if $z = (x, y)$ with $x \geq 0$, and $m(z) > 0$ if $z = (x, y)$ with $x < 0$.*

4 Outline of Paper

In Section 6, we show how the existence of a super Lyapunov function leads to a strong regularization of moments. In Section 7, we discuss further the properties of the deterministic model problem. In Section 8, we perform an asymptotic analysis of the generator associated with (3.1). In Section 9, we use associated Poisson equations to construct local super Lyapunov functions in the different regions whose boundaries are determined by the asymptotic analysis. In Section 10, we patch the local Lyapunov functions together to construct the global Lyapunov function and thereby prove Theorem 3.1. In Section 11, we prove, under various assumptions,the existence of a smooth transition density with various positivity properties. Our approach here invokes methods from geometric control theory and Malliavin calculus in a manner which, we hope, will be of independent interest. In Section 11.3, we transfer the smoothness and positivity results to the invariant measure and in doing so prove Theorem 3.4. In Section 12, we prove that Theorem 3.1, when combined with a standard minorization condition, implies Theorem 3.2. In Section 12.1, we show how in the uniformly elliptic setting, namely $\sigma_x, \sigma_y > 0$, the needed minorization condition follows immediately from the strong from of positivity which holds in that setting. In Section 12.2, we show how the weaker positivity properties which hold when $\sigma_x = 0, \sigma_y > 0$ are sufficient to prove the minorization condition. In Section 13, we make a few concluding remarks. The Appendix contains a relatively standard comparison result which we include for completeness. It is used in Section 6 about the Super Lyapunov Structure.

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6 Consequences of Super Lyapunov Structure

We begin with a lemma, whose proof is given at the end of the section, which is a simple translation of the bound on the generator for the definition of a global super Lyapunov function to a bound on the action of the semigroup. Despite its simplicity, it is nonetheless the key to all of the enhanced results that are a consequence of the existence of a super Lyapunov function (as opposed to merely a standard Lyapunov function).

Lemma 6.1. *Suppose that $V : \mathbb{R}^2 \to (0, \infty)$ is a super Lyapunov function for the SDE corresponding to a Markov semi-group P_t. Then for every $t > 0$, there exists a positive constant K_t, such that $t \mapsto K_t$ is a continuous, monotone decreasing function on $(0, \infty)$ with $K_t \to (2b/m)^{1/\gamma}$ as $t \to \infty$, and*

$$(P_t V)(z) \leq K_t \quad \text{for all} \quad z \in \mathbb{R}^2 \text{ and } t > 0\,.$$

Recalling the definition of ρ_β from (3.3), Lemma 6.1 implies the following result.

Proposition 6.2. *If V is a super Lyapunov function and K_t is the constant defined in Lemma 6.1 then for any $t > 0$, $\beta > 0$, test function ϕ, and probability measures μ_1 and μ_2, we have*

$$\|P_t\phi\|_0 \leq (1 + \beta K_t)\|\phi\|_\beta \qquad \text{and} \qquad \rho_\beta(\mu_1 P_t, \mu_2 P_t) \leq (1 + \beta K_t)\|\mu_1 - \mu_2\|_{TV}\,.$$

Remark 6.3. *It is clear that $\rho_0(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{TV}$ and furthermore if $0 \leq \alpha$, $\beta > 0$ and $K = \sup_x \frac{1+\alpha V(x)}{1+\beta V(x)}$, then one has $\|\phi\|_\beta \leq K\|\phi\|_\alpha$, which implies*

$$\{\phi : \|\phi\|_\alpha \leq 1/K\} \subset \{\phi : \|\phi\|_\beta \leq 1\},$$

which in turn implies $\rho_\alpha(\mu_1, \mu_2) \leq K\rho_\beta(\mu_1, \mu_2)$. Thus as long as α and β are both positive, the associated norms and metrics are equivalent. However, if one of them is zero, the needed inequalities only go in one direction. Nonetheless, Proposition 6.2 allows us to use the action of P_t to recover the missing inequality.

Proof of Proposition 6.2. By similar reasoning to that used in the second part of Remark 6.3, we see that if one assumes that $\|P_t\phi\|_0 \leq (1 + \beta K_t)\|\phi\|_\beta$ for some constant K_t, then $\{\phi : \|\phi\|_\beta \leq 1/(1 + \beta K_t)\} \subset \{\phi : \|P_t\phi\|_0 \leq 1\}$ which then implies that $\rho_\beta(\mu_1 P_t, \mu_2 P_t) \leq (1 + \beta K_t)\rho_0(\mu_1, \mu_2)$. Since as noted in Remark 6.3 $\rho_0(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{TV}$, the proof of the second quoted inequality is now complete provided we prove the first.

Now since $|\phi(z)| \leq \|\phi\|_\beta(1 + \beta V(z))$ for all z, one has

$$|(P_t\phi)(z)| \leq \|\phi\|_\beta\big(1 + \beta(P_t V)(z)\big) \leq \|\phi\|_\beta(1 + \beta K_t)\,.$$

Since the right-hand side is independent of z, we obtain the desired result by taking the supremum over z. □

Remark 6.4. *In light of Remark 6.3 and Proposition 6.2, to prove Theorem 3.2 we need only prove the more standard Harris chain-type geometric convergence result of $\rho_\beta(\mu_1 P_t, \mu_2 P_t) \leq C \exp(-\eta t)\rho_\beta(\mu_1, \mu_2)$ for some $\beta > 0$.*

Proof of Lemma 6.1. Let $V_t = V(Z_t)$, where Z_t is the solution to the SDE corresponding to P_t. Let L denote the generator associated to the SDE corresponding to P_t. Since V is a super Lyapunov function, there exist constants $m, b > 0$ and $\gamma > 1$ such that

$$LV_t \leq -mV_t^\gamma + b \quad \text{for all} \quad t \geq 0\,.$$

By Dynkin's formula,

$$\begin{aligned}(P_t V)(z) = \mathbb{E}_z[V_t] = V(z) + \mathbb{E}_z\left[\int_0^t LV_s ds\right] &\leq V(z) - m\int_0^t \mathbb{E}_z[V_s^\gamma]ds + bt \\ &\leq V(z) - m\int_z^t \mathbb{E}_z[V_s]^\gamma + bt \qquad \text{by convexity.}\end{aligned}$$

For simplicity of notation, let $\phi_z(t) = (P_t V)(z) = \mathbb{E}_z[V_t]$. Then $\phi_z(t)$ satisfies the following differential inequality:

$$\begin{aligned}\phi_z'(t) &\leq -m[\phi_z(t)]^\gamma + b \\ &\leq -\frac{m}{2}[\phi_z(t)]^\gamma \quad \text{if} \quad \phi_z(t) \geq \left(\frac{2b}{m}\right)^{\frac{1}{\gamma}}\,.\end{aligned}$$

Let $R = \left(\frac{2b}{m}\right)^{\frac{1}{\gamma}}$ and let $\tau = \inf\{t > 0 : \phi_z(t) \leq R\}$. Since $\phi_z'(t) < 0$ if $\phi_z(t) \geq R$, this implies that once $\phi_z(t) \leq R$, $\phi_z(t)$ remains less than or equal to R for all times afterward. Thus, for all $t \geq \tau$, $\phi_z(t) \leq R$. Now suppose $\psi_z(t)$ satisfies the following differential equation:

$$\begin{cases}\psi_z'(t) = -\frac{m}{2}[\psi_z(t)]^\gamma \quad \text{for all} \quad t \in [0, \tau] \\ \psi_z(0) = \phi_z(0) = V(z)\,.\end{cases}$$

Then by Proposition A.1 in the Appendix, $\phi_z(t) \leq \psi_z(t)$ for all $t \in [0, \tau]$. Now the differential equation for $\psi_(t)$ can be solved explicitly to obtain that for all $t \in [0, \tau]$:

$$\psi_z(t) = \left(\frac{m(\gamma-1)t}{2} + V(z)^{-(\gamma-1)}\right)^{-\frac{1}{\gamma-1}} \leq \left(\frac{m(\gamma-1)t}{2}\right)^{-\frac{1}{\gamma-1}}\,.$$

Defining the constants K_t as follows

$$K_t = \max\left\{\left(\frac{2b}{m}\right)^{\frac{1}{\gamma}}, \left(\frac{m(\gamma-1)t}{2}\right)^{-\frac{1}{\gamma-1}}\right\}\,,$$

we conclude that $\phi_z(t) \leq K_t$ for all $t > 0$, which completes the proof. □

7 Deterministic Equation

To better understand the context of our results for the stochastically perturbed system, we pause for a moment and highlight some properties of the underlying deterministic dynamics:

$$\begin{aligned}\dot{x}_t &= x_t^2 - y_t^2 \\ \dot{y}_t &= 2x_t y_t\,.\end{aligned} \tag{7.1}$$

The trajectories of the system are shown in Figure 1, from which the dynamics of the system can be quickly and easily understood.

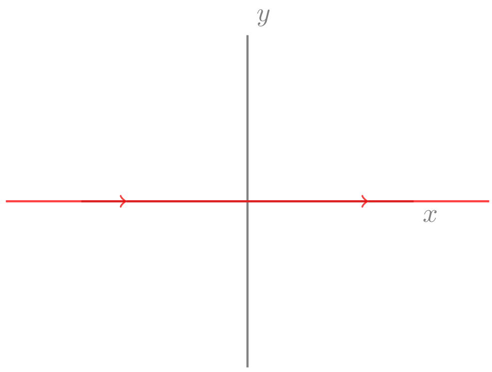

Figure 1: A number of representative orbits of the deterministic dynamics governed by (7.1).

For any initial condition (x_0, y_0), the solution (x_t, y_t) to this system is given by

$$\begin{aligned} x_t &= \frac{x_0 - (x_0^2 + y_0^2)t}{(1 - x_0 t)^2 + (y_0 t)^2} \\ y_t &= \frac{y_0}{(1 - x_0 t)^2 + (y_0 t)^2} \,. \end{aligned} \tag{7.2}$$

In particular, the system exhibits finite-time blow-up (at time $t = \frac{1}{x_0}$) for initial conditions $(x_0, 0)$ on the positive x-axis. For all other initial conditions, the ω-limit set $\omega(x_0, y_0)$ is simply the origin, which is the unique fixed point of the system. We note that the origin is not reached in finite time by any trajectory with initial condition $(x_0, y_0) \neq (0, 0)$.

Now, for any choice of initial condition (x_0, y_0) not on the x-axis, the trajectories of the deterministic system are circles centered at the point $C(x_0, y_0)$ with radius $R(x_0, y_0)$ given as follows:

$$C(x_0, y_0) = \Big(0, \frac{x_0^2 + y_0^2}{2y_0}\Big), \qquad R(x_0, y_0) = \frac{x_0^2 + y_0^2}{2|y_0|} \,. \tag{7.3}$$

Furthermore, for all choices of initial conditions (x_0, y_0) not on the positive x-axis, the time to return to a fixed ball of radius R about the origin is uniformly bounded by $\frac{2}{R}$. In Section 12.2, we employ this uniform bound to prove a positivity and minorization condition on the transition density for the stochastically-perturbed system.

8 Dominant Balances of Generator

We now begin the program laid out in Section 2.1. We begin by considering the dominant operators in various regions of the state space.

Associated to the SDE (3.1) is the generator L defined by

$$L = (x^2 - y^2)\partial_x + 2xy\partial_y + \sigma_x \partial_{xx} + \sigma_y \partial_{yy} \,. \tag{8.1}$$

In order to prove that the addition of noise arrests the blow-up on the x-axis sufficiently to produce an invariant probability measure, we need to understand the behavior of the dynamics at infinity. There are many different routes to infinity and we now consider the various possible dominant balances associated with different routes.

To help identify the relevant scaling, consider the behavior of L under the scaling map $(x, y) \mapsto (\ell x, \ell^p y)$ which produces

$$\ell\, x^2 \partial_x - \ell^{2p-1}\, y^2 \partial_x + \ell\, 2xy \partial_y + \ell^{-2}\, \sigma_x \partial_{xx} + \ell^{-2p}\, \sigma_y \partial_{yy} \,.$$

If $p = 1$ the first three terms balance and dominate the remaining terms as $\ell \to \infty$. If $p > 1$ then the second term dominates. If $p = -\frac{1}{2}$ then the first, third and fifth balance and dominate all other terms as $\ell \to \infty$. These balances cover all of the routes to infinity except for those which approach or rest on the y-axis and identify $p = -1/2$ as a critical scaling. (The routes near the y-axis are captured by $p = -1/2$ and $\ell \to 0$ but these will not play an important role in our analysis.)

If $|x|y^2 < \infty$ as $|(x, y)| \to \infty$ with $x > a > 0$, the dominant part of L is contained in

$$A = x^2\partial_x + 2xy\partial_y + \sigma_y\partial_{yy}\,. \tag{8.2}$$

If $|x|y^2 \to 0$ as $|(x, y)| \to \infty$ with $x > a > 0$, then the dominant part is only ∂_{yy}. Notice that ∂_{yy} is contained in A, so we can still choose to use A this region. In all other relevant cases as $|(x, y)| \to \infty$, the dominant part of L is contained in

$$T = (x^2 - y^2)\partial_x + 2xy\partial_y\,.$$

We have neglected the term $\sigma_x\partial_{xx}$ in the operator T which scaling analysis suggests might be relevant in neighborhood of the y-axis. However its inclusion does not qualitatively change the behavior in a neighborhood of the y-axis. The same can not be said of the term $\sigma_y\partial_{yy}$ in a neighborhood of the x-axis.

8.1 Scaling

To better understand the structure of the solutions in the various regimes, we investigate the scaling properties of the various operators introduced in the previous section. We introduce the scaling transformations

$$S_\ell^{(1)}\colon (x, y) \mapsto (\ell x, \ell^{-\frac{1}{2}}y) \quad \text{and} \quad S_\ell^{(2)}\colon (x, y) \mapsto (\ell x, \ell y)\,.$$

Observe that operator A scales homogeneously under the scaling $S_\ell^{(1)}$, while the operator T scales homogeneously under the scaling $S_\ell^{(2)}$. We would also like the operator T to scale homogeneously under the scaling $S_\ell^{(1)}$; however, this does not hold for all of the terms in T. We remedy this by introducing a non-negative parameter λ and defining the family of operators

$$T_\lambda = (x^2 - \lambda y^2)\partial_x + 2xy\partial_y \tag{8.3}$$

and extending the definition of the scaling operators by

$$S_\ell^{(1)}\colon (x, y, \lambda) \mapsto (\ell x, \ell^{-\frac{1}{2}}y, \ell^3\lambda) \quad \text{and} \quad S_\ell^{(2)}\colon (x, y, \lambda) \mapsto (\ell x, \ell y, \lambda)\,.$$

Now T_λ scales homogeneously under the scaling map $S_\ell^{(1)}$ and A remains invariant under $S_\ell^{(2)}$. This gambit of introducing an extra parameter to produce a homogeneous scaling was also used in a similar way in [12].

Given a function $\phi : \mathcal{R} \times [0, \infty) \to \mathbb{R}$, where $\mathcal{R} \subset \mathbb{R}^2$, we say that ϕ scales homogeneously under the scaling $S_\ell^{(i)}$ if $\phi \circ S_\ell^{(i)} = \ell^\delta\phi$ for some δ. In this case, we say that ϕ scales like ℓ^δ under the i-th scaling. We write this compactly as $\phi \overset{i}{\sim} \ell^\delta$.

Proposition 8.1. *If $\phi \overset{1}{\sim} \ell^\delta$ then $\partial_x\phi \overset{1}{\sim} \ell^{\delta-1}$ and $\partial_y\phi \overset{1}{\sim} \ell^{\delta+\frac{1}{2}}$. Similarly, if $\phi \overset{2}{\sim} \ell^\delta$ then $\partial_x\phi \overset{2}{\sim} \ell^{\delta-1}$ and $\partial_y\phi \overset{2}{\sim} \ell^{\delta-1}$. In both cases, if one side is infinite, then so is the other.*

Proof of Proposition 8.1. We only show one case; all others follow similarly. If $\phi \overset{1}{\sim} \ell^\delta$, then $\phi(\ell x, \ell^{-\frac{1}{2}}y, \ell^3\lambda) = \ell^\delta\phi(x, y, \lambda)$. Differentiating in x, we obtain

$$\ell(\partial_x\phi)(\ell x, \ell^{-\frac{1}{2}}y, \ell^3\lambda) = \ell^\delta(\partial_x\phi)(x, y, \lambda)\,.$$

Dividing through by ℓ, we conclude that $\partial_x\phi \overset{1}{\sim} \ell^{\delta-1}$. □

In the next section, we decompose the plane into regions where the different dominant balances hold. These regions are defined by boundary curves which are well-behaved under one or both of the scalings. To facilitate the construction of these regions, given $x_0 > 0, y_0 > 0$, $\lambda \geq 0$ and $p \in \mathbb{R}$, we define the following "elementary" regions:

$$\begin{aligned}\Lambda(x_0, y_0, \lambda) &= \Big\{ \frac{x^2 + \lambda y^2}{|y|} \geq \frac{x_0^2 + \lambda y_0^2}{y_0} \Big\} \\ \Gamma_p^{\pm}(x_0, y_0) &= \{ \pm x \geq x_0, |x|^p |y| \leq x_0^p y_0 \} \,.\end{aligned} \tag{8.4}$$

Observe that for any $\ell > 0$, we have to following scaling relations

$$\begin{aligned} S_\ell^{(1)}(\Gamma_p^{\pm}(x_0, y_0)) &= \Gamma_p^{\pm}(\ell x_0, \ell^{-\frac{1}{2}} y_0), & S_\ell^{(2)}(\Gamma_p^{\pm}(x_0, y_0)) &= \Gamma_p^{\pm}(\ell x_0, \ell y_0), \\ S_\ell^{(1)}(\Lambda(x_0, y_0, \lambda)) &= \Lambda(\ell x_0, \ell^{-\frac{1}{2}} y_0, \ell^3 \lambda), & S_\ell^{(2)}(\Lambda(x_0, y_0, \lambda)) &= \Lambda(\ell x_0, \ell y_0, \lambda), \end{aligned}$$

and lastly $\Gamma_p^{\pm}(\ell x_0, \ell^{-p} y_0) \subset \Gamma_p^{\pm}(x_0, y_0)$ for $\ell > 1$.

9 Construction of Local Lyapunov Functions

Based on the discussion in the previous section, we will divide the plane into three regions $\mathcal{R}_i(\alpha)$, where α is a positive parameter that we specify later. As described in Section 2.1, we call these regions the "priming," "transport," and "diffusive" regions, respectively. We now describe the placement of these various regions which are indicated pictorially in Figure 2.

Our priming region, $\mathcal{R}_1(\alpha)$, is a subset of the left-half plane, and here there exists a very natural Lyapunov function, because in this region, the deterministic drift is directed toward the origin. On the other hand, the diffusive region, $\mathcal{R}_3(\alpha)$, is a funnel-like region around the positive x-axis where there is finite-time blow-up in the deterministic setting. Demonstrating the existence of a local Lyapunov function in the diffusive region is a key piece in proving noise-induced stabilization in our model problem. The transport region $\mathcal{R}_2(\alpha)$ is governed primarily by deterministic transport from the diffusive region to the priming region. In this section, we focus on the construction of a local Lyapunov function in each of these three regions.

9.1 The Priming Region

When looking for a priming Lyapunov function, it is natural to consider the norm to some power. In this specific example, we expect the norm to some power to be a Lyapunov function in the left-half plane since the drift vector field points at least partially towards the origin; see Figure 1.

For $\delta > 0$, we define $v_1(x, y) = (x^2 + y^2)^{\frac{\delta}{2}}$ and observe that

$$\begin{aligned} Lv_1(x, y) =& \delta x (x^2 + y^2)^{\frac{\delta}{2}} + \delta (\frac{\delta}{2} - 1)(x^2 + y^2)^{\frac{\delta}{2} - 2}(2\sigma_x x^2 + 2\sigma_y y^2) \\ &+ (\sigma_x + \sigma_y)\delta (x^2 + y^2)^{\frac{\delta}{2} - 1} \,. \end{aligned} \tag{9.1}$$

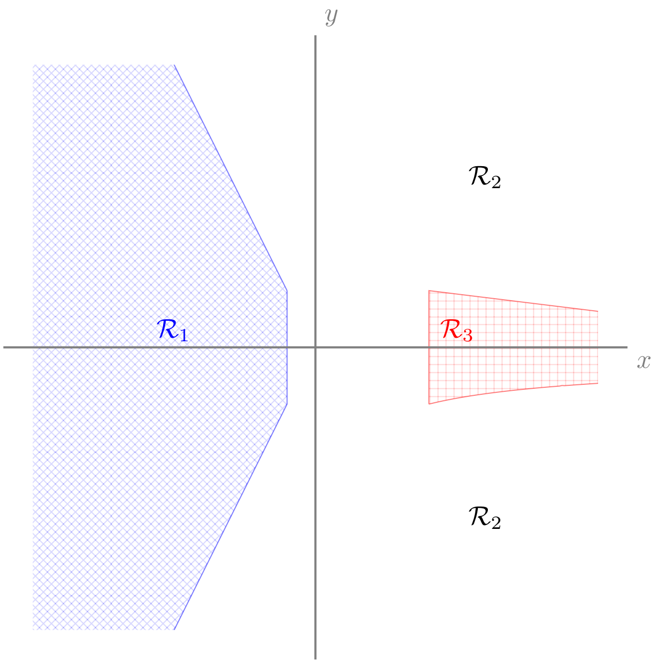

Figure 2: The different regions in which local Lyapunov functions are constructed. $\mathcal{R}_1$ is the priming region. The two regions labeled $\mathcal{R}_2$ are transport regions. And $\mathcal{R}_3$ is the diffusive region which connects the two transport regions in which information is propagating in different directions.

In particular, if $(x, y) \in \Gamma^-_{-1}(\frac{\alpha}{2}, 1)$, we get that

$$\begin{aligned}(Lv_1)(x,y) \leq &- \frac{\alpha\delta}{\sqrt{\alpha^2+4}}(x^2+y^2)^{\frac{\delta+1}{2}} + \delta(\delta-2)\frac{\sigma_x x^2 + \sigma_y y^2}{(x^2+y^2)^{2-\frac{\delta}{2}}} + \delta\frac{\sigma_x+\sigma_y}{(x^2+y^2)^{1-\frac{\delta}{2}}} \\ = &- \frac{\alpha\delta}{\sqrt{\alpha^2+4}}(x^2+y^2)^{\frac{\delta+1}{2}} \\ &\times\left[1 - \frac{\sqrt{\alpha^2+4}}{\alpha}\left(\frac{[\delta-2]\,(\sigma_x x^2+\sigma_y y^2)}{(x^2+y^2)^{\frac{5}{2}}} + \frac{\sigma_x+\sigma_y}{(x^2+y^2)^{\frac{3}{2}}}\right)\right].\end{aligned}$$

This implies that for any $\delta > 0$ and $\alpha > 0$, there exists an R_1 sufficiently large so that if $|(x, y)| > R_1$, then the term in the square brackets is greater than $\frac{1}{2}$. Hence v_1 is a super Lyapunov function in the region

$$\mathcal{R}_1(\alpha) = \Gamma^-_{-1}(\frac{\alpha}{2}, 1)$$

with exponent $\frac{\delta+1}{\delta}$. As we will see later, we will have to restrict δ to the interval $(0, \frac{2}{5})$, and this automatically implies that $\delta \in (0, 2)$. In turn, this guarantees that $\delta - 2 < 0$ and that the term in the square brackets above is greater that $\frac{1}{2}$ provided

$$\frac{\sigma_x+\sigma_y}{(x^2+y^2)^{\frac{3}{2}}} < \frac{1}{2}\frac{\alpha}{\sqrt{\alpha^2+4}}\,.$$

We formalize this observation in the following proposition.

Proposition 9.1. *For any $\alpha > 0$ and $\delta \in (0, 2)$, if $(x, y) \in \mathcal{R}_1(\alpha)$ with $|(x, y)| \geq R_1$, then v_1 satisfies*

$$(Lv_1)(x,y) \leq -m_1\, v_1^{\gamma_1}(x,y)$$

where $m_1 = \frac{\alpha\delta}{2\sqrt{\alpha^2+4}} > 0$, $\gamma_1 = \frac{\delta+1}{\delta} > 1$, $R_1 = \left[2\left(\sigma_x + \sigma_y\right)\frac{\sqrt{\alpha^2+4}}{\alpha}\right]^{\frac{1}{3}}$.

Our choice of the region $\mathcal{R}_1(\alpha)$ is motivated by the following. From (9.1), it is clear that we need to define a region in the negative half-plane bounded away from the y-axis. Furthermore, in order to guarantee that v_1 is super Lyapunov, we need to ensure a region in which $|(x,y)| \to \infty$ implies $|x| \to \infty$. Note that $\mathcal{R}_1(\alpha)$ is a subset of the left half-plane, in which the dominant dynamics at infinity are given by T and hence the relevant scaling transformation is $S_\ell^{(2)}$. For this reason, it is desirable to define the boundary of the region so that it behaves well under $S_\ell^{(2)}$. From the previous section, we see that

$$S_\ell^{(2)}(\Gamma_{-1}^-(x_0,y_0)) = \Gamma_{-1}^-(\ell x_0, \ell y_0) \subset \Gamma_{-1}^-(x_0,y_0) \quad \text{for} \quad \ell > 1$$

which motivates our choice of $\mathcal{R}_1(\alpha)$ and the shape of its boundary in particular.

9.2 Decomposition of Remainder of Plane

We will now propagate the priming Lyapunov function through a sequence of regions until all of the routes to infinity are covered.

As mentioned above, near the boundary of $\mathcal{R}_1(\alpha)$ and away from the x-axis, the operator T is dominant. This holds true until one enters the region defined by the curves $xy^2 = c$ where c is a sufficiently large positive constant and $x > 0$ is sufficiently large. At this point, the dominant balance changes and the operator A becomes dominant. Hence we define the transport region, $\mathcal{R}_2(\alpha)$, with one boundary inside the region $\mathcal{R}_1(\alpha)$ which is invariant under the scaling $S_\ell^{(2)}$, and one boundary which is defined by the curve $|x|y^2 = c$ for some constant. As we make precise in the definition below, we will choose $c = \alpha$.

We set $\mathcal{R}_2(\alpha) = \mathcal{R}_2(\alpha, 1)$ where for $\alpha, \lambda \geq 0$, we define

$$\mathcal{R}_2(\alpha,\lambda) = \overline{\Gamma_{\frac{1}{2}}^+(\alpha\sqrt{\lambda}, 1)^c \cap \Lambda(\alpha\sqrt{\lambda}, 1, \lambda) \cap \Gamma_{-1}^-(\alpha\sqrt{\lambda}, 1)^c}\,.$$

Now, observe that outside of $\mathcal{R}_1(\alpha) \cup \mathcal{R}_2(\alpha)$ all of the routes to infinity have $|x|y^2 < \infty$. Hence the operator A is dominant in this entire region and we do not need to further subdivide the remainder of the plane. To define $\mathcal{R}_3(\alpha)$, recall that we need nontrivial overlap with the transport region $\mathcal{R}_2(\alpha)$. Hence we again chose a boundary curve of the form $|x|y^2 = c$ but with $c > \alpha$. In particular, we define $\mathcal{R}_3(\alpha) = \Gamma_{\frac{1}{2}}^+(2\alpha, 1)$. Note that $\mathcal{R}_3(\alpha)$ is the diffusive region: the diffusion term in the operator A is critical to the stabilization of the process here.

In summary, for each $\alpha > 0$, we have defined three regions

$$\begin{aligned} \text{Priming Region: } \mathcal{R}_1(\alpha) &= \Gamma_{-1}^-(\tfrac{\alpha}{2}, 1) \\ \text{Transport Region: } \mathcal{R}_2(\alpha) &= \mathcal{R}_2(\alpha, 1) \\ \text{Diffusive Region: } \mathcal{R}_3(\alpha) &= \Gamma_{\frac{1}{2}}^+(2\alpha, 1)\,. \end{aligned}$$

Notice that $\mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha)$ and $\mathcal{R}_2(\alpha) \cap \mathcal{R}_3(\alpha)$ are nonempty and that $\mathbb{R}^2 \setminus (\mathcal{R}_1(\alpha) \cup \mathcal{R}_2(\alpha) \cup \mathcal{R}_3(\alpha))$ is a bounded set. We construct a local super Lyapunov function in each of the three regions and then smoothly patch them together to form one global super Lyapunov function on the entire plane.

9.3 The Associated Poisson Equations

We now propagate the priming Lyapunov function v_1 which is defined in $\mathcal{R}_1(\alpha)$ to the other regions by solving a succession of Poisson equations. Throughout most of our construction, α will remain a free parameter; we specify α later to ensure a number of necessary estimates. We begin with the transport region $\mathcal{R}_2(\alpha)$.

9.3.1 The Transport Region $\mathcal{R}_2(\alpha)$

For $\delta > 0$ and $\alpha > 0$, we define $v_2(x, y)$ as the solution to the following Poisson equation:

$$\begin{cases} (Tv_2)(x,y) = -\left(\dfrac{x^2+y^2}{|y|}\right)^{\delta+1} & \text{on } \mathcal{R}_2(\alpha) \\ v_2(x,y) = v_1(x,y) & \text{on } \partial B_1(\alpha) \end{cases} \tag{9.2}$$

where $\partial B_1(\alpha) = \left\{x \le -\alpha, |y| = \dfrac{1}{\alpha}|x|\right\}$.

The rationale for this is as follows. We wish to propagate the priming Lyapunov function through the region $\mathcal{R}_2(\alpha)$, so we need to take it as the boundary condition. Since the operator T represents one of the dominate balances, it necessarily scales homogeneously. In this case, T scales like ℓ^1 under the scaling transformation $S_\ell^{(2)}$. Hence if v_2 is to scale homogeneously under $S_\ell^{(2)}$ as ℓ^p for some power p then the righthand side must scale like ℓ^{p+1} and the boundary conditions must scale like ℓ^p both under $S_\ell^{(2)}$. (Notice the boundary ∂B_1 is invariant under $S_\ell^{(2)}$.)

Notice that our choice of right-hand side scales as $\ell^{\delta+1}$ and the boundary conditions scale as would be consistent with a solution which scales like ℓ^δ under $S_\ell^{(2)}$. The form of the boundary conditions are dictated by our choice of v_1. The exact from of the righthand side was chosen so that it was constant along trajectories of the limiting dynamics in $\mathcal{R}_2(\alpha)$ which are the characteristics of T.

9.3.2 The Diffusive Region $\mathcal{R}_3(\alpha)$

For $\delta > 0$ and $\alpha > 0$, we define $v_3(x, y)$ by the following Poisson equation

$$\begin{cases} (Av_3)(x,y) = -c_1 x^{\hat{\delta}+1} & \text{on } \mathcal{R}_3(\alpha) \\ v_3(x,y) = c_2 x^{\hat{\delta}} & \text{on } \partial B_2(\alpha) \end{cases} \tag{9.3}$$

where $\partial B_2(\alpha) = \{x \ge \alpha, xy^2 = 2\alpha\}$,

$$\hat{\delta} = \frac{5}{2}\delta + \frac{3}{2} \tag{9.4}$$

and $c_1, c_2 > 0$ are constants which will be chosen later. We remark that the values of c_1 and c_2 do not affect the local super Lyapunov property of v_3, but rather are chosen in order to facilitate the patching of the local super Lyapunov functions into one global super Lyapunov function in Section 10. As before, we have chosen a right-hand side which is negative definite, scales homogeneously under the appropriate scaling, namely $S_\ell^{(1)}$, and has unbounded growth in the region. We use a constant multiple of $x^{\hat{\delta}}$ as the boundary condition rather than the function v_2 from the neighboring region because we want a function which scales homogeneously under $S_\ell^{(1)}$. However, $x^{\hat{\delta}}$ is in fact the asymptotic behavior (up to a constant multiple) of $v_2(x, y)$ as $|(x, y)| \to \infty$ on the specified boundary.

In Section 9.5, we verify that v_2 and v_3 are super Lyapunov functions in the regions in which they are defined. However, we first establish a number of preliminary results.

9.4 Existence of Solutions and Their Properties

The scaling properties of the solutions to the above Poisson equations are one of main tools we use to show that they are local Lyapunov functions. This is because, with one exception, points at infinity in a given region can be scaled back to points in the same region by the scaling transformation under which the associated differential operator is homogeneous. As we discuss below, the exception is the subregion of $\mathcal{R}_2(\alpha)$ which lies near the boundary of $\mathcal{R}_3(\alpha)$.

9.4.1 Properties of the Solution in the Transport Region

Care must be taken when scaling the points in the subregion of $\mathcal{R}_2(\alpha)$ which lie close to the boundary of $\mathcal{R}_3(\alpha)$. The points in this region naturally scale with $S_\ell^{(1)}$ while the operator T which is associated to $\mathcal{R}_2(\alpha)$ scales homogeneously under $S_\ell^{(2)}$. This issue was also addressed in Section 8.1 where we introduced the parameter λ to generate a family of operators T_λ which scale homogeneously with $S_\ell^{(1)}$.

With this mind, it is natural to introduce the function $v_2(x,y,\lambda)$ which, for a given $\lambda \in (0,1]$, solves the following family of auxiliary Poisson equations in $\mathcal{R}_2(\alpha,\lambda)$:

$$\begin{cases} (T_\lambda v_2)(x,y,\lambda) = -h(x,y,\lambda) & \text{on } \mathcal{R}_2(\alpha,\lambda) \\ v_2(x,y,\lambda) = f(x,y,\lambda) & \text{on } \partial B_1(\alpha\sqrt{\lambda}) \end{cases} \tag{9.5}$$

where we define

$$h(x,y,\lambda) = \left(\frac{x^2+\lambda y^2}{|y|}\right)^{\delta+1} \qquad\qquad f(x,y,\lambda) = \lambda^{\frac{\delta+1}{2}}(x^2+\lambda y^2)^{\frac{\delta}{2}}\,.$$

For ease of notation, we write

$$h(x,y) = h(x,y,1) \quad \text{and} \quad f(x,y) = f(x,y,1)\,.$$

Notice that $h \overset{1}{\sim} \ell^{\hat\delta+1}$, $f \overset{1}{\sim} \ell^{\hat\delta}$, $h \overset{2}{\sim} \ell^{\delta+1}$, and $f \overset{2}{\sim} \ell^{\delta}$ where $\hat\delta$ was defined in (9.4). Also observe that $v_2(x,y,1)$ coincides with the $v_2(x,y)$ defined by (9.2).

9.4.2 Properties of the Solution in the Diffusive Region

The dynamics associated to the operator A, which is dominant in $\mathcal{R}_3(\alpha)$, should be understood as having one diffusive direction and one deterministic direction which is uncoupled from the diffusion and acts as the "clock" of the diffusion. To see this, observe that A is the operator associated to the system of SDEs given by

$$\begin{aligned} d\hat X_t &= \hat X_t^2 dt & \hat X_0 &= x \\ d\hat Y_t &= 2\hat X_t \hat Y_t dt + \sqrt{2\sigma_y}\, dW_t & \hat Y_0 &= y\,. \end{aligned} \tag{9.6}$$

Now, let $(\hat X_0, \hat Y_0) = (x,y)$ lie in $\mathcal{R}_3(\alpha)$ and define $\hat\tau = \inf\{t>0 : (\hat X_t, \hat Y_t) \in \partial B_2(\alpha)\}$. Then $v_3(x,y)$, which was defined in (9.3), can be represented probabilistically as

$$\begin{aligned} v_3(x,y) &= c_2 \mathbb{E}_{(x,y)}\big[\hat X_{\hat\tau}^{\hat\delta}\big] + c_1 \mathbb{E}_{(x,y)}\Big[\int_0^{\hat\tau} \hat X_s^{\hat\delta+1} ds\Big] \\ &= \big(\tfrac{c_1}{\hat\delta} + c_2\big)\mathbb{E}_{(x,y)}\big[\hat X_{\hat\tau}^{\hat\delta}\big] - \tfrac{c_1}{\hat\delta} x^{\hat\delta} \end{aligned} \tag{9.7}$$

provided that, first, the expectation is finite; and second, that the right-hand side of equation (9.7) depends in a C^2 fashion on $(x,y) \in \mathcal{R}_3(\alpha)$. Both of these facts will follow from Proposition 9.2, which we present below, and are made formal in Proposition 9.3, which appears in the next section.

Since $\hat X_t$ is deterministic, this representation of v_3 amounts to a deterministic function of $\hat\tau$. To better understand the properties of $\hat\tau$, we introduce the time change $T(t) = \int_0^t \hat X_s ds = -\ln|1-xt|$ and the process $Z_{T(t)} = \hat X_t^{\frac{1}{2}}\,\hat Y_t$. Due to the scaling of the boundary of $\mathcal{R}_3(\alpha)$, if we define $\tau = \inf\{T > 0 : |Z_T| \geq \sqrt{2\alpha}\}$ then $\hat{\tau} = \frac{1}{x}(1 - e^{-\tau})$, $\hat{X}_t = xe^{T(t)}$, and Z_T satisfies the SDE

$$dZ_T = \frac{5}{2} Z_T dT + \sqrt{2\sigma_y}\, dW_T\,, \qquad Z_0 = x^{\frac{1}{2}} y\,. \tag{9.8}$$

Since Z_T is the solution to a Gaussian SDE, the following proposition follows easily.

Proposition 9.2. *For $\hat{\delta} < \frac{5}{2}$ and $(x, y) \in \mathcal{R}_3(\alpha)$, $\mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big] < \infty$ and the map $(x, y) \mapsto \mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big]$ is C^2.*

Proof of Proposition 9.2. To see the finiteness of the expectation, observe that

$$\begin{aligned}\mathbb{P}_{(x,y)}(e^{\hat{\delta}\tau} > s) &= \mathbb{P}_{(x,y)}\Big(\sup_{0 \leq T \leq \frac{\ln s}{\hat{\delta}}} |Z_T| < \sqrt{2\alpha}\Big) \leq \mathbb{P}_{(x,y)}\left(|Z_{\frac{1}{\hat{\delta}}\ln s}| < \sqrt{2\alpha}\right) \\ &= \mathbb{P}\Big(\Big|\sqrt{2\sigma_y}\, s^{\frac{5}{2\hat{\delta}}} \int_0^{\frac{1}{\hat{\delta}}\ln s} e^{-\frac{5}{2}r} dW_r\Big| < \sqrt{2\alpha}\Big) \leq \Big(\frac{10\alpha}{\sigma_y \pi (s^{5/\hat{\delta}} - 1)}\Big)^{\frac{1}{2}}\,.\end{aligned}$$

Hence for $\hat{\delta} < \frac{5}{2}$, this decays sufficiently rapidly in order to guarantee that $\mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big]$ is finite. The continuity properties now follow from the continuity properties of τ. Specifically, $\mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big] = g(\sqrt{x}\, y)$ where $g(z)$ solves the following ordinary differential equation

$$\begin{cases} \sigma_y g''(z) + \frac{5}{2} z g'(z) + \hat{\delta} g(z) = 0 & \text{for } g \in (-\sqrt{2\alpha}, \sqrt{2\alpha}) \\ g(\sqrt{2\alpha}) = g(-\sqrt{2\alpha}) = 1\,. \end{cases} \tag{9.9}$$

Since by standard results on the regularity of ODEs, $g(z) \in C^2([-\sqrt{2\alpha}, \sqrt{2\alpha}])$, we conclude that $\mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big] = g(\sqrt{x}\; y) \in C^2(\mathcal{R}_3(\alpha))$ as desired. ☐

We remark that this proposition imposes a further restriction on the size of the parameter δ, which previously was only required to be positive. Observe that in light of (9.4) the requirement that $\hat{\delta} < \frac{5}{2}$ forces $\delta \in (0, \frac{2}{5})$.

9.4.3 Principal Result on Existence and Scaling of Solutions

We consolidate these observations and now state and prove our principal existence and scaling result.

Proposition 9.3. *For every $\delta \in (0, \frac{2}{5})$, there exists a strictly positive C^2 function $v_3 \colon \mathcal{R}_3(\alpha) \to (0, \infty)$ which solves* (9.3)*. For every $\lambda \in (0, 1]$, there exists a strictly positive C^2 function $v_2 \colon \mathcal{R}_2(\alpha, \lambda) \to (0, \infty)$ which solves* (9.5)*. In addition, $v_2 \overset{1}{\sim} \ell^{\hat{\delta}}$, $v_2 \overset{2}{\sim} \ell^{\delta}$, $v_3 \overset{1}{\sim} \ell^{\hat{\delta}}$ and $(x, y, \lambda) \mapsto v_2(x, y, \lambda)$ is continuous on $\mathcal{R}_2^*(\alpha) \times [0, 1]$ where $\mathcal{R}_2^*(\alpha) = \cap_{\lambda \in [0,1]} \mathcal{R}_2(\alpha, \lambda)$. In fact, v_2 has an explicit formula given in* (9.11) *below and v_3 a semi-explicit formula given in* (9.10) *also below.*

Proof of Proposition 9.3. We begin with v_3. The preceding discussion all but gives the existence proof. In particular, it shows that if g is defined by (9.9) and $\hat{\delta}$ by (9.4) then the map

$$(x, y) \mapsto \mathbb{E}_{(x,y)}\big[\hat{X}_{\hat{\tau}}^{\hat{\delta}}\big] = x^{\hat{\delta}} \mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big] = x^{\hat{\delta}} g(\sqrt{x} y)$$

is well-defined, positive, and C^2 for $\delta \in (0, \frac{2}{5})$ and $(x, y) \in \mathcal{R}_3(\alpha)$. Returning to (9.7), classical results (see, for example, [4]) allow us to justify the stochastic representation formula for v_3, which now can be rewritten as

$$v_3(x, y) = x^{\hat{\delta}} \Big[\big(\tfrac{c_1}{\hat{\delta}} + c_2\big) \mathbb{E}_{(x,y)}\big[e^{\hat{\delta}\tau}\big] - \tfrac{c_1}{\hat{\delta}}\Big] = x^{\hat{\delta}} \Big[\big(\tfrac{c_1}{\hat{\delta}} + c_2\big) g(\sqrt{x} y) - \tfrac{c_1}{\hat{\delta}}\Big]\,. \tag{9.10}$$

As a consequence of this formula, to prove the scaling it suffices to show that

$$\mathbb{E}_{(x,y)}\left[e^{\hat{\delta}\tau}\right] \overset{1}{\sim} \ell^0 .$$

This is clear, since the only dependence of τ upon x and y results from Z_0, and $Z_0 = x^{\frac{1}{2}}y = (\ell x)^{\frac{1}{2}}(y\ell^{-\frac{1}{2}})$ is invariant under $S_\ell^{(1)}$. We now turn to v_2. While in light of the scaling and continuity properties of T_λ, h, and f this result could be obtained by abstract means, we employ the method of characteristics to produce an explicit solution. Namely,

$$v_2(x,y,\lambda) = \Big(\frac{x^2+\lambda y^2}{|y|}\Big)^\delta \Big[\frac{x}{|y|} + \alpha\sqrt{\lambda} + \sqrt{\lambda}\Big(\frac{1}{\alpha^2+1}\Big)^{\frac{\delta}{2}}\Big] . \tag{9.11}$$

The scaling properties, regularity, and positivity follow by direct calculation with (9.11). □

Remark 9.4. *As $\lambda \to 0$, $v_2(x,y,\lambda)$ given in* (9.11) *converges to $x^{2\delta+1}|y|^{-(\delta+1)}$. This was expected since formally taking $\lambda \to 0$ in* (9.5) *produces the equation*

$$\begin{cases} (x^2\partial_x + 2xy\partial_y)v = -\dfrac{x^{2(\delta+1)}}{|y|^{\delta+1}} & \text{on } \mathcal{R}_2(\alpha,0) \\ v(x,y) = 0 & \text{on } \partial B_1(\alpha\sqrt{\lambda}) . \end{cases} \tag{9.12}$$

The solution to this simplified equation is easily seen to be $x^{2\delta+1}|y|^{-(\delta+1)}$. Even in a setting where (9.5) *cannot be solved explicitly, this simplified equation may well be easier to solve. We will see in Section 10 that the most delicate parts of the patching require precise information about the limit of v_2 as $\lambda \to 0$. This suggests that the analysis may be feasible even when* (9.5) *is not solvable.*

Remark 9.5. *For both v_2 and v_3 we have used specifics of the solutions to verify the scaling. It is possible to derive the results just from the scaling of the operators, right-hand sides, and boundary conditions. The positivity for both solutions also follows from the positivity of the boundary data and the negative definiteness of the right-hand sides.*

9.5 Proof of Local Super Lyapunov Property

Letting $B_R(z) = \{(x,y) \in \mathbb{R}^2 : |(x,y) - z| \le R\}$, we state the following proposition which establishes that v_2 and v_3 are local super Lyapunov functions.

Proposition 9.6. *Fix any $\delta \in (0, \frac{2}{5})$ then for all $\alpha > 0$ sufficiently large, there exist constants $m_i > 0$ and $R_i > 0$ so that if $(x,y) \in \mathcal{R}_i(\alpha)$ with $|(x,y)| \ge R_i$, then v_i satisfies*

$$(Lv_i)(x,y) \le -m_i v_i^{\gamma_i}(x,y)$$

for $i = 2,3$ where $\gamma_2 = \gamma_3 = \frac{5\delta+5}{5\delta+3}$. In addition, we have the following refined estimate in the second region which emphasizes its transitional nature and which will be of later use. Defining

$$\mathcal{R}_2^{(1)}(\alpha) = \overline{\mathcal{R}_2(\alpha) \cap \mathcal{R}_2^{(2)}(\alpha)^c} \quad \text{and} \quad \mathcal{R}_2^{(2)}(\alpha) = \overline{\mathcal{R}_2(\alpha) \cap \Gamma_{-1}^+(\alpha,1)^c} , \tag{9.13}$$

if $j = 1,2$ and $(x,y) \in \mathcal{R}_2^{(j)}$, we have

$$(Lv_2)(x,y) \le -m_2 v_i^{\gamma_2^{(j)}}(x,y)$$

where $\gamma_2^{(1)} = \gamma_3 = \frac{5\delta+5}{5\delta+3}$ and $\gamma_2^{(2)} = \gamma_1 = \frac{\delta+1}{\delta}$.

Proof of Proposition 9.6. We begin with v_3 since it is more straightforward. Observe that for any $\gamma > 1$, using equation (9.3) and the positivity of v_3, one has

$$\begin{aligned}(Lv_3)(x,y) &= (Av_3)(x,y) - y^2\partial_x v_3(x,y) + \sigma_x \partial_{xx} v_3(x,y) \\ &\le -v_3^\gamma(x,y)\left[\frac{c_1 x^{\hat\delta+1} - y^2|\partial_x v_3(x,y)| - \sigma_x|\partial_{xx}v_3(x,y)|}{v_3^\gamma(x,y)}\right] \\ &\le -m v_3^\gamma(x,y)\end{aligned}$$

where we define

$$m = \inf_{(x,y)\in\mathcal{R}_3(\alpha)\cap B_R^c(0)}\left[\frac{c_1 x^{\hat\delta+1} - y^2|\partial_x v_3(x,y)| - \sigma_x|\partial_{xx}v_3(x,y)|}{v_3^\gamma(x,y)}\right].$$

If for some choice of $\gamma > 1$ and $R > 0$, one has $m > 0$, then it is clear that v_3 is a local super Lyapunov function. To prove that such γ and R exist, we use the scaling and continuity properties of v_3 which were proven in Proposition 9.3. Observe that every point $(x,y) \in \mathcal{R}_3(\alpha)$ can be mapped back to a point $(2\alpha, b)$, where $(x,y) = S_\ell^{(1)}(2\alpha, b)$, $\ell = \frac{x}{2\alpha}$, and $b = \sqrt{\ell}y \in [-1,1]$. Therefore v_3 satisfies the following scaling relations:

$$\begin{aligned} v_3(x,y) &= \ell^{\hat\delta} v_3(2\alpha,b) \qquad & (\partial_x v_3)(x,y) &= \ell^{\hat\delta-1}(\partial_x v_3)(2\alpha,b) \\ x^{\hat\delta+1} &= \ell^{\hat\delta+1}(2\alpha)^{\hat\delta+1} \qquad & (\partial_{xx} v_3)(x,y) &= \ell^{\hat\delta-2}(\partial_{xx} v_3)(2\alpha,b)\,. \end{aligned}$$

These scaling relations lead us to choose $\gamma = \frac{5\delta+5}{5\delta+3}$, which is the ratio of the exponents of ℓ in Av_3 and v_3. With this choice of γ, we obtain

$$\frac{c_1 x^{\hat\delta+1} - y^2|\partial_x v_3(x,y)| - \sigma_x|\partial_{xx}v_3(x,y)|}{v_3^\gamma(x,y)} = \frac{c_1(2\alpha)^{\hat\delta+1} - \ell^{-3}(b^2|\partial_x v_3(2\alpha,b)| + \sigma_x|\partial_{xx}v_3(2\alpha,b)|)}{v_3^\gamma(2\alpha,b)}\,.$$

Hence if we define $\ell_* = \inf\{x/(2\alpha) : (x,y) \in \mathcal{R}_3(\alpha)\cap B_R^c(0)\}$ and

$$M = \sup_{b\in[-1,1]} b^2|\partial_x v_3(2\alpha,b)| + \sigma_x|\partial_{xx}v_3(2\alpha,b)| + v_3(2\alpha,b)\,,$$

the preceding estimate and the strict positivity of v_3 imply that

$$m \ge \frac{c_1(2\alpha)^{\hat\delta+1} - \ell_*^{-3}M}{M^\gamma}\,.$$

Since v_3 is C^2, we know that $M < \infty$ (the supremum is over a compact set). Furthermore, observe that $\ell_* \to \infty$ as $R \to \infty$. Combining these last two observations with the above estimate, we can choose R sufficiently large to ensure that

$$c_1(2\alpha)^{\hat\delta+1} - \ell_*^{-3}M > 0$$

and hence that $m > 0$. We define R_3 and γ_3 to be the above choices of R and γ, respectively, which are valid in $\mathcal{R}_3(\alpha)$. Substituting these values in the expression for m, we obtain m_3. This completes the proof of the local super Lyapunov property for v_3.

We now turn to proving the corresponding property for v_2. We start as we did for v_3, by noting that for any $\gamma > 1$

$$\begin{aligned}(Lv_2)(x,y) &= -h(x,y) + (\sigma_x\partial_{xx}v_2 + \sigma_y\partial_{yy}v_2)(x,y) \\ &\le -v_2^\gamma(x,y)\left[\frac{h - \sigma_x|\partial_{xx}v_2| - \sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](x,y) \le -m v_2^\gamma(x,y)\,.\end{aligned}$$

where in this case we define

$$m = \inf_{\substack{(x,y)\in\mathcal{R}_2(\alpha)\\ |(x,y)|>R}} \left[\frac{h-\sigma_x|\partial_{xx}v_2|-\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](x,y)\,.$$

As before, we need to show that there exist $\gamma > 1$ and $R > 0$ such that $m > 0$. Since $\mathcal{R}_2(\alpha)$ has two different natural scalings, we decompose this region and handle each subregion separately. Recall the definition of $\mathcal{R}_2^{(1)}$ and $\mathcal{R}_2^{(2)}$ from (9.13) and observe that $\mathcal{R}_2^{(1)}(\alpha)$ scales well under $S_\ell^{(1)}$ and $\mathcal{R}_2^{(2)}(\alpha)$ under $S_\ell^{(2)}$. We define $m^{(i)}$ as

$$m^{(i)} = \inf_{\substack{(x,y)\in\mathcal{R}_2^{(i)}(\alpha)\\ |(x,y)|>R}} \left[\frac{h-\sigma_x|\partial_{xx}v_2|-\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](x,y)\,.$$

We begin with $\mathcal{R}_2^{(2)}(\alpha)$ since the analysis in this subregion is very similar to the previous analysis for v_3.

First, note that the circle of radius $r = 2(\alpha^2+1)$ centered at the origin is contained in $\Lambda(\alpha,1,1)$. Hence any point in $\mathcal{R}_2^{(2)}(\alpha)$ can be connected by a radial line contained in $\mathcal{R}_2^{(2)}(\alpha)$ to the part of this circle contained in $\mathcal{R}_2^{(2)}(\alpha)$. It follows from this that any $(x,y)\in\mathcal{R}_2^{(2)}(\alpha)$ can be written in the form $(x,y) = S_\ell^{(2)}(a,b)$ where $\ell = |(x,y)|r^{-1}$ and (a,b) is a point on the circle of radius r centered at the origin. Therefore,

$$\begin{aligned} v_2(x,y) &= \ell^\delta v_2(a,b) & (\partial_{xx}v_2)(x,y) &= \ell^{\delta-2}(\partial_{xx}v_2)(a,b)\\ h(x,y) &= \ell^{\delta+1}h(a,b) & (\partial_{yy}v_2)(x,y) &= \ell^{\delta-2}(\partial_{yy}v_2)(a,b)\,. \end{aligned}$$

Again, by analogy to the previous case, these scaling relations lead us to choose $\gamma = \frac{\delta+1}{\delta}$, which is the ratio of the exponents of ℓ in Tv_2 and v_2. With this choice of γ, if $(x,y) = S_\ell^{(2)}(a,b)$, we have that

$$\left[\frac{h-\sigma_x|\partial_{xx}v_2|-\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](x,y) = \left[\frac{h-\ell^{-3}\sigma_x|\partial_{xx}v_2|-\ell^{-3}\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](a,b)\,.$$

Setting $\tilde{\mathcal{R}}_2^{(2)} = \{(a,b)\in\mathcal{R}_2^{(2)}(\alpha) : |(a,b)| = r\}$, we define

$$\rho = \inf_{(a,b)\in\tilde{\mathcal{R}}_2^{(2)}} \frac{h(a,b)}{v_2^\gamma(a,b)} \quad\text{and}\quad M = \sup_{(a,b)\in\tilde{\mathcal{R}}_2^{(2)}} \left[\frac{\sigma_x|\partial_{xx}v_2|+\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](a,b)\,.$$

Letting $\ell_* = \inf\{|(x,y)|/r : (x,y)\in\mathcal{R}_2^{(2)}(\alpha)\cap B_R^c(0)\} = R/r$, we observe that $m^{(2)} \geq \rho - \ell_*^{-3}M$. Because h and v_2 are C^2 in (x,y) and strictly positive and $\tilde{\mathcal{R}}_2^{(2)}$ is a compact set, we conclude that $\rho > 0$ and $M < \infty$. Hence one can choose R sufficiently large in order to ensure that $m^{(2)} \geq \rho - \ell_*^{-3}M > 0$. Again, we denote these specific choices of R and γ, which are valid in $\mathcal{R}_2^{(2)}(\alpha)$, by $R_2^{(2)}$ and $\gamma_2^{(2)}$. Substituting these values in the expression for $m^{(2)}$, we obtain $m_2^{(2)}$. This completes the proof that v_2 is a local super Lyapunov property in the subregion $\mathcal{R}_2^{(2)}(\alpha)$.

We now turn to region $\mathcal{R}_2^{(1)}(\alpha)$. Every point $(x,y)\in\mathcal{R}_2^{(1)}(\alpha)$ can be mapped back to a point (a,b) on the curve $\{\alpha b = a\}$ such that $(x,y) = S_\ell^{(1)}(a,b)$, where $\ell = \left(\frac{x}{\alpha y}\right)^{\frac{2}{3}}$, $a = \alpha^{\frac{2}{3}}(xy^2)^{\frac{1}{3}}$, and $b = \alpha^{-\frac{1}{3}}(xy^2)^{\frac{1}{3}}$. Hence we get the scaling relations

$$\begin{aligned} v_2(x,y,1) &= \ell^{\hat\delta}v_2(a,b,\ell^{-3}) & (\partial_{xx}v_2)(x,y,1) &= \ell^{\hat\delta-2}(\partial_{xx}v_2)(a,b,\ell^{-3})\\ h(x,y,1) &= \ell^{\hat\delta+1}h(a,b,\ell^{-3}) & (\partial_{yy}v_2)(x,y,1) &= \ell^{\hat\delta+1}(\partial_{yy}v_2)(a,b,\ell^{-3})\,. \end{aligned}$$

Now using the scaling map $S_\ell^{(1)}$ to map $(a,b)\mapsto(\alpha,1)$ we obtain

$$\begin{aligned} v_2(a,b,\ell^{-3}) &= b^\delta v_2(\alpha,1,\ell^{-3}) & (\partial_{xx}v_2)(a,b,\ell^{-3}) &= b^{\delta-2}(\partial_{xx}v_2)(\alpha,1,\ell^{-3}) \\ h(a,b,\ell^{-3}) &= b^{\delta+1}h(\alpha,1,\ell^{-3}) & (\partial_{yy}v_2)(a,b,\ell^{-3}) &= b^{\delta-2}(\partial_{yy}v_2)(\alpha,1,\ell^{-3})\,. \end{aligned}$$

Again, these scaling relations in $\mathcal{R}_2^{(1)}(\alpha)$ lead us to choose $\gamma=\frac{5\delta+5}{5\delta+3}$, which is the ratio of the exponents of ℓ in Tv_2 and v_2. Combining these two sets of scaling estimates and setting $\bar\gamma=\delta(1-\gamma)+1$, we get that

$$\begin{aligned} &\Big[\frac{h-\sigma_x|\partial_{xx}v_2|-\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\Big](x,y,1) \\ &\qquad = b^{\bar\gamma}\left[\frac{h-(b\ell)^{-3}\sigma_x|\partial_{xx}v_2|-b^{-3}\sigma_y|\partial_{yy}v_2|}{v_2^\gamma}\right](\alpha,1,\ell^{-3}) \\ &\qquad = b^{\bar\gamma}\Big[\frac{h}{v_2^\gamma}\Big(1-(b\ell)^{-3}\sigma_x\frac{|\partial_{xx}v_2|}{h}-b^{-3}\sigma_y\frac{|\partial_{yy}v_2|}{h}\Big)\Big](\alpha,1,\ell^{-3})\,. \end{aligned}$$

We have organized this calculation a bit differently for reasons which will become clear momentarily. Based on the preceding calculation, we define

$$\rho(\lambda)=\inf_{\lambda'\in[0,\lambda]}\frac{h(\alpha,1,\lambda')}{v_2^\gamma(\alpha,1,\lambda')}\quad\text{and}\quad M_1(\lambda)=\inf_{\lambda'\in[0,\lambda]}\frac{|\partial_{xx}v_2(\alpha,1,\lambda')|}{h(\alpha,1,\lambda')}$$
$$\text{and}\quad M_2(\lambda)=\inf_{\lambda'\in[0,\lambda]}\frac{|\partial_{yy}v_2(\alpha,1,\lambda')|}{h(\alpha,1,\lambda')}\,.$$

Notice that, unlike the previous calculations, we have made the constants ρ, M_1, and M_2 depend on the maximal value of λ. This is because in our current setting we require more precise information about these constants than merely that they are finite and positive.

We set $\ell_*=\inf\{(x/\alpha y)^{\frac{2}{3}}:(x,y)\in\mathcal{R}_2^{(1)}(\alpha)\cap B_R^c(0)\}$ and we observe that since $b\geq 1$,

$$m^{(1)}\geq\rho(\ell_*^{-3})(1-\ell_*^{-3}\sigma_xM_1(\ell_*^{-3})-\sigma_yM_2(\ell_*^{-3}))\,.$$

We wish to conclude that the right-hand side of the above expression is strictly positive. To conclude this, however, we need to understand the behavior of $M_1(\lambda)$ and $M_2(\lambda)$ as $\lambda\to 0$. By direct calculation from the explicit formula for v_2, we see that

$$M_1(0)=\frac{2\delta(2\delta+1)}{\alpha^3}\quad\text{and}\quad M_2(0)=\frac{(\delta+1)(\delta+2)}{\alpha}\,.$$

Since $M_1(\lambda)$ and $M_2(\lambda)$ are both continuous functions of λ on $(0,1]$ with finite limits as $\lambda\to 0$, and since ℓ_* can be made arbitrarily large by choosing R sufficiently large, for any $\epsilon>0$ we can choose R large enough to ensure

$$1-\ell_*^{-3}\sigma_xM_1(\ell_*^{-3})-\sigma_yM_2(\ell_*^{-3})\geq 1-\frac{\sigma_y(\delta+1)(\delta+2)}{\alpha}-\epsilon\,.$$

We conclude that as long as $\frac{\sigma_y(\delta+1)(\delta+2)}{\alpha}<1$, we can always choose R large enough to guarantee that $m^{(1)}$ is positive. This last inequality holds whenever α is sufficiently large. Again, we denote these specific choices of R and γ , which are valid in $\mathcal{R}_2^{(1)}(\alpha)$, by $R_2^{(1)}$ and $\gamma_2^{(1)}$. Substituting these values in the expression for $m^{(1)}$, we obtain $m_2^{(1)}$. Choosing $m_2=\min\{m_2^{(1)},m_2^{(2)}\}$, $\gamma_2=\min\{\gamma_2^{(1)},\gamma_2^{(2)}\}$, and $R_2=\max\{R_2^{(1)},R_2^{(2)}\}$ completes the proof that v_2 is a local super Lyapunov function in the entire region $\mathcal{R}_2(\alpha)$. The more detailed statements of the behavior in $\mathcal{R}_2^{(1)}$ and $\mathcal{R}_2^{(2)}$ merely serve to summarize the above points. □

10 Construction of a Global Super Lyapunov Function

We now patch together the three local Lyapunov functions that are defined in distinct regions of the plane in order to produce one smooth, global Lyapunov function defined on the entire plane. To do this, we use the standard mollifier $\phi(t)$, a smooth, increasing function which varies from 0 to 1 and is suitably normalized to integrate to unity on the whole real line. Specifically, we take $\phi(t) = \frac{1}{m}\int_{-\infty}^{t}\psi(s)ds$ with $m = \int_{-\infty}^{\infty}\psi(s)ds$ where

$$\psi(t) = \begin{cases} \exp\left(\frac{-1}{1-(2t-1)^2}\right) & \text{for } 0 < t < 1 \\ 0 & \text{otherwise } . \end{cases}$$

Next, we define the functions $h_1(x,y)$ and $h_2(x,y)$ as follows:

$$h_1(x,y) = 2 + \frac{\alpha|y|}{x} \qquad h_2(x,y) = 2 - \frac{xy^2}{\alpha}\,.$$

The function $h_1(x,y) = 0$ on one boundary of the wedge-shaped region $\mathcal{R}_1(\alpha)\cap\mathcal{R}_2(\alpha)$; $h_1(x,y) = 1$ on the other boundary of this region; and h_1 varies smoothly between 0 and 1 in the interior. Similarly, $h_2(x,y) = 0$ on one boundary of the funnel-like region $\mathcal{R}_2(\alpha)\cap\mathcal{R}_3(\alpha)$ and $h_2(x,y) = 1$ on the other boundary. Thus, outside of a fixed ball, we define our global Lyapunov function V to agree with the local Lyapunov functions in subregions of their domains of definition and to be a smooth, convex combination of the two local Lyapunov functions in regions of intersection. In particular, let $\tilde{V}(x,y)$ be given by

$$\tilde{V}(x,y) = \begin{cases} v_1(x,y) & \text{for } (x,y)\in\mathcal{R}_1(\alpha)\cap\mathcal{R}_2(\alpha)^c \\ V_1(x,y) & \text{for } (x,y)\in\mathcal{R}_1(\alpha)\cap\mathcal{R}_2(\alpha) \\ v_2(x,y) & \text{for } (x,y)\in\mathcal{R}_2(\alpha)\cap\mathcal{R}_1(\alpha)^c\cap\mathcal{R}_3(\alpha)^c \\ V_2(x,y) & \text{for } (x,y)\in\mathcal{R}_2(\alpha)\cap\mathcal{R}_3(\alpha) \\ v_3(x,y) & \text{for } (x,y)\in\mathcal{R}_3(\alpha)\cap\mathcal{R}_2(\alpha)^c \\ 0 & \text{otherwise} \end{cases}$$

where

$$V_1(x,y) = [1-\phi(h_1(x,y))]v_2(x,y) + \phi(h_1(x,y))v_1(x,y)$$
$$V_2(x,y) = [1-\phi(h_2(x,y))]v_2(x,y) + \phi(h_2(x,y))v_3(x,y)\,.$$

We then choose $V(x,y)\in C^2(\mathbb{R}^2)$ to satisfy

$$V(x,y) = \begin{cases} \tilde{V}(x,y) & \text{for } x^2+y^2>\rho^2 \\ \text{arbitrary positive and smooth} & \text{for } x^2+y^2\le\rho^2 \end{cases}$$

where ρ will be specified below.

Remark 10.1. *At the start of the Lyapunov construction in Section 9.1, we fix a choice of $\delta > 0$ when defining v_1. This choice is then propagated through our construction and is explicitly present in the definition of v_2 and v_3. During the analysis of v_3, we note in Proposition 9.2 and Proposition 9.3 that we must choose $\delta \in (0, \frac{2}{5})$. Except for this one restriction, we are free to choose δ. Hence our construction of V depends on two parameters δ and ρ. As we will summarize in Proposition 10.2 below, δ gives the power of the polynomial growth in all but the pure, positive x-direction. On the other hand, ρ sets the distance from the origin past which the Lyapunov estimates are valid.*

Consolidating our results on the scaling behavior of the functions v_i, $i = 1, 2, 3$, we obtain the following:

Proposition 10.2. *Fixing a $\delta \in (0, \frac{2}{5})$, there exists positive constants c and C, so that* $c|(x,y)|^{\delta} \le V(x,y) \le C|(x,y)|^{\hat{\delta}} = C|(x,y)|^{\frac{5}{2}\delta + \frac{3}{2}}$.

Proof of Proposition 10.2. After observing that $\delta < \hat{\delta} = \frac{5}{2}\delta + \frac{3}{2}$ on $(0, \frac{2}{5})$, the result follows quickly from Proposition 9.3 and the definition of v_1 from Section 9.1. On the right half-plane, the result follows from the definition of v_1. On the left half-plane, one can either use the the scaling relations or the explicit representations given in (9.11) and (9.10) to obtain the desired inequalities. □

The following proposition states that V is a super Lyapunov function. Therefore, one of our main theorems, Theorem 3.1 from Section 3, is an immediate consequence of this proposition.

Proposition 10.3. *For any $\delta \in (0, \frac{2}{5})$, there exists a ρ from the definition of V so that $V(x, y)$ is a global super Lyapunov function on $\mathbb{R}^2$.*

In light of Proposition 9.1 and Proposition 9.6, the main missing ingredient in the proof of Proposition 10.3 is the verification that V is a local Lyapunov function in the patching regions. This is the content of the next proposition; we prove it before returning to the proof of Proposition 10.3 at the end of the section.

Lemma 10.4. *For any $\delta \in (0, \frac{2}{5})$, $V_1(x, y)$ is a local super Lyapunov function on $\mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha)$ and $V_2(x, y)$ is a local super Lyapunov function on $\mathcal{R}_2(\alpha) \cap \mathcal{R}_3(\alpha)$.*

Proof of Lemma 10.4. Let m_i, R_i and γ_i for $i = 1, 2, 3$ be the constants from Proposition 9.1 and Proposition 9.6. Next define $m_* = \min\{m_1, m_2, m_3\}$, $R^* = \max\{R_1, R_2, R_3\}$ and $\gamma_* = \min\{\gamma_1, \gamma_2, \gamma_3\} = \frac{5\delta+5}{5\delta+3}$. We further increase R_* so that if $(x, y) \in \mathcal{R}_i$ and $|(x, y)| \ge R_*$ then $v_i(x, y) > 1$. In proving that V_2 is a local super Lyapunov function we need to recall the more refined version of the super Lyapunov property in $\mathcal{R}_2$ given in Proposition 9.6.

We address each of the claims in the lemma separately. We begin with the proof that V_1 is super Lyapunov since it is the most straightforward. If $\rho > R^*$, we have that for all $(x, y) \in \mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha) \cap B^c_\rho(0)$

$$\begin{aligned}(LV_1)(x,y) = {} & (1 - \phi(h_1(x,y)))(Lv_2)(x,y) + \phi(h_1(x,y))(Lv_1)(x,y) + E_1(x,y)\\ & \le -m_*[(1 - \phi(h_1(x,y)))v_2^{\gamma_1}(x,y) + \phi(h_1(x,y))v_1^{\gamma_1}(x,y)] + E_1(x,y)\\ & \le -m_*[V_1(x,y)]^{\gamma_1} + E_1(x,y) \text{ by convexity}\\ & \le -m_*(1 - M_1)[V_1(x,y)]^{\gamma_1}\end{aligned}$$

where M_1 and $E_1(x, y)$ are defined as

$$M_1 = \sup_{\substack{(x,y)\in\mathcal{R}_1(\alpha)\cap\mathcal{R}_2(\alpha)\\ |(x,y)|>\rho}} \left[\frac{E_1(x,y)}{m_*[V_1(x,y)]^{\gamma_1}}\right]$$

and

$$\begin{aligned}E_1(x,y) = {} & L[\phi(h_1(x,y))](v_1(x,y) - v_2(x,y))\\ & + 2\sigma_x \frac{\partial}{\partial x}[\phi(h_1(x,y))]\frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)]\\ & + 2\sigma_y \frac{\partial}{\partial y}[\phi(h_1(x,y))]\frac{\partial}{\partial y}[v_1(x,y) - v_2(x,y)]\,.\end{aligned}$$

If we can choose ρ sufficiently large so that $M_1 < 1$, then $V_1(x,y)$ will be a local super Lyapunov function on $\mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha)$. To show $M_1 < 1$, we use the scaling properties of v_1 and v_2 to map back to a circular arc of radius $r = \sqrt{\alpha^2+4}$. Let

$$\ell = \frac{\sqrt{x^2+y^2}}{r}, \qquad a = \frac{x}{\ell}, \qquad \text{and} \quad b = \frac{y}{\ell}\,.$$

Then

$$(x,y) \in \mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha) \cap B_\rho^c(0) \quad \Longrightarrow \quad (x,y) = S_\ell^{(2)}(a,b) \quad \text{with} \quad \ell \geq 1$$

Note that $h_1(x,y) = h_1(a,b)$, so

$$V_1(x,y) = \ell^\delta V_1(a,b) \quad \text{and} \quad [V_1(x,y)]^{\gamma_1} = \ell^{\delta+1}[V_1(a,b)]^{\gamma_1}\,.$$

As a consequence of these scaling relations, we get that for all $(x,y) \in \mathcal{R}_1(\alpha) \cap \mathcal{R}_2(\alpha) \cap B_\rho^c(0)$,

$$\begin{aligned}
E_1(x,y) = {} & \ell^{\delta+1}\alpha\phi'(h_1(a,b))|b|\left(1+\frac{b^2}{a^2}\right)[v_1(a,b)-v_2(a,b)] \\
& + \ell^{\delta-1}\alpha\phi''(h_1(a,b))\left(\frac{-\sigma_x|b|}{a^2} + \frac{\sigma_y \mathrm{sgn}(b)}{a}\right)[v_1(a,b)-v_2(a,b)] \\
& + \ell^{\delta-2}\alpha\phi'(h_1(a,b))\frac{2\sigma_x|b|}{a^3}[v_1(a,b)-v_2(a,b)] \\
& + \ell^{\delta-2}\alpha\phi'(h_1(a,b))\frac{-2\sigma_x|b|}{a^2}\frac{\partial}{\partial x}[v_1(a,b)-v_2(a,b)]) \\
& + \ell^{\delta-2}\alpha\phi'(h_1(a,b))\frac{2\sigma_y \mathrm{sgn}(b)}{a}\frac{\partial}{\partial y}[v_1(a,b)-v_2(a,b)]\,.
\end{aligned}$$

Hence we have that

$$M_1 \leq \sup_{\substack{(a,b)\in\mathcal{R}_1(\alpha)\cap\mathcal{R}_2(\alpha) \\ |(x,y)|\leq r}} \left[\frac{e_{1,1}(a,b)}{m_*[V_1(a,b)]^{\gamma_1}} + \frac{e_{1,2}(a,b)}{\ell^2 m_*[V_1(a,b)]^{\gamma_1}}\right] \tag{10.1}$$

where

$$\begin{aligned}
e_{1,1}(a,b) &= \alpha\phi'(h_1(a,b))|b|\Big(1+\frac{b^2}{a^2}\Big)[v_1(a,b)-v_2(a,b)] \\
e_{1,2}(a,b) &= \alpha\phi''(h_1(a,b))\Big(\frac{-\sigma_x|b|}{a^2} + \frac{\sigma_y \mathrm{sgn}(b)}{a}\Big)[v_1(a,b)-v_2(a,b)] \\
&\quad + \alpha\phi'(h_1(a,b))\frac{2\sigma_x|b|}{a^3}[v_1(a,b)-v_2(a,b)] \\
&\quad + \alpha\phi'(h_1(a,b))\frac{-2\sigma_x|b|}{a^2}\frac{\partial}{\partial x}[v_1(a,b)-v_2(a,b)] \\
&\quad + \alpha\phi'(h_1(a,b))\frac{2\sigma_y \mathrm{sgn}(b)}{a}\frac{\partial}{\partial y}[v_1(a,b)-v_2(a,b)]\,.
\end{aligned}$$

By explicit computation with v_1 and v_2, we can show that $e_{1,1}(a,b)$ is always negative for (a,b) in the desired region. The second term of the sum in (10.1), the upper bound for M_1, can then be made arbitrarily small by choosing ℓ large enough; this corresponds to choosing ρ sufficiently large. This establishes that $M_1 < 1$, which completes the proof of the lemma.

We now turn to proving that V_2 is super Lyapunov. If $\rho > R^*$, then for all $(x,y) \in \mathcal{R}_2(\alpha) \cap \mathcal{R}_3(\alpha) \cap B^c_\rho(0)$

$$\begin{aligned}(LV_2)(x,y) = &\ (1-\phi(h_2(x,y)))(Lv_2)(x,y) + \phi(h_2(x,y))(Lv_3)(x,y) + E_2(x,y)\\ &\le -m_*[(1-\phi(h_2(x,y)))v_2^{\gamma_3}(x,y) + \phi(h_2(x,y))v_3^{\gamma_3}(x,y)] + E_2(x,y)\\ &\le -m_*[V_2(x,y)]^{\gamma_3} + E_2(x,y) \quad \text{by convexity}\\ &\le -m_*(1-M_2)[V_2(x,y)]^{\gamma_3}\end{aligned}$$

where

$$M_2 = \sup_{(x,y)\in\mathcal{R}_2(\alpha)\cap\mathcal{R}_3(\alpha)\cap B^c_\rho(0)} \left[\frac{E_2(x,y)}{m_*[V_2(x,y)]^{\gamma_3}}\right]$$

and

$$\begin{aligned}E_2(x,y) = &\ L[\phi(h_2(x,y))](v_3(x,y) - v_2(x,y))\\ &+ 2\sigma_x \frac{\partial}{\partial x}[\phi(h_2(x,y))]\frac{\partial}{\partial x}[v_3(x,y) - v_2(x,y)]\\ &+ 2\sigma_y \frac{\partial}{\partial y}[\phi(h_2(x,y))]\frac{\partial}{\partial y}[v_3(x,y) - v_2(x,y)]\,.\end{aligned}$$

If $M_2 < 1$, then $V_2(x,y)$ will be a super Lyapunov function on $\mathcal{R}_2(\alpha) \cap \mathcal{R}_3(\alpha)$. To show $M_2 < 1$, we use the scaling properties of v_2 and v_3 to map back to a vertical line. Let

$$\ell = \frac{x}{2\alpha}, \qquad a = 2\alpha, \qquad \text{and} \qquad b = y\sqrt{\ell}\,.$$

Then

$$(x,y) \in \mathcal{R}_2(\alpha)\cap\mathcal{R}_3(\alpha)\cap B^c_\rho(0) \quad\Longrightarrow\quad (x,y,1) = S^{(1)}_\ell(a,b,\ell^{-3})$$

where $|b| \in \left[\frac{1}{\sqrt{2}}, 1\right]$ and $\ell \ge 1$. Note that $h_2(x,y) = h_2(a,b)$, so $V_2(x,y) = V_2(x,y,1)$ satisfies

$$V_2(x,y,1) = \ell^{\hat{\delta}} V_2(a,b,\ell^{-3}) \quad \text{and} \quad [V_2(x,y,1)]^{\gamma_3} = \ell^{\hat{\delta}+1}[V_2(a,b,\ell^{-3})]^{\gamma_3}$$

where

$$V_2(x,y,\lambda) = [1-\phi(h_2(x,y))]v_2(x,y,\lambda) + \phi(h_2(x,y))v_3(x,y)\,.$$

Now we have that for all $(x,y) \in \mathcal{R}_2(\alpha)\cap\mathcal{R}_3(\alpha)\cap B^c_\rho(0)$,

$$\begin{aligned}E_2(x,y) = &\ \ell^{\hat{\delta}+1}\frac{\phi'(h_2(a,b))}{\alpha}(-5a^2b^2 - 2a\sigma_y)[v_3(a,b) - v_2(a,b,\ell^{-3})]\\ &+ \ell^{\hat{\delta}+1}\frac{\phi'(h_2(a,b))}{\alpha}(-4ab\sigma_y)\frac{\partial}{\partial y}[v_3(a,b) - v_2(a,b,\ell^{-3})]\\ &+ \ell^{\hat{\delta}+\frac{1}{2}}\frac{\phi''(h_2(a,b))}{\alpha}(-2ab\sigma_y)[v_3(a,b) - v_2(a,b,\ell^{-3})]\\ &+ \ell^{\hat{\delta}-1}\frac{\phi''(h_2(a,b))}{\alpha}(-b^2\sigma_x)[v_3(a,b) - v_2(a,b,\ell^{-3})]\\ &+ \ell^{\hat{\delta}-2}\frac{\phi'(h_2(a,b))}{\alpha}b^4[v_3(a,b) - v_2(a,b,\ell^{-3})]\\ &+ \ell^{\hat{\delta}-2}\frac{\phi'(h_2(a,b))}{\alpha}(-2b^2\sigma_x)\frac{\partial}{\partial x}[v_3(a,b) - v_2(a,b,\ell^{-3})]\,.\end{aligned}$$

Define $N(\lambda^*)$ as follows:

$$N(\lambda^*) = \sup_{\substack{|b|\in[\frac{1}{\sqrt{2}},1]\\ \lambda\in(0,\lambda^*]}} \left[\frac{e_{2,1}(a,b,\lambda)}{m_*[V_2(a,b,\lambda)]^{\gamma_3}} + \frac{e_{2,2}(a,b,\lambda)}{\sqrt{\ell}m_*[V_2(a,b,\lambda)]^{\gamma_3}}\right] \tag{10.2}$$

where

$$\begin{aligned}e_{2,1}(a,b,\lambda) &= \frac{\phi'(h_2(a,b))}{\alpha}(-5a^2b^2 - 2a\sigma_y)[v_3(a,b) - v_2(a,b,\lambda)] \\ &\quad + \frac{\phi'(h_2(a,b))}{\alpha}(-4ab\sigma_y)\frac{\partial}{\partial y}[v_3(a,b) - v_2(a,b,\lambda)]\,, \\ e_{2,2}(a,b,\lambda) &= \frac{\phi''(h_2(a,b))}{\alpha}(-2ab\sigma_y)[v_3(a,b) - v_2(a,b,\lambda)] \\ &\quad + \frac{\phi''(h_2(a,b))}{\alpha}(-b^2\sigma_x)[v_3(a,b) - v_2(a,b,\lambda)] \\ &\quad + \frac{\phi'(h_2(a,b))}{\alpha}b^4[v_3(a,b) - v_2(a,b,\lambda)] \\ &\quad + \frac{\phi'(h_2(a,b))}{\alpha}(-2b^2\sigma_x)\frac{\partial}{\partial x}[v_3(a,b) - v_2(a,b,\lambda)]\,.\end{aligned}$$

Note that for any $\lambda^* > 0$, we can choose ρ sufficiently large to force M_2 (which, we recall, depends on ρ) to be less than $N(\lambda^*)$. Ultimately, we will choose λ^* sufficiently small so that $N(\lambda^*) < 1$. The second term of the sum in (10.2) can be made arbitrarily small by increasing the size of ℓ; again, increasing the size of ℓ corresponds to increasing ρ. We now address the first term of the sum in (10.2). From Lemma 10.5 which is stated and proven bellow, we see that we can chose the parameters to make this term negative.

Combining all of these observations, we have demonstrated that $M_2 < 1$, which completes the proof of the lemma. □

Lemma 10.5. *There exist positive constants c_1 and c_2 in the definition of the Poisson equation for $v_3(x,y)$, and positive α and λ^* such that for all $\lambda \in [0,\lambda^*]$, the following inequalities hold for $a = 2\alpha$ and $|b| \in \big[\frac{1}{\sqrt{2}}, 1\big]$:*

$$v_3(a,b) - v_2(a,b,\lambda) > 0 \tag{10.3}$$

$$b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,\lambda)\Big] > 0 \tag{10.4}$$

Proof of Lemma 10.5. Recall that, from (9.10), $v_3(x,y)$ can be represented as

$$v_3(x,y) = x^{\hat\delta}\Big[\big(\tfrac{c_1}{\hat\delta} + c_2\big)\mathbb{E}_{(x,y)}\big[e^{\hat\delta\tau}\big] - \tfrac{c_1}{\hat\delta}\Big] \tag{10.5}$$

where $\tau = \inf\{t > 0 : |Z_t| \notin [-\sqrt{2\alpha}, \sqrt{2\alpha}]\}$ and Z_t is the process given in (9.8). Note that the expectation in (10.5) can be written as the solution to a second-order ODE, namely:

$$\mathbb{E}_{(x,y)}\big[e^{\hat\delta\tau}\big] = g_\epsilon(\sqrt{\epsilon x}\;y)$$

where $g_\epsilon(z)$ solves the following boundary value problem with $\epsilon = \frac{1}{2\alpha}$:

$$\begin{cases}\epsilon\sigma_y g_\epsilon''(z) + \frac{5}{2}zg_\epsilon'(z) + \hat\delta g_\epsilon(z) = 0 \quad \text{for } z \in (-1,1) \\ g_\epsilon(-1) = g_\epsilon(1) = 1\,.\end{cases} \tag{10.6}$$

Define $g_0(z)$ to be the solution to the limiting ODE in (10.6) when $\epsilon = 0$ and note that $g_0(z)$ can be computed exactly for initial conditions $z \neq 0$:

$$g_0(z) = \frac{1}{|z|^{\delta + \frac{3}{5}}}\,. \tag{10.7}$$

Now, let $v_0(x,y)$ be defined as

$$v_3^0(x,y) = x^{\hat\delta}\Big[\big(\tfrac{c_1}{\hat\delta} + c_2\big)g_0(\sqrt{\epsilon x}\;y) - \tfrac{c_1}{\hat\delta}\Big]\,. \tag{10.8}$$

We address the first difference between v_3 and v_2 in (10.3) as follows. We write

$$v_3(a,b) - v_2(a,b) = v_3(a,b) - v_3^0(a,b) \tag{10.9}$$
$$+ v_3^0(a,b) - v_2(a,b,0) \tag{10.10}$$
$$+ v_2(a,b,0) - v_2(a,b,\lambda)\,. \tag{10.11}$$

To show that this difference is positive, we will show that $v_3^0(a,b) - v_2(a,b,0) > 0$ and that the other two differences are small in comparison. Similarly, for the difference between the y-derivatives of v_3 and v_2 in (10.4), we write

$$b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,\lambda)\Big] = b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_3^0}{\partial y}(a,b)\Big] \tag{10.12}$$
$$+ b\Big[\frac{\partial v_3^0}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,0)\Big] \tag{10.13}$$
$$+ b\Big[\frac{\partial v_2}{\partial y}(a,b,0) - \frac{\partial v_2}{\partial y}(a,b,\lambda)\Big] \tag{10.14}$$

and again, we will show that $b\big[\frac{\partial v_3^0}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,0)\big] > 0$ and the other two differences are small in comparison. Specifically, we demonstrate that there exist positive constants c_1 and c_2 in the Poisson equation for v_3 such that the differences in (10.10) and (10.13) are positive; and then, that there exists α sufficiently large such that the differences on the righthand sides of (10.9) and (10.12) are comparatively small; and last, that there exists a λ^* such that (10.11) and (10.14) are comparatively small for all $\lambda \in [0, \lambda^*]$. For the first of these claims, note that

$$v_3^0(a,b) - v_2(a,b,0) = \frac{(2\alpha)^{2\delta+1}}{|b|^{\delta+1}} q(b) \tag{10.15}$$
$$b\Big[\frac{\partial v_3^0}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,0)\Big] = \frac{(2\alpha)^{2\delta+1}}{|b|^{\delta+1}} \tilde{q}(b)$$

where q and $\tilde{q}$ are given by

$$q(b) = 2^{\frac{1}{2}\delta+\frac{1}{2}}\Big[\big(\tfrac{\tilde{c_1}}{\hat{\delta}} + \tilde{c_2}\big)|b|^{\frac{2}{5}} - \frac{\tilde{c_1}}{\hat{\delta}}|b|^{\delta+1}\Big] - 1$$
$$\tilde{q}(b) = -\big(\delta + \tfrac{3}{5}\big)2^{\frac{1}{2}\delta+\frac{1}{2}}\big(\tfrac{\tilde{c_1}}{\hat{\delta}} + \tilde{c_2}\big)|b|^{\frac{2}{5}} + \delta + 1$$

and $c_1 = \frac{\tilde{c_1}}{\alpha^{\frac{1}{2}\delta+\frac{1}{2}}}$ and $c_2 = \frac{\tilde{c_2}}{\alpha^{\frac{1}{2}\delta+\frac{1}{2}}}$. We note that c_1 and c_2 are chosen to scale with α so that v_2 and v_3 have identical scaling in α. Moreover, as we demonstrate below, $\tilde{c_1}$ and $\tilde{c_2}$ can be chosen independently of α. It is clear that $\tilde{q}$ is a monotone decreasing function of $|b|$; hence it is minimized at the right endpoint of the interval for $|b|$, that is, $|b| = 1$. Thus if we can show $\tilde{q}(1) > 0$, then it follows that

$$b\Big[\frac{\partial v_3^0}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,0)\Big] > 0 \tag{10.16}$$

for all $|b| \in [2^{-\frac{1}{2}}, 1]$. If we can also show $q(2^{-\frac{1}{2}}) > 0$, then from (10.15), we conclude that

$$v_3^0(a, 2^{-\frac{1}{2}}) - v_2(a, 2^{-\frac{1}{2}}, 0) > 0\,.$$

Combining this with (10.16) gives the desired positivity of (10.10) on the whole interval $|b| \in [2^{-\frac{1}{2}}, 1]$. Hence, we need only verify that there exist positive values of $\tilde{c_1}$ and $\tilde{c_2}$ such that

$$q(2^{-\frac{1}{2}}) = 2^{\frac{1}{2}\delta+\frac{1}{2}}\Big[\big(\tfrac{\tilde{c_1}}{\hat{\delta}} + \tilde{c_2}\big)2^{-\frac{1}{5}} - \tfrac{\tilde{c_1}}{\hat{\delta}}2^{-(\frac{1}{2}\delta+\frac{1}{2})}\Big] - 1 > 0 \tag{10.17}$$
$$\tilde{q}(1) = -\big(\delta + \tfrac{3}{5}\big)2^{\frac{1}{2}\delta+\frac{1}{2}}\big(\tfrac{\tilde{c_1}}{\hat{\delta}} + \tilde{c_2}\big) + \delta + 1 > 0\,. \tag{10.18}$$

The verification of this is elementary and we omit the details.

We remark that $\tilde{c_1}$ and $\tilde{c_2}$ can be chosen independently of α, since the above inequalities have no dependence on α. Thus, choosing positive $\tilde{c_1}$ and $\tilde{c_2}$ such that (10.17) and (10.18) are both satisfied, we get that for all $|b| \in [2^{-\frac{1}{2}}, 1]$,

$$v_3^0(a,b) - v_2(a,b,0) \geq \alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} q(2^{-\frac{1}{2}}) > 0$$
$$b\Big[\frac{\partial v_3^0}{\partial y}(a,b) - \frac{\partial v_2}{\partial y}(a,b,0)\Big] \geq \alpha^{2\delta+1} 2^{2\delta+1} \tilde{q}(1) > 0\,.$$

We turn our attention to making the differences

$$v_3(a,b) - v_3^0(a,b) \quad \text{and} \quad b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_3^0}{\partial y}(a,b)\Big]$$

comparatively small. Note that

$$v_3(a,b) - v_3^0(a,b) = \alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} \big(\tfrac{\tilde{c_1}}{\delta} + \tilde{c_2}\big)[g_\epsilon(b) - g_0(b)]$$
$$b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_3^0}{\partial y}(a,b)\Big] = \alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} b\big(\tfrac{\tilde{c_1}}{\delta} + \tilde{c_2}\big)[g_\epsilon'(b) - g_0'(b)]\,.$$

To be precise, we will show that

$$|v_3(a,b) - v_3^0(a,b)| < \tfrac{1}{3}\big[\alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} q(2^{-\frac{1}{2}})\big]$$
$$\Big|b\Big[\frac{\partial v_3}{\partial y}(a,b) - \frac{\partial v_3^0}{\partial y}(a,b)\Big]\Big| < \tfrac{1}{3}\left[\alpha^{2\delta+1} 2^{2\delta+1} \tilde{q}(1)\right]\,.$$

This is equivalent to establishing that

$$\Big|\alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} \big(\tfrac{\tilde{c_1}}{\delta} + \tilde{c_2}\big)[g_\epsilon(b) - g_0(b)]\Big| < \tfrac{1}{3}\big[\alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} q(2^{-\frac{1}{2}})\big] \tag{10.19}$$
$$\Big|\alpha^{2\delta+1} 2^{\frac{5}{2}\delta+\frac{3}{2}} b\big(\tfrac{\tilde{c_1}}{\delta} + \tilde{c_2}\big)[g_\epsilon'(b) - g_0'(b)]\Big| < \tfrac{1}{3}\left[\alpha^{2\delta+1} 2^{2\delta+1} \tilde{q}(1)\right]\,. \tag{10.20}$$

Observe that the same powers of α appear on both sides of each of the above inequalities. Therefore, to prove (10.19) and (10.20), it suffices to show that $g_\epsilon(b)$ and $g_\epsilon'(b)$ converge uniformly to $g_0(b)$ and $g_0'(b)$, respectively, for $|b| \in [2^{-\frac{1}{2}}, 1]$ as $\epsilon = \frac{1}{2\alpha} \to 0$. Both of these uniform convergences follow from classical results; see, for example, [1].

Since $\epsilon = \frac{1}{2\alpha}$, we can choose α sufficiently large to guarantee that both (10.19) and (10.20) hold and that v_2 remains a local super Lyapunov function on $\mathcal{R}_2(\alpha)$ (recall that in Proposition 9.6, a lower bound on the size of α was imposed). Finally, by choosing λ^* sufficiently small, the differences in (10.11) and (10.14) can be made small for all $\lambda \in [0, \lambda*]$. This is an immediate consequence of the fact that $v_2(a,b,\lambda)$ is a C^2 function of $\lambda \in [0,1]$. ☐

Having established the super Lyapunov property in the patching region, we return to the proof of the main result of this section.

Proof of Proposition 10.3. The local super Lyapunov condition has now been proven in regions; namely, for v_1 in Proposition 9.1, for v_2 and v_3 in Proposition 9.6, and for the patched functions V_1 and V_2 in Proposition 10.4. All that remains is to make a global choice of constants. The constant ρ from Proposition 10.4 was chosen to be valid in all regions. It is sufficient to choose

$$M = \min\big\{m_*(1-M_1), m_*(1-M_2)\big\} < m_*,$$
$$b = \sup\big\{|(LV)(x,y)| : x^2+y^2 \leq \rho^2\big\}, \text{ and}$$
$$\gamma = \min\{\gamma_1,\gamma_2,\gamma_3\} = \frac{5\delta+5}{5\delta+3}.$$

These choices guarantee that for all $(x, y) \in \mathbb{R}^2$

$$(LV)(x, y) \le -M\,[V(x, y)]^{\gamma} + b\,.$$

□

11 Existence and Positivity of Transition Density

Having established the existence of a global super Lyapunov function, we now make a small detour to prove the existence of a smooth density with appropriate positivity properties. These results provide the missing ingredient to prove the ergodic result stated in Theorem 3.2, namely a minorization condition. It is worth noting, however, that proving the minorization condition is not our sole goal. Indeed, if it were, we would not need all of the results of this section: we could simply use smoothness and appropriate open set controllability results. See for example [25, 24].

Instead, our interest is larger, motivated by two concerns. First, we wish to understand the general structure of the invariant measure, not merely its uniqueness. Second, we want to take this simple example to highlight some techniques, different than those often used, which can be applied in more general situations to address questions of positivity. We feel that these methods convey more intuition and better allow for the inclusion of *a priori* facts about the dynamics.

11.1 Positivity when $\sigma_x > 0$

When $\sigma_x > 0$ (since we always assume $\sigma_y > 0$), the system is uniformly elliptic, and everything is relatively straightforward. Since the diffusion associated with (3.1) has a constant, positive definite diffusion matrix, classical results guarantee the existence of a function $p : (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2 \to (0, \infty)$ such that p is jointly continuous, $p_t(z, z')$ is strictly positive for all (t, z, z'), and such that for all measurable sets A

$$P_t(z, A) = \int_A p_t(z, z')dz'\,. \tag{11.1}$$

We summarize these results for future reference in the following Proposition.

Proposition 11.1. *If $\sigma_y > 0$ and $\sigma_x > 0$ then for all $t > 0$, P_t has an everywhere positive density $p_t(z, z')$ with respect to Lebesgue measure which is smooth in both z and z'.*

11.2 Positivity when $\sigma_x = 0$

When $\sigma_x = 0$ (but σ_y still positive), the situation is more delicate. We begin by observing that the generator of the associated diffusion is still hypoelliptic; to see this, we write the generator of the diffusion as

$$L = X + \frac{1}{2}\sigma_y \partial_y^2\,.$$

Observe that $[[X, \partial_y], \partial_y] = -2$ and hence the relevant ideal in the algebra generated by X and ∂_y is of full rank. In turn, this ensures the existence of a continuous function p so that (11.1) holds. The main difference between this and the setting of Section 11.1 is that it is no longer immediate that $p_t(z, z')$ is positive for all $t > 0$ and $z, z' \in \mathbb{R}^2$. In fact it is not true.

Intuitively, it is clear that if z is in the left-half plane and z' in the right-half plane then $p_t(z, z')$ should be zero, since there is no way to move across the y-axis. On the other hand, it is reasonable to expect that given any z' in the left-half plane, there exists a $T = T(z')$ such that $p_t(z, z') > 0$ for all $t > T$ and $z \in \mathbb{R}^2$.

There are a number of ways to prove such a result. The most generally applicable and powerful technique is to leverage geometric control theory to show that the support of $P_t(z,\ \cdot\ )$ contains a sufficiently large, bounded region of the left-half plane for any z and t sufficiently large. From this, for example, one can show that (X_t, Y_t) is sufficiently smooth in the Malliavin sense (which it is), and deduce that $p_t(z, z')$ is strictly positive in the interior of the support.

Alternatively, one can use sufficiently quantitative open-set controllability results to extend the very local positivity which follows just from the joint-continuity of $(z, z') \mapsto p_t(z, z')$. This is the method employed in [25, 24] in various forms.

Here we take an approach most consonant with the first option. However, rather than merely citing the appropriate geometric control theory results, we construct an explicit series of simple controls to prove the positivity condition we require. The subsequent discussion is lengthier, but we feel that it is more intuitive and is a useful complement to more general control theoretic arguments, especially for the uninitiated.

Before turning to the existence of a positive density for (3.1), we first consider an analogous calculation in a simpler setting. Consider a smooth map $\phi\colon \mathbb{R}^m \to \mathbb{R}^d$ where $m > d$ and let Γ be a non-degenerate Gaussian probability measure on $\mathbb{R}^m$. Consider the push forward of Γ by ϕ, denoted by $\Gamma\phi^{-1}$ and defined by $\Gamma\phi^{-1}(A) = \Gamma(\phi^{-1}(A))$. For the measure $\Gamma\phi^{-1}$ to be absolutely continuous with respect to Lebesgue measure, it is necessary and sufficient that

$$\Gamma\{x \in \mathbb{R}^m : \mathrm{Det}|(D\phi)(x)(D\phi^T)(x)| = 0\} = 0\,.$$

(See [5, 6] for more details.) Supposing that this condition holds, we let $\hat{\gamma}$ denote the density of $\Gamma\phi^{-1}$ with respect to Lebesgue measure. We are interested in when $\hat{\gamma}(z)$ is positive at a given point $z \in \mathbb{R}^d$. It is a simple exercise in calculus to see that $\hat{\gamma}(z) > 0$ if and only if there exists a $x \in \mathbb{R}^m$ with $\gamma(x) = z$ and for which $(D\phi(x))(D\phi^T(x))$ is a non-degenerate matrix. The first condition ensures that there is a way to reach z; that is to say, z is the image of R^d under ϕ. The second ensures that an infinitesimal piece of volume, and hence probability, is brought with x when it is mapped by ϕ.

This intuitive fact has a counterpart in stochastic analysis. While these ideas rest on the foundation of Malliavin calculus, the closest analogue is found in the work of Ben Arous and Leandre [8, 7] and the subsequent presentation by Nualart[3]. We begin by identifying the map in question.

To any $U \in L^2([0,T], \mathbb{R}^d)$, we associate $\{(X_t^U, Y_t^U) : t \in [0,T]\}$ which solves the system of equations

$$\begin{aligned} \dot{X}_t^U &= (X_t^U)^2 - (Y_t^U)^2 \\ \dot{Y}_t^U &= 2X_t^U Y_t^U + U_t \end{aligned}\ . \tag{11.2}$$

In the following discussion, we will refer to U as the control and denote by (X_t^d, Y_t^d) the solution to the deterministic system of equations (7.1); that is, the system (11.2) with $U(x,y)$ identically zero. It is also worth mentioning that for $U \in L^2([0,T], \mathbb{R})$, $t \mapsto (X_t^U, Y_t^U)$ is continuous on $[0,T]$.

In analogy to the discussion at the start of the section, for $T > 0$ and $z \in \mathbb{R}^2$, we will consider the map $\Phi_{T,z}\colon L^2((0,T]; \mathbb{R}) \to \mathbb{R}^2$ defined by $U \mapsto (\Phi_{T,z}^{(1)}, \Phi_{T,z}^{(2)}) = (X_T^U, Y_T^U)$ and $(X_0^U, Y_0^U) = z$. Translating [3] to our current setting, we obtain the following theorem:

Theorem 11.2. *$p_T(z, z') > 0$ if and only if there exists a $U \in L^2([0,T], \mathbb{R})$ so that $\Phi_{T,z}(U) = z'$ and furthermore the matrix*

$$M_{T,z}(U) = \begin{pmatrix} \|D\Phi_{T,z}^{(1)}(U)\|_{L^2}^2 & \langle D\Phi_{T,z}^{(1)}(U), D\Phi_{T,z}^{(2)}(U)\rangle_{L^2} \\ \langle D\Phi_{T,z}^{(1)}(U), D\Phi_{T,z}^{(2)}(U)\rangle_{L^2} & \|D\Phi_{T,z}^{(2)}(U)\|_{L^2}^2 \end{pmatrix} \tag{11.3}$$

is nondegenerate. Here D represents the Frechet derivative.

The condition from [8, 7, 3] that the SDE under consideration have coefficients which are bounded with all derivatives bounded can be removed by a standard localization argument. The key step is knowing that $|(X_t, Y_t)|$ is almost surely finite which follows from the Lyapunov function we constructed in the preceding sections.

This matrix M is just the product of the Jacobians introduced in the motivating discussion at the start of this section, translated to our current setting. To facilitate calculations and to better see this connection, it is useful to observe that for any $\eta \in \mathbb{R}^2$

$$\langle M_{T,z}(U)\eta, \eta\rangle = \int_0^T \langle J^U_{s,T} e_2, \eta\rangle^2 ds \tag{11.4}$$

where $e_2 = (0, 1)$ and $J^U_{s,t}$ is the Jacobi flow (the linearization of the SDE/controlled ODE) defined by

$$J^U_{s,t} = \begin{pmatrix} \frac{\partial \Phi^{(1)}_{t,z}(U)}{\partial \Phi^{(1)}_{s,z}(U)} & \frac{\partial \Phi^{(1)}_{t,z}(U)}{\partial \Phi^{(2)}_{s,z}(U)} \\ \\ \frac{\partial \Phi^{(2)}_{t,z}(U)}{\partial \Phi^{(1)}_{s,z}(U)} & \frac{\partial \Phi^{(2)}_{t,z}(U)}{\partial \Phi^{(2)}_{s,z}(U)} \end{pmatrix} .$$

We now state a simple condition which ensures the nondegeneracy of M. It captures the fact that as long as there is some twist in $J_{s,s+\epsilon}$ then $J_{s,s+\epsilon}e_2$ and e_2 will not be co-linear, and hence $\langle J_{s,s+\epsilon}e_2, \eta\rangle + \langle e_2, \eta\rangle \neq 0$ for any $\eta \neq 0$. Combining this with the continuity of $s \mapsto J_{s,T}$ produces the desired nondegeneracy of (11.4). The following lemma follows this outline, providing a condition which ensure such that the system has the desired twist.

Proposition 11.3. *To ensure the nondegeneracy of $M_{T,z}(U)$, it is sufficient that there exist a $t_0 \in [0, T]$ so that $\Phi_{t_0,z}(U) \neq 0$.*

Proof of Proposition 11.3. Since $t \mapsto \Phi_{t,z}(U)$ is continuous, we can without loss of generality assume that $t_0 \in (0, T)$ and pick an $\epsilon > 0$ so $[t_0 - \epsilon, t_0] \subset (0, T)$ and $\Phi_{t,z}(U) > 0$ for all $t \in [t_0 - \epsilon, t_0]$.

The nondegeneracy of $M_{T,z}(U)$ is equivalent to

$$\inf_{\eta \in \mathbb{R}^2 : |\eta|=1} \langle M_{T,z}(U)\eta, \eta\rangle > 0 .$$

In light of (11.4), we see that

$$\langle M_{T,z}(U)\eta, \eta\rangle \geq \int_{t_0-\epsilon}^{t_0} \langle J^U_{t,T} e_2, \eta\rangle^2 dt = \int_{t_0-\epsilon}^{t_0} \langle J^U_{t,t_0} e_2, (J^U_{t_0,T})^* \eta\rangle^2 dt$$

where $(J^U_{t_0,T})^*$ is the adjoint of the matrix $J^U_{t_0,T}$. Combining these two observations, we see that for some positive constant c, depending on U,

$$\inf_{\eta \in \mathbb{R}^2 : |\eta|=1} \langle M_{T,z}(U)\eta, \eta\rangle \geq c \inf_{\eta \in \mathbb{R}^2 : |\eta|=1} \int_{t_0}^{t_0+\epsilon} \langle J^U_{t,t_0} e_2, \eta\rangle^2 dt .$$

Since $(t, \eta) \mapsto \langle J^U_{t,t_0} e_2, \eta\rangle$ is jointly continuous, it is sufficient to show that

$$\langle J^U_{t,t_0} e_2, e_2^\perp\rangle \neq 0 \quad \text{for all } t \in [t_0 - \epsilon, t_0) \tag{11.5}$$

for some positive ϵ where $e_2^\perp$ is the standard choice of vector perpendicular to e_2. Since $J^U_{t_0,t_0} e_2 = e_2$, this guarantees that for every given $\eta \neq 0$, one has $\langle J^U_{t,t_0} e_2, \eta\rangle \neq 0$ for t in some open interval of $[t_0 - \epsilon, t_0]$. The continuity in η then ensures the infimum over all η with $|\eta| = 1$ is still positive.

To establish (11.5), we appeal directly to the equations. We see that as long as $\Phi_{t,z}(U) \neq 0$ to $t \in [t_0 - \epsilon, t_0]$ then $\langle J_{s,r}\eta, \eta^\perp \rangle \neq 0$ for all s, r with $t_0 - \epsilon \leq s < r \leq t_0$ and all $\eta \neq 0$. This is due to the fact that as long as the trajectory is not at zero, the linearization rotates any vector a nontrivial amount over any interval of time. In particular, the linearization satisfies the equation $\partial_t J_{s,t} = A_t J_{s,t}$ for $t \geq s$ with $J_{s,s}$ equal to the identity matrix and

$$A_t = \begin{pmatrix} 2X_t & -2Y_t \\ 2Y_t & 2X_t \end{pmatrix} = 2R_t \begin{pmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{pmatrix} \tag{11.6}$$

where $(X_t, Y_t) = (R_t \cos\theta_t, R_t \sin\theta_t)$. □

Remark 11.4. *The proof of Proposition 11.3 gives very appealing intuition concerning the positivity of the transition density. Stochastic variation is injected at every moment of time in the y-direction. However, the deterministic part of the flow is rotating at every point except the origin, as is seen from the calculation in* (11.6)*. This rotation ensures that there is stochastically independent variation in two linearly independent directions.*

As a consequence of Proposition 11.3, to invoke Theorem 11.2 to prove the positivity of $p_t(z, z')$ for two given points $z, z' \in \mathbb{R}^2$, we simply need to find a control U for which $\Phi_{t,z}(U) = z'$ and for which the path does not spend all of its time at the origin. Since the path is continuous in time, this last condition poses no restriction if either z or z' is not the origin. If both z and z' are the origin, it is still elementary to find a control satisfying the second condition which still also satisfies $\Phi_{t,z}(U) = z'$.

We collect these last observations in the following lemma which combines Proposition 11.3 and Theorem 11.2.

Corollary 11.5. *The transition density p_t satisfies $p_T(x, y) > 0$ for a given $T > 0$ and $x, y \in \mathbb{R}^2$ if there exist a $U \in L^2([0,T], \mathbb{R})$ such that $\Phi_{T,x}(U) = y$ and there exists an $s \in [0,T]$ so that $\Phi_{s,x}(U) \neq 0$. Similarly, $p_T(x, y) = 0$ if there exists no $U \in L^2([0,T], \mathbb{R})$ such that $\Phi_{T,x}(U) = y$.*

We now build the required controls. All the controls we design will take the form $U_t = u(X_t^U, Y_t^U, t)$ for some piecewise smooth $u : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$. At first glance, this might seem an implicit specification which risks being ill-defined, since (X_t^U, Y_t^U) depends on the function U_t we define through (11.2). However, a moment's reflection shows this not to be the case, since in this setting (X_t^U, Y_t^U) can be defined in a self-contained way as the solution to the ODE

$$\begin{aligned} \dot{X}_t^U &= (X_t^U)^2 - (Y_t^U)^2 \\ \dot{Y}_t^U &= 2X_t^U Y_t^U + u(X_t^U, Y_t^U, t)\,. \end{aligned}$$

Then, with this solution in hand, one can define $U_t = u(X_t^U, Y_t^U, t)$.

Lemma 11.6. *Let $z_* = (x_*, y_*)$ with $x_* < 0$ be fixed. There exists a finite $T_*(z_*) > 0$ such that for all $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and for all $T > T_*(z_*)$, there exists a control $U \in L^2([0,T], \mathbb{R})$ for which the controlled system* (11.2) *satisfies*

$$(X_0^U, Y_0^U) = z_0, \quad (X_T^U, Y_T^U) = z_*$$

and such that $M_{T,z_0}(U)$ is nondegenerate.

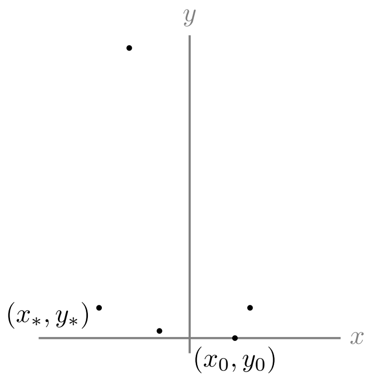

Figure 3: A representative example of a control path used to move from any point in $\mathbb{R}^2$ to any point in the left-half plane.

Proof of Lemma 11.6. We begin with the case of $z_* = (x_*, y_*)$ with $y_* \neq 0$. The remaining case will be considered at the end of the proof. The critical feature of z_* with $y_* \neq 0$ is that there exists a deterministic orbit which begins on the y-axis and flows in finite time to the point z_*.

Let $\mathcal{B}$ be the closed ball of radius $|x_*|/3$ about the origin intersected with the negative half-plane $\{(x, y) : x \leq 0\}$. We begin by setting $U_t = \text{sgn}^+(y_0)$ for $t \in [0, 1]$. (Here $\text{sgn}^+(x) = x/|x|$ for $x \neq 0$ and 1 if $x = 0$.) Define T_1 to be the first time $t \geq 1$ such that $(X_t^U, Y_t^U) \in \mathcal{B}$ where $(X_0^U, Y_0^U) = z_0$. We will set $U_t = 0$ for $t \in [1, T_1 + s]$ where $s \geq 0$ is a parameter we will vary in our construction. By driving with $|U_t| = 1$ on the time interval $[0, 1]$, we have ensured that $Y_1^U \neq 0$, which in turn implies that $T_1 \leq T_1^*$ for some finite, z_0-independent constant T_1^*. (In light of the discussion at the end of Section 7, $T_1^* \leq 1 + \frac{6}{|x_*|}$.) Let $\mathcal{C}_*$ be the orbit of the deterministic system which passes through the point z_* but which is not contained in the set $\mathcal{B}$. Recall that this orbit is a circle tangent to the origin. We now define T_2 to be the infimum over time $t \geq T_1 + s$ such that $(X_t^U, Y_t^U) \in \mathcal{C}_*$. If we define $U_t = \text{sgn}^+(Y_t^U)M - 2X_t^U Y_t^U$ for $t \in [T_1 + s, T_2]$, then $\dot{Y}_t^U = \text{sgn}^+(Y_t^U)M$, and $Y_t^U = Y_{T_1+s}^U + M\text{sgn}^+(Y_{T_1+s}^U)\,(t - T_1 - s)$ for $t \in [T_1 + s, T_2]$. Hence for M large enough, we can ensure that X_t moves very little in the time it takes Y_t to grow sufficiently to cross $\mathcal{C}_*$. This allows us to prove that $T_2 - (T_1 + s)$ is bounded from above for any sufficiently large fixed choice of M with a bound which is independent of s and z_0 since $(X_{T_1+s}^U, Y_{T_1+s}^U) \in \mathcal{B}$. For the same reason, by choosing M large enough we can ensure that $|X_{T_2}^U| \leq 2|x_*|/3$. Fixing such an M, we define T_3 to be the infimum of times $t > T_2$ such that $(X_t^U, Y_t^U) = z_*$. For $t \in [T_2, T_3]$, we set $U_t = 0$. Since M is fixed and $|X_{T_2}^U| \leq 2|x_*|/3$, clearly $T_3 - T_2$ is bounded uniformly for all $s \geq 0$ and $z_0 \in \mathbb{R}^2$. If we view $T_3 - (T_1 + s)$ as a function of $(X_{T_1+s}^U, Y_{T_1+s}^U)$, then it is continuous, since the governing ODEs depend continuously on their initial conditions. Since $(X_{T_1+s}^U, Y_{T_1+s}^U) \in \mathcal{B}$, which is a compact set, we know that there exists a finite bound τ^* so that $T_3 - (T_1 + s) \leq \tau^*$ for all $(X_{T_1+s}^U, Y_{T_1+s}^U) \in \mathcal{B}$. Because $(X_{T_1+s}^U, Y_{T_1+s}^U)$ is a continuous function of (z_0, s) and T_1 an continuous function of z_0, we see that $(z_0, s) \mapsto T_3$ is a continuous function. Since T_1 and $T_3 - T_2$ are bounded uniformly in (z_0, s), we conclude that if $T : (z_0, s) \mapsto T_3$, then there exists a T_* so that $T(z_0, 0) \leq T_*$ for all z_0. Since $s \mapsto T(z_0, s)$ is continuous and $T(z_0, s) \to \infty$ as $s \to \infty$, we conclude that for any $t \geq T_*$ and $z_0 \in \mathbb{R}^2$ there exists a $s(t, z_0)$ so that $T(z_0, s(t, z_0)) = t$. The control U constructed corresponding to this choice of s is the desired control. It is clearly in $L^2([0, t]; \mathbb{R})$ since it is uniformly bounded.

We now return to the case of $z_* = (x_*, y_*)$ where $y_* = 0$. Let (X_t^U, Y_t^U) be the solution with control $U_t = -2X_tY_t - 1$ and initial condition $(X_0^U, Y_0^U) = z_*$. Since this is in fact a flow, its solutions also exist backward in time. It is clear from the choice U that $Y^U_{-1} = 1 \neq 0$. For future reference, define $z_1 = (X^U_{-1}, Y^U_{-1})$. Hence by the first part of the proof, there exists a T_* so that for any $s \geq T_*$ there exists a control U so that $(X^U_0, Y^U_0) = z_0$ and $(X^U_s, Y^U_s) = z_1$. Thus if we define

$$\tilde{U}_t = \begin{cases} U_t & \text{if } t \in [0,s] \\ -2X_tY_t - 1 & \text{if } t \in (s, s+1] \end{cases} .$$

By the choice of z_1, $(X^{\tilde{U}}_T, Y^{\tilde{U}}_T) = z_*$ as required if we set $T = s+1$.

Lastly we observe that none of the control paths employed are identically zero for all time. Hence Lemma 11.3, implies that $M_{T,z_0}(\tilde{U})$ is non-degenerate. □

Combining the results from this section, we get the following Proposition which is the analogue of Proposition 11.1.

Proposition 11.7. *If $\sigma_y > 0$ but $\sigma_x = 0$, then for any $z_* = (x_*, y_*) \in \mathbb{R}^2$ with $x_* < 0$ there exists a T_* so that for any $t > T_*$ one has $p_t(z, z_*) > 0$ for any $z \in \mathbb{R}^2$. Furthermore, if $z_0 = (x_0, y_0) \in \mathbb{R}^2$ with $x_0 < 0$ and $z_1 = (x_1, y_1) \in \mathbb{R}^2$ with $x_1 \geq 0$ than one has $p_t(z_0, z_1) = 0$ for any $t > 0$.*

Proof of Proposition 11.7. As already outlined, the positivity claim follows from Theorem 11.2, because Proposition 11.6 guarantees the existence of a control with the needed properties. The fact that $p_t(z_0, z_1) = 0$ when x_1 is strictly positive also follows from Theorem 11.2, provided we can show that there is no control which moves one from z_0 to z_1. To see this, observe that except for the fixed point at the origin, the vector field for any control always points toward the left half-plane along the y-axis. Hence it is impossible to leave the left half-plane. The fact that $p_t(z_0, z_1) = 0$ when $x_1 = 0$ follows from the strict positivity of $p_t(z_0, z_1)$ when $x_1 < 0$ and from the continuity of $p_t(z_0, z_1)$. □

11.3 Positivity of the Invariant Measure: Proof of Theorem 3.4

Assuming that $\sigma_y > 0$, we know that $P_t(z, \cdot)$ is absolutely continuous with respect to Lebesgue measure and has a smooth density. Hence if μ is an invariant measure (and therefore we have $\mu = \mu P_t$ for any $t > 0$), we see that μ also has a smooth density, m, with respect to Lebesgue measure. The invariance implies that for all $z \in \mathbb{R}^2$ and $t > 0$

$$m(z) = \int_{\mathbb{R}^2} p_t(z', z) m(z') dz' \tag{11.7}$$

where p_t is the density of P_t.

Let z be a point such that for some $t > 0$, $p_t(z', z) > 0$ for all $z' \in \mathbb{R}^2$. Since m integrates to one and is smooth, there must exist some open set A such that $m(z') > 0$ for all $z' \in A$. Combining this observation with (11.7), we get

$$m(z) \geq \int_A p_t(z', z) m(z') dz' > 0 .$$

Hence we deduce that $m(z)$ is positive at any point z which satisfies the stated assumption. Applying this result to the information on the positivity of p_t in Proposition 11.1 and Proposition 11.7, we obtain the conclusions about the positivity of $m(z)$ in Theorem 3.4.

To deduce the statements that $m(z) = 0$ for $z = (x, y)$ with $x \geq 0$ if $\sigma_y >$ and $\sigma_x = 0$, we use Proposition 11.7, which states that if $w \in H_+ = \{z = (x, y) \in \mathbb{R}^2 : x \geq 0\}$, then

$p_t(\zeta, w) = 0$ for all $t > 0$ if $\zeta \in H_-$, where H_- is defined to be the complement of H_+. This implies that if $z \in H_+$, then

$$m(z) = \int_{H_+} p_t(w, z) m(w) dw \,.$$

Integrating this expression over H_+ and interchanging the order of integration, we get

$$\int_{H_+} m(z) dz = \int_{H_+} \phi(w) m(w) dw \,.$$

where $\phi(w) = \int_{H_+} p_t(w, z) dz$. Since $m(z), \phi(z) \geq 0$ for all z, this implies that for Lebesgue-almost-every $z \in H_+$, either $m(z) = 0$ or $\phi(z) = 1$. Yet from Proposition 11.7, we know that given any $w \in H_-$, there exists a time $t > 0$ so that $p_t(z, w) > 0$ for all $z \in \mathbb{R}^2$. Because $z \mapsto p_t(z, w)$ is continuous, we know $\phi(w) < 1$ for every $w \in H_+$. This implies that $m(z) = 0$ for almost every $z \in H_+$. Since $m(z)$ is continuous, this forces $m(z) = 0$ for all $z \in H_+$. This completes the proof of Theorem 3.4.

12 Minorization and Geometric Ergodicity

We now establish the minorization condition we need to complete the proof of Theorem 3.2. Specifically, we seek a probability measure ν and positive constants α, R, and T so that

$$\inf_{\{z \in \mathbb{R}^2 : |z| \leq R\}} P_T(z, \,\cdot\,) \geq \alpha \nu(\,\cdot\,) \tag{12.1}$$

and $R > K_T$ where K_T is the constant from Lemma 6.1. This condition is a localized version of the classical Doeblin condition and central to the theory of Harris chains [21, 26, 18]. While the Lyapunov condition ensures the existence of an invariant measure and guarantees sufficiently rapid returns to the "center of phase space" to produce geometric mixing, the minorization condition ensures the existence of probabilistic mixing.

To summarize our current situation, we pause to prove the following intermediate result.

Theorem 12.1. *If the minorization condition holds from* (12.1)*, then the Markov semigroup P_t generated by* (3.1) *satisfies the conclusions of Theorem 3.2.*

Proof of Theorem 12.1. By Theorem 1.3 in [18], there exist constants $\bar{\alpha} \in (0, 1)$ and $\beta > 0$ such that $\rho_\beta(\mu_1 P_T, \mu_2 P_T) \leq \bar{\alpha} \rho_\beta(\mu_1, \mu_2)$. Results such as this are quite classical. Other proofs can be found, for example, in [26]. Combining this estimate with Proposition 6.2 immediately implies that for any $n \in \{0\} \cup \mathbb{N}$

$$\rho_\beta(\mu_1 P_{nT}, \mu_2 P_{nT}) \leq \bar{\alpha}^{n-1} \rho_\beta(\mu_1 P_T, \mu_2 P_T) \leq \bar{\alpha}^n \Big(\frac{1 + \beta K_T}{\bar{\alpha}}\Big) \rho_0(\mu_1, \mu_2) \,. \tag{12.2}$$

To extend this estimate to an arbitrary $t \geq 0$, we define a nonnegative integer n and $\tau \in (0, 1)$ so that $t = nT + \tau$ and observe that

$$\begin{aligned} \rho_\beta(\mu_1 P_t, \mu_2 P_t) &= \rho_\beta(\mu_1 P_\tau P_{nT}, \mu_2 P_\tau P_{nT}) \leq \bar{\alpha}^n \Big(\frac{1 + \beta K_T}{\bar{\alpha}}\Big) \rho_0(\mu_1 P_\tau, \mu_2 P_\tau) \\ &\leq \bar{\alpha}^n \Big(\frac{1 + \beta K_T}{\bar{\alpha}}\Big) \rho_0(\mu_1, \mu_2) \leq \bar{\alpha}^{\frac{t}{T}} \Big(\frac{1 + \beta K_T}{\bar{\alpha}^2}\Big) \rho_0(\mu_1, \mu_2) \end{aligned}$$

As noted in Remark 6.3, for any $\beta' \geq 0$ there exists a constant C so that $\rho_{\beta'}(\nu_1, \nu_2) \leq C \rho_\beta(\nu_1, \nu_2)$ for all probability measure ν_i. This completes the proof. $\square$

12.1 Minorization when $\sigma_x > 0$

Now since for each $t > 0$, $(z, z') \mapsto p_t(z, z')$ is continuous and everywhere positive, it is elementary that for any $R > 0$ there exists a positive constant $\alpha = \alpha(R, t) > 0$ so that $\inf\{p_t(z, z') : z, z' \in \mathbb{R}^d, |z|, |z'| \leq R\} \geq \alpha$. The minorization condition follows immediately from this, since for any measurable set A

$$P_t(z, A) = \int_A p_t(z, z')dz' \geq \alpha \text{Leb}(A \cap B_R(0)) = \alpha \text{Leb}(B_R(0))\nu(A)$$

where Leb is Lebesgue measure and $\nu(A) = \text{Leb}(A \cap B_R(0))/\text{Leb}(B_R(0))$.

12.2 Minorization when $\sigma_x = 0$

We now state and prove a lemma which shows that the needed minorization condition follows quickly from continuity and a relaxed positivity assumption. In Section 11.2, this relaxed positivity assumption was shown to hold by using a very explicit control theory argument coupled with some stochastic analysis.

Lemma 12.2. *Let $P(z, dz')$ be a Markov transition kernel on $\mathbb{R}^d$ such that $P(z, dz') = p(z, z')dz'$ with $p\colon \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ jointly continuous. If there exists $z^* \in \mathbb{R}^d$ and a compact set B such that for all $z \in B$, $p(z, z^*) > 0$ there exist $\alpha \in (0, 1)$ and a probability measure ν such that*

$$\inf_{z \in B} P(z, \cdot) \geq \alpha\, \nu(\cdot)\,.$$

Proof. Since $z \mapsto p(z, z^*)$ is continuous, it achieves its minimum on the compact set B. Since $p(z, z^*)$ is strictly positive for all $z \in B$, we know that for some $\alpha \in (1/2, 0)$, $p(z, z^*) > 2\alpha$ for all $z \in B$. By the joint continuity of p, for each $z \in B$ there exists a $\delta_z > 0$ be such that $p(y, x) > \alpha$ for all $(y, x) \in B_{\delta_z}(z) \times B_{\delta_z}(z^*)$. Since $\{B_{\delta_z}(z) \colon z \in B\}$ is an open cover of the compact set B, we can extract a finite subcover. Let K be the collection of points z associated with this finite subcover. If $\delta = \min\{\delta_z : z \in K\}$, then $\delta > 0$ since $\delta_z > 0$ and K is finite. Now since

$$B \subset \bigcup_{z \in K} B_{\delta_z}(z)$$

we have that $p(y, x) > \alpha$ for all $(y, x) \in B \times B_\delta(z^*)$. Then $P(y, A) \geq \alpha \text{Leb}(B_{\delta(z)}(z))\nu(A)$, where $\nu(A) = \text{Leb}(A \cap B_\delta(z^*))/\text{Leb}(B_{\delta(z)}(z))$ and Leb is Lebesgue measure on $\mathbb{R}^d$, since

$$P(y, A) = \int_A p(y, x)dx \geq \int_{A \cap B_\delta(z^*)} p(y, x)dx \geq \alpha \text{Leb}(A \cap B_\delta(z^*))\,.$$

□

13 Conclusion

We describe a general methodology for building a Lyapunov function in a setting where the global stability of the systems requires flux of probability into regions which are clearly dissipative from the rest of phase space. We are most interested in problems, like the example considered here, where the noise plays an essential role in creating this transport is some regions. The algorithm makes use of local Lyapunov functions, which are constructed as solutions to Poisson equations in different regions, and are then patched together to form one global Lyapunov function. We apply these techniques to one specific example in the plane to illustrate how the addition of noise gives rise to an invariant probability measure for a system whose purely deterministic dynamics exhibit instability. Furthermore, our resulting "super" Lyapunov function enables us to extract a stronger convergence than what is usually proved in the Harris-chain setting—indeed, an exponential convergence independent of initial condition—to this equilibrium measure. En route to proving this convergence, we employ explicit control-theoretic constructions and we rely on the tools of Malliavin calculus. It is our hope that the simple and specific applications of control theory and Malliavin calculus in our model problem will be of independent interest.

Of course, further work remains to be done: the application of these methods to other examples, for instance, and the development of more general theorems about noise-induced stabilization. In particular, in the planar system we consider, patching the local Lyapunov functions turns out to be one of the most delicate and important parts of the proof. Therefore, it would be especially interesting to find more general approaches to the problem of patching Lyapunov functions, and more general conditions under which it can be done successfully.

When our construction works, it is likely to produce a Lyapunov function which provides strong control over the excursions towards infinity and a nearly sharp rate for the convergence to equilibrium. However, the construction of such a function is laborious. It would be interesting to obtain a simpler "partial fluid limit" which captures only the minimal stochasticity at infinity needed to stabilize the system. This may arise as an extension to our work in the direction of [14, 23, 15, 27]. Such an approach might allow simpler proofs of stabilization without necessarily proving the existence of strong Lyapunov function.

A Comparison Proposition

Proposition A.1. *Suppose $f \in C(\mathbb{R})$ is a non-increasing function and that $\phi(t)$ and $\psi(t)$ are C^1 functions on $\mathbb{R}$ satisfying $\phi(0) = \psi(0)$ and $\phi'(t) \leq f(\phi(t))$, $\psi'(t) = f(\psi(t))$ for all $t \geq 0$ then $\phi(t) \leq \psi(t)$ for all $t \geq 0$.*

Proof of Proposition A.1. For all $0 \leq r \leq t$, we have that

$$\phi(t) \leq \phi(r) + \int_r^t f(\phi(s))ds \quad \text{and} \quad \psi(t) = \psi(r) + \int_r^t f(\psi(s))ds$$

which implies

$$\psi(t) - \phi(t) \geq (\psi(r) - \phi(r)) + \int_r^t (f(\psi(s)) - f(\phi(s)))ds\,.$$

Let $T_1 = \inf\{t > 0 : \psi(t) - \phi(t) < 0\}$. Suppose for contradiction that $T_1 < \infty$. Then by continuity, $\psi(T_1) - \phi(T_1) = 0$ and there exists $T_2 \in (T_1, \infty)$ such that for all $t \in (T_1, T_2)$, $\psi(t) - \phi(t) < 0$. Then for all $t \in (T_1, T_2)$,

$$\psi(t) - \phi(t) \geq (\psi(T_1) - \phi(T_1)) + \int_{T_1}^t (f(\psi(s)) - f(\phi(s)))ds\,. \tag{A.1}$$

Now since f is non-increasing, $\psi(t) < \phi(t)$ implies that $f(\psi(t)) \geq f(\phi(t)$. Hence this combined with (A.1) implies that for all $t \in (T_1, T_2)$, $\psi(t) - \phi(t) \geq 0$. This is a contradiction. Hence T_1 must be infinite and $\phi(t) \leq \psi(t)$ for all $t \geq 0$. $\square$

References

[1] Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York, 1992, Reprint of the 1972 edition. MR-1225604

[2] Ludwig Arnold and Wolfgang Kleimann, *Qualitative theory of stochastic systems*, Probabilistic analysis and related topics, Vol. 3, Academic Press, New York, 1983, pp. 1–79. MR-748853

[3] M. T. Barlow and D. Nualart, *Lectures on probability theory and statistics*, Lecture Notes in Mathematics, vol. 1690, Springer-Verlag, Berlin, 1998, Lectures from the 25th Saint-Flour Summer School held July 10–26, 1995, Edited by P. Bernard. MR-1668107

[4] Richard F. Bass, *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, 1998. MR-1483890

[5] Denis R. Bell, *Degenerate stochastic differential equations and hypoellipticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 79, Longman, 1995.

[6] ______, *Stochastic differential equations and hypoelliptic operators*, Trends Math., p. 9âĂŞ42, BirkhÃď'user Boston, 2004.

[7] G. Ben Arous and R. Léandre, *Décroissance exponentielle du noyau de la chaleur sur la diagonale. I*, Probab. Theory Related Fields **90** (1991), no. 2, 175–202. MR-1128069

[8] ______, *Décroissance exponentielle du noyau de la chaleur sur la diagonale. II*, Probab. Theory Related Fields **90** (1991), no. 3, 377–402. MR-1133372

[9] Carl M. Bender and Steven A. Orszag, *Advanced mathematical methods for scientists and engineers*, McGraw-Hill Book Co., New York, 1978, International Series in Pure and Applied Mathematics. MR-538168

[10] J. Birrell, D. P. Herzog, and J. Wehr, *Transition from ergodic to explosive behavior in a family of stochastic differential equations*, ArXiv e-prints (2011).

[11] Katarínn Bodová and Charles R. Doering, *Noise-induced statistically stable oscillations in a deterministically divergent nonlinear dynamical system*, Communication Mathematical Science **10** (2012), no. 1, 137–157.

[12] B. Cooke, J. C. Mattingly, S. A. McKinley, and S. C. Schmidler, *Geometric Ergodicity of Two–dimensional Hamiltonian systems with a Lennard–Jones–like Repulsive Potential*, ArXiv e-prints (2011).

[13] Randal Douc, Gersende Fort, and Arnaud Guillin, *Subgeometric rates of convergence of f-ergodic strong Markov processes*, Stochastic Process. Appl. **119** (2009), no. 3, 897–923. MR-2499863

[14] Paul Dupuis and Ruth J. Williams, *Lyapunov functions for semimartingale reflecting Brownian motions*, Ann. Probab. **22** (1994), no. 2, 680–702. MR-1288127

[15] Gersende Fort, Sean Meyn, Eric Moulines, and Pierre Priouret, *The ODE method for stability of skip-free Markov chains with applications to MCMC*, Ann. Appl. Probab. **18** (2008), no. 2, 664–707. MR-2399709

[16] K. Gawedzki, D. P. Herzog, and J. Wehr, *Ergodic properties of a model for turbulent dispersion of inertial particles*, ArXiv e-prints (2010).

[17] Moshe Gitterman, *The noisy oscillator*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005, The first hundred years, from Einstein until now. MR-2313138

[18] M. Hairer and J. C. Mattingly, *Yet another look at Harris' ergodic theorem for Markov chains*, ArXiv e-prints (2008).

[19] Martin Hairer, *P@W course on the convergence of markov processes*, `http://www.hairer.org/notes/Convergence.pdf`, 2010.

[20] Martin Hairer and Jonathan C. Mattingly, *Slow energy dissipation in anharmonic oscillator chains*, Comm. Pure Appl. Math. **62** (2009), no. 8, 999–1032. MR-2531551

[21] T. E. Harris, *The existence of stationary measures for certain Markov processes*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II (Berkeley and Los Angeles), University of California Press, 1956, pp. 113–124. MR-0084889

[22] R. Z. Has′minskiĭ, *Stochastic stability of differential equations*, Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, vol. 7, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, Translated from the Russian by D. Louvish. MR-600653

[23] Jianyi Huang, Ioannis Kontoyiannis, and Sean P. Meyn, *The ODE method and spectral theory of Markov operators*, Stochastic theory and control (Lawrence, KS, 2001), Lecture Notes in Control and Inform. Sci., vol. 280, Springer, Berlin, 2002, pp. 205–221. MR-1931654

[24] J. C. Mattingly, S. A. McKinley, and N. S. Pillai, *Geometric ergodicity of a bead-spring pair with stochastic Stokes forcing*, ArXiv e-prints (2009).

[25] J. C. Mattingly, A. M. Stuart, and D. J. Higham, *Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise*, Stochastic Process. Appl. **101** (2002), no. 2, 185–232. MR-1931266

[26] S. P. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*, Communications and Control Engineering Series, Springer-Verlag London Ltd., London, 1993. MR-1287609

[27] Sean Meyn, *Control techniques for complex networks*, Cambridge University Press, Cambridge, 2008. MR-2372453

[28] M. Scheutzow, *Stabilization and destabilization by noise in the plane*, Stochastic Anal. Appl. **11** (1993), no. 1, 97–113. MR-1202675

[29] A. Yu. Veretennikov, *On polynomial mixing bounds for stochastic differential equations*, Stochastic Process. Appl. **70** (1997), no. 1, 115–127. MR-1472961

[30] ——, *On polynomial mixing and the rate of convergence for stochastic differential and difference equations*, Teor. Veroyatnost. i Primenen. **44** (1999), no. 2, 312–327. MR-1751475

[31] Roscoe B. White, *Asymptotic analysis of differential equations*, revised ed., Imperial College Press, London, 2010. MR-2724458