Electronic Journal of Probability

Vol. 3 (1998) Paper no. 4, pages 1–36.

Journal URL
http://www.math.washington.edu/˜ejpecp/
Paper URL
http://www.math.washington.edu/˜ejpecp/EjpVol3/paper4.abs.html

GEOMETRIC EVOLUTION UNDER ISOTROPIC STOCHASTIC FLOW

M. Cranston and Y. LeJan

University of Rochester
Rochester, New York, USA
cran@math.rochester.edu

Université de Paris, Sud
Orsay 91405, France
Yves.LeJan@math.u-psud.fr

Abstract. Consider an embedded hypersurface M in $\mathbb{R}^3$. For Φ_t a stochastic flow of differomorphisms on $\mathbb{R}^3$ and $x \in M$, set $x_t = \Phi_t(x)$ and $M_t = \Phi_t(M)$. In this paper we will assume Φ_t is an isotropic (to be defined below) measure preserving flow and give an explicit descripton by SDE's of the evolution of the Gauss and mean curvatures, of M_t at x_t. If $\lambda_1(t)$ and $\lambda_2(t)$ are the principal curvatures of M_t at x_t then the vector of mean curvature and Gauss curvature, $(\lambda_1(t)+\lambda_2(t), \lambda_1(t)\lambda_2(t))$, is a recurrent diffusion. Neither curvature by itself is a diffusion. In a separate addendum we treat the case of M an embedded codimension one submanifold of $\mathbb{R}^n$. In this case, there are $n-1$ principal curvatures $\lambda_1(t), \dots, \lambda_{n-1}(t)$. If $P_k, k = 1, n-1$ are the elementary symmetric polynomials in $\lambda_1, \dots, \lambda_{n-1}$, then the vector $(P_1(\lambda_1(t), \dots, \lambda_{n-1}(t)), \dots, P_{n-1}(\lambda_1(t), \dots, \lambda_{n-1}(t))$ is a diffusion and we compute the generator explicitly. Again no projection of this diffusion onto lower dimensions is a diffusion. Our geometric study of isotropic stochastic flows is a natural offshoot of earlier works by Baxendale and Harris (1986), LeJan (1985, 1991) and Harris (1981).

1991 *Mathematics Subject Classification.* 60H10, 60J60.

Key words and phrases. Stochastic flows, Lyapunov exponents, principal curvatures.

Research supported by NSA and Army Office of Research through MSI at Cornell. Submitted to EJP on February 19, 1996. Final version accepted on February 12, 1998.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1 Isotropic Flows and Geometric Setting.

We begin with a nonnegative measure $F(d\rho)$ on $[0,+\infty)$ with finite moments of all orders. The pth moment of F will be denoted by μ_p. To F we can associate a covariance

$$C_{ij}(z)=\int_0^\infty\int_{S^{n-1}}e^{i\rho\langle z,t\rangle}(\delta^i_j-t^it^j)\sigma_{n-1}(dt)F(d\rho), \tag{1.1}$$

where σ_{n-1} is normalized Lebesgue measure on S^{n-1}. By a result of Kolmogorov, to C_{ij} is associated a smooth (in x) vector-field valued Brownian motion $U_t(x)$ such that

$$EU^i_t(x)U^j_s(y)=(t\wedge s)C^{ij}(x-y),\quad 1\le i,j\le n. \tag{1.2}$$

A Brownian flow on $\mathbb{R}^n$ is constructed by solving the equation

$$\Phi_t(x)=x+\int_0^t\partial U_s(\Phi_s(x)) \tag{1.3}$$

(∂ denotes Stratonovich differential.) Existence and uniqueness of such flows was proved in Baxendale (1984), LeJan and Watanabe (1984). Then if T_a is translation by $a\in\mathbb{R}^n$, $T_a\Phi_tT_{-a}$ and Φ_t are identical in law so Φ is stationary. If R is unitary, $R\Phi_tR^{-1}$ and Φ_t are identical in law. When these two properties hold we say Φ is isotropic. These properties both follow from the nature of C_{ij} so our Φ is isotropic. The field $U_t(x)$ is smooth so we differentiate it writing

$$W^i_j=\frac{\partial}{\partial x^j}U^i \tag{1.4}$$

$$B^i_{jk}=\frac{\partial^2}{\partial x^j\partial x^k}U^i. \tag{1.5}$$

Then we have as direct consequence of (1.1) and (1.2) (using $\langle,\rangle$ to denote both Euclidean inner product as well as quadratic variation for martingales, d will denote the Itô differential.)

$$\langle dW^i_j(t,y),dW^k_\ell(t,y)\rangle=\frac{\mu_2}{n(n+2)}[(n+1)\delta^i_k\delta^j_\ell-\delta^i_j\delta^k_\ell-\delta^i_\ell\delta^k_j]dt \tag{1.6}$$

$$\langle dB^i_{jk}(t,y),dW^p_q(t,y)\rangle=0 \tag{1.7}$$

and for vectors $u,v\in\mathbb{R}^n$,
(1.8)
$$\langle\langle dB(u,u),v\rangle,\langle dB(u,u),v\rangle\rangle=\frac{3\mu_4}{n(n+2)(n+4)}[(n+3)||u||^4||v||^2-4\langle u,v\rangle^2||u||^2]dt$$

By isotropy, (1.6), (1.7) and (1.8) remain valid under any unitary change of coordinates. We define,

$$C^{ik}_{j\ell}dt = \langle dW^i_j, dW^k_\ell \rangle = \frac{\mu_2}{n(n+2)}[(n+1)\delta^i_k\delta^j_\ell - \delta^i_j\delta^k_\ell - \delta^i_\ell\delta^k_j]dt$$

where both dW^i_j, dW^k_ℓ are evaluated at (t, y), and

$$C^{ij}_{k\ell pq}dt = \langle dB^i(e_k, e_\ell), dB^j(e_p, e_q) \rangle$$

again both dB terms are evaluated at (t, y). A quick calculation shows $E\left(\sum_{i=1}^n dW^i_i\right)^2 = 0$ so div $U = 0$ and therefore Φ is a measure-preserving flow. Also, it's rather easy to check that $C^{nn}_{1111} = 3C^{nn}_{1122}$ and $C^{nn}_{1122} = \frac{(n+3)\mu_4}{n(n+2)(n+4)}$.

The most general form of $\langle dW^i_j, (t, y), dW^k_\ell(t, y) \rangle = C^{ik}_{j\ell}dt$ for isotropic flows is $C^{ik}_{j\ell} = a\delta^{i,k}\delta_{j,\ell} + b\delta^i_j\delta^k_\ell + c\delta^i_\ell\delta^k_j$ where $a + c$, $a - c$, $a + c + db$ are nonnegative (see LeJan (1985) and the references therein). We retrict our attention to the present covariance structure; $a = (n+1)\frac{\mu_2}{n(n+2)}$, $b = -\frac{\mu_2}{n(n+2)}$, $c = -\frac{\mu_2}{n(n+2)}$, as computations are massively simplified by the fact that certain Ito correction terms vanish. The works of Baxendale and Harris (1986) and LeJan (1985) focused on first order properties of the general isotropic flows in Euclidean space. That is, on properties of the derivative flow $D\Phi_t(x)$. Perhaps the most fundamental information about the derivative flow is its Lyapunov spectrum. To describe the Lyapunov spectrum, first extend $D\Phi_t$ to p-forms $\alpha = v_1 \wedge \cdots \wedge v_p$ (we're in Euclidean space so we can identify forms and vectors) by defining $D\Phi_t(x)\alpha = D\Phi_t(x)v_1 \wedge \cdots \wedge D\Phi_t(x)v_p$. Then if $v_1, \ldots, v_p$ are linearly independent, a.s.

$$\lim_{t\to\infty} \frac{1}{t} \log \|D\Phi_t(x)\alpha\| = \gamma_1 + \cdots + \gamma_p$$

is the sum of the first p Lyapunov exponents. In the case of the isotropic, measure-preserving flow on $\mathbb{R}^n$, $\gamma_1 + \cdots + \gamma_p = \frac{np(n-p)\mu_2}{2(n+2)}$ (see LeJan 1985). The problem dealt with in this work involves second order behavior of the flow. The evolving curvature of a submanifold moving under the flow would depend on the properties of the second order flow,

$$(x, u, v) \to (\Phi_t(x), D\Phi_t(x)u,\ D^2\Phi_t(x)(D\Phi_t(x)u,\ D\Phi_t(x)u) + D\Phi_t(x)v)\,.$$

In particular, our results rely on the properties given by (1.6), (1.7) and (1.8).

Initially, we will consider M an embedded hypersurface in $\mathbb{R}^n$, later specializing to the case $n = 3$. Take $\Pi_t : T_{x_t}\mathbb{R}^n \to T_{x_t}M_t$ to be the orthogonal projection and ∇ to be the canonical connection on Euclidean space. Then S_t, the second fundamental form of M_t at x_t, is defined for $u, v \in T_{x_t}M_t$ by $S_t(u, v) = (I - \Pi_t)\nabla_u v$ where v is extended in an arbitrary but smooth manner in a neighborhood of x_t. Now S_t induces a linear map from the exterior algebra $\Lambda^k(T_{x_t}M_t) \to \Lambda^k(T_{x_t}M_t)$

for $k = 1, 2, \ldots, n-1$. This arises as follows. First observe that since we are in Euclidean space we can identify vectors and forms. Next take $\nu_t \in T^{\perp}_{x_t} M_t$ to be a unit vector (there are two choices and we may select the process ν_t to be continuous in t.) Now in the case $k = 1$, $u \to \langle S_t(u, \cdot), \nu_t \rangle$ is a map from $\Lambda^1(T_{x_t} M_t)$ to $\Lambda^1(T_{x_t} M_t)$. When $k > 1$ we need to introduce the index set

$$I_k = \{\vec{m} \in \{1, \ldots, n-1\}^k : m_1 < m_2 < \cdots < m_k\}$$

and for $\sigma \in S_k$, the set of permutations on k letters, define $(-1)^\sigma$ to be the sign of σ. Then for $u_1, \ldots, u_k, v_1, \ldots, v_k \in T_{x_t} M_t$, set

$$S^{(k)}(u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{j=1}^{k} \langle S_t(u_{\sigma(j)}, v_j), \nu_t \rangle$$

This gives rise to the linear map

$$u_1 \wedge \cdots \wedge u_k \to S^{(k)}(u_1 \wedge \cdots \wedge u_k, \cdot)$$

from $\Lambda^k(T_{x_t} M_t) \to \Lambda^k(T_{x_t} M_t)$.

The object of our study will be the trace of $S^{(k)}, TrS^{(k)}$, for $k = 1, 2, \ldots, n-1$. This is obtained via contraction on indices (see Bishop and Goldberg (1980)). Take $\{u_1, \ldots, u_{n-1}\}$ to be any basis for $T_{x_t} M_t$. For $k \in \{1, \ldots, n-1\}$, $\vec{\ell} \in I_k$, set $|\vec{\ell}| = \ell_1 + \cdots + \ell_k$,

$$\alpha_{\vec{\ell}} = u_{\ell_1} \wedge u_{\ell_2} \wedge \cdots \wedge u_{\ell_k}$$

$$\alpha^{\vec{\ell}} = (-1)^{|\vec{\ell}|+k} u_1 \wedge \cdots \wedge \hat{u}_{\ell_1} \wedge \cdots \wedge \hat{u}_{\ell_k} \wedge \cdots \wedge u_{n-1}$$

($\hat{}$ indicates the indicated vector is omitted from the wedge product)

$$\alpha = u_1 \wedge \cdots \wedge u_{n-1}.$$

Define an inner product

$$\langle \alpha_{\vec{\ell}}, \alpha_{\vec{m}} \rangle = det \langle u_{\ell_i}, u_{mj} \rangle.$$

Then we may express $TrS^{(k)}$ as

$$TrS^{(k)} = S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle . ||\alpha||^{-2}, \tag{1.9}$$

where the sum (Einstein's convention) is over $\vec{\ell}, \vec{m} \in I_k$. Now this is invariant under change of basis so we may select $\{u_1, \ldots, u_{n-1}\}$ to be the unit principal directions of curvature (see Spivak (1975).) Namely, the eigenvectors of $S^{(1)} : \Lambda^1(T_{x_t} M_t) \to$

$\Lambda^1(T_{x_t}M_t)$. Let $\lambda_1, \dots, \lambda_{n-1}$ be the corresponding eigenvalues, otherwise known as the principal curvatures. Now the u_i are orthonormal so

$$\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2} = \delta^{\vec{\ell}, \vec{m}}$$

and as $\langle S_t(u_j, v), \nu_t \rangle = \lambda_j \langle u_j, v \rangle$

$$\sum_{\sigma \in S_k} (-1)^\sigma \prod_{j=1}^{k} \langle S_t(u_{\ell_{\sigma(j)}}, u_{\ell_j}), \nu_t \rangle = \lambda_{\ell_1} \dots \lambda_{\ell_k} \equiv \lambda_{\vec{\ell}}$$

Thus

$$TrS^{(k)} = \sum_{\vec{\ell} \in I_k} \lambda_{\vec{\ell}} \equiv P_k(\lambda_1, \dots, \lambda_{n-1}). \tag{1.10}$$

P_k is the elementary symmetric polynomial of degree k.

The process $\xi_t = (x_t, \Pi_t, \nu_t, P_1(\lambda_1, \dots, \lambda_{n-1}), \cdots P_{n-1}(\lambda_1, \dots, \lambda_{n-1}))$ will, in our case be a diffusion. By the translation invariance part of isotropy, it is clear that x_t is not needed, i.e., $(\Pi_t, \nu_t, P_1(\lambda_1, \dots, \lambda_{n-1}), \cdots, P_{n-1}(\lambda_1, \dots, \lambda_{n-1}))$ has the Markov property. By the invariance of the law of the flow under unitary actions, it follows that Π_t and ν_t are also superfluous. Thus, for isotropic flows the vector $(P_1(\lambda_1, \dots, \lambda_{n-1}), \dots, P_{n-1}(\lambda_1, \dots, \lambda_{n-1}))$ is a diffusion. The point of the present article is to describe this diffusion explicitly in the measure-preserving isotropic case. We can derive consequences about the behavior of this diffusion from the explicit description. These properties suggest what sort of behavior one might have with more general flows. What we expect for more general flows is that the process of curvatures for a k-dimensional submanifold evolving under a general flow will be recurrent (though not necessarily a diffusion) if $\gamma_k > 0$. That is, generically the k dimensional tangent space $T_{x_t}M_t$ will 'see' only stretching directions and this will tend to smooth out the curvatures. This condition may even be too strong.

§2 Preliminary Results.

Recall the correlations of dW given by (1.6). The flow $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ diffeomorphism whose first derivative satisfies the Stratonovich stochastic differential equation, where $x \in M$,

$$dD\Phi_t(x) = \partial W(x_t) D\Phi_t(x) \tag{2.1}$$

Given $u \in T_xM$, $D\Phi_t(x)u = u(t) \in T_{x_t}M_t$ satisfies

$$du = \partial W u \tag{2.2}$$

where we have suppressed the variable (x_t) in ∂W. In fact, due to the choice of C in (1.1)

$$du = dWu, \tag{2.3}$$

that is, the Itô corrections vanish. Furthermore, if we select a basis $\{u_1, \dots, u_{n-1}\}$ for T_xM, then $\{u_1(t), \dots, u_{n-1}(t)\}$, where $u_j(t) = D\Phi_t(x)u_j$, forms a basis for $T_{x_t}M_t$. We will write u_j for $u_j(t)$ for simplicity of notation.

We now proceed to derive a Stratonovich equation for the second fundamental form S_t of M_t at x_t. Given vectors $u, v \in T_xM$ move them via the flow, $u(t) = D\Phi_t(x)u, v(t) = D\Phi_t(x)v$. Extend v to a smooth vector field V in a neighborhood of x. Set $V_t(y) = D\Phi_t(\Phi_t^{-1}(y))V(\Phi_t^{-1}(y))$ so that V_t is a vector field near x_t such that $V_t(x_t) = v(t)$. Then if $Z_t \equiv \nabla_{u(t)}V_t(x_t)$ we claim

$$dZ(u,v) = \partial W Z(u,v) + \partial B(u,v). \tag{2.4}$$

For proof of (2.4), take γ to be a curve on M with $\gamma(0) = x, \gamma'(0) = u$ and define $\gamma_t = \Phi_t(\gamma)$ so that $\gamma_t'(0) = u(t) = D\Phi_t u$. Then,

$$\begin{aligned}
\nabla_{u_t}V_t &= \lim_{s\to 0} s^{-1}(V_t(\gamma_t(s)) - V_t(\gamma_t(0))) \\
&= \lim_{s\to 0} s^{-1}(D\Phi_t(\gamma(s))V(\gamma(s)) - D\Phi_t(x)v(x)) \\
&= \lim_{s\to 0} s^{-1}[(D\Phi_t(\gamma(s)) - D\Phi_t(x))V(\gamma(s)) \\
&+ D\Phi_t(x)(V(\gamma(s)) - v(x))] \\
&= D^2\Phi_t(x)(u,v) + D\Phi_t(x)\nabla_u v
\end{aligned}$$

Since

$$\begin{aligned}
\Phi_t(x) &= x + \int_0^t \partial U(\Phi_s(x)) \\
D\Phi_t(x) &= I + \int_0^t \partial W(\Phi_s(x))D\Phi_s(x) \\
D^2\Phi_t(x)(\cdot,\cdot) &= \int_0^t \partial B(\Phi_s(x))(D\Phi_s(x)\cdot, D\Phi_s(x)\cdot) \\
&+ \int_0^t \partial W(\Phi_s(x))D^2\Phi_s(x)(\cdot,\cdot)
\end{aligned}$$

(2.4) follows. A word about notation: $\partial W_t Z$ is an abbreviation for $\partial W_j^i(\Phi_t(x))(Z_t^j)$ and $\partial B(u,v)$ is short for $(\partial B_{jk}^i(\Phi_t(x))u_j(t)v_k(t)$.

Next we need an equation for $\Pi_t : T_{x_t}\mathbb{R}^n \to T_{x_t}M_t$. For any $\xi \in T_{x_t}M_t$, let $\{u_1, \dots, u_{n-1}\}$ be the moving frame mentioned above, set $\alpha = u_1 \wedge \dots \wedge u_{n-1}$ and defining

$$\alpha^i = (-1)^{i+1}u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_{n-1} \tag{2.5}$$

we have

$$\Pi_t\xi = \sum_{i=1}^{n-1} \langle \xi \wedge \alpha^i, \alpha\rangle ||\alpha||^{-2} u_i. \tag{2.6}$$

The next two lemmas are elementary but crucial.

Lemma 2.1. *If ξ_t is a $T_{x_t}M_t$-valued continuous semimartingale, then*

$$(I - \Pi_t) \cdot \partial W \xi_t = \partial \Pi_t \xi_t.$$

Proof. Suppose $\Pi_t \xi_t = \xi_t$ is as in the statement of the lemma with $\xi_t = x_t^i u_i$. Then

$$\begin{aligned}(I - \Pi_t)\partial W \xi_t - \partial \Pi_t \xi_t = &\, \partial W \xi_t - \Pi_t \partial W \xi_t \\ &+ 2\frac{\langle \partial \alpha_t, \alpha_t \rangle}{||\alpha_t||^2} \xi_t - \partial W \xi_t \\ &- \frac{\langle \xi_t \wedge \alpha_t^i, \partial \alpha_t \rangle}{||\alpha_t||^2} u_i - \frac{\langle \xi_t \wedge \partial \alpha_t^i, \alpha_t \rangle}{||\alpha_t||^2} u_i \\ &= -\Pi_t \partial W \xi_t + 2\frac{\langle \partial \alpha_t, \alpha_t \rangle}{||\alpha_t||^2} \xi_t \\ &- x_t^i \frac{\langle \alpha_t, \partial \alpha_t \rangle}{||\alpha_t||^2} u_i - x_t^i \frac{\langle u_i \wedge \partial \alpha_i, \alpha_t \rangle}{||\alpha_t||^2} u_i\end{aligned}$$

Since

$$d\alpha_t = -u_i \wedge \partial \alpha^i + \partial W u_i \wedge \alpha^i$$

the last term becomes

$$\begin{aligned}& - x_t^i \frac{\langle u_i \wedge \partial \alpha^i, \alpha \rangle}{||\alpha_t||^2} u_i \\ &= -\frac{\langle \partial \alpha_t, \alpha_t \rangle}{||\alpha_t||^2} \xi_t + \frac{\langle \partial W u_i \wedge \alpha_t^i, \alpha_t \rangle}{||\alpha_t||^2} x_t^i u_i \\ &= -\frac{\langle \partial \alpha_t, \alpha_t \rangle}{||\alpha_t||^2} \xi_t + \Pi_t \partial W \xi_t\end{aligned}$$

From this we get the conclusion of the lemma.

□

Lemma 2.2. *If $\eta_t \in T_{x_t}M_t^{\perp}$ is a continuous semimartingale, then*

$$\partial \Pi_t \eta_t = \Pi_t \partial \tilde{W} \eta_t$$

where $\tilde{W}$ is the transpose of W.

Proof. Recalling that

$$\begin{aligned}\partial \Pi_t(v) = &-2\frac{\langle \partial \alpha_t, \alpha_t \rangle}{||\alpha_t||^2} \Pi_t v + \partial W \Pi_t v \\ &+ \frac{\langle v \wedge \alpha_t^i, \partial \alpha_t \rangle}{||\alpha_t||^2} u_i + \frac{\langle v \wedge \partial \alpha_t^i, \alpha_t \rangle}{||\alpha_t||^2} u_i\end{aligned}$$

we see that since $\Pi_t \eta_t = 0$,

$$\partial \Pi(\eta_t) = \frac{\langle \eta_t \wedge \alpha_t^i, \partial \alpha_t \rangle}{||\alpha_t||^2} u_i.$$

But

$$\partial \alpha_t = \partial W u_k \wedge \alpha_t^k$$

so

$$\begin{aligned}
\partial \Pi_t(\eta_t) &= \frac{\langle \eta_t \wedge \alpha_t^i, \partial W u_k \wedge \alpha_t^k \rangle}{||\alpha_t||^2} u_i \\
&= \frac{\langle \partial \tilde{W} \eta_t, u_k(t) \rangle \langle \alpha_t^k(t), \alpha_t^k(t) \rangle u_i(t)}{||\alpha(t)||^2} \\
&= \frac{\langle \partial \tilde{W} \eta_t \wedge \alpha_t^i, \alpha_t \rangle}{||\alpha_t||^2} u_t^i \\
&= \Pi_t \partial \tilde{W} \eta_t
\end{aligned}$$

and the lemma is proved. □

Using $S = (I - \Pi)Z$ together with Lemmas 2.1 and 2.2 we obtain the following.

Proposition 2.3. *For $u, v \in T_x M$, $u(t) = D\Phi_t(x)u, v(t) = D\Phi_t(x)v$,*

$$dS(u,v) = (I - \Pi)dB(u,v) + \partial W S(u,v) - \Pi(\partial W + \partial \tilde{W})S(u,v) \tag{2.7}$$

where $\tilde{W}$ denotes the transpose of W.

Proof. We shall abbreviate by suppressing the dependence on u and v. So,

$$\begin{aligned}
dS &= (I - \Pi)\partial Z - \partial \Pi Z \\
&= (I - \Pi)dB + (I - \Pi)\partial W Z - \partial \Pi Z, \text{ by (2.4)}, \\
&= (I - \Pi)dB + (I - \Pi)\partial W S \\
&+ (I - \Pi)\partial W R - \partial \Pi R - \partial \Pi S \quad , \Pi Z \equiv R \\
&= (I - \Pi)dB + (I - \Pi)\partial W S - \partial \Pi S, \quad \text{by Lemma 2.1} \\
&= (I - \Pi)dB + \partial W S - \Pi(\partial W + \partial \tilde{W})S, \text{by Lemma 2.2.}
\end{aligned}$$

□

The equation (2.7) can be simplified. Set

$$\begin{aligned}
dP &= (I - \Pi)dW \\
dQ &= \Pi d\tilde{W}
\end{aligned} \tag{2.8}$$

and define λ and μ so that

$$\begin{aligned}
(I - \Pi)\partial W &= dP + d\lambda \\
\Pi \partial \tilde{W} &= dQ + d\mu.
\end{aligned} \tag{2.9}$$

Lemma 2.4.

$$d\Pi = dP\Pi + d\lambda\Pi + dQ(I-\Pi) + d\mu(I-\Pi). \tag{2.10}$$

$$\begin{aligned} dS(u,v) = (I-\Pi)dB(u,v) + (dP-dQ)S(u,v) + \frac{1}{2}(dP-dQ)^2S(u,v) \\ + d(\lambda-\mu)S(u,v) \end{aligned} \tag{2.11}$$

Proof. Let ζ_t be an $\mathbb{R}^n$-valued semimartingale. Then

$$d\Pi = (I-\Pi)\partial W\Pi + \Pi\partial\tilde{W}(I-\Pi) \tag{2.12}$$

holds since

$$\begin{aligned} \partial\Pi_t\zeta_t &= \partial\Pi_t(\Pi_t\zeta_t) + \partial\Pi_t((I-\Pi_t)\zeta_t) \\ &= (I-\Pi_t)\partial W_t\Pi_t\zeta_t + \Pi_t\partial\tilde{W}_t(I-\Pi_t)\zeta_t, \end{aligned}$$

by Lemmas 2.1 and 2.2. Thus, we have the following quadratic variations,

$$\begin{aligned} d\lambda &= \frac{1}{2}(-(I-\Pi)dW\Pi dW - \Pi d\tilde{W}(I-\Pi)dW) \\ &= \frac{1}{2}(-dP\Pi dW - dQ(I-\Pi)dW)\,, \\ d\mu &= \frac{1}{2}((I-\Pi)dW\Pi d\tilde{W} + \Pi d\tilde{W}(I-\Pi)d\tilde{W}) \\ &= \frac{1}{2}(dP\Pi d\tilde{W} + dQ(I-\Pi)d\tilde{W}). \end{aligned}$$

Consequently, performing Itô corrections on (2.12), we get

$$d\Pi = dP\Pi + d\lambda\Pi + dQ(I-\Pi) + d\mu(I-\Pi)$$

which proves (2.10). In order to derive (2.11), rewrite (2.7) as

$$dS(u,v) = (I-\Pi)dB(u,v) + (I-\Pi)\partial WS(u,v) - \Pi\partial\tilde{W}S(u,v).$$

Then (2.11) follows from the definitions of $dP, dQ, d\lambda$ and $d\mu$ in (2.9).

□

Lemma 2.5. *If ζ_t is a $T_{x_t}M_t^{\perp}$-valued semi-martingale, then*

$$d\lambda\zeta = \frac{(n-1)\mu_2}{2n(n+2)}\zeta dt \tag{2.12}$$

$$d\mu\zeta = \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\zeta dt \tag{2.13}$$

$$\frac{1}{2}(dP - dQ)^2\zeta = -\frac{(n-1)n\mu_2}{2n(n+2)}\zeta dt \tag{2.14}$$

Proof. Select an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_{x_t}M_t$ such that $e_n \in T_{x_t}M_t^{\perp}$. Then,

$$\begin{aligned}
2d\lambda\zeta &= -d\Pi dW\zeta - dP\Pi dW\zeta - dQ(I-\Pi)dW\zeta \\
&= -(I-\Pi)dW\Pi dW\zeta - \Pi d\tilde{W}(I-\Pi)dW\zeta \\
&= -(\sum_{i=1}^{n-1} C_{ni}^{in})\zeta dt, \text{ since } \langle dW_i^n, dW_n^n\rangle = 0, 1 \le i \le n-1, \\
&= \frac{(n-1)\mu_2}{n(n+2)}\zeta dt, \text{ by (1.6)} \\
2d\mu\zeta &= dP\Pi d\tilde{W}\zeta + dQ(I-\Pi)d\tilde{W}\zeta \\
&= (\sum_{i=1}^{n-1} C_{nn}^{ii})\zeta dt \\
&= \frac{(n-1)(n+1)\mu_2}{n(n+2)}\zeta dt, \text{ by (1.6).}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2}(dP-dQ)^2\zeta &= \frac{1}{2}(dP-dQ)((I-\Pi)dW\zeta - \Pi d\tilde{W}\zeta) \\
&= \frac{1}{2}(dP-dQ)(dW_n^n\zeta - \sum_{i=1}^{n-1} d\tilde{W}_i^n e_i\langle\zeta, e_n\rangle) \\
&= \frac{1}{2}(dW_n^n dW_n^n\zeta - \sum_{i=1}^{n-1} dW_n^i d\tilde{W}_i^n\zeta) \\
&= \frac{1}{2}(C_{nn}^{nn}\zeta - (\sum_{i=1}^{n-1} C_{nn}^{ii})\zeta)dt \\
&= \frac{\mu_2}{2n(n+2)}((n-1) - (n-1)(n+1))\zeta dt\,, \text{ by (1.6),} \\
&= -\frac{(n-1)\mu_2}{2(n+2)}\zeta dt.
\end{aligned}$$

□Recall that we have selected an initial basis for T_xM with $u_n \in T_xM^{\perp}$ and $u_i(t) = D\Phi_t(x)u_i$ gives a basis for $T_{x_t}M_t$ when i runs through $\{1, \dots, n-1\}$.

Theorem 2.6. *If $\nu(t)$ denotes the unit vector $\nu = \frac{(I-\Pi)u_n}{||(I-\Pi)u_n||}$ perpendicular to M_t at x_t, then*

$$d\nu = -dQ\nu - \frac{(n^2-1)\mu_2}{2n(n+2)}\nu dt.$$

Proof. Set $v_n = (I-\Pi)u_n$. Then by Lemmas 2.2 and 2.3,

$$\begin{aligned} dv_n &= -\Pi\partial\tilde{W}v_n + (I-\Pi)\partial W v_n \\ &= (dP-dQ)v_n + \frac{1}{2}(dP-dQ)^2 v_n + (d\lambda - d\mu)v_n \\ &= (dP-dQ)v_n - \frac{(n-1)\mu_2}{(n+2)}v_n dt, \text{ by Lemma 2.5} \end{aligned}$$

Then, applying

$$\begin{aligned} d||v_n||^2 &= 2\langle v_n, dv_n\rangle + \langle (dP-dQ)v_n\rangle \\ &= 2\langle v_n, (dP-dQ)v_n\rangle - \frac{2(n-1)\mu_2}{(n+2)}||v_n||^2 dt + \langle (dP-dQ)v_n\rangle \\ &= 2||v_n||^2 dW_n^n - \frac{2(n-1)\mu_2}{(n+2)}||v_n||^2 dt + \frac{(n-1)\mu_2}{n}||v_n||^2 dt \\ &= 2||v_n||^2 dW_n^n - \frac{(n-1)(n-2)\mu_2}{n(n+2)}||v_n||^2 dt \end{aligned}$$

and so

$$d||v_n||^{-1} = -||v_n||^{-1}dW_n^n + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}||v_n||^{-1}dt.$$

Consequently, as $\nu = \frac{v_n}{||v_n||}$,

$$\begin{aligned} d\nu = (dP-dQ)\nu &- \frac{(n-1)\mu_2}{(n+2)}\nu dt - (I-\Pi)dW\nu + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\nu dt \\ &- \langle (dP-dQ)\nu, dW_n^n\rangle \\ = -dQ\nu &- \frac{(n-1)^2\mu_2}{2n(n+2)}\nu dt - \frac{(n-1)\mu_2}{n(n+2)}\nu dt \\ = dQ\nu &- \frac{(n+1)(n-1)\mu_2}{2n(n+2)}\nu dt\,, \quad \text{as desired.} \end{aligned}$$

□

We can now derive the equation for $\langle S(u,v),\nu\rangle$.

Theorem 2.7. *Suppose $u(t) = D\Phi_t u$, $v(t) = D\Phi_t v$ and ν is as in Theorem 2.6. Then*

$$d\langle S(u,v),\nu\rangle = \langle dB(u,v),\nu\rangle + \langle S(u,v),\nu\rangle\langle dW\nu,\nu\rangle - \frac{(n-1)^2\mu_2}{2n(n+2)}\langle S(u,v),\nu\rangle dt.$$

Moreover, if $u_i(t) = D\Phi_t u_i$, then

$$\begin{aligned}\langle d\langle S(u_i,u_j),\nu\rangle, d\langle S(u_k,u_\ell),\nu\rangle\rangle = C^{33}_{1122}[&\langle u_i,u_j\rangle\langle u_k,u_\ell\rangle \\ &+ \langle u_i,u_k\rangle\langle u_j,u_\ell\rangle + \langle u_i,u_\ell\rangle\langle u_j,u_k\rangle]dt \\ &+ \frac{\mu_2(n-1)}{n(n+2)}\langle S(u_i,u_j),\nu\rangle\langle S(u_k,u_\ell),\nu\rangle dt.\end{aligned}$$

Proof. From Lemmas 2.4 and 2.5 we have

$$dS(u,v) = (I-\Pi)dB(u,v) + (dP - dQ)S(u,v) - \frac{(n-1)\mu_2}{(n+2)}S(u,v)dt.$$

Also

$$\begin{aligned}-\langle (dP-dQ)S(u,v), dQ\nu\rangle &= \langle dQS(u,v), dQ\nu\rangle \\ &= \langle S(u,v),\nu\rangle \sum_{i=1}^{n-1} C^{ii}_{nn}dt \\ &= \frac{(n^2-1)\mu_2}{n(n+2)}\langle S(u,v),\nu\rangle dt.\end{aligned}$$

So,

$$\begin{aligned}d\langle S(u,v),\nu\rangle &= \langle dS(u,v),\nu\rangle + \langle S(u,v),d\nu\rangle + \langle dS(u,v),d\nu\rangle \\ &= \langle dB(u,v),\nu\rangle + \langle S(u,v),\nu\rangle dW^n_n - \frac{(n-1)\mu_2}{(n+2)}\langle S(u,v),\nu\rangle dt \\ &- \frac{(n^2-1)\mu_2}{2n(n+2)}\langle S(u,v),\nu\rangle dt - \langle (dP-dQ)S(u,v), dQ\nu\rangle \\ &= \langle dB(u,v),\nu\rangle + \langle S(u,v),\nu\rangle dW^n_n - \frac{(n-1)(n-1)^2\mu_2}{2n(n+2)}\langle S(u,v),\nu\rangle dt\end{aligned}$$

For the quadratic variation term, since B and W are independent and $C^{nn}_{111} = 3C^{nn}_{1122}$,

$$\begin{aligned}\langle dS(u_i,u_j),\nu\rangle\langle dS(u_k,u_\ell),\nu\rangle &= \langle\langle dB(u_i,u_j),\nu\rangle\langle dB(u_k,u_\ell),\nu\rangle\rangle \\ &+ \langle S(u_i,u_j),\nu\rangle\langle S(u_k,u_\ell),\nu\rangle C^{nn}_{nn}dt \\ &= C^{nn}_{1122}[\langle u_i,u_j\rangle\langle u_k,u_\ell\rangle + \langle u_i,u_k\rangle\langle u_j,u_\ell\rangle \\ &+ \langle u_i,u_\ell\rangle\langle u_j,u_k\rangle]dt \\ &+ \frac{(n-1)\mu_2}{n(n+2)}\langle S(u_i,u_j),\nu\rangle\langle S(u_k,u_\ell),\nu\rangle dt.\end{aligned}$$

□

We now split the exposition. The case of a hypersurface in $\mathbb{R}^3$ will be developed below. The case of a hypersurface in $\mathbb{R}^n$, $n > 3$, will be treated in an addendum. The latter involves lengthy computations together with combinatorial arguments, whereas the former is more readable.

Theorem 2.8. *Let $u, v \in T_xM$ with u and v linearly independent and define $u(t) = D\Phi_t u$, $v(t) = D\Phi_t v$, (however in our computations we shall suppress the t dependence). The process $(Tr\, S^{(1)},\ Tr\, S^{(2)})$ satisfies the equations*

$$\begin{aligned}
d\,Tr\, S^{(1)} &= [\langle dB(u,u),\nu\rangle\|v\|^2 + \langle dB(v,v),\nu\rangle\|u\|^2 - 2\langle dB(u,v),\nu\rangle\langle u,v\rangle]\|\alpha\|^{-2} \\
&\quad + Tr\, S^{(1)}(dW_3^3 - 2(dW_1^1 + dW_2^2)) \\
&\quad + 2[S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)]\|\alpha\|^{-2} \\
&\quad + \frac{4\mu_2}{15} Tr\, S^{(1)}\, dt
\end{aligned}$$

$$\begin{aligned}
d\,Tr\, S^{(2)} &= [S(v,v)\langle dB(u,u),\nu\rangle + S(u,u)\langle dB(v,v),\nu\rangle - 2S(u,v)\langle dB(u,v),\nu\rangle]\|\alpha\|^{-2} \\
&\quad + 2\,Tr\, S^{(2)}(dW_3^3 - (dW_1^1 + dW_2^2)) \\
&\quad + \frac{4\mu_2}{15} Tr\, S^{(2)}\, dt\,.
\end{aligned}$$

$$\begin{aligned}
\langle d\,Tr\, S^{(1)}\rangle &= \left[\frac{16\mu_4}{35} + \frac{\mu_2}{15}[14(Tr\, S^{(1)})^2 - 24\,Tr\, S^{(2)}]\right] dt \\
\langle d\,Tr\, S^{(2)}\rangle &= \left[\frac{2\mu_4}{35}(3(Tr\, S^{(1)})^2 - 4\,Tr\, S^{(2)}) + \frac{32\mu_2}{15}(Tr\, S^{(2)})^2\right] dt \\
\langle d\,Tr\, S^{(1)}, d\,Tr\, S^{(2)}\rangle &= \left[\frac{8\mu_4}{35}\, Tr\, S^{(1)} + \frac{16\mu_2}{15} Tr\, S^{(1)} Tr\, S^{(2)}\right] dt\,.
\end{aligned}$$

Proof. Recall we have selected an orthomal basis $\{e_1, e_2, e_3\}$ for $T_{x_t}M_t$ such that $\nu = e_3$. Then with $\alpha = u \wedge v$, from LeJan (1986) we have

$$\frac{d\|\alpha\|^{-2}}{\|\alpha\|^{-2}} = -2(dW_1^1 + dW_2^2) - \frac{2\mu_2}{15}\, dt\,. \tag{2.15}$$

Using (1.6), it follows that

$$d\langle u,v\rangle = \langle dWu,v\rangle + \langle u,dWv\rangle + \frac{10\mu_2}{15}\langle u,v\rangle dt\,. \tag{2.16}$$

Now by (1.9), the mean curvature is

$$Tr\, S^{(1)} = [S(u,u)\|v\|^2 + S(v,v)\|u\|^2 - 2S(u,v)\langle u,v\rangle]\|\alpha\|^{-2} \tag{2.17}$$

so by Itô's formula,

$$\begin{aligned}
d\,Tr\,S^{(1)} &= [\|v\|^2 dS(u,u) + \|u\|^2 dS(v,v) - 2\langle u,v\rangle dS(u,v) \\
&\quad + S(u,u)d\|v\|^2 + S(v,v)d\|u\|^2 - 2S(u,v)d\langle u,v\rangle \\
&\quad + \langle dS(u,u), d\|v\|^2\rangle + \langle dS(v,v), d\|u\|^2\rangle - 2\langle dS(u,v), d\langle (u,v)\rangle]\|\alpha\|^{-2} \\
&\quad + Tr\,S^{(1)}\frac{d\|\alpha\|^{-2}}{\|\alpha\|^{-2}} + \langle d(Tr\,S^{(1)}\|\alpha\|^2), d\|\alpha\|^{-2}\rangle \\
&= [\langle dB(u,u),\nu\rangle\|v\|^2 + \langle dB(v,v),\nu\rangle\|u\|^2 - 2\langle dB(u,v),\nu\rangle\langle u,v\rangle]\|\alpha\|^{-2} \\
&\quad + Tr\,S^{(1)}dW_3^3 - \frac{2\mu_2}{15}Tr\,S^{(1)}dt \\
&\quad + [S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)]\|\alpha\|^{-2} \\
&+ \frac{10\mu_2}{15}Tr\,S^{(1)}dt - \frac{2\mu_2}{15}Tr\,S^{(1)}dt \\
&- 2\,Tr\,S^{(1)}(dW_1^1 + dW_2^2) - \frac{2\mu_2}{15}Tr\,S^{(1)}dt\ ,
\end{aligned}$$

using Theorem 2.7, (2.15), (2.16) and the fact that

$$\begin{aligned}
\langle d(Tr\,S^{(1)}\|\alpha\|^2),\ d\|\alpha\|^{-2}\rangle &= -2\|\alpha\|^{-2}[\langle \|\alpha\|^2 Tr\,S^{(1)}dW_3^3,\ dW_1^1 + dW_2^2\rangle \\
&\quad + 2\langle S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle \\
&- S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle), dW_1^1 + dW_2^2\rangle] \\
&= -2\left[\frac{-2\mu_2}{15} + \frac{2\mu_2}{15}\right] Tr\,S^{(1)}dt \\
&= 0\ .
\end{aligned}$$

Thus,

(2.18)

$$\begin{aligned}
d\,Tr\,S^{(1)} &= [\langle dB(u,u),\nu\rangle\|v\|^2 + \langle dB(v,v),\nu\rangle\|u\|^2 - 2\langle dB(u,v),\nu\rangle\ \langle u,v\rangle]\|\alpha\|^{-2} \\
&+ Tr\,S^{(1)}(dW_3^3 - 2(dW_1^1 + dW_2^2)) \\
&+ 2[S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle \\
&+ \langle u,dWv\rangle)]\|\alpha\|^{-2} \\
&+ \frac{4\mu_2}{15}Tr\,S^{(1)}dt\ .
\end{aligned}$$

Consequently, the quadratic variation of $Tr\,S^{(1)}$ is given by

(2.19)

$$\begin{aligned}
\langle d\,Tr\,S^{(1)}\rangle &= [\ \langle dB(u,u),\nu\rangle\|v\|^2 + \langle dB(v,v),\nu\rangle\|u\|^2 - 2\langle dB(u,v),\nu\rangle\ \langle u,v\rangle\ ]\|\alpha\|^{-4} \\
&+ (Tr\,S^{(1)})^2\langle dW_3^3 - 2(dW_1^1 + dW_2^2)\rangle \\
&+ 4[S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)]\|\alpha\|^{-4} \\
&+ 4Tr\,S^{(1)}\langle dW_3^3 - 2(dW_1^1 + dW_2^2), S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle \\
&- S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)\rangle\|\alpha\|^{-2}
\end{aligned}$$

Setting

$$q_n = \frac{(n+3)\mu_4}{n(n+2)\,(n+4)} = C_{1122}^{nn}$$

and if u_i, u_j, u_k, u_ℓ are all orthogonal to ν, then

$$\begin{aligned}\langle dB(u_i,u_j),\nu\rangle \langle dB(u_k,u_\ell),\nu\rangle = q_n[&\langle u_i,u_j\rangle \langle u_k,u_\ell\rangle + \langle u_i,u_k\rangle \langle u_j,u_\ell\rangle \\ &+ \langle u_i,u_\ell\rangle \langle u_j,u_k\rangle]dt\ .\end{aligned} \tag{2.20}$$

Then

$$\begin{aligned}&\langle\, \langle dB(u,u),\nu\rangle \|v\|^2 + \langle dB(v,v),\nu\rangle \|u\|^2 - 2\langle dB(u,v),\nu\rangle \langle u,v\rangle\rangle \|\alpha\|^{-4} \\ &= q_3[3\|u\|^4\|v\|^4 + 3\|v\|^4\|u\|^4 + 4(2\langle u,v\rangle^2 + \|u\|^2\|v\|^2)\langle u,v\rangle^2 \\ &\quad + 2(\|u\|^2\|v\|^2 + 2\langle u,v\rangle^2)\|u\|^2\|v\|^2 - 4(3\|u\|^2\langle u,v\rangle)\|v\|^2\langle u,v\rangle \\ &\quad - 4(3\|v\|^2\langle u,v\rangle)\|u\|^2\langle u,v\|]\|\alpha\|^{-4}dt \\ &= q_3[8\|u\|^4 - 16\|u\|^2\|v\|^2\langle u,v\rangle^2 + 8\langle u,v\rangle^4]\|\alpha\|^{-4}dt \\ &= 8q_3 dt\ .\end{aligned} \tag{2.21}$$

Next, observe

$$\begin{aligned}\langle dW_3^3 - 2(dW_1^1 + dW_2^2)\rangle &= \langle dW_3^3\rangle - 4\langle W_3^3,\, dW_1^1 + dW_2^2\rangle + 4\langle dW_1^1 + dW_2^2\rangle \\ &= \frac{18\mu_2}{15}dt\end{aligned}$$

and

$$\begin{aligned}\langle\langle dWv,v\rangle\rangle &= \frac{2\mu_2}{15}\|v\|^4 dt \\ \langle\langle dWv,v\rangle\,\langle dWu,u\rangle\rangle &= \frac{\mu_2}{15}(3\langle u,v\rangle^2 - \|u\|^2\|v\|^2)dt \\ \langle\langle dWu,v\rangle,\langle u,dWv\rangle\rangle &= \frac{\mu_2}{15}(3\langle u,v\rangle^2 - \|v\|^2)dt \\ \langle\langle dWu,v\rangle,\langle dWu,v\rangle\rangle &= \frac{\mu_2}{15}(4\|u\|^2\|v\|^2 - 2\langle u,v\rangle^2)dt \\ \langle\langle dWu,v\rangle,\langle dWv,v\rangle\rangle &= \frac{2\mu_2}{15}\langle u,v\rangle\|v\|^2 dt \\ \langle\langle dWv,u\rangle,\langle dWu,u\rangle\rangle &= \frac{2\mu_2}{15}\langle u,v\rangle\|u\|^2 dt \\ \langle dWv,u\rangle\,\langle v,v\rangle &= \frac{2\mu_2}{15}\langle u,v\rangle\|v\|^2 dt\ .\end{aligned}$$

Thus,

$$
\begin{aligned}
&[S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)] \\
&= \frac{\mu_2}{15}[2S(u,u)^2\|v\|^4 + 2S(v,v)^2\|u\|^4 \\
&\quad + 2S(u,v)^2(3\|u\|^2\|v\|^2 + \langle u,v\rangle^2) + 2S(u,u)S(v,v)(3\langle u,v\rangle^2 - \|u\|^2\|v\|^2) \\
&\quad - 2S(u,u)S(u,v)(1\langle u,v\rangle\|u\|^2 + 2\langle u,v\rangle\|u\|^2)\|v\|^2 \\
&\quad - 2S(v,v)S(u,v)(2\langle u,v\rangle\|u\|^2 + 2\langle u,v\rangle\|u\|^2)\|u\|^2]dt \\
&= \frac{\mu_2}{15}[2\{S(u,u)^2\|v\|^4 + S(v,v)^2\|u\|^4 + 2S(u,u)S(v,v)\|u\|^2\|v\|^2 \\
&\qquad - 4S(u,u)S(u,v)\|u\|^2\|v\|^2\langle u,v\rangle \\
&\qquad - 4S(v,v)S(u,v)\|u\|^2\|v\|^2\langle u,v\rangle \\
&\qquad 4S(u,v)^2\langle u,v\rangle^2\} \\
&\quad - 6(S(u,u)S(v,v) - S(u,v)^2)\|\alpha\|^2]dt \\
&= \frac{\mu_2}{15}[2(Tr\,S^{(1)})^2 - 6Tr\,S^{(2)}]\|\alpha\|^4 dt
\end{aligned}
$$

It's easy to verify that

$$
\begin{aligned}
\langle dW_3^3, \langle dWv,v\rangle\rangle &= -\frac{\mu_2}{15}\|v\|^2 dt \\
\langle dW_3^3, \langle dWu,v\rangle\rangle &= \langle dW_3^3, \langle u,dWv\rangle\rangle = -\frac{\mu_2}{15}\langle u,v\rangle dt \\
\langle dW_1^1 + dW_2^2, \langle dWu,v\rangle\rangle &= \frac{\mu_2}{15}\langle u,v\rangle dt
\end{aligned}
$$

so

$$
\begin{aligned}
&\langle dW_3^3, S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)\rangle \\
&= -\frac{\mu_2}{15}[S(u,u)\|v\|^2 + S(v,v)\|u\|^2 - 2S(u,v)\langle u,v\rangle]dt \\
&= -\frac{\mu_2}{15}Tr\,S^{(1)}\|\alpha\|^2 dt\,, \\
&\langle dW_1^1 + dW_2^2\,, S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)\rangle \\
&= \frac{\mu_2}{15}[S(u,u)\|v\|^2 + S(v,v)\|u\|^2 - 2S(u,v)\langle u,v\rangle]dt \\
&= \frac{\mu_2}{15}Tr\,S^{(1)}\|\alpha\|^2 dt\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&4\,Tr\,S^{(1)}\|\alpha\|^{-2}\langle dW_3^3 - 2(dW_1^1 + dW_2^2), S(u,u)\langle dWv,v\rangle + S(v,v)\langle dWu,u\rangle \\
&\quad - S(u,v)(\langle dWu,v\rangle + \langle u,dWv\rangle)\rangle \\
&\quad = 4(Tr\,S^{(1)})^2\left[-\frac{\mu_2}{15} - \frac{2\mu_2}{15}\right]dt \\
&\quad = -\frac{12\mu_2}{15}(Tr\,S^{(1)})^2 dt\,.
\end{aligned}
\tag{2.23}
$$

Combining (2.20), (2.21) and (2.22) we arrive at

$$\langle d\,Tr S^{(1)}\rangle = \frac{16\mu_4}{35}dt + \frac{\mu_2}{15}[14(Tr\,S^{(1)})^2 - 24\,Tr\,S^{(2)}]dt\ . \tag{2.24}$$

Turning now to the Gauss curvature, by (1.9) we have,

$$Tr\,S^{(2)} = [S(u,u)S(v,v) - S(u,v)^2]\|\alpha\|^{-2}\ . \tag{2.25}$$

So, by Itô's formula,

$$\begin{aligned}
d\,Tr\,S^{(2)} &= (S(v,v)dS(u,u) + S(u,u)dS(v,v) - 2S(u,v)dS(u,v)\\
&\quad + \langle dS(u,u), dS(v,v)\rangle - \langle dS(u,v), dS(u,v)\rangle]\|\alpha\|^{-2}\\
&\quad + [S(u,u)S(v,v) - S(u,v)^2]d\|\alpha\|^{-2}\\
&\quad + \langle S(v,v)dS(u,u) + S(u,u)dS(v,v) - 2S(u,v)dS(u,v), d\|\alpha\|^{-2}\rangle\\
&= [S(v,v)\langle dB(u,u),\nu\rangle + S(u,u)\langle dB(v,v),\nu\rangle\\
&\quad - 2S(u,v)\langle dB(u,v),\nu\rangle]\|\alpha\|^{-2}\\
&\quad + 2Tr\,S^{(2)}dW_3^3 - \frac{4\mu_2}{15}Tr\,S^{(2)}dt\\
&\quad + [\langle\langle dB(u,u),\nu\rangle, \langle dB(v,v),\nu\rangle\rangle - \langle dB(u,v),\nu\rangle, \langle dB(u,v),\nu\rangle\rangle]\|\alpha\|^{-2}\,dt\\
&\quad + \frac{2\mu_2}{15}Tr\,S^{(2)}dt\\
&\quad - 2\,Tr\,S^{(2)}(dW_1^1 + dW_2^2) - \frac{2\mu_2}{15}Tr\,S^{(2)}dt\\
&+ \frac{8\mu_2}{15}Tr\,S^{(2)}dt\ .
\end{aligned}$$

Thus,

$$\begin{aligned}
dTr\,S^{(2)} &= [S(v,v)\langle dB(u,u),\nu\rangle + S(u,u)\langle dB(v,v),\nu\rangle\\
&\qquad - 2S(u,v)\langle dB(u,v),\nu\rangle]\|\alpha\|^{-2}\\
&\quad + 2Tr\,S^{(2)}(dW_3^3 - (dW_1^1 + dW_2^2)) + \frac{4\mu_2}{15}Tr\,S^{(2)}dt\,,\\
&\text{since}\quad \langle\langle dB(u,u),\nu\rangle, \langle dB(v,v),\nu\rangle\rangle = \langle(dB(u,v),\nu\rangle, \langle dB(u,v),\nu\rangle\rangle\ .
\end{aligned} \tag{2.26}$$

Using (2.20), (2.26) and our remarks above on W correlations, we get

(2.27)
$$\begin{aligned}
\langle d\,Tr\,S^{(2)}\rangle &= \langle S(v,v)\langle dB(u,u),\nu\rangle + S(u,u)\langle dB(v,v),\nu\rangle \\
&\quad - 2S(u,v)\langle dB(u,v),\nu\rangle\rangle\|\alpha\|^{-4} + 4(Tr\,S^{(2)})^2\langle dW^3 - (dW_1^1 + dW_2^2)\rangle \\
&= q_3[3S(v,v)^2\|u\|^4 + 3S(u,u)^2\|v\|^4 + 4S(u,v)^2(\|u\|^2\|v\|^2 + 2\langle u,v\rangle^2) \\
&\quad + 2S(u,u)S(v,v)(\|u\|^2\|v\|^2 + 2\langle u,v\rangle^2) \\
&\quad - 12S(v,v)S(u,v)\|u\|^2\langle u,v\rangle - 12S(u,u)S(u,v)\|v\|^2\langle u,v\rangle]\|\alpha\|^{-2}dt \\
&+ \frac{32\mu_2}{15}(Tr\,S^{(2)})^2dt \\
&= 3q_3(Tr\,S^{(1)})^2dt - 4q_3Tr\,S^{(2)}dt + \frac{32\mu_2}{15}(Tr\,S^{(2)})^2dt \\
&= \frac{2\mu_4}{35}(3(Tr\,S^{(1)})^2 - 4\,Tr\,S^{(2)})dt + \frac{32\mu_2}{15}(Tr\,S^{(2)})^2dt\,.
\end{aligned}$$

Finally we need

(2.28)
$$\begin{aligned}
\langle dTr\,S^{(1)}, dTr\,S^{(2)}\rangle &= 2Tr\,S^{(1)}Tr\,S^{(2)}\langle dW_3^3 - 2(dW_1^1 + dW_2^2), dW_3^3 \\
&\quad - (dW_1^1 + dW_2^2)\rangle + 4Tr\,S^{(2)}\langle S(u,u)\langle dWv,v\rangle \\
&\quad + S(v,v)\langle Wu,u\rangle - S(u,v)(\langle dWu,v\rangle \\
&\quad + \langle u,dWv\rangle), dW_3^3 - (dW_1^1 + dW_2^2)\rangle\|\alpha\|^{-2} \\
&\quad + \|\alpha\|^{-4}\langle\langle dB(u,u),\nu\rangle\|v\|^2 + \langle dB(v,v),\nu\rangle\|u\|^2 \\
&\quad - 2\langle dB(u,v),\nu\rangle\langle u,v\rangle\ , S(v,v)\langle dB(u,u),\nu\rangle \\
&\quad + S(u,u)\langle dB(v,v),\nu\rangle - 2S(u,v)\langle dB(u,v),\nu\rangle\rangle \\
&= \frac{24\mu_2}{15}Tr\,S^{(1)}Tr\,S^{(2)}dt - \frac{8\mu_2}{15}Tr\,S^{(1)}Tr\,S^{(2)}dt + 4q_3\,Tr\,S^{(1)}dt \\
&= \frac{16\mu_2}{15}Tr\,S^{(1)}Tr\,S^{(2)}dt + \frac{8\mu_4}{35}Tr\,S^{(1)}dt\,.
\end{aligned}$$

This completes the proof. □

Theorem 2.9. *The process $(Tr\,S_t^{(1)}, Tr\,S_t^{(2)})$ is a recurrent diffusion with generator*

$$\begin{aligned}
Lf(\kappa,m) &= \frac{1}{2}\left(\frac{\mu_2}{15}(14\kappa^2 - 24m) + \frac{16\mu_4}{35}\right)\frac{\partial^2 f}{\partial\kappa^2}(\kappa,m) \\
&+ \left(\frac{16\mu_2}{15}\kappa m + \frac{8\mu_4}{35}\kappa\right)\frac{\partial^2 f}{\partial\kappa\,\partial m}(\kappa,m) \\
&+ \frac{1}{2}\left(\frac{32\mu_2}{15}m^2 + \frac{2\mu_4}{35}(3\kappa^2 - 4m)\right)\frac{\partial^2 f}{\partial m^2}(\kappa,m) \\
&+ \frac{4\mu_2}{15}\kappa\frac{\partial f}{\partial\kappa}(\kappa,m) + \frac{4\mu_2}{15}m\frac{\partial f}{\partial m}(\kappa,m)\,.
\end{aligned}$$

This diffusion never enters the region $\kappa^2 \leq 4m$ at strictly positive times a.s. .

Remarks.

(1) In $n = 3$, the three Lyapunov exponents are (see LeJan 1986) $\gamma_1 = 3$, $\gamma_2 = 0$, $\gamma_3 = -3$. Thus there is always a stretching direction, a neutral direction and a compressing direction. A curve will always "see" the stretching direction. This roughly explains the positive recurrence of the curvature of a curve proved in LeJan 1986. In the case of a surface, the tangent plane will "see" the stretching and the neutral direction. This also explains recurrence. The authors plan a complete account of the situation in higher dimensions in a forthcoming paper.

(2) Since $X = Tr\, S^{(1)} = \lambda_1 + \lambda_2$ and $Y = Tr\, S^{(2)} = \lambda_1\lambda_2$, it follows that (X, Y) stays in the region $\kappa^2 \geq 4m$. The boundary of this region corresponds to $\lambda_1 = \lambda_2$. That is $\kappa^2 = 4m$ when the surface in question has an umbilic (i.e. a point where $\lambda_1 = \lambda_2$). Thus, x may be an umbilic for M but x_t, $t > 0$, will never be an umbilic for M_t.

One easily checks that the matrix a appearing in the generator for this diffusion (2^{nd} order derivative coefficients) degenerates on the curve $\kappa^2 - 4m = 0$. Moreover, the two eigenvectors of this matrix are, respectively, perpendicular to and tangent to the curve $\kappa^2 - 4m = 0$. The eigenvector perpendicular to this curve has eigenvalue zero. Notice the drift vector $\left(\frac{4\mu_2}{15}\kappa, \frac{4\mu_2}{15}m\right)$ points away from the region $\kappa^2 - 4m < 0$. Thus, when the point x on the initial manifold M is an umbilic point, the diffusion (X_t, Y_t) has diffusion component only tangential the curve $\kappa^2 = 4m$ and drifts away from this curve never to return.

(3) Neither $Tr\, S^{(1)}$ nor $Tr\, S^{(2)}$ is a diffusion by itself.

(4) In the case $n = 3$ one can assert that (X_t, Y_t) does not spend a lot of time in the region $\lambda = \{(\kappa, m) : \kappa > M_1, m > M_2\}$ with M_1 and M_2 large positive constants. Recall the following results of LeJan (1986): draw a curve γ on the initial manifold M with $\gamma(0) = x$. Set $\gamma_t = \Phi_t(\gamma)$. Then the curvature τ_t of γ_t at $\gamma_t(0)$ is a positive recurrent diffusion. Thus, the average amount of time that τ_t spends above a large positive constant is small. Since it is impossible to draw a curve on M_t with zero curvature (or even small curvature) when $(X_t, Y_t) \in \Lambda$, (X_t, Y_t) can not stay for long in Λ. More precise statements can be made by referring to the article of LeJan where the exact form of the invariant measure for τ_t is given.

Proof (of Theorem 2.9). The claim that (X_t, Y_t) is a diffusion follows immediately from Theorem 2.8. Define $h = \sqrt{\kappa^2 - 4m}$. Then with $a = \frac{\mu_2}{15}$, $b = \frac{\mu_4}{35}$, in (m, h)-coordinates

$$\begin{aligned} Lf(\kappa, h) = &(4a\kappa^2 + 3ah^2 + 8b)\frac{\partial^2 f}{\partial \kappa^2} + 14a\kappa h \frac{\partial^2 f}{\partial \kappa \partial h} \\ &+ (3a\kappa^2 + 4ah^2 + 4b)\frac{\partial^2 f}{\partial h^2} \\ &+ 4a\kappa \frac{\partial f}{\partial \kappa} + h^{-1}(3a\kappa^2 + ah^2 + 4b)\frac{\partial f}{\partial h}\ . \end{aligned}$$

We prove the claim regarding lack of umbilics first as it eases a technical point which might arise in the proof of the recurrence. For this, it suffices to show that H_t does not hit zero a.s. where (X_t, H_t) is diffusion given in (κ, h) coordinates. First define $\eta_t = 3aX_t^2 + 4aH_t^2 + 4b$ and then $\tau_t = \int_0^t \frac{dx}{\eta s}$. Then

$$\begin{aligned} H_{\tau_t} &= h + b_t + \int_0^{\tau_t} H_s^{-1}(3aX_s^2 + aH_s^2 + 4b)ds \\ &= h + b_t + \int_0^t H_{\tau_s}^{-1}(3aX_{\tau_s}^2 + aH_{\tau_s}^2 + 4b)\eta_{\tau_s}^{-1}\, ds \end{aligned}$$

with b_t a one-dimensional Brownian motion. Fix now an $\epsilon \in (0, 1/6)$ and suppose $0 < h < 1/6$ as well. Set

$$\rho_t^\epsilon = h + b_t + \left(\frac{1}{2} + \epsilon\right) \int_0^t \frac{ds}{\rho_s^\epsilon}\, .$$

If $\sigma_\epsilon = \inf\{t > 0 : H_{\tau_t} > 2\sqrt{\frac{(\frac{1}{2}-\epsilon)b}{(1+4\epsilon)a}}\}$, then since for $h \leq \sqrt{\frac{(\frac{1}{2}-\epsilon)b}{(1+4\epsilon)a}}$, one has $(\frac{1}{2} + \epsilon) < \frac{3a\kappa^2 + ah^2 + 4b}{3a\kappa^2 + 4ah^2 + 4b}$ for all κ. By an elementary comparison theorem (see Ikeda-Watanabe, 1989), it follows that $\rho_t^\epsilon \leq H_{\tau_t}$ for $t < \sigma_\epsilon$. Since ρ_t^ϵ can not hit zero (it is a Bessel process above the critical dimension), H_t can't hit zero either.

For recurrence, we shall show (X_t, H_t) is recurrent which will suffice. Set $f(\kappa, h) = \ell n(\kappa^6 + h^6)$. Then

$$\begin{aligned} f_\kappa &= \frac{6\kappa^5}{\kappa^6 + h^6}\ , \quad f_h = \frac{6h^5}{\kappa^6 + h^6} \\ f_{\kappa\kappa} &= \frac{30\kappa^4 h^6 - 6\kappa^{10}}{(\kappa^6 + h^6)^2}\ , \quad f_{\kappa h} = -\frac{36\kappa^5 h^5}{(\kappa^6 + h^6)^2}\ ; \quad f_{hh} = \frac{30\kappa^6 h^4 - 6h^{10}}{(\kappa^6 + h^6)^2} \end{aligned}$$

and

$$\begin{aligned} Lf &= 6(\kappa^6 + h^6)^{-2}[(4a\kappa^2 + 3ah^2 + 8b)(5\kappa^4 h^6 - \kappa^{10}) - 84a\kappa^6 h^6 \\ &\quad + (3a\kappa^2 + 4ah^2 + 4b)(5\kappa^6 h^4 - h^{10}) + 4a\kappa^{12} + 4a\kappa^6 h^6 \\ &\quad + h^4(3a\kappa^2 + ah^2 + 4b)(\kappa^6 + h^6)] \\ &= 6(\kappa^6 + h^6)^{-2}[-a(3\kappa^{10}h^2 - 18\kappa^8 h^4 + 39\kappa^6 h^6 - 15\kappa^4 h^8 + 3h^{12}) \\ &\quad - b(8\kappa^{10} - 24\kappa^6 h^4 - 40\kappa^4 h^6)] \\ &\quad \kappa = uh \\ &= -6(\kappa^6 + h^6)^{-2} h^{10}[3ah^2(u^{10} - 6u^8 + 13u^6 - 5u^4 + 1) \\ &\quad + 8bu^4(u^6 - 3u^2 - 5)] \end{aligned}$$

The polynomial $P(u) = u^{10} - 6u^8 + 13u^6 - 5u^4 + 1 > 0$ for all u.

Also, there is a $u_0 > 0$ such that $Q(u) = u^6 - 3u^2 - 5 \geq 0$ whenever $|u| \geq u_0$. Thus,

$$Lf = -6(\kappa^6 + h^6)^{-2} h^{10} [3ah^2 P(u) + 8bu^4 Q(u)]$$

has

$$\{Lf \leq 0\} \supseteq \{(\kappa, h) : |\kappa| \geq u_0 h\} \ .$$

Also, denote $M = \sup_u \frac{Q(u)}{P(u)}$, one sees

$$\{Lf \leq 0\} \supseteq \left\{ (\kappa, h) : h \geq \sqrt{\frac{86M}{3a}} \right\}$$

Consequently, off of the compact triangle

$$T = \{(\kappa, h) : |\kappa| \leq u_0 h\} \cap \left\{ (\kappa, h) : h \leq \sqrt{\frac{86M}{3a}} \right\}$$

we have $Lf \leq 0$. Now, define

$$C(r) = \{(\kappa, h) : h > 0, \kappa^6 + h^6 < r\}$$

and select r_0 large enough so that $C(r_0) \supset T$. Put for $R > r_0$

$$u_R(\kappa, h) = \frac{\ell n(\kappa^6 + h^6) - \ell n R}{\ell n r_0 - \ell n R} \ .$$

Then on $C(R) \setminus C(r_0)$, $Lu_R \geq 0$ and $u_R\big|_{\partial C(r_0) \cap \{h>0\}} = 1$, $u_R\big|_{\partial C(R) \cap \{h>0\}} = 0$. Thus, if $\tau_r = \inf\{t > 0 : (X_t, H_t) \in \partial C(r)\}$ (recall, H_t never hits 0), then as u_R is L-subharmonic and has the same boundary values as $P_{(\kappa,w)}(\tau_{r_0} < T_R)$

$$u_R(\kappa, h) \leq P_{(\kappa,h)}(\tau_{r_0} < \tau_R) \ .$$

But $\lim_{R \uparrow \infty} u_R(\kappa, h) = 1$ which implies the recurrence of (X_t, H_t) and therefore the recurrence of (X_t, Y_t). $\square$

References

P. Baxendale, (1984), *Brownian motions in the diffeomorphism Group. I,* Compositio Math. **53**, 19–50.

P. Baxendale and T.Harris, (1986), *Isotropic stochastic flows,* Annals of Prob., **Vol 14**, 1155–1179.

R. Bishop and S. Goldberg, (1980), *Tensor Analysis on Manifolds*, Dover, Toronto.

Hardy, Littlewood, Polya, (1973), *Inequalities*, Cambridge University Press, Cambridge.

T. Harris, (1981), *Brownian motions on the homeomorphisms of the plane,* Annals of Prob. **Vol. 9**, 232–254.

N. Ikeda, S. Watanabe, (1989), *Stochastic Differential Equations and Diffusion Processes*, North-Holland, New York.

Y. LeJan, (1985), *On isotropic Brownian motions,* Zeit. für Wahr., **70**, 609–620.

Y. LeJan, (1991), *Asymototic properties of isotropic Brownian flows,* in Spatial Stochastic Processes, ed. K. Alexander, J. Watkins, Birkhäuser, Boston, 219–232.

Y. LeJan, S. Watanabe (1984), *Stochastic flows of diffeomorphisms*, Proc. of the Taniguchi Symposium, 307–332, North Holland, New York.

M. Spivak, (1979), *A comprehensive introduction to differential geometry*, Publish or Perish, Berkeley.

Addendum. Higher dimensional hypersurfaces.

Recall $\alpha_{\vec{\ell}} = u_{\ell_1} \wedge \cdots \wedge u_{\ell_k}, \alpha_{\vec{m}} = u_{m_1} \wedge \cdots \wedge u_{m_k}$, for $\vec{\ell}, \vec{m} \in I_k$. Define

$$\alpha_{\vec{\ell}_p} = (-1)^{p+1} u_{\ell_1} \wedge \cdots \wedge \hat{u}_{\ell_p} \wedge \cdots \wedge u_{\ell_k}, \quad \alpha_{\vec{m}_i} = (-1)^{i+1} u_{m_1} \wedge \cdots \wedge \hat{u}_{m_i} \wedge \cdots \wedge u_{m_k}.$$

Then $\alpha_{\vec{\ell}_p} = (-1)^{p+1} u^{(p)}_{\ell_1} \wedge \cdots \wedge u^{p}_{\ell_{k-1}}$ $\quad \alpha^{(i)}_{\vec{m}_i} = (-1)^{p+1} u^{(i)}_{m_1} \wedge \cdots \wedge u^{(i)}_{m_{k-1}}$ defines $u^{(p)}_{\ell}$ and $u^{(i)}_{m}$, and set $S^{(0)}(\cdot, \cdot) \equiv 1$.

Theorem A.1.

$$dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) = \sum_{i,p=1}^{k} \quad S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i}) \langle dB(u_{\ell_p}, u_{m_i}), \nu \rangle$$
$$+ kS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) dW^n_n - \frac{k(n-k)(n-1)\mu_2}{2n(n+2)} S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) dt\,,$$

or, in a more synthetic form

$$dS^{(k)}_{(\alpha)} = S^{(k-1)} \wedge dB^{\nu}(\alpha) + kS^{(k)}(\alpha) \langle d \langle W\nu, \nu \rangle - \frac{-k(n-k)(n-1)}{2n(n+2)} S^{(k)}(\alpha)\ ,$$

with $\langle B^{\nu} u, v \rangle = \langle B(u,v), \nu \rangle$.

Proof. By Theorem 2.7 and Itô's formula,

$$dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) = \sum_{\sigma \in S_k} (-1)^{\sigma} \sum_{i=1}^{k} [\prod_{\substack{j=1 \\ j \neq i}}^{k} \langle S(u_{\ell_{\sigma(j)}}, u_{m_j}), \nu \rangle] \langle dB(u_{\ell_{\sigma(i)}}, u_{m_i}), \nu \rangle$$
$$+ kS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) dW^n_n - \frac{k(n-1)(n-1)\mu_2}{2n(n+2)} S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) dt$$
$$+ \frac{1}{2} \sum_{\sigma \in S_k} (-1)^{\sigma} \sum_{1 \le i \neq p \le k} [\prod_{\substack{j=1 \\ j \neq \{i,p\}}}^{k} \langle S(u_{\ell_{\sigma(j)}}, u_{m_j}), \nu \rangle][dB^n(u_{\ell_{\sigma(i)}}, u_{m_i}), dB^n(u_{\ell_{\sigma(p)}}, u_{m_p}) \rangle$$
$$+ \langle dB^n(u_{\ell_{\sigma(i)}}, u_{m_p}), dB^n(u_{\ell_{\sigma(p)}}, u_{m_i}) \rangle + \langle dB^n(u_{\ell_{\sigma(i)}}, u_{\ell_{\sigma(p)}}), dB^n(u_{m_i}, u_{m_p}) \rangle]$$
$$+ \frac{k(k-1)(n-1)\mu_2}{2n(n+2)} S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) dt.$$

The first term may be written

$$\sum_{i,p=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(i) = p}} (-1)^{\sigma} [\prod_{\substack{j=1 \\ j \neq i}}^{k} \langle S(u_{\ell_{\sigma(j)}}, u_{m_j}), \nu \rangle] \langle dB(u_{\ell_p}, u_{m_i}), \nu \rangle$$
$$= \sum_{i,p=1}^{k} [\sum_{\sigma = S_{k-1}} (-1)^{p+i} (-1)^{\sigma} [\prod_{\substack{j=1 \\ j \neq i}}^{k} \langle S(u^{(p)}_{\ell_{\sigma(j)}}, u^{(i)}_{m_j}), \nu \rangle]] \langle dB(u_{\ell_p}, u_{m_i}), \nu \rangle$$
$$= \sum_{i,p=1}^{k} S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i}) \langle dB(u_{\ell_p}, u_{m_i}), \nu \rangle.$$

The third term combines with the fifth term to give

$$-\frac{k(n-k)(n-1)\mu_2}{2n(n+2)} S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})dt.$$

Finally, in the fourth term, notice that given $\sigma \in S_k$ and $i, p \in \{1, \dots, k\}$ there is exactly one other $\eta \in S_k$ such that both $\{\sigma(i), \sigma(p)\} = \{\eta(i), \eta(p)\}$ and $\sigma(j) = \eta(j)$ for $j \notin \{i, p\}$. Furthermore, $(-1)^\sigma = -(-1)^\eta$, and

$$\begin{aligned}
&[\prod_{\substack{j=1 \\ j\notin\{i,p\}}}^{k} \langle S(u_{\ell_{\sigma(j)}}, u_{m_j}), \nu\rangle][\langle B^n(u_{\ell_{\sigma(i)}}, u_{m_i}), dB^n(u_{\ell_{\sigma(p)}}, u_{m_p})\rangle \\
&\quad + \langle dB^n(u_{\ell_{\sigma(i)}}, u_{m_p}), dB^n(u_{\ell_{\sigma(p)}}, u_{m_i})\rangle + \langle dB^n(u_{\ell_{\sigma(i)}}, u_{\ell_{\sigma(p)}}), dB^n(u_{m_i}, u_{m_p})\rangle] \\
&\quad = [\prod_{\substack{j=1 \\ j\notin\{i,p\}}}^{k} \langle S(u_{\ell_{\eta(j)}}, u_{m_j}), \nu\rangle][\langle dB^n(u_{\ell_{\eta(i)}}, u_{m_i}), dB^n(u_{\ell_{\eta(p)}}, u_{m_p})\rangle \\
&\quad + \langle dB^n(u_{\ell_{\eta(i)}}, u_{m_p}), dB^n(u_{\ell_{\eta(p)}}, u_{m_i})\rangle \\
&\quad\quad + \langle dB^n(u_{\ell_{\eta(i)}}, u_{\ell_{\eta(p)}}), dB^n(u_{m_i}, u_{m_p})\rangle]
\end{aligned}$$

Thus exchanging the sums in the fourth term reveals the cancellation of terms from these $\sigma - \eta$ pairs and so the whole term vanishes. This finishes the proof. $\square$

From LeJan (1985) we recall the notation

$$\tau_\ell^j \beta = e^j \wedge i(e^\ell)\beta$$

where $i(e^\ell)\beta = \Sigma(-1)^{k+1}\langle \beta_k, e^\ell\rangle \beta_1 \wedge \cdots \wedge \hat{\beta}_k \wedge \cdots \wedge \beta_m$. The situation for general n is given by the following Theorem which we prove in the Addendum.

Theorem A.2. *For* $1 \le k \le n-1$,

$$\begin{aligned}
dTrS^{(k)} &= [\sum_{i,p=1}^{k} S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i})\langle dB(u_{\ell_p}, u_{m_i}), \nu\rangle]\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle ||\alpha||^{-2} \\
&\quad + TrS^{(k)}[kdW_n^n - 2\sum_{i=1}^{n-1} dW_i^i] \\
&\quad + S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}}\rangle) dW_j^i ||\alpha||^{-2} \\
&\quad - \frac{(n+1)k(n-k)\mu_2}{2n(n+2)} TrS^{(k)}dt, \text{ or}
\end{aligned}$$

$$\begin{aligned}
dTrS^{(k)} &= d\langle S^{(k)}\alpha, \alpha\rangle/\|\alpha\|^2 \\
&= \langle S^{(k-1)} \wedge dB^\nu(\alpha) + \left(\langle dW\nu, \nu\rangle - 2\frac{\langle dW\alpha, \alpha\rangle}{\|\alpha\|^2}\right) S^{(k)}(\alpha) \\
&\quad + S^{(k)} \wedge dW(\alpha), \alpha\rangle/\|\alpha\|^2 - \frac{(n+1)k(n-k)\mu_2}{2n(n+2)}\langle S^{(k)}(\alpha), \alpha\rangle/\|\alpha\|^2 .
\end{aligned}$$

Proof. For $\vec{\ell} \in I_k$,

$$d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle = (\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}} \rangle) dW_j^i + \frac{(k+1)(n-1-k)\mu_2}{n} \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle dt \tag{A.1}$$

$$\frac{d||\alpha||^{-2}}{||\alpha||^{-2}} = -2 \sum_{i=1}^{n-1} dW_i^i - \frac{(n-2)(n-1)\mu_2}{n(n+2)} dt \tag{A.2}$$

$$d||\alpha||^{-2} = -2||\alpha||^{-2} dW_i^i + A_t, \quad A_t \text{ has bounded variation} \tag{A.3}$$

Since

$$T_r S^{(k)} = S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2}$$

by Itô's formula,

$$\begin{aligned} dTrS^{(k)} &= (dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})) \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2} \\ &+ S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) (d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle) ||\alpha||^{-2} \\ &+ TrS^{(k)} \frac{d||\alpha||^{-2}}{||\alpha||^{-2}} \\ &+ \langle dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}), d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle \rangle ||\alpha||^{-2} \\ &+ \langle dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}), d||\alpha||^{-2} \rangle \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle \\ &+ S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) \langle d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle, d||\alpha||^{-2} \rangle \end{aligned}$$

Proceeding in order through the terms, we have using Theorem A.1

$$\begin{aligned} & dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2} \\ &= [\sum_{i,p=1}^{k} \mathcal{S}^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i}) \langle dB(u_{\ell_p}, u_{m_i}), \nu \rangle] \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2} \\ &+ kTr\mathcal{S}^{(k)} dW_n^n - \frac{k(2n-k)(n-1)\mu_2}{2n(n+2)} TrS^{(k)} dt. \end{aligned} \tag{A.4}$$

From (A.1) follows,

$$\begin{aligned} & S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle ||\alpha||^{-2} \\ &= S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}) (\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}} \rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}} \rangle) dW_j^i ||\alpha||^{-2} \\ &+ \frac{(k+1)(n-k-1)\mu_2}{n} TrS^{(k)} dt. \end{aligned} \tag{A.5}$$

By (A.2)

$$TrS^{(k)}\frac{d||\alpha||^{-2}}{||\alpha||^{-2}} = -2TrS^{(k)}\sum_{i=1}^{n-1} dW_i^i - \frac{(n-2)(n-1)\mu_2}{n(n+2)}TrS^{(k)}dt. \tag{A.6}$$

Combining Theorem A.1 and (A.1),

$$\begin{aligned}
&\langle dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}), d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle\rangle ||\alpha||^{-2} \\
&= kS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}}\rangle)\langle dW_j^i, dW_n^n\rangle \\
&= \frac{k\mu_2}{n(n+2)}S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}}\rangle)[(n+1)\delta_n^i\delta_n^j - \delta_j^i\delta_n^n - \delta_n^i\delta_n^j]dt \\
&= -\frac{k\mu_2}{n(n+2)}S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^i \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^i \alpha^{\vec{m}}\rangle)dt \\
&= -\frac{2k(n-k-1)\mu_2}{n(n+2)}TrS^{(k)}dt.
\end{aligned}$$

By Theorem A.1 and (A.3)

$$\begin{aligned}
&\langle dS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}}), d||\alpha||^{-2}\rangle\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle \\
&= -2kS^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})\langle dW_n^n, dW_i^i\rangle\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle ||\alpha||^{-2} \\
&= \frac{2k(n-1)\mu_2}{n(n+2)}TrS^{(k)}dt.
\end{aligned} \tag{A.7}$$

From (A.1) and (A.3) we get

$$\begin{aligned}
&S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})\langle d\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle, d||\alpha||^{-2}\rangle \\
&= -2S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}}\rangle)\langle dW_p^p, dW_j^i\rangle ||\alpha||^{-2} \\
&= \frac{2\mu_2}{n(n+2)}\mathcal{S}^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^i \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^i \alpha^{\vec{m}}\rangle)||\alpha||^{-2}dt \\
&= -\frac{4(n-k-1)\mu_2}{n(n+2)}TrS^{(k)}dt
\end{aligned} \tag{A.8}$$

Summing (A.4)–(A.8) shows

$$\begin{aligned}
dT_rS^{(k)} &= [\sum_{i,p=1}^{k}(-1)^{i+p+1}\mathcal{S}^{(k-1)}(\alpha_{\vec{\ell}}^{\ell_p}, \alpha_{\vec{m}}^{m_i})\langle dB(u_{\ell_p}, u_{m_i}), \nu\rangle]\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle ||\alpha||^{-2} \\
&+ TrS^{(k)}[kdW_n^n - 2dW_i^i] \\
&+ S^{(k)}(\alpha_{\vec{\ell}}, \alpha_{\vec{m}})(\langle \tau_i^j \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle + \langle \alpha^{\vec{\ell}}, \tau_i^j \alpha^{\vec{m}}\rangle)dW_j^i \cdot \|\alpha\|^{-2} \\
&+ \frac{k(n-k)(n+1)\mu_2}{2n(n+2)}TrS^{(k)}dt
\end{aligned}$$

and the proof is complete. □

Theorem A.3. *For* $n-1 \geq k \geq \ell \geq 1$,

$$\langle dTrS^{(k)}, dTrS^{(\ell)}\rangle = \frac{\mu_2}{n(n+2)}[(k\ell+2(k+\ell)+4)(n-1)$$
$$-2(k+2)(n-\ell-1)-2(\ell+2)(n-k-1)-4(n-\ell-1)(n-k-1)]TrS^{(k)}TrS^{(\ell)}dt$$
$$+(1-\delta^k_{n-1})\frac{4\mu_2}{(n+2)}\sum_{j=0}^{(n-k-1)\wedge\ell} d(n,(k,\ell);j)TrS^{(k+j)}TrS^{(\ell-j)}dt$$
$$+C^{nn}_{1122}\sum_{j=0}^{(n-k)\wedge(\ell-1)} a(n,(k,\ell);j)TrS^{(k+j-1)}TrS^{(\ell-j-1)}dt$$

where

$$d(n,(k,\ell);0) = (n-1-k)$$
$$d(n,(k,\ell);j) = (n-1-k-j)\binom{k-\ell+2j}{j} - \sum_{r=0}^{j-1} d(n,(k,\ell);r)\binom{k-\ell+2j}{j-r}\ ,j\geq 1$$
$$b(n,(k,\ell);0) = (n-k)(n-\ell+2)$$
$$b(n,(k,\ell);j) = 3(n-k-j)\binom{k-\ell+2j}{j} + (k-\ell+2j)(k-\ell+2j-1)\binom{k-\ell+2j-2}{j-1}$$
$$+(k-\ell+2j)(n-k-j)[\binom{k-\ell+2j-1}{j}+\binom{k-\ell+2j-1}{j-1}]$$
$$+(n-k-j)(n-k-j-1)\binom{k-\ell+2j}{j}, j\geq 1$$
$$a(n,(k,\ell);0) = b(n,(k,\ell);0)$$
$$a(n,(k,\ell);j) = b(n,(k,\ell);j) - \sum_{m=0}^{j-1} a(n,(k,\ell);m)\binom{k-\ell+2j}{j-m}$$
$$C^{nn}_{1122}(n^2-1) = \frac{(n+3)\mu_4}{n(n+2)(n+4)}.$$

Proof. We compute the quadratic variation terms in $\langle dTrS^{(k)}, dTrS^{(\ell)}\rangle$, $k\geq \ell$, which do not arise from dB. These are

$$\langle TrS^{(k)}[kdW^n_n - 2\sum_{i=1}^{n-1} dW^i_i] + S^{(k)}(\alpha_{\vec{\ell}},\alpha_{\vec{m}})(\langle\tau^j_h\alpha^{\vec{\ell}},\alpha^{\vec{m}}\rangle + \langle\alpha^{\vec{\ell}},\tau^j_h\alpha^{\vec{m}}\rangle)||\alpha||^{-2}dW^h_j,$$
$$TrS^{(\ell)}[\ell dW^n_n - 2\sum_{m=1}^{n-1} dW^m_m] + S^{(\ell)}(\alpha_{\vec{r}},\alpha_{\vec{s}})(\langle\tau^p_q\alpha^{\vec{r}},\alpha^{\vec{s}}\rangle + \langle\alpha^{\vec{r}},\tau^p_q\alpha^{\vec{s}}\rangle)||\alpha||^{-2}dW^q_p\rangle$$

unless $k=n-1$ or both $\ell=k=n-1$ in which case, the term $S^{(k)}(\alpha_{\vec{\ell}},\alpha_{\vec{m}})$ $(\langle\tau^j_i\alpha^{\vec{\ell}},\alpha^{\vec{m}}\rangle + \langle\alpha^{\vec{\ell}},\tau^j_i\alpha^{\vec{m}}\rangle)||\alpha||^{-2}dW^i_j = 0$ or both this and the corresponding $S^{(\ell)}$ term vanishes. Thus we get

$$TrS^{(k)}TrS^{(\ell)}\langle kdW^n_n - 2\sum_{i=1}^{n-1} dW^i_i, \ell dW^n_n - 2\sum_{j=1}^{n-1} dW^j_j\rangle$$

$$+TrS^{(k)}S^{(\ell)}(\alpha_{\vec r},\alpha_{\vec s})(\langle\tau^p_q\alpha^{\vec r},\alpha^{\vec s}\rangle+\langle\alpha^{\vec r},\tau^p_q\alpha^{\vec s}\rangle)||\alpha||^{-2}(1-\delta^\ell_{n-1})$$
$$\langle kdW^n_n-2\sum_{i=1}^{n-1}dW^i_i,dW^q_p\rangle$$
$$+TrS^{(\ell)}S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})(\langle\tau^j_h\alpha^{\vec \ell},\alpha^{\vec m}\rangle+\langle\alpha^{\vec \ell},\tau^j_h\alpha^{\vec m}\rangle)||\alpha||^{-2}(1-\delta^k_{n-1})$$
$$\langle \ell dW^n_n-2\sum_{i=1}^{n-1}dW^i_i,dW^h_j\rangle$$
$$+S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})S^{(\ell)}(\alpha_{\vec r},\alpha_{\vec s})(\langle\tau^j_h\alpha^{\vec \ell},\alpha^{\vec m}\rangle+\langle\alpha^{\vec \ell},\tau^j_h,\alpha^{\vec m}\rangle)(\langle\tau^p_q\alpha^{\vec r},\alpha^{\vec s}\rangle$$
$$+\langle\alpha^{\vec r},\tau^p_q\alpha^{\vec s}\rangle)(1-\delta^\ell_{n-1})$$
$$\cdot(1-\delta^k_{n-1})||\alpha||^{-4}\langle dW^h_j,dW^q_p\rangle$$
$$=\frac{\mu_2}{n(n+2)}TrS^{(k)}TrS^{(\ell)}[k\cdot\ell(n-1)+2(k+\ell)(n-1)+4(n-1)]dt$$
$$+\frac{\mu_2}{n(n+2)}TrS^{(k)}S^{(\ell)}(\alpha_{\vec r},\alpha_{\vec s})(\langle\tau^p_p\alpha^{\vec r},\alpha^{\vec s}\rangle+\langle\alpha^{\vec r},\tau^p_p\alpha^{\vec s}\rangle)||\alpha||^{-2}(1-\delta^\ell_{n-1})[kC^{np}_{np}-2\sum_{i=1}^{n-1}C^{ip}_{ip}]dt$$
$$+\frac{\mu_2}{n(n+2)}TrS^{(\ell)}S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})(\langle\tau^j_j\alpha^{\vec \ell},\alpha^{\vec m}\rangle+\langle\alpha^{\vec \ell},\tau^j_j\alpha^{\vec m}\rangle)||\alpha||^{-2}(1-\delta^k_{n-1})[\ell C^{nj}_{nj}-2\sum_{i=1}^{n-1}C^{ij}_{ij}]dt$$
$$+\frac{\mu_2}{n(n+2)}S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})S^{(k)}(\alpha_{\vec r},\alpha_{\vec s})(\langle\tau^j_h\alpha^{\vec \ell},\alpha^{\vec m}\rangle+\langle\alpha^{\vec \ell},\tau^j_h\alpha^{\vec m}\rangle)(\langle\tau^p_q\alpha^{\vec r},\alpha^{\vec s}\rangle$$
$$+\langle\alpha^{\vec r},\tau^p_q\alpha^{\vec s}\rangle)(1-\delta^k_{n-1})||\alpha||^{-4}[(n+1)\delta^h_q\delta^j_p-\delta^h_j\delta^q_p-\delta^h_p\delta^q_j]dt$$

We now proceed singly through the last three terms. Using the invariance of our expressions under a change of basis we may take the u_i to be the unit principal curvature directions. Then

$$\begin{aligned}
S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})&=\lambda_{\vec \ell}\delta^{\vec \ell}_{\vec m}\\
S^{(\ell)}(\alpha_{\vec r},\alpha_{\vec s})&=\lambda_{\vec r}\delta^{\vec r}_{\vec s}\\
\langle\tau^p_q\alpha^{\vec r},\alpha^{\vec s}\rangle\delta^{\vec r}_{\vec s}&=\delta^p_q\delta^{\vec r}_{\vec s}\ \text{ if and only if } p\in\vec r\ ,=0\quad\text{otherwise,}\\
\langle\tau^j_h\alpha^{\vec \ell},\alpha^m\rangle\delta^{\vec \ell}_{\vec m}&=\delta^j_h\delta^{\vec \ell}_{\vec m}\ \text{ if and only if } j\in\vec \ell,=0\quad\text{otherwise.}
\end{aligned}\tag{A.9}$$

Thus,

(A.10)
$$\frac{\mu_2}{n(n+2)}TrS^{(k)}S^{(\ell)}(\alpha_{\vec r},\alpha_{\vec s})(\langle\tau^p_p\alpha^{\vec r},\alpha^{\vec s}\rangle+\langle\alpha^{\vec r},\tau^p_p\alpha^{\vec s}\rangle)(1-\delta^\ell_{n-1})||\alpha||^{-2}[kC^{np}_{np}-2\sum_{i=1}^{n-1}C^{ip}_{ip}]dt$$
$$=\frac{\mu_2}{n(n+2)}TrS^{(k)}\lambda_{\vec r}\langle\tau^p_p\alpha^{\vec r},\alpha^{\vec r}\rangle(1-\delta^\ell_{n-1})[k(n\delta^n_p-\delta^n_n\delta^p_p)-2]dt$$
$$=-\frac{2(k+2)(n-\ell-1)\mu_2}{n(n+2)}TrS^{(k)}TrS^{(\ell)}dt.$$

Similarly,

(A.11)
$$\frac{\mu_2}{n(n+2)}TrS^{(\ell)}S^{(k)}(\alpha_{\vec \ell},\alpha_{\vec m})(\langle\tau^j_j\alpha^{\vec \ell},\alpha^{\vec m}\rangle+\langle\alpha^{\vec \ell},\tau^j_j\alpha^{\vec m}\rangle)||\alpha||^{-2}(1-\delta^k_{n-1})(\ell C^{nj}_{nj}-2\sum_{i=1}^{n-1}C^{ij}_{ij})dt$$
$$=\frac{-2(\ell+2)(n-k-1)\mu_2}{n(n+2)}TrS^{(k)}TrS^{(\ell)}dt.$$

Finally, the last term is

(A.12)

$$
\begin{aligned}
&(1-\delta^k_{n-1})\frac{4\mu_2}{n(n+2)}\lambda_{\vec{\ell}}\lambda_{\vec{r}}\langle\tau^j_h\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^p_q\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle(1-\delta^{\ell}_{n-1})[(n+1)\delta^h_q\delta^j_p-\delta^h_j\delta^q_p-\delta^h_p\delta^q_j]dt\\
=&(1-\delta^k_{n-1})\frac{4\mu_2}{n(n+2)}\lambda_{\vec{\ell}}\lambda_{\vec{r}}[(n+1)\langle\tau^j_h\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^j_h\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle-\langle\tau^j_j\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^p_p\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle\\
&-\langle\tau^j_h\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^h_j\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle]dt\\
=&(1-\delta^k_{n-1})\frac{4\mu_2}{n(n+2)}\lambda_{\vec{\ell}}\lambda_{\vec{r}}[n\langle\tau^j_j\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^j_j\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle-\langle\tau^j_j\alpha^{\vec{\ell}},\alpha^{\vec{\ell}}\rangle\langle\tau^p_p\alpha^{\vec{r}},\alpha^{\vec{r}}\rangle]dt\\
=&(1-\delta^k_{n-1})\frac{4\mu_2}{n(n+2)}\lambda_{\vec{\ell}}\lambda_{\vec{r}}[n\delta_{j\not\in\vec{\ell}}\delta_{j\not\in\vec{r}}-\delta_{j\not\in\vec{\ell}}\delta_{p\not\in\vec{r}}]dt\quad(\delta_{j\not\in\vec{\ell}}=\begin{cases}1, & \text{when } j\not\in\vec{\ell},\\ 0, & \text{otherwise}\end{cases})\\
=&(1-\delta^k_{n-1})(\frac{4n\mu_2}{n(n+2)}\lambda_{\vec{\ell}}\lambda_{\vec{r}}\cdot\delta_{j\not\in\vec{\ell}}\delta_{j\not\in\vec{r}}dt-\frac{4(n-k-1)(n-\ell-1\mu_2)}{n(n+2)}TrS^{(k)}TrS^{(\ell)}dt).
\end{aligned}
$$

However, by summing on j we see that

$$
\lambda_{\vec{\ell}}\lambda_{\vec{r}}\delta_{j\not\in\vec{\ell}}\delta_{j\not\in\vec{r}}=\sum_{\substack{m=0\\|\vec{\ell}^c\cap\vec{r}^c|=m}}^{n-\ell-1}m\lambda_{\vec{\ell}}\lambda_{\vec{r}}\tag{A.13}
$$

which is a symmetric homogeneous polynomial in $\lambda_1,\ldots,\lambda_{n-1}$ of degree $k+\ell$ so we seek an expansion for (A.13) of the form

$$
\sum_{j=0}^{(n-k-1)\wedge\ell}d(n,(k,\ell);j)TrS^{(k+j)}TrS^{(\ell-j)}.\tag{A.14}
$$

The principal term in $TrS^{(k+j)}TrS^{(\ell-j)}$ (which doesn't appear in $TrS^{(k+m)}TrS^{(\ell-m)}$ for $m>j$) is

$$
\lambda_1^2\ldots\lambda_{\ell-j}^2\lambda_{\ell-j+1}\ldots\lambda_{k+j}.
$$

In $\sum_{\substack{m=0\\|\vec{\ell}\cap\vec{r}|=m}}^{n-\ell-1}m\lambda_{\vec{\ell}}\lambda_{\vec{r}}$ this term appears $(n-1-k-j)\binom{k-\ell+2j}{j}$ times.

In $TrS^{(k+r)}TrS^{(\ell-r)}$ for $r<j$ this term appears $\binom{k-\ell+2j}{j-r}$ times. Thus in the expansion for $\sum_{\substack{m=0\\|\vec{\ell}^c\cap\vec{r}^c|=m}}^{n-\ell-1}m\lambda_{\vec{\ell}}\lambda_{\vec{r}}$

$$
\begin{aligned}
&d(n,(k,\ell);0)=(n-1-k)\\
&d(n,(k,\ell);j)=(n-1-k-j)\binom{k-\ell+2j}{j}-\sum_{r=0}^{j-1}d(n,(k,\ell);r)\binom{k-\ell+2j}{j-r},j\geq 1\,.
\end{aligned}
$$

Putting (A.10), (A.11) and (A.12) together results in

(A.15)

$$
\begin{aligned}
&[\frac{\mu_2}{n(n+2)}TrS^{(k)}TrS^{(\ell)}[(k\ell+2(k+\ell)+4)(n-1)-2(k+2)(n-\ell-1)\\
&-2(\ell+2)(n-k-1)-4(n-\ell-1)(n-k-1)]\\
&+(1-\delta^k_{n-1})\frac{4\mu_2}{n(n+2)}\sum_{j=0}^{(n-k-1)\wedge\ell}d(n,(k,\ell);j)TrS^{(k+j)}TrS^{(\ell-j)}]dt.
\end{aligned}
$$

The quadratic variation term in $\langle dTrS^{(k)}, dTrS^{(\ell)}\rangle, k \geq \ell$, arising from dB is,

$$
\begin{aligned}
&Q(n,(k,\ell))dt\\
&\quad = \langle \sum_{i,p=1}^{k} S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i})\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle \langle dB(u_{\ell_p}, u_{m_i}), \nu\rangle,\\
&\quad \sum_{j,q=1}^{\ell} S^{(\ell-1)}(\alpha_{\vec{r}_q}, \alpha_{\vec{s}_j})\langle \alpha^{\vec{r}}, \alpha^{\vec{s}}\rangle \langle dB(u_{r_q}, u_{s_j}), \nu\rangle\rangle ||\alpha||^{-4}\\
&\quad = C^{nn}_{1122} \sum_{i,p=1}^{k} \sum_{j,q=1}^{\ell} S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i}) S^{(\ell-1)}(\alpha_{\vec{r}_q}, \alpha_{\vec{s}_j})[\langle u_{\ell_p}, u_{m_i}\rangle \langle u_{r_q}, u_{s_j}\rangle\\
&\quad + \langle u_{\ell_p}, u_{r_q}\rangle\langle u_{m_i}, u_{s_j}\rangle + \langle u_{\ell_p}, u_{s_j}\rangle \langle u_{r_q}, u_{m_i}\rangle] \langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle \langle \alpha^{\vec{r}}, \alpha^{\vec{s}}\rangle ||\alpha||^{-4} dt
\end{aligned}
$$

and taking advantage of the invariance of this expression under a change of basis, we take u_i to be the ith unit principal direction of curvature. Then

$$
\begin{aligned}
S^{(k-1)}(\alpha_{\vec{\ell}_p}, \alpha_{\vec{m}_i})\langle \alpha^{\vec{\ell}}, \alpha^{\vec{m}}\rangle &= \lambda_{\vec{\ell}_p} \delta^{\vec{\ell}}_{\vec{m}} \delta^i_p\\
S^{\ell-1}(\alpha_{\vec{r}_q}, \alpha_{\vec{s}_j})\langle \alpha^{\vec{r}}, \alpha^{\vec{s}}\rangle &= \lambda_{\vec{r}_q} \delta^{\vec{r}}_{\vec{s}} \delta^j_q\\
[\langle u_{\ell_p}, u_{m_i}\rangle\langle u_{r_q}, u_{s_j}\rangle + \langle u_{\ell_p}, u_{r_q}\rangle\langle u_{m_i}, u_{s_j}\rangle + \langle u_{\ell_p}, u_{s_j}\rangle\langle u_{r_q}, u_{m_i}\rangle]\delta^{\vec{\ell}}_{\vec{m}}\delta^{\vec{r}}_{\vec{s}}&\\
= (\delta^{\ell_p}_{m_i}\delta^{r_q}_{s_j} + \delta^{\ell_p}_{r_q}\delta^{m_i}_{s_j} + \delta^{\ell_p}_{s_j}\delta^{r_q}_{m_i})\delta^{\vec{\ell}}_{\vec{m}}\delta^{\vec{r}}_{\vec{s}}.&
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Q(n,(k,\ell)) &= C^{nn}_{1122} \sum_{p=1}^{k}\sum_{q=1}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q}(1 + 2\delta^{\ell_p}_{r_q})dt\\
&= C^{nn}_{1122}(3 \sum_{p=1}^{k} \sum_{\substack{q=1\\ \ell_p = r_q}}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q} + \sum_{p=1}^{k} \sum_{\substack{q=1\\ \ell_p \neq r_q}}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q}).
\end{aligned}
$$

Since $Q(n,(k,\ell))$ is a symmetric homogeneous polynomial of degree $k+\ell-2$ in $\lambda_1, \dots, \lambda_{n-1}$, it may be written

$$
Q(n,(k,\ell)) = C^{nn}_{1122} \sum_{j=1}^{(n-k)\wedge(\ell-1)} a(n,(k,\ell);j) TrS^{(k+j-1)} TrS^{(\ell-j-1)}.
$$

We now employ some elementary counting arguments to compute the coefficients $a(n,(k,\ell);j)$.

In $TrS^{(k+j-1)}TrS^{(\ell-j-1)}$ the term

(A.16) $$\lambda_1^2 \dots \lambda_{\ell-j-1}^2 \lambda_{\ell-j} \dots \lambda_{k+j-1} \quad \text{appears}$$

but no such term (with $\ell - j - 1$ squares) appears in $TrS^{(k+m-1)}TrS^{(\ell-m-1)}$ for $m > j$. However, if $r < j$,

$$\lambda_1^2 \dots \lambda_{\ell-j-1}^2 \lambda_{\ell-j} \dots \lambda_{k+j-1} \quad \text{appears} \tag{A.17}$$

$\binom{k-\ell+2j}{j-r}$ times in $TrS^{(k+r-1)}TrS^{(\ell-r-1)}$.

We now count the number of appearances of (A.16) in $Q(n,(k,\ell))$. From $3\sum_{p=1}^{k}\sum_{\substack{q=1\\ \ell_p=r_q}}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q}$ we must fill in the empty spaces in $\lambda_{\vec{r}}$ and $\lambda_{\vec{\ell}}$

$$\begin{aligned} \lambda_{\vec{r}} &= \lambda_1 \dots \lambda_{\ell-j-1} - \cdots - \\ \lambda_{\vec{\ell}} &= \lambda_1 \dots \lambda_{\ell-j-1} - \cdots -_{\ell} \cdots -_k \end{aligned} \tag{A.18}$$

with the terms $\lambda_{\ell-j}, \dots, \lambda_{k+j-1}, \lambda_x, \lambda_x$, where $x = \ell_p = r_q$. Notice λ_t appears before λ_u only if $t < u$. One λ_x must go in $\lambda_{\vec{r}}$ the other in $\lambda_{\vec{\ell}}$. If the order is not to be violated, this forces $x > k - 1 + j$ and λ_x must end each of the terms $\lambda_{\vec{r}}$ and $\lambda_{\vec{\ell}}$. There are $(n-1) - (k-1+j) = n-k-j$ such choices for x. This leaves

$$\binom{k-1+j-(\ell-j-1)}{j} = \binom{k-\ell+2j}{j}$$

remaining possibilities for filling in the empty spaces in (A.18) in $\lambda_{\vec{r}}$ and $\lambda_{\vec{\ell}}$. Note that if we interpret $\binom{0}{0} = 1$ this is still correct for $k = \ell$ and $j = 0$. Thus

$$3\sum_{p=1}^{k}\sum_{\substack{q=1\\ \ell_p=r_q}}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q} \text{ has } 3(n-k-j)\binom{k-\ell+2j}{j} \tag{A.19}$$

terms of the form $\lambda_1^2 \dots \lambda_{\ell-j-1}^2 \lambda_{\ell-j} \dots \lambda_{k+j-1}$.

In $\sum_{p=1}^{k}\sum_{\substack{q=1\\ \ell_p\neq r_q}}^{\ell} \lambda_{\vec{\ell}_p}\lambda_{\vec{r}_q}$ we must fill in the blanks of

$$\begin{aligned} \lambda_{\vec{r}} &= \lambda_1 \dots \lambda_{\ell-j-1} - \cdots - \\ \lambda_{\vec{\ell}} &= \lambda_1 \dots \lambda_{\ell-j-1} - \cdots -_{\ell} \cdots -_k \end{aligned}$$

with $\lambda_{\ell-j}, \dots, \lambda_{k+j-1}, \lambda_x, \lambda_y$ with $x \neq y$, λ_x going into $\lambda_{\vec{r}}, \lambda_y$ going into $\lambda_{\vec{\ell}}$.

If $\ell - j \leq x, y \leq k + j - 1$, then λ_x and λ_y appear on the list $\lambda_{\ell-j}, \dots, \lambda_{k+j-1}$. Since identical terms can't appear twice in $\lambda_{\vec{r}}$ or $\lambda_{\vec{\ell}}$ the twin of λ_x must go in $\lambda_{\vec{\ell}}$, the twin of λ_y must go in $\lambda_{\vec{r}}$. There are thus $\binom{k-\ell+2j-2}{j-1}$ ways to fill in the spaces once λ_x and λ_y are given with $\ell - j \leq x, y \leq k + j - 1$. There are $(k + j - 1 - (\ell -$

$j-1))(k+j-1-(\ell-j-1)-1)$ choices for the ordered pair (x,y). Notice that when $j=0$ there are no such terms. Thus

$$\begin{aligned}&\ell-j\le x,y\le k+j-1 \text{ accounts for}\\ &(k-\ell+2j)(k-\ell+2j-1)\binom{k-\ell+2j-2}{j-1}\end{aligned} \tag{A.20}$$

terms of the form $\lambda_1^2\dots\lambda_{\ell-j-1}^2\lambda_{\ell-j}\dots\lambda_{k+j-1}$ in $Q(n,(k,\ell))$ (interpreting $\binom{z}{-1}=0$.)

If $\ell-j\le x\le k+j-1<y\le n-1$, then λ_y must terminate the sequence $\lambda_{\vec{\ell}}$ and the twin of λ_x must go into $\lambda_{\vec{\ell}}$. There are then $\begin{pmatrix}k-\ell+2j-1\\ j\end{pmatrix}$ ways to fill in the spaces in (A.18) once such a pair (x,y) is given. We interpret this expression as 0 when $k=\ell$ and $j=0$. There are $(k-\ell+2j)(n-k-j)$ such pairs so we have

$$\begin{aligned}&\text{corresponding to } \ell-j\le x\le k+j-1<x\le n-1\\ &\text{there are } (k-\ell+2j)(n-k-j)\begin{pmatrix}k-\ell+2j-1\\ j\end{pmatrix}\\ &\text{terms of the form } \lambda_1^2\cdots\lambda_{\ell-j-1}^2\lambda_{\ell-j}\cdots\lambda_{k+j-1} \text{ in}\\ &Q(n,(k,\ell))\ .\end{aligned} \tag{A.21}$$

If $\ell-j\le y\le k+j-1<x\le n-1$, then the twin of λ_y must go in $\lambda_{\vec{r}}$. When $j=0$, this is impossible. There are thus $\begin{pmatrix}k-\ell+2j-1\\ j-1\end{pmatrix}$ ways to fill the spaces in (A.18) once (x,y) is given and $(k-\ell+2j)(n-k-j)$ such pairs so

$$\begin{aligned}&\text{corresponding to } \ell-j\le y\le k+j-1<x\le n-1,\\ &\text{there are } (k-\ell+2j)(n-k-j)\begin{pmatrix}k-\ell+2j-1\\ j-1\end{pmatrix}\\ &\text{terms of the form } \lambda_1^2\cdots\lambda_{\ell-j-1}^2\lambda_{\ell-j}\cdots\lambda_{k+j-1} \text{ in}\\ &Q(n,(k,\ell)), \text{ for } j\ge 0, \text{ (interpreting } \tbinom{z}{-1}=0.)\end{aligned} \tag{A.22}$$

Finally, when $k+j-1<x,y\le n-1$, there are $\begin{pmatrix}k-\ell+2j\\ j\end{pmatrix}$ ways to fill in the spaces in (A.18) once (x,y) is given and $(n-k-j)(n-k-j-1)$ such pairs (x,y). Thus

$$\begin{aligned}&\text{corresponding to } k+j-1<x,y\le n-1, \text{ there are}\\ &(n-k-j)(n-k-j-1)\begin{pmatrix}k-\ell+2j\\ j\end{pmatrix} \text{ terms of the}\\ &\text{form } \lambda_1^2\cdots\lambda_{\ell-j-1}^2\lambda_{\ell-j}\cdots\lambda_{k+j-1} \text{ in } Q(n,(k,\ell)),\\ &\text{interpreting } \tbinom{0}{0}=1.\end{aligned} \tag{A.23}$$

Totalling (A.18)–(A.23), we see

$$
\text{(A.24)} \qquad Q(n,k) \text{ has}
$$

$$
\begin{aligned}
b(n,(k,\ell);j) &= 3(n-k-j)\binom{k-\ell+2j}{j} \\
&+ (k-\ell+2j)(k-\ell+2j-1)\binom{k-\ell+2j-2}{j-1} \\
&+ (k-\ell+2j)(n-k-j)\binom{k-\ell+2j-1}{j} \\
&+ (k-\ell+2j)(n-k-j)\binom{k-\ell+2j-1}{j-1} \\
&+ (n-k-j)(n-k-j-1)\binom{k-\ell+2j}{j}
\end{aligned}
$$

terms with exactly $\ell-j-1$ squares. Notice that we interpret $\binom{z}{-1}=0$, $\binom{-1}{0}=0$, $\binom{0}{0}=1$. Thus, $b(n,(k,\ell);0)=(n-k)(n-k-2)$. Upon decomposing $Q(n,(k,\ell))$ into sums of expressions with exactly $\ell-j-1$ squares summed over j, we are lead to, using (A.16) and (A.23),

$$
Q(n,k) = C^{nn}_{1122} \sum_{j=0}^{(n-k)\wedge(\ell-1)} a(n,(k,\ell);j) \text{ Tr } S^{(k+j-1)} \text{ Tr } S^{(\ell-j-1)}
$$

where

$$
\begin{aligned}
a(n,(k,\ell);j) &= b(n,(k,\ell);j) - \sum_{m=0}^{j-1}\binom{k-\ell+2j}{j-m} a(n,(k,\ell);m)\ , \\
a(n,(k,\ell);0) &= b(n,(k,\ell);0) = (n-k)(n-\ell+2)\ .
\end{aligned}
$$

From Theorems A.2 and A.3, we get immediately

Theorem A.4. *Suppose Φ_t is the isotropic, measure preserving flow on $\mathbb{R}^n$ given by (1.3). Let M be a smooth hypersurface in $\mathbb{R}^n$ with $x\in M$. If $(\lambda_1(t),\ldots,\lambda_{n-1}(t))$ are the principal curvatures of $\Phi_t(M)$ at $\Phi_t(x)$, then*

$$
Tr\ S^{(k)} = \sum_{\vec{\ell}\in I_k} \lambda_{\vec{\ell}}
$$

and $(Tr\ S^{(1)},\ Tr\ S^{(2)},\ldots,\ Tr\ S^{(n-1)})$ is an $(n-1)$-dimensional diffusion. The *generator is given by*

$$
\begin{aligned}
Lf(x_1,\dots,x_{n-1}) = \sum_{1\le \ell\le k\le n-1} & \left(\frac{1}{2}+\frac{1}{2}(1-\delta^k_\ell)\right) \\
& \left\{\frac{\mu_2}{n(n+2)}[((k\ell+2(k+\ell)+4)(n-1)\right. \\
& -2(k+2)(n-\ell-1)-2(\ell+2)(n-k-1) \\
& \quad -4(n-\ell-1)(n-k-1))x_k x_\ell \\
& + \left(1-\delta^k_{n-1}\right)4n \sum_{j=0}^{(n-k-1)\wedge\ell} d(n,(k,\ell);j)x_{k+j}x_{\ell-j}] \\
& \left. +C^{nn}_{1122}\sum_{j=0}^{(n-k)\wedge(\ell-1)} a(n,(k,\ell);j)x_{k+j-1}x_{\ell-j-1}\right\}\frac{\partial^2 f}{\partial x_k\partial x_\ell} \\
& + \sum_{k=1}^{n-1}\frac{(n+1)k(n-k))\mu_2}{2n(n+2)}\,x_k\,\frac{\partial f}{\partial x_k}
\end{aligned}
$$

where, for $k\ge\ell$

$$
\begin{aligned}
d(n,(k,\ell);0) &= (n-1-k) \\
d(n,(k,\ell);j) &= (n-1-k-j)\binom{k-\ell+2j}{j} - \sum_{r=0}^{j-1} d(n,(k,\ell);r)\binom{k-\ell+2j}{j-r}\ ,j\ge 1 \\
b(n,(k,\ell);j) &= 3(n-k-j)\binom{k-\ell+2j}{j} + (k-\ell+2j)(k-\ell+2j-1)\binom{k-\ell+2j-2}{j-1} \\
&\quad + (k-\ell+2j)(n-k-j)\left[\binom{k-\ell+2j-1}{j}+\binom{k-\ell+2j-1}{j-1}\right] \\
&\quad + (n-k-j)(n-k-j-1)\binom{k-\ell+2j}{j}\ ;\ j\ge 0\,,\ \text{with } \binom{z}{-1}=\binom{-1}{0},\ \binom{0}{0}=1, \\
a(n,(k,\ell);0) &= b(n,(k,\ell);0) = (n-k)(n-\ell+2), \\
a(n,(k,\ell);j) &= b(n,(k,\ell);j) - \sum_{r=0}^{j-1} a(n,(k,\ell);r)\binom{k-\ell+2j}{j-r}\ ,j>0 \\
C^{nn}_{1122} &= \frac{(n+3)\mu_4}{n(n+2)(n+4)}\ ,
\end{aligned}
$$

and $x_{-1}=x_0=1$.

Remarks.

(1) Straightforward computations show that for $n = 3$

$$
\begin{aligned}
k = \ell = 1; \qquad & d(3,(1,1);0) = 1 \\
& d(3,(1,1);1) = -2 \\
& a(3,(1,1);0) = 8
\end{aligned}
$$

$$
k = 2, \ell = 1; \qquad a(3,(2,1);0) = 4
$$

$$
\begin{aligned}
k = 2, \ell = 2; \qquad & a(3,(2,2);0) = 3 \\
& a(3,(2,2);1) = -4
\end{aligned}
$$

Thus, when M is a hypersurface in $\mathbb{R}^3$, the pair (Tr $S^{(1)}$, Tr $S^{(2)}$) is a diffusion with generator

$$
\begin{aligned}
Lf(\kappa, m) = {} & \frac{1}{2}\left(\frac{\mu_2}{15}(14\kappa^2 - 24m) + \frac{16\mu_4}{35}\right)\frac{\partial^2 f}{\partial \kappa^2}(\kappa, m) \\
& + \left(\frac{16\mu_2}{15}\kappa m + \frac{8\mu_4}{35}\kappa\right)\frac{\partial^2 f}{\partial \kappa \partial m}(\kappa, m) \\
& + \frac{1}{2}\left(\frac{32\mu_2}{15}m^2 + \frac{2\mu_4}{35}(3\kappa^2 - 4m)\right)\frac{\partial^2 f}{\partial m^2}(\kappa, m) \\
& + \frac{4\mu_2}{15}\kappa\frac{\partial f}{\partial \kappa}(\kappa, m) + \frac{4\mu_2}{15}mm\frac{\partial f}{\partial m}(\kappa, m)
\end{aligned}
$$

which agrees with Theorem 2.9.

(2) For $n = 4$, $a = \frac{\mu_2}{24}$, $b = C^{44}_{1122} = \frac{7\mu_4}{192}$,

$$
\begin{aligned}
Lf(x, y, z) = {} & \frac{1}{2}(a(19x^2 - 32y) + 15b)\frac{\partial^2 f}{\partial x^2} \\
& + \frac{1}{2}(a(44y^2 - 32xz) + b(8x^2 - 4y))\frac{\partial^2 f}{\partial y^2} \\
& + \frac{1}{2}(a75z^2 + b(3y^2 - 4xz))\frac{\partial^2 f}{\partial z^2} \\
& + (a(22xy - 48z) + b10x)\frac{\partial^2 f}{\partial x \partial y} + (a25xz + b5y)\frac{\partial^2 f}{\partial x \partial z} \\
& + (a50yz + b(4xy - 6z))\frac{\partial^2 f}{\partial y \partial z} \\
& + a\frac{15}{2}x\frac{\partial f}{\partial x} + a10y\frac{\partial f}{\partial y} + a\frac{15}{2}z\frac{\partial f}{\partial z}\ .
\end{aligned}
$$

(3) For any n, no proper subset of $\{\text{Tr } S^{(1)}, \dots, \text{ Tr } S^{(n-1)}\}$ can be a diffusion.

(4) The authors intend to resolve such issues as transience or recurrence for higher dimensional cases.
(5) Since $x = \lambda_1 + \lambda_2 + \lambda_3$, $y = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, $z = \lambda_1\lambda_2\lambda_3$, Newton's inequalities, Hardy-Littlewood- Polya (1973), imply $x^2 \geq 3y$, $y^2 \geq 3xz$ with equality if and only if $\lambda_1 = \lambda_2 = \lambda_3$. As in the case $n = 3$, the diffusion matrix degenerates on $\lambda_1 = \lambda_2 = \lambda_3$ and one expects that the process will not reach this set at positive times.