

The probability that planar loop-erased random walk uses a given edge*

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Abstract

We give a new proof of a result of Rick Kenyon that the probability that an edge in the middle of an $n \times n$ square is used in a loop-erased walk connecting opposite sides is of order $n^{-3/4}$. We, in fact, improve the result by showing that this estimate is correct up to multiplicative constants.

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1 Introduction

Loop-erased random walk is a process obtained by erasing loops from simple random walk. Although the process can be defined for arbitrary Markov chains, we will discuss the process only on the planar integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$. We start this paper by stating our main result which is an improvement of a result of Rick Kenyon [2].

Let

$$A_n = \{j + ik \in \mathbb{Z} + i\mathbb{Z} : -n < j < n + 1, -n < k < n\},$$
$$\partial A_n = \{z \in \mathbb{Z}^2 : \text{dist}(z, A_n) = 1\}.$$

Let \mathcal{K}_n denote the set of nearest neighbor paths $\omega = [\omega_0, \dots, \omega_k]$ with $\text{Re}[\omega_0] = -n$, $\text{Re}[\omega_k] = n + 1$ and $\{\omega_1, \dots, \omega_{k-1}\} \subset A_n$. We write $|\omega| = k$ for the number of steps, and let $p(\omega) = 4^{-|\omega|}$ be the simple random walk measure. Let

$$f(n) = \sum_{\omega \in \mathcal{K}_n} p(\omega).$$

It is known that $\lim_{n \rightarrow \infty} f(n) = c_1 \in (0, \infty)$ (see, e.g., [6, Proposition 8.1.3]), where the constant c_1 can be given in terms of the Green's function of Brownian motion on a domain bounded by a square.

A path in \mathcal{K}_n is a *self-avoiding walk (SAW)* if it does not visit any lattice point more than once. Let \mathcal{W}_n denote the set of SAWs $\eta = [\eta_0, \dots, \eta_k] \in \mathcal{K}_n$. For each $\omega \in \mathcal{K}_n$ there is a unique self-avoiding walk $L(\omega) \in \mathcal{W}_n$ obtained by chronological loop-erasing (see [6, Chapter 9] for appropriate definitions). The loop-erased measure $\hat{p}_n(\eta)$ is defined by

$$\hat{p}_n(\eta) = \sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} p(\omega).$$

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Note that

$$\sum_{\eta \in \mathcal{W}_n} \hat{p}_n(\eta) = f(n).$$

Let \mathcal{W}_n^+ denote the set of $\eta \in \mathcal{W}_n$ that contain the directed edge $[0, 1]$ and \mathcal{W}_n^- those that contain $[1, 0]$. Let $\mathcal{W}_n^* = \mathcal{W}_n^- \cup \mathcal{W}_n^+$ be the set of $\eta \in \mathcal{W}_n$ that contain the edge $[0, 1]$ in either direction. We write $a_n \asymp b_n$ to mean that a_n/b_n and b_n/a_n are uniformly bounded. In this paper we prove the following theorem.

Theorem 1.1. As $n \rightarrow \infty$,

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \asymp n^{-3/4}. \tag{1.1}$$

With a little more work, we could establish the existence of the limit

$$\lim_{n \rightarrow \infty} n^{3/4} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta),$$

but we will not do it here. Our estimate is developed further in [1] to show convergence in general simply connected domains, so for ease we will only prove (1.1) here. Our result is a strengthening of a result of Kenyon [2] who proved that

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \approx n^{-3/4}, \tag{1.2}$$

where \approx indicates that the logarithms of both sides are asymptotic. Actually, his proof shows more than this but it does not establish the up-to-constants result (1.1). His proof used the relationship between loop-erased walks and two other models, dimers and uniform spanning trees. Another proof of (1.2) was given by Masson [12] using the relationship between loop-erased walk and the Schramm-Loewner evolution (*SLE*). We do not need to make reference to any of these models in our proof of (1.1). There are two main steps.

- A combinatorial identity is proved which writes the left-hand side of (1.1) in terms of simple random walk quantities.
- The simple random walk quantities are estimated.

Our computation to obtain the exponent $3/4$ uses the Brownian loop measure to estimate the random walk loop measure. This is in the spirit of Kenyon's calculations [2] since the loop measure is closely related to the determinant of the Laplacian.

Although the proof is self-contained (other than some estimates for simple random walk) it does use a key idea from Kenyon's work as discussed in [3, Section 5.7]. For each random walk path ω , we let $J(\omega)$ be the number of times that the path crosses any edge of the form $[-ki, -ki + 1]$ or $[-ki + 1, -ki]$ where k is a positive integer. Let $q(\omega) = (-1)^{J(\omega)} p(\omega)$. Let $Y_+(\omega)$ denote the number of times that ω uses the directed edge $[0, 1]$, $Y_-(\omega)$ the number of times that ω uses the directed edge $[1, 0]$, and $Y(\omega) = Y_+(\omega) - Y_-(\omega)$. The combinatorial identity is obtained by writing the quantity

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega) Y(\omega) = \sum_{\omega \in \mathcal{K}_n} p(\omega) (-1)^{J(\omega)} Y(\omega). \tag{1.3}$$

in two different ways.

The paper is written using the perspective of loop-erased walk in terms of the random walk loop measure as in [6, Chapter 9]. We start by reviewing this perspective in Section 2 and then we prove the identity in Section 3. Section 4 discusses the random walk estimates. One of the main motivations for doing the estimates in this paper is to

show that the loop-erased random walk converges to SLE_2 in the *natural parametrization* [7, 9]. Up-to-constant estimates for the loop-erased walk probability can be viewed as a step in the program to establish this result.

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2 Random walk loop measure

The *random walk loop measure* is a measure on unrooted random walk loops. A *rooted loop* is a nearest neighbor path

$$l = [l_0, l_1, \dots, l_{2k}] \tag{2.1}$$

with $k \geq 0$ and $l_0 = l_{2k}$. We call l_0 the root of the loop and write $|l| = 2k$ for the number of steps. An *unrooted loop* \bar{l} is an equivalence class of rooted loops with $k > 0$ under the equivalence relation

$$[l_j, l_{j+1}, \dots, l_{2k}, l_1, l_2, \dots, l_j] \sim [l_0, l_1, \dots, l_{2k}]$$

for all j . Note that the orientation of the loops is maintained. The random walk loop measure m is defined by

$$m(\bar{l}) = 4^{-|\bar{l}|} \frac{d(\bar{l})}{|\bar{l}|},$$

where $|\bar{l}| = 2k$ is the number of steps and $d(\bar{l})$ is the number of rooted loops in the equivalence class of the unrooted loop \bar{l} . Note that $d(\bar{l})$ is always an integer dividing $|\bar{l}|$. In a slight abuse of notation, if l is a loop and $A \subset \mathbb{Z}^2$, we write $l \subset A$ to mean that the vertices of l are contained in A and $l \cap A$ for the set of vertices in A that l visits.

One way to get the random walk loop measure is to consider the measure on *rooted* loops that assigns measure

$$\hat{m}(l) = \frac{p(l)}{|l|} = \frac{1}{4^{|l|} |l|}, \tag{2.2}$$

to each rooted loop. We then write

$$m(\bar{l}) = \sum_l \hat{m}(l),$$

where the sum is over all rooted loops l that are representatives of \bar{l} . We can view \hat{m} as the measure on rooted loops obtained from m by assigning the root uniformly over all vertices.

There is an equivalent way of defining this measure that we will also use. Enumerate $\mathbb{Z}^2 = \{v_1, v_2, \dots\}$ and let $V_n = \{v_1, \dots, v_n\}$. We define a different measure on rooted loops by assigning to each unrooted loop a rooted loop by choosing uniformly over all visits to the vertex of highest index in the loop. More precisely, we define for each (rooted) loop as in (2.1) with $k > 0, l \subset V_n, l \not\subset V_{n-1}$ measure $s^{-1} 4^{-2k}$ where $s = \#\{j : 1 \leq j \leq 2k, l_j = v_n\}$. This induces a measure on unrooted loops by summing over rooted loops that generate an unrooted loop. One can check that the induced measure on unrooted loops is the same as the loop measure above regardless of which enumeration of \mathbb{Z}^2 is chosen. (The factor s^{-1} compensates for the fact that several rooted loops give the same unrooted loop.) We will use an enumeration in which $|v_j|$ is nondecreasing.

If $V = \{v_1, \dots, v_k\} \subset A \subsetneq \mathbb{Z}^2$, we define

$$F_V(A) = \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) \right\} = \prod_{j=1}^k G_{U_j}(v_j, v_j). \tag{2.3}$$

Here $U_j = A \setminus \{v_1, \dots, v_{j-1}\}$ and G_U denotes the usual random walk Green's function in the set U . That is, $G_U(v, v)$ is the expected number of visits to v of a simple random walk starting at v before leaving U . It is well known that we can write

$$G_U(v, v) = \sum_l p(l) = \left[1 - \sum_l p(l) \right]^{-1}, \tag{2.4}$$

where the first sum is over all loops with $|l| \geq 0$ rooted at v that stay in U and the second sum is over all such loops with $|l| > 0$ that return only once to v .

The second equality in (2.3) is obtained by associating to each unrooted loop, a rooted loop rooted at the vertex v_j of smallest index. If there are multiple choices, that is, if the loop visits the vertex of smallest index multiple times, the root is chosen uniformly over the possibilities. See [6, Proposition 9.3.1, Proposition 9.3.2] for more details. The loop-erased measure satisfies [6, Proposition 9.5.1]

$$\hat{p}_n(\eta) = p(\eta) F_\eta(A_n). \tag{2.5}$$

We can also define a loop measure m^q using the signed weight $q(\omega) = (-1)^{J(\omega)} p(\omega)$. In other words, the measure on paths is

$$q[\omega_0, \dots, \omega_n] = \prod_{j=1}^n q(\omega_{j-1}, \omega_j) = \prod_{j=1}^n (-1)^{J([\omega_{j-1}, \omega_j])} = (-1)^{J(\omega)} p(\omega).$$

The definition is the same as m except replacing p with q in (2.2). The quantities $J(l), Y(l)$ as defined in the introduction are functions of the unrooted loop \bar{l} . Note that $J(l)$ is just the winding number of the loop around $\frac{1}{2} - \frac{i}{2}$. Also note that $Y(l)$ does depend on the orientation of l , so it is important that we are considering oriented, unrooted loops. Let \mathcal{J}_A denote the set of unrooted loops $\bar{l} \subset A$ such that $J(\bar{l})$ is odd. If $V \subset A$, let $\mathcal{J}_{A,V}$ denote the set of unrooted loops $\bar{l} \in \mathcal{J}_A$ that intersect V . Let

$$\begin{aligned} Q_V(A) &= \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m^q(\bar{l}) \right\} \\ &= \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) (-1)^{J(\bar{l})} \right\} \\ &= \exp \left\{ \sum_{\bar{l} \subset A, \bar{l} \cap V \neq \emptyset} m(\bar{l}) - 2 \sum_{\bar{l} \in \mathcal{J}_{A,V}} m(\bar{l}) \right\} = F_V(A) \exp \{-2m(\mathcal{J}_{A,V})\}. \end{aligned}$$

As in the case for F , if $V = \{v_1, \dots, v_k\} \subset A$, then by associating to each unrooted loop a rooted loop of smallest index, we get

$$Q_V(A) = \prod_{j=1}^k g_{U_j}(v_j, v_j). \tag{2.6}$$

Here $U_j = A \setminus \{v_1, \dots, v_{j-1}\}$ and

$$g_U(v_j, v_j) = \sum_l q(l) = \sum_l (-1)^{J(l)} p(l)$$

where the sum is over all (rooted) loops l from v_j to v_j staying in U . In particular, if $\eta \in \mathcal{W}_n$, then when the algebraic computation which gives (2.5) is applied to q , we get

$$\sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} q(\omega) = q(\eta) Q_\eta(A_n).$$

This implies that

$$\sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} (-1)^{J(\omega) - J(\eta)} p(\omega) = p(\eta) Q_\eta(A_n).$$

As in (2.4), we can write $g_{A_n}(0, 0) = [1 - s_n]^{-1}$ where

$$s_n = \sum_l q(l) = \sum_l p(l) (-1)^{J(l)}.$$

and the sum is over all loops l in A_n rooted at 0 with $|l| = 1$ and having no other returns to the origin. Since simple random walk in two dimensions is recurrent, we can see that $s_n \rightarrow s$ where $s = \mathbb{E}[J']$ where $J' = (-1)^{J(S[0, T_0])}$, S is a simple random walk starting at the the origin, and $T_0 = \min\{j \geq 1 : S_j = 0\}$. Since $\mathbb{P}\{J' = 1\} > 0$ and $\mathbb{P}\{J' = -1\} > 0$, we see that $|s| < 1$ and hence $0 < (1 - s)^{-1} < \infty$. Using (2.6) with $V = \{0\}$, we get

$$\lim_{n \rightarrow \infty} Q_V(A_n) = \lim_{n \rightarrow \infty} g_{A_n}(0, 0) = \lim_{n \rightarrow \infty} (1 - s_n)^{-1} = (1 - s)^{-1} > 0.$$

A similar argument shows that if $\mathbb{Z}^2 \setminus U$ is finite and non-empty, and v is in the unbounded component of U , then $g_U(v, v)$ is finite and strictly positive. Given this and (2.6), it is straightforward to show that if V is finite, then

$$Q_V = Q_V(\mathbb{Z}^2) = \lim_{n \rightarrow \infty} Q_V(A_n) \tag{2.7}$$

exists and is strictly positive. We will only use this with $V = \{0, 1\}$ and we write $Q_{01}(A_n)$ for $Q_{\{0,1\}}(A_n)$

For the important computation of the random walk loop measure, we will use the Brownian loop measure as introduced in [11] which was shown to be the scaling limit of the random walk loop measure in [10]. We discuss the nature of the limit in Section 4. Consider the measure \hat{m} on rooted loops as in (2.2). We will write this in an equivalent way. For each rooted loop $l = [l_0, \dots, l_{2n}]$, we associate a triple (z, n, l') where $z = l_0$, $|l| = 2n$ and l' is a loop of time duration $2n$ rooted at the origin. Then \hat{m} is the same as the measure on triples (z, n, l') where z is chosen according to counting measure, n is chosen according to the measure $\lambda_n = \mathbb{P}\{S_{2n} = 0\}/(2n)$ and given n , l' is chosen according to the probability measure on random walk starting at the origin conditioned to return to the origiun at time n . As a scaling limit we consider the measure on continuous loops $\gamma(t), 0 \leq s \leq t_\gamma$. We write each such loop as a triple (z, t_γ, γ') where z is the root, t_γ is the time duration, and γ' is a Brownian bridge of time duration t_γ rooted at the origin. The bridge distribution can be gotten from the bridges of time duration 1 by scaling. Using $\mathbb{P}\{S_{2n} = 0\} \sim (\pi n)^{-1}$, we see that the candidate for the scaling limit is

$$\text{area} \times \left(\frac{1}{2\pi t^2} dt \right) \times (\text{Brownian bridge}). \tag{2.8}$$

(Here we use the fact that a random walk of time duration $2n$ corresponds to a Brownian motion of time duration n .) This is the definition of the Brownian loop measure in the plane, and the loop measure in bounded domains is obtained by restriction. The important fact is that the Brownian loop measure, considered as a measure on unrooted loops, is conformally invariant, that is, if μ_D denotes the Brownian loop measure restricted to a domain D and $f : D \rightarrow f(D)$ is a conformal transformation, then the image of μ_D is the same as $\mu_{f(D)}$. (If we want a measure on parametrized loops, then we must change the parametrization of the loops as usual for Brownian motion.)

We will be interested in the set of Brownian loops contained in the disk $\{|z| < e^s\}$ that are not contained in $\{|z| < e^{s-r}\}$ and have odd winding number about the origin. Conformal invariance of the loop measure implies that the Brownian loop measure of

this set of loops depends only on r , say $\phi(r)$. It is also finite, nonzero, increasing in r and satisfies $\phi(r + s) = \phi(r) + \phi(s)$. In particular, $\phi(r) = \alpha r$ for some α . We will show that $\alpha = 1/8$ by using a different form of the loop measure.

For computations, it is more convenient to use the Brownian (boundary) bubble measure which we now describe. Let us first consider bubbles rooted at the origin in the upper half plane \mathbb{H} . It is the limit as $\epsilon \downarrow 0$ of a measure on paths from $i\epsilon$ to 0 in \mathbb{H} which we now describe. Let $H_{\mathbb{H}}(z, x) = |\operatorname{Im}(z)|/|z|^2$ be π times the usual Poisson kernel in the upper half plane. In other words, the probability that a Brownian motion starting at z exits \mathbb{H} at an interval $I \subset \mathbb{R}$ is

$$\frac{1}{\pi} \int_I H_{\mathbb{H}}(z, x) dx.$$

For each ϵ consider the measure of total mass $1/\epsilon = H_{\mathbb{H}}(\epsilon i, 0)$ on paths whose normalized probability measure is that of a Brownian h -process to 0. (An h -process can be viewed roughly as a Brownian motion conditioned to leave \mathbb{H} at 0.) As $\epsilon \downarrow 0$, the limit measure is a σ -finite measure $\nu_{\mathbb{H}}(0)$ on loops from 0 to 0 and otherwise in \mathbb{H} . (More precisely, for each $r > 0$, we consider the limit of the measure restricted to loops that reach the disk of radius r . This defines the bubble measure restricted to loop that reach the disk of radius r . The limit measure is a concatenation of an excursion in $r\mathbb{D}^+ := r(\mathbb{D} \cap \mathbb{H})$ from 0 to $\zeta \in \partial(r\mathbb{D}^+) \cap \mathbb{H}$ with a path in \mathbb{H} from ζ to 0. See [5, Section 5.5].) The normalization is such that the measure of bubbles that hit the unit circle equals one as can be seen by noting that the measure of this set is given by

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^\pi [\pi^{-1} H_{\mathbb{D}^+}(i\epsilon, e^{i\theta})][\pi^{-1} H_{\mathbb{H}}(e^{i\theta}, 0)] d\theta = \int_0^\pi [(2/\pi) \sin \theta] [(1/\pi) \sin \theta] d\theta = 1.$$

This definition can be extended to simply connected domains with smooth boundaries either by the analogous definition or by the following conformal covariance rule: if $f : \mathbb{H} \rightarrow D$ is a conformal transformation, then

$$f \circ \nu_{\mathbb{H}}(0) = |f'(0)|^2 \nu_D(f(0)).$$

(In the definition of $f \circ \nu_{\mathbb{H}}(0)$, we need to modify the parametrization of the curve using Brownian scaling, but the parametrization is not important in this paper.)

For each unrooted Brownian loop, we can focus on the (unique except for an set of loops of zero measure) point z of minimal imaginary part. The corresponding rooted loop is a “bubble” in the domain $\mathbb{H} + z$ rooted at z . Using this, an alternative expression for the loop measure (on unrooted loops) is

$$\frac{1}{\pi} \int_{\mathbb{C}} \nu_{\mathbb{H}+z}(s) dA(s).$$

There is a similar expression that one can obtain for the half-infinite cylinder obtained from the equivalence relation $z \sim z + 2\pi$ for all $z \in \mathbb{H}$. The Brownian bubble measure $\nu_{\mathbb{H}}(0)$ is replaced by $\tilde{\nu}_{\mathbb{H}}(0)$ which is the limit of the measures of total mass

$$\sum_{k=-\infty}^{\infty} H_{\mathbb{H}}(\epsilon i, 2\pi k),$$

whose probability measure is that of an h -process conditioned to leave \mathbb{H} at $\{2\pi k : k \in \mathbb{Z}\}$. Again, to take the limit for each $r \leq 1$ we restrict to loops of diameter at least r . Then the Brownian loop measure on the half-infinite cylinder can be written as

$$\frac{1}{\pi} \int_0^\infty \int_0^{2\pi} \tilde{\nu}_{\mathbb{H}+iy}(x + iy) dx dy.$$

This could also be written in the form (2.8), but the roots would be restricted to points in the cylinder (or, equivalently, to $0 \leq \operatorname{Re}(z) < 2\pi$) and the term $1/(2\pi t^2)$ is replaced by

$$\frac{1}{2\pi t^2} \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{(2\pi k)^2}{2t}\right\}.$$

The half-infinite cylinder is conformally equivalent to $\mathbb{D} \setminus \{0\}$ and using this we get the following description for the Brownian loop measure restricted to curves in the unit disk \mathbb{D} :

$$\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \nu_{r\mathbb{D}}(re^{i\theta}) r \, d\theta \, dr, \tag{2.9}$$

To be more precise, the loop measure is the measure on *unrooted* loops induced by the above measure on rooted loops. (This representation of the measure on unrooted loops focuses on the rooted representative with root as far from the origin as possible.) The Brownian loop measure is the scaling limit of the random walk loop measure in a sense made precise in [10]. We discuss this more in Section 4.

3 A combinatorial identity

Let \mathcal{K}'_n denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \dots, \omega_k]$ with $\operatorname{Re}[\omega_0] = -n, \omega_k = 0$ and $\{\omega_1, \dots, \omega_{k-1}\} \subset A_n \setminus [0, \infty)$. Let \mathcal{K}''_n denote the set of nearest neighbor paths $\omega = [\omega_0, \omega_1, \dots, \omega_k]$ with $\operatorname{Re}[\omega_0] = n + 1, \omega_k = 1$ and $\{\omega_1, \dots, \omega_{k-1}\} \subset A_n \setminus (-\infty, 1]$. There is a natural bijection between \mathcal{K}'_n and \mathcal{K}''_n obtained by reflection about the line $\{\operatorname{Re}(z) = 1/2\}$. Let

$$R_n = \sum_{\omega \in \mathcal{K}'_n} p(\omega) = \sum_{\omega \in \mathcal{K}''_n} p(\omega).$$

By considering the reversed path, we can see that $R_n = \mathbb{P}\{\operatorname{Re}(S_\tau) = -n\}$ where S is a simple random walk starting at the origin and $\tau = \min\{j > 0 : S_j \in \partial A_n \cup [0, \infty)\}$. It is known (see e.g., [6, Proposition 5.3.2]) that

$$R_n \asymp n^{-1/2}, \quad n \rightarrow \infty. \tag{3.1}$$

The goal of this section is to prove the following combinatorial identity which relates the probability that loop-erased walk uses the undirected edge $\{0, 1\}$ to some simple random walk quantities.

Theorem 3.1.

$$4 \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) = Q_{01}(A_n) R_n^2 \exp\{2m(\mathcal{J}_{A_n})\}.$$

Proof. We start by making the following topological observation:

$$(-1)^{J(\eta)} Y(\eta) = 1 \quad \text{if } \eta \in \mathcal{W}_n^*. \tag{3.2}$$

To see this, consider the path η as a continuous path from $\{\operatorname{Re}(z) = -n\}$ to $\{\operatorname{Re}(z) = n + 1\}$ in the domain $D = \{x + iy \in \mathbb{C} : -n < x < n + 1, -n < y < n\}$. Then η is a crosscut of D such that $D \setminus \eta$ consists of two components, the “top” component D^+ and the “bottom” component D^- . Each ordered edge $[w, w']$ in η can be considered as subsets of ∂D^+ and ∂D^- . If we traverse the edge from w to w' , the left-hand side of $[w, w']$ (considered as a prime end) is in ∂D^+ and the right-hand side is in ∂D^- . Let N_+ be the set of integers k such that the ordered edge $[ki, ki + 1]$ is contained in η , N_- the set of integers k such that the ordered edge $[ki + 1, ki]$ is contained in η , and $N = N_+ \cup N_-$. We claim that if $j \in N_+$ and k is the largest integer less than j with

$k \in N$, then $k \in N_-$. Indeed, since $j \in N_+$, the open line segment from $ji + (1/2)$ to $ki + (1/2)$ is contained in D^- which implies that $k \in N_-$. We now consider the smallest k such that $k \in N$. The line segment from $-ni + (1/2)$ to $ki + (1/2)$ is contained in D^- and hence $k \in N_+$. As we continue up the line $\{\text{Re}(z) = 1/2\}$ we see that when we intersect edges in η , they alternate being in N_+ or N_- , with the first in N_+ , the second in N_- , the third in N_+ , etc. When we reach the unordered edge $\{0, 1\}$, we see that if $0 \in N_+$, then there have been an even number of edges before $\{0, 1\}$ and if $0 \in N_-$, there have been an odd number of edges. In other words, $(-1)^{J(\eta)} = 1$ if $\eta \in \mathcal{W}_n^+$ and $(-1)^{J(\eta)} = -1$ if $\eta \in \mathcal{W}_n^-$. This gives (3.2).

Let Λ_n be defined as in (1.3). We claim that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega) Y(L(\omega)) = \sum_{\eta \in \mathcal{W}_n^*} \sum_{L(\omega)=\eta} p(\omega) (-1)^{J(\omega)-J(\eta)}. \tag{3.3}$$

To see this, suppose that $L(\omega) = \eta = [\eta_0, \dots, \eta_k]$. Then we can write ω uniquely as

$$\omega = [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \oplus \dots \oplus [\eta_{k-2}, \eta_{k-1}] \oplus l_{k-1} \oplus [\eta_{k-1}, \eta_k],$$

where l_j is a loop rooted at η_j that does not enter $\{\eta_1, \dots, \eta_{j-1}\}$. We write

$$J(\omega) = J(\eta) + J_L(\omega), \quad Y(\omega) = Y(\eta) + Y_L(\omega),$$

where J_L, Y_L denote the contributions from the loops. Then

$$Y(\omega) = Y(\eta) + \sum_{j=1}^{k-1} Y(l_j).$$

For each loop l_j there is the corresponding reversed loop l_j^R for which $Y(l_j^R) = -Y(l_j)$. Since $J(l_j^R) = J(l_j)$ and $Y(l_j^R) = -Y(l_j)$, we get cancellation. Doing this for all the loops, we see that

$$\sum_{\omega \in \mathcal{K}_n, L(\omega)=\eta} q(\omega) [Y(\omega) - Y(\eta)] = 0.$$

This gives the first equality in (3.3). The second equality uses (3.2) and the fact that $Y(\eta) = 0$ if $\eta \notin \mathcal{W}_n^*$.

If $\eta \in \mathcal{W}_n^*$, then

$$\begin{aligned} \sum_{L(\omega)=\eta} p(\omega) (-1)^{J(\omega)-J(\eta)} &= p(\eta) \sum_{L(\omega)=\eta} \frac{p(\omega)}{p(\eta)} (-1)^{J_L(\omega)} \\ &= p(\eta) Q_\eta(A_n) \\ &= p(\eta) \exp \left\{ \sum_{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset} (-1)^{J(\bar{l})} m(\bar{l}) \right\} \\ &= p(\eta) F_\eta(A_n) \exp \left\{ -2 \sum_{\substack{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset, \\ J(\bar{l}) \text{ odd}}} m(\bar{l}) \right\}. \end{aligned}$$

If $J(\bar{l})$ is odd, then \bar{l} must include at least one unordered edge $\{ki, ki + 1\}$ with $k \geq 0$ and at least one unordered edge $\{ki, ki + 1\}$ with $k < 0$. Therefore, topological considerations imply that if $\eta \in \mathcal{W}_n^*$, then $\eta \cap \bar{l} \neq \emptyset$. Hence

$$\sum_{\bar{l} \subset A_n, \bar{l} \cap \eta \neq \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) = \sum_{\bar{l} \subset A_n, J(\bar{l}) \text{ odd}} m(\bar{l}) = m(\mathcal{J}_{A_n}).$$

Combining this with (3.3), we see that

$$\Lambda_n = \sum_{\eta \in \mathcal{W}_n^*} p(\eta) Q_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} p(\eta) F_\eta(A_n) = e^{-2m(\mathcal{J}_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta). \quad (3.4)$$

We will now compute Λ_n as defined in (1.3) in a different way. Let $\omega = [\omega_0, \dots, \omega_\tau] \in \mathcal{K}_n$. If ω does not visit 0 or ω does not visit 1, then $Y(\omega) = 0$. Hence, we only need to consider the sum over $\omega \in \mathcal{K}_n$ that visit both 0 and 1. For such ω , we define $T_0 = \min\{j : \omega_j = 0\}$, $T'_0 = \max\{j < \tau : \omega_j = 0\}$, and we define T_1, T'_1 similarly.

Suppose that $T_0 < T_1, T'_0 > T'_1$. In this case we write

$$\omega = \omega^- \oplus l \oplus \omega^+, \quad (3.5)$$

where l is the loop $[\omega_{T_0}, \dots, \omega_{T'_0}]$. Note that $Y(\omega) = Y(l)$. For any such loop l , there is the corresponding reversed loop $l^R = [\omega_{T'_0}, \omega_{T'_0-1}, \dots, \omega_{T_0}]$ for which $Y(l^R) = -Y(l)$. These terms cancel and hence the sum in (1.3) over ω with $T_0 < T_1, T'_0 > T'_1$ is zero. Similarly, the sum over ω with $T_1 < T_0 \leq T'_0 < T'_1$ is zero.

Suppose that $T_0 > T_1, T'_0 > T'_1$. Then we can write ω uniquely as

$$\omega = \omega^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+,$$

with the following conditions. Here l_0 is a loop in A_n rooted at 0, l_1 is a loop in $A_n \setminus \{0\}$ rooted at 1, ω' is a path from 1 to 0 whose other vertices are in $A_n \setminus \{0, 1\}$, ω^- is a path from $\{\operatorname{Re}(z) = -n\}$ to 1 whose other vertices are in $A_n \setminus \{0, 1\}$, and ω^+ is a path from 0 to $\{\operatorname{Re}(z) = n + 1\}$ whose other vertices are in $A_n \setminus \{0, 1\}$. Let $\tilde{\omega}^-$ be the reflection of ω^- about the real axis, and $\tilde{\omega} = \tilde{\omega}^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+$. Then $J(\omega^-) + J(\tilde{\omega}^-)$, and hence $J(\omega) + J(\tilde{\omega})$, are odd and these terms will cancel in the sum. Hence the sum over all ω with $T_0 > T_1, T'_0 > T'_1$ is zero.

Let \mathcal{K}_n^1 be the set of paths in \mathcal{K}_n that visit both 0 and 1 and satisfy $T_0 < T_1, T'_0 < T'_1$. We have shown that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^1} q(\omega) Y(\omega).$$

If $\omega \in \mathcal{K}_n^1$, let $\rho = \min\{j > T'_0 : \omega_j = 1\}$. Then we can write ω uniquely as

$$\omega = \omega^- \oplus l_0 \oplus \omega' \oplus l_1 \oplus \omega^+. \quad (3.6)$$

Here $l_0 = [\omega_{T_0}, \dots, \omega_{T'_0}]$ is a loop in A_n rooted at 0, $l_1 = [\omega_\rho, \dots, \omega_{T'_1}]$ is a loop in $A_n \setminus \{0\}$ rooted at 1, $\omega' = [\omega_{T'_0}, \dots, \omega_\rho]$ is a path from 0 to 1, $\omega^- = [\omega_0, \dots, \omega_{T_0}]$ is a path from $\{\operatorname{Re}(z) = -n\}$ to 0, $\omega^+ = [\omega_{T'_1}, \dots, \omega_\tau]$ is a path from 1 to $\{\operatorname{Re}(z) = n + 1\}$. All of the vertices of $\omega', \omega^-, \omega^+$ other than the endpoints are in $A_n \setminus \{0, 1\}$. Note that $Y(\omega) = Y(l_0) + Y(\omega')$. As in the previous arguments, we can replace l_0 with the reversed loop l_0^R , to see that

$$\sum_{\omega \in \mathcal{K}_n^1} (-1)^{J(\omega)} Y(l_0) p(\omega) = 0.$$

Also $Y(\omega') \in \{0, 1\}$ with $Y(\omega') = 1$ if and only if $T'_0 + 1 = \rho$, that is, if $\omega' = [0, 1]$. Therefore, if \mathcal{K}_n^2 denotes the set of paths in \mathcal{K}_n^1 with $\omega' = [0, 1]$, then

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega)} p(\omega) = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega^-) + J(l_0) + J(l_1) + J(\omega^+)} p(\omega). \quad (3.7)$$

If $\omega \in \mathcal{K}_n^2$, let ξ be the smallest j such that ω_j is on the positive real axis. Suppose for the moment that $\xi < T_0$. Then we can write

$$\omega^- = \omega^{-,1} \oplus \omega^{-,2},$$

by splitting the path at time ξ . The path $\omega^{-,2}$ is a path from the positive real axis to 0 that does not go through the point 1. Hence, $J(\omega^{-,2}) + J(\tilde{\omega}^{-,2})$ is odd, where $\tilde{\omega}^{-,2}$ is the reflection of $\omega^{-,2}$ about the real axis. These terms will cancel in the sum (3.7), and hence it suffices to sum over ω^- such that $\omega^- \cap [1, \infty) = \emptyset$. For these ω^- , we can see by topological reasons that $(-1)^{J(\omega^-)} = 1$. By a similar argument, it suffices to sum over ω^+ satisfying $\omega^+ \cap (-\infty, 0] = \emptyset$, and for these walks $(-1)^{J(\omega^+)} = 1$. Therefore, if \mathcal{K}_n^3 denote the set of paths in \mathcal{K}_n^2 satisfying

$$\omega^- \cap [1, \infty) = \emptyset, \quad \omega^+ \cap (-\infty, 0] = \emptyset,$$

we see that

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} p(\omega).$$

Let us write any $\omega \in \mathcal{K}_n^3$ as in (3.6). We must choose $\omega^- \in \mathcal{K}'_n, (\omega^+)^R \in \mathcal{K}''_n$ and $\omega' = [0, 1]$. Summing over all of these possibilities, gives a factor of $R_n^2/4$. The choices of l_0, l_1 are independent of the choices of ω^- and ω^+ . The only restriction is that the loops lie in A_n and l_1 does not contain the origin. By our definition,

$$\sum_{l_0, l_1} (-1)^{J(l_0) + J(l_1)} p(l_0) p(l_1) = g_{A_n}(0, 0) g_{A_n \setminus \{0\}}(1, 1) = Q_{01}(A_n).$$

Therefore,

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} p(\omega) = \frac{1}{4} R_n^2 Q_{01}(A_n).$$

Comparing this with (3.4) gives the theorem. □

4 Estimate on the random walk loop measure

Using Theorem 3.1 and the estimates (2.7) and (3.1), we see that

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \asymp n^{-1} \exp\{2m(\mathcal{J}_{A_n})\}.$$

The proof of (1.1) is finished with the following proposition.

Proposition 4.1. *There exists $c < \infty$ such that for all n ,*

$$\left| m(\mathcal{J}_{A_n}) - \frac{1}{8} \log n \right| \leq c.$$

Proof. Let $C_n = \{z \in \mathbb{Z}^2 : |z| < e^n\}$. We will prove the stronger fact that the limit

$$\lim_{n \rightarrow \infty} \left[m(\mathcal{J}_{C_n}) - \frac{n}{8} \right] \tag{4.1}$$

exists by showing that

$$\sum_{n=1}^{\infty} \left| m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) - \frac{1}{8} \right| < \infty. \tag{4.2}$$

Let μ denote the Brownian loop measure, and let $\tilde{\mathcal{J}}$ denote the set of unrooted Brownian loops γ in the unit disk that intersect $\{|z| \geq e^{-1}\}$ and such that the winding number of γ about the origin is odd. We will establish (4.2) by showing that $\mu(\tilde{\mathcal{J}}) = 1/8$ and

$$\left| m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) - \mu(\tilde{\mathcal{J}}) \right| = O(n^{-2}). \tag{4.3}$$

For the Brownian loop measure, we do a computation similar to that in [4, Proposition 3.9]. Using (2.9), we write

$$\mu(\tilde{\mathcal{J}}) = \frac{1}{\pi} \int_{e^{-1}}^1 \int_0^{2\pi} \phi(r, \theta) d\theta r dr,$$

where $\phi(r, \theta)$ denotes the Brownian bubble measure of loops in $r\mathbb{D}$ rooted at $re^{i\theta}$ with odd winding number about the origin. Rotational symmetry implies that $\phi(r, \theta) = \phi(r, 0)$ and conformal covariance implies that $\phi(r, 0) = r^{-2} \phi$ where $\phi = \phi(1, 0)$. Hence,

$$\mu(\tilde{\mathcal{J}}) = \frac{\phi}{\pi} \int_{e^{-1}}^1 \int_0^{2\pi} r^{-2} d\theta r dr = 2\phi. \tag{4.4}$$

By considering the (multi-valued) covering map $f(z) = i \log z$ which satisfies $|f'(1)| = 1$, we see that

$$\phi = \sum_{k \text{ odd}} H_{\partial\mathbb{H}}(0, 2\pi k),$$

where $H_{\partial\mathbb{H}}$ denotes the boundary Poisson kernel (normal derivative of the Poisson kernel) in the upper half-plane \mathbb{H} normalized as before so that $H_{\partial\mathbb{H}}(0, x) = x^{-2}$. Therefore,

$$2\phi = 2 \sum_{k=-\infty}^{\infty} \frac{1}{[2\pi(2k+1)]^2} = \frac{1}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] = \frac{1}{8}.$$

If $s > 2$, let $\tilde{\mathcal{U}}_s$ denote the set of Brownian loops (both odd and even winding number) in \mathbb{D} that intersect both $\{|z| \geq e^{-1}\}$ and $\{|z| \leq e^{-s}\}$. We claim that as $s \rightarrow \infty$,

$$\mu(\tilde{\mathcal{U}}_s) = s^{-1} + O(s^{-2}), \tag{4.5}$$

$$\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{U}}_s] = (2s)^{-1} + O(s^{-2}). \tag{4.6}$$

To see this, we first consider the boundary bubble measure λ_s of loops in \mathbb{D} rooted at 1 that enter $\{|z| \leq e^{-s}\}$. An exact expression is given as follows. Let B_t be a Brownian motion and $\sigma_s = \inf\{t : |B_t| = e^{-s}\}$. Then,

$$\lambda_s = \lim_{\epsilon \downarrow 0} \epsilon^{-1} \mathbb{E}^{1-\epsilon} [H_{\mathbb{D}}(B_{\sigma_s}, 1); \sigma_s < \sigma_0].$$

(We write $\mathbb{E}^z, \mathbb{P}^z$ for expectations and probabilities assuming $B_0 = z$.) The Poisson kernel in the disk is well known; for our purpose we need only know that

$$H_{\mathbb{D}}(e^{-s+xi}, 1) = \sum_{k \in \mathbb{Z}} H_{\mathbb{H}}(x + 2\pi k + is, 0) = \sum_{k \in \mathbb{Z}} \frac{s}{(x + 2\pi k)^2 + s^2} = \frac{1}{2} + O(e^{-s}),$$

(recall that $H_{\mathbb{D}}(z, 1)$ is π times the hitting density which is uniform on the circle), and a standard estimate for Brownian motion gives

$$\mathbb{P}^{1-\epsilon}\{\sigma_s < \sigma_0\} = \frac{\log(1-\epsilon)}{-s} \sim \frac{\epsilon}{s}.$$

Therefore, $\lambda_s = (2s)^{-1} + O(e^{-s})$. Using rotational invariance, and conformal covariance, if $r \geq e^{-1}$ and $\lambda(r, \theta, s)$ denotes the bubble measure of bubbles in $r\mathbb{D}$ rooted at $re^{i\theta}$ that enter $\{|z| \leq e^{-s}\}$, then

$$\lambda(r, \theta, s) = r^{-2} (2s)^{-1} [1 + O(s^{-1})].$$

If we compute as in (4.4), we get (4.5). The relation (4.6) is done similarly except that we have to worry about the winding number of the loop. Here we use

$$\sum_{k \text{ even}} H_{\mathbb{H}}(x + 2\pi k + is, 0) = \sum_{k \text{ even}} \frac{s}{(x + 2\pi k)^2 + s^2} = \frac{1}{4} + O(e^{-s}),$$

to see that

$$\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{U}}_s] = \frac{1}{2} \mu[\tilde{\mathcal{U}}_s] [1 + O(e^{-s})]. \tag{4.7}$$

For each unrooted random walk loop $\bar{l} \in \mathcal{J}_{C_n} \setminus \mathcal{J}_{C_{n-1}}$, there is a corresponding continuous unrooted loop $\bar{l}^{(n)}$ in \mathbb{D} obtained from linear interpolation and Brownian scaling. We will write $d(\bar{l}, \gamma) \leq \delta$, if we can parametrize and root the loops $\bar{l}^{(n)}$ and γ such that the loops are within δ in the supremum norm. In [10] it was shown that there exists $\alpha > 0$ and a coupling of the random walk and Brownian loop measures in D , restricted to loops of diameter at least $1/e$, so that the total masses agree up to $O(e^{-n\alpha})$ and such that in the coupling, except for a set of paths of size $O(e^{-n\alpha})$, we have $d(\bar{l}, \gamma) < e^{-n\alpha}$. (Actually, a more precise estimate is given in [10], but this is all we need for this paper.) We would like to say that in the coupling, the Brownian loop has odd winding number if and only if $J(\bar{l})$ is odd. If the loops stay away from the origin, this holds. However, if the loops are near the origin, it is possible for the winding numbers of the continuous and the discrete walks to be different. However, and this is why we can prove what we need, it is also true that if a macroscopic loop (either continuous or discrete) gets close to the origin, then it is just about equally likely to have an odd as an even winding number. Let us be more precise.

Let $\beta < \alpha$ and let \mathcal{U}^n denote the set of random walk loops contained in C_{n+1} , that intersect both $C_{n+1} \setminus C_n$ and $\{|z| \leq e^{-\beta n} e^{n+1}\}$. Using the coupling, random walk estimates, and (4.6), we see that

$$m(\mathcal{U}^n) = \mu(\tilde{\mathcal{J}}'_{\beta n}) + O(n^{-2}) = (\beta n)^{-1} + O(n^{-2}).$$

Let us split \mathcal{U} into two sets: loops for which $\text{dist}(0, \gamma) \leq 2e^{-n\alpha}$ and those for which $\text{dist}(0, \gamma) > 2e^{-n\alpha}$. If $\text{dist}(0, \gamma) > 2e^{-n\alpha}$ and $d(\bar{l}, \gamma) \leq e^{-n\alpha}$, then $J(\bar{l})$ is odd if and only if the winding number of γ is odd. Therefore

$$m((\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) \setminus \mathcal{U}^n) = \mu(\tilde{\mathcal{J}} \setminus \tilde{\mathcal{J}}'_{\beta n}) + O(n^{-2}).$$

(The error term $O(n^{-2})$ is comparable to the measure of loops γ such that $e^{-n\beta} \leq \text{dist}(0, \gamma) \leq 2e^{-n\beta}$.)

A coupling argument can be used to give a random walk analogue of (4.7),

$$m[\mathcal{U}^n \cap (\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n})] = \frac{1}{2} m(\mathcal{U}^n) [1 + o(n^{-1})].$$

We sketch the proof which, in fact, gives an error of $O(e^{-un})$ for some u . We use the definition of the loop measure using an enumeration of $\mathbb{Z}^2 = \{z_1, z_2, \dots\}$ such that $|z_j|$ increases. Then an unrooted loop in \mathcal{U}^n is obtained from a loop rooted in $C_{n+1} \setminus C_n$. Let us call the root z_k and so the loops lies in $V_k = \{z_1, \dots, z_k\}$. Let us stop the walk at the first time it reaches a point, say z' , in $\{|z| \leq e^{-\beta n} e^{n+1}\}$. The remainder of the loop acts like a random walk started at z' conditioned to reach z_k without before leaving V_k . Let J' denote the number of times such a walk crosses the half line $\{(1/2) + iy : y < 0\}$. We claim that the probability that J' is odd equals $\frac{1}{2} + O(e^{-un})$ for some $u > 0$. Indeed, we can couple two walks starting at the point so that each walk has the distribution of random walk conditioned to reach z_k before leaving V_k and that, except for an event of probability $O(e^{-\delta})$, the parity of J' is different for the two walks. This uses a standard technique. The key estimate is the following. There exists $c > 0$ such that if S is a simple random walk starting at $z \in C_{j-1}$ and $T = \min\{j : S_j \in C_j\}$, then for all $w \in \partial C_j$ with $\text{Im}(w) > 0$,

$$\mathbb{P}\{S(T) = w, J' \text{ odd}\} \geq ce^{-j},$$

$$\mathbb{P}\{S(T) = w, J' \text{ even}\} \geq ce^{-j}.$$

Without the restriction of the parity of J' , see, for example, [6, Lemma 6.3.7]. To get the result with the restriction, we just note that there is a positive probability of making a loop in the annulus $C_j \setminus C_{j-1}$, and this increases J' by one. Hence, we can find a coupling and a $\rho > 0$ such that at each annulus there is a probability ρ of a successful coupling given that the walks have not yet been coupled. Since there are of order βn annuli, we can couple the processes so that the probability of not being coupled is $(1 - \rho)^{\beta n} = O(e^{-un})$ for some u .

From the last two estimates and (4.6), we see that

$$\left| \mu(\tilde{\mathcal{J}}) - m(\mathcal{I}_{C_{n+1}} \setminus \mathcal{I}_{C_n}) \right| \leq cn^{-2}.$$

This gives (4.3). □

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