

On uniform positivity of transition densities of small noise constrained diffusions

Amarjit Budhiraja* Zhen-Qing Chen†

Abstract

Constrained diffusions in convex polyhedral cones with a general oblique reflection field, and with a diffusion coefficient scaled by a small parameter $\varepsilon > 0$, are considered. Using an interior Dirichlet heat kernel lower bound estimate for second order elliptic operators in bounded domains from [14], certain uniform in ε lower bounds on transition densities of such constrained diffusions are established. These lower bounds together with results from [2] give, under additional stability conditions, an exponential leveling property as $\varepsilon \rightarrow 0$ for exit times from suitable bounded domains.

Keywords: Exponential leveling; reflected diffusions; Dirichlet heat kernel estimates; Skorohod problem; exit time estimates; Friedlin-Wentzell asymptotics.

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1 Introduction

Diffusions in polyhedral domains have been extensively studied in the heavy traffic limit theory for stochastic processing networks (see for example, [12, 9, 11, 13, 4]). In a recent work [2] small noise asymptotics for a general family of such constrained diffusions have been studied. The precise setting there is as follows. Let $G \subset \mathbb{R}^k$ be convex polyhedral cone with a nonempty interior with the vertex at origin given as the intersection of half spaces $G_i, i = 1, 2, \dots, N$. Let n_i be the unit inward normal vector associated with G_i via the relation

$$G_i = \{x \in \mathbb{R}^k : \langle x, n_i \rangle \geq 0\}.$$

We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i . With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the *direction of constraint* associated with the face F_i . Precise definition of constrained diffusions considered here is given in Section 2, but roughly speaking such a process evolves infinitesimally as a diffusion in \mathbb{R}^k and is instantaneously pushed back using the oblique reflection direction d_i upon reaching the face F_i . Formally, such a process can be represented as a solution of a stochastic integral equation of the form

$$X_x^\varepsilon(t) = \Gamma \left(x + \int_0^\cdot b(X^\varepsilon(s))ds + \varepsilon \int_0^\cdot \sigma(X^\varepsilon(s))dW(s) \right) (t), \quad (1.1)$$

*University of North Carolina at Chapel Hill, USA. E-mail: budhiraj@email.unc.edu

†University of Washington at Seattle, USA. E-mail: zqchen@uw.edu

where Γ is the Skorohod map (see below Definition 2.1) taking trajectories with values in \mathbb{R}^k to those with values in G , consistent with the constraint vectors $\{d_i, i = 1, \dots, N\}$. Under certain regularity assumptions on the Skorohod map (see Condition 2.2) and the usual Lipschitz conditions on the coefficients b and σ , the above integral equation has a unique pathwise solution. One of the main results of [2] is an ‘exponential leveling’ property of exit times from bounded domains for such small noise diffusions. Such results for diffusions in \mathbb{R}^k have been obtained in [6] which is concerned with asymptotics of Dirichlet problems in bounded domains associated with diffusions with infinitesimal generator of the form

$$\mathcal{L}_\varepsilon f(x) = \frac{\varepsilon^2}{2} \text{Tr}(\sigma(x)D^2 f(x)\sigma(x)^{tr}) + b(x) \cdot \nabla f(x), \quad f \in C_b^2(\mathbb{R}^k). \quad (1.2)$$

Here $\sigma(x)^{tr}$ is the transpose of the matrix $\sigma(x)$. The precise result in [6] is as follows. Denote by Z_x^ε the diffusion process governed by the generator \mathcal{L}_ε and initial distribution δ_x . Let B be a bounded domain in \mathbb{R}^k and K be an arbitrary compact subset of B . Suppose that all solutions of the ODE $\dot{\xi} = b(\xi)$ with $x = \xi(0) \in B$ converge, without leaving B , to a single linearly asymptotically stable critical point in B . Then, with suitable conditions on the coefficients of the diffusion, for all bounded measurable f

$$\sup_{x,y \in K} |\mathbb{E}(f(Z_x^\varepsilon(\tau_x^\varepsilon))) - \mathbb{E}(f(Z_y^\varepsilon(\tau_y^\varepsilon)))|$$

converges to 0 at an exponential rate. Here, $\tau_x^\varepsilon = \inf\{t : Z_x^\varepsilon(t) \in B^c\}$. This property is a statement on the long time behavior of the diffusion and says that although, as $\varepsilon \rightarrow 0$, the exit time of the process from the domain approaches ∞ , the expected values of functionals of exit location, corresponding to distinct initial conditions, coalesce asymptotically, at an exponential rate. The key ingredient in the proof is the gradient estimate

$$\sup_{x \in K} |\nabla u^\varepsilon(x)| \leq c\varepsilon^{-1/2}, \quad (1.3)$$

where u^ε is the solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon(x) = 0, & x \in B \\ u(x) = f(x), & x \in \partial B. \end{cases}$$

Diffusions of interest in [2] and in the current work are constrained to take values in domains with corners and where the constraining mechanism is governed by an oblique reflection field that changes discontinuously from one face of the boundary to another. To the best of our knowledge there are no regularity (e.g. C^1 solutions) results known for the associated partial differential equations(PDE) with oblique reflecting boundary condition. In view of this, a probabilistic approach for the study of exponential leveling property for such constrained settings, that ‘almost’ bypasses all PDE estimates was developed in [2]. The main step in the proof is the construction of certain (uniform in ε) Lyapunov functions under a suitable stability condition which are then used to construct a coupling of the processes $X_x^\varepsilon, X_y^\varepsilon$ with explicit uniform estimates on exponential moments of time to coupling. The key ingredient in this coupling construction is a, uniform in ε , minorization condition on transition densities of the reflected diffusions (see Condition 2.4). The paper [2] gave one simple example with constant drift and diffusion coefficients where such a condition is satisfied. However the question of when such a minorization property is available was left as an open problem.

The objective of this work is to answer this question and give general conditions under which Condition 2.4 holds. The main result of the paper is Theorem 2.7 that shows that under Conditions 2.2 and 2.3 the minorization statement in Condition 2.4 holds.

This result, together with Theorem 3.2 in [2] then gives general sufficient conditions for an exponential leveling property to hold for a broad family of constrained diffusion processes. This is noted in Corollary 2.8.

Once the estimate in Theorem 2.7 is available, the proof of the exponential leveling property does not use any PDE results, however the proof of Theorem 2.7 itself crucially relies on an interior lower bound estimate for the Dirichlet heat kernel of \mathcal{L}_ε over bounded domains ([14], see Theorem 3.1). In this sense the proofs are not fully probabilistic.

The rest of the paper is organized as follows. In Section 2 we introduce the precise mathematical setting and state our main result (Theorem 2.7). In Corollary 2.8 we present the exponential leveling result that follows on combining Theorem 2.7 with the results in [2]. Finally, in Section 3 we present the proof of Theorem 2.7.

The following notation will be used. Closure, complement, boundary and interior of a subset B of a topological space S will be denoted by \bar{B} , B^c , ∂B and B° , respectively. For a set $B \in \mathbb{R}^k$ and $a \in \mathbb{R}$, we denote by aB the set $\{ax : x \in B\}$. The Borel σ field on a metric space S will be denoted as $\mathcal{B}(S)$. Given a metric space (S, d) , and subsets A, B of S , we will define $\text{dist}(A, B) = \inf_{x \in A, y \in B} d(x, y)$. If $A = \{x\}$ for some $x \in S$, we write $\text{dist}(A, B)$ as $\text{dist}(x, B)$. Denote by $C([0, \infty); S)$ (resp. $C([0, T]; S)$) the space of continuous functions from $[0, \infty)$ (resp. $[0, T]$) to a metric space S . This space is endowed with the usual local uniform topology. For $\eta \in C([0, \infty); \mathbb{R}^k)$, and $t > 0$, define $|\eta|_t = \sup \sum_{i=1}^n |\eta(t_{i+1}) - \eta(t_i)|$, where sup is taken over all partitions $0 = t_1 < t_2 < \dots < t_{n+1} = t$. Lebesgue measure on \mathbb{R}^k will be denoted by λ . We denote by $C_b^2(\mathbb{R}^k)$ the space of bounded twice continuously differentiable functions with bounded derivatives.

2 Main Result

We begin by making precise the constraining mechanism, in terms of a suitable Skorohod problem, that keeps the diffusion in the polyhedral cone G . Recall the half spaces G_i and vectors $n_i, d_i, i = 1, \dots, N$ that were introduced in Section 1.. For $x \in \partial G$ define

$$d(x) = \{v \in \mathbb{R}^k : v = \sum_{i \in \text{In}(x)} \alpha_i d_i; \alpha_i \geq 0; |v| = 1\},$$

where

$$\text{In}(x) = \{i \in \{1, 2, \dots, N\} : \langle x, n_i \rangle = 0\}.$$

Definition 2.1. (Skorokhod Problem). Let $\psi \in C([0, \infty), \mathbb{R}^k)$ be given such that $\psi(0) \in G$. Then $(\phi, \eta) \in C([0, \infty), G) \times C([0, \infty), \mathbb{R}^k)$ solves the Skorokhod Problem (SP) for ψ (with respect to the data $\{(d_i, n_i), i = 1, \dots, N\}$), if $\phi(0) = \psi(0)$ and if for all $t \in [0, \infty)$: (1) $\phi(t) = \psi(t) + \eta(t)$, (2) $|\eta|(t) < \infty$, (3) $|\eta|(t) = \int_0^t \mathbb{1}_{\{\phi(s) \in \partial G\}} d|\eta|(s)$, and there exists a Borel Measurable function $\gamma : [0, \infty) \rightarrow \mathbb{R}^k$ such that $\gamma(t) \in d(\phi(t))$, for $d|\eta|$ -a.e. t and $\eta(t) = \int_0^t \gamma(s) d|\eta|(s), t \geq 0$.

Let $C_G([0, \infty); \mathbb{R}^k)$ be the collection of $\psi \in C([0, \infty); \mathbb{R}^k)$ such that $\psi(0) \in G$. The domain $D \subseteq C_G([0, \infty); \mathbb{R}^k)$ on which there is a unique solution to the Skorohod problem we define the Skorohod map (SM) Γ as $\Gamma(\psi) \doteq \phi$ if (ϕ, η) is the unique solution of the Skorohod problem posed by ψ . We will make the following assumption on the regularity of the Skorohod map defined by the data $\{(d_i, n_i); i = 1, 2, \dots, N\}$.

Condition 2.2. The Skorohod map is well defined on all of $C_G([0, \infty); \mathbb{R}^k)$, that is, $D = C_G([0, \infty); \mathbb{R}^k)$, and the SM is Lipschitz continuous in the following sense: There exists

a constant $K \in (0, \infty)$ such that for all $\phi_1, \phi_2 \in C_G([0, \infty); \mathbb{R}^k)$:

$$\sup_{0 \leq t < \infty} |\Gamma(\phi_1)(t) - \Gamma(\phi_2)(t)| < K \sup_{0 \leq t < \infty} |\phi_1(t) - \phi_2(t)|. \tag{2.1}$$

We refer the reader to [9, 7, 8] for sufficient conditions under which Condition 2.2 holds.

The following scaling property of the SM is an immediate consequence of the definition: For $f \in C([0, \infty), \mathbb{R}^k)$ and $\gamma > 0$, let $f^\gamma(t) = \frac{1}{\gamma}f(\gamma t)$, $t \geq 0$. Then

$$(\phi^\gamma, \eta^\gamma) \text{ solves the SP for } \psi^\gamma \text{ if and only if } (\phi, \eta) \text{ solves the SP for } \psi. \tag{2.2}$$

Now we introduce the small noise constrained diffusion process that will be considered in this work. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is given a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses. Let $(W(t), \mathcal{F}_t)$ be a k -dimensional standard Wiener process on the above probability space. Let $\sigma : G \rightarrow \mathbb{R}^{k \times k}$, $b : G \rightarrow \mathbb{R}^k$ be mappings satisfying the following condition.

Condition 2.3. (i) *There exists $\gamma_1 \in (0, \infty)$ such that*

$$|b(x) - b(y)| \leq \gamma_1|x - y| \text{ and } |b(x)| \leq \gamma_1 \text{ for every } x, y \in G \tag{2.3}$$

(ii) *There exists $\gamma_2 \in (0, \infty)$ such that*

$$|\sigma(x) - \sigma(y)| \leq \gamma_2|x - y| \text{ and } |\sigma(x)| \leq \gamma_2 \text{ for } x, y \in G. \tag{2.4}$$

(iii) *There exists $\underline{\sigma} \in (0, \infty)$ such that for all $x \in G$ and $v \in \mathbb{R}^k$*

$$v' \sigma(x) \sigma(x)' v \geq \underline{\sigma} |v|^2.$$

Given $\varepsilon > 0$, let X^ε be the unique strong solution of the following stochastic integral equation:

$$X(t) = \Gamma \left(x + \int_0^t b(X(s)) ds + \varepsilon \int_0^t \sigma(X(s)) dW(s) \right) (t), \quad t \geq 0, \tag{2.5}$$

Existence of strong solutions and pathwise uniqueness for (2.5) is a consequence of the Lipschitz property of the coefficients and of the Skorohod map (see [7]). It is convenient to have the process X for various initial conditions and values of ε to be defined on a common canonical space. Indeed, one can find a filtered measurable space, which we denote again as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, on which is given a family of probability measures, $\{\mathbb{P}_x^\varepsilon\}_{x \in G}$, and continuous adapted stochastic processes Z, Y and W such that for all $x \in G$, under \mathbb{P}_x^ε , $\{W(t), \{\mathcal{F}_t\}_{t \geq 0}\}$ is a k -dimensional standard Wiener process and (Z, W, Y) satisfy \mathbb{P}_x^ε a.s. the integral equation

$$\begin{aligned} Z(t) &= \Gamma \left(x + \int_0^t b(Z(s)) ds + \varepsilon \int_0^t \sigma(Z(s)) dW(s) \right) (t), \\ &= x + \int_0^t b(Z(s)) ds + \varepsilon \int_0^t \sigma(Z(s)) dW(s) + \mathbb{D}Y(t), \quad t \geq 0, \end{aligned} \tag{2.6}$$

where $\mathbb{D} = (d_1, \dots, d_N)$. Also, for every $\varepsilon > 0$, $(Z, \{\mathbb{P}_x^\varepsilon\}_{x \in G})$ is a strong Markov family (cf. [5]).

It can be shown that for $t > 0$ and $x \in G$, the measure $\mathbb{P}_x^\varepsilon(Z(t) \in \cdot)$ is absolutely continuous with respect to the Lebesgue measure on G (see Lemma 5.7 in [5]). Denote by $p_\varepsilon(t, x, y)$ the probability density of $Z(t)$ under \mathbb{P}_x^ε , namely

$$\mathbb{P}_x^\varepsilon(Z(t) \in A) = \int_A p_\varepsilon(t, x, y) dy, \quad A \in \mathcal{B}(G).$$

Let, for $r > 0$ and $x_0 \in G$, $\mathbb{B}_r(x_0) = \{x \in G : |x - x_0| \leq r\}$. When $x_0 = 0$, we simply write \mathbb{B}_r .

The following minorization condition was introduced in [2] (see Condition 3.1 therein).

Condition 2.4. For every $t_1 > 0$, $\varepsilon_0 > 0$ and $0 < M_1 < M < \infty$, there exists a Borel $E \subset \mathbb{B}_{M_1}$ with $\lambda(E) > 0$ and $\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon^{2k} p_\varepsilon(t_1 \varepsilon^2, x, z) \geq \kappa \quad \text{for all } x \in \varepsilon^2 \mathbb{B}_M \text{ and } \lambda\text{-a.e. } z \in \varepsilon^2 E.$$

The above condition played a key role in [2] in proving an exponential leveling result for exit times from bounded domains which we now describe. Let

$$\mathcal{C} = \left\{ - \sum_{i=1}^N \alpha_i d_i : \alpha_i \geq 0; i \in \{1, \dots, N\} \right\}.$$

The paper [3] shows that under Conditions 2.2 and 2.3 and Condition 2.5 below the constrained diffusion X^ε is positive recurrent.

Condition 2.5. For some $\delta > 0$

$$b(x) \in \mathcal{C}(\delta) = \{v \in \mathcal{C} : \text{dist}(v, \partial \mathcal{C}) \geq \delta\} \quad \text{for all } x \in G. \tag{2.7}$$

Let B be a bounded relatively open subset of G . Suppose that $0 \in B$ and $\partial B = \partial \bar{B}$. For $x \in G$ we denote by ξ_x the unique solution of the integral equation

$$\xi_x(t) = \Gamma \left(x + \int_0^t b(\xi_x(s)) ds \right) (t), \quad t \geq 0. \tag{2.8}$$

Also, let $\mathcal{S}_x = \{\xi_x(t), t \geq 0\}$. For $\gamma > 0$, let $B_\gamma = \{x \in B : \text{dist}(\mathcal{S}_x, \partial B) \geq \gamma\}$ and let $B_0 = \bigcup_{\gamma > 0} B_\gamma$. Clearly $B_0 \subset B$ and under Condition 2.5 one can show that for every γ sufficiently small, there is a $r > 0$ such that $\mathbb{B}_r \subset B_\gamma$ and for every $x \in B_0$, $\xi_x(t)$ converges to 0, as $t \rightarrow \infty$, without leaving B (cf. Lemma 2.1 in [2]).

Denote by \mathcal{H} the collection of all $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\sup_{t > 0} \frac{|\psi(t)|}{(1 + \log^+(t))^q} < \infty$, for some $0 < q < 1$, and $\sup_{r > 0} \frac{\sup_{s, t \geq 0: |t-s| \leq r} |\psi(t) - \psi(s)|}{r^{m+1}} < \infty$ for some $m \geq 1$. Let $\tau \equiv \tau(B) = \inf\{t \geq 0 : Z(t) \in B^c\}$.

The following is the exponential leveling result from [2].

Theorem 2.6. Suppose that Conditions 2.2, 2.3, 2.4 and 2.5 holds. Let \mathbb{K} be a compact subset of B_0 . Then the following hold.

(i) For every bounded measurable map $f : \bar{B} \rightarrow \mathbb{R}$ there is a $\delta_1 \in (0, \infty)$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{K}} e^{\delta_1/\varepsilon} |\mathbb{E}_x^\varepsilon f(Z(\tau)) - \mathbb{E}_y^\varepsilon f(Z(\tau))| = 0.$$

(ii) For every $\psi \in \mathcal{H}$

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{K}} |\mathbb{E}_x^\varepsilon \psi(\tau) - \mathbb{E}_y^\varepsilon \psi(\tau)| < \infty.$$

The main result of the current work says that Condition 2.4 in Theorem 2.6 can be dropped. Specifically, we prove the following result.

Theorem 2.7. Suppose Conditions 2.2 and 2.3 hold. Then Condition 2.4 holds.

As an immediate consequence of this result we have the following.

Corollary 2.8. Suppose that Conditions 2.2, 2.3 and 2.5 hold. Then properties (i) and (ii) in the statement of Theorem 2.6 hold.

Rest of this paper is devoted to the proof of Theorem 2.7.

3 Proof of Theorem 2.7.

For $x \in \mathbb{R}^k$, denote by $Q_{\mathcal{L},x}$ the probability law of the diffusion with initial distribution δ_x and infinitesimal generator \mathcal{L} :

$$\mathcal{L}f(x) = \frac{1}{2}\text{Tr}(\sigma(x)D^2f(x)\sigma(x)^{tr}) + b(x) \cdot \nabla f(x), \quad f \in C_b^2(\mathbb{R}^k), \quad (3.1)$$

where b, σ satisfy Condition 2.3 and $\sigma(x)^{tr}$ is the transpose of the matrix $\sigma(x)$. Denote by ξ the canonical coordinate process on $C([0, \infty); \mathbb{R}^k)$ and given any bounded C^2 -smooth open set B in \mathbb{R}^k , let

$$\tau_B = \inf\{t \geq 0 : \xi(t) \in B^c\}.$$

Then, for every $t > 0$ and $\varepsilon > 0$, there is a jointly continuous function $p_{\mathcal{L},B}^\dagger(t, \cdot, \cdot) : B \times B \rightarrow \mathbb{R}_+$ such that for all $x \in B$ and Borel $A \subset B$

$$Q_{\mathcal{L},x}(\xi(t) \in A; t < \tau_B) = \int_A p_{\mathcal{L},B}^\dagger(t, x, y) dy.$$

The function $p_{\mathcal{L},B}^\dagger(t, x, y)$ is called Dirichlet heat kernel for \mathcal{L} in B . In some PDE literature (e.g., in [1, 14]), $p_{\mathcal{L},B}^\dagger(t, x, y)$ is also called the Green function for \mathcal{L} in B . Following the terminology of many other authors in PDE and in probability theory, we reserve the name of Green function for $\int_0^\infty p_{\mathcal{L},B}^\dagger(t, x, y) dt$, which is the fundamental solution for solving the Poisson equation of \mathcal{L} in B with zero Dirichlet boundary condition.

For $\varepsilon > 0$, define $\widehat{\mathcal{L}}_\varepsilon$ by replacing σ and b in (3.1) with $b_\varepsilon(\cdot) = b(\varepsilon^2 \cdot)$ and σ with $\sigma_\varepsilon(\cdot) = \sigma(\varepsilon^2 \cdot)$. Let $a(x) = (a_{ij}(x)) = \sigma(x)\sigma(x)^{tr}$ and $\widehat{b}(x) = (\widehat{b}_1(x), \dots, \widehat{b}_k(x))$ with $\widehat{b}_k(x) = \sum_{j=1}^k \frac{\partial a_{kj}(x)}{\partial x_j}$. Note that \widehat{b} is defined as a bounded function almost everywhere on \mathbb{R}^k , as each a_{ij} is a bounded Lipschitz function. We can rewrite $\widehat{\mathcal{L}}_\varepsilon$ almost everywhere on \mathbb{R}^k as follows:

$$\widehat{\mathcal{L}}_\varepsilon = \frac{1}{2}\nabla(a(\varepsilon^2x)\nabla) + (b(\varepsilon^2x) - \varepsilon^2\widehat{b}(\varepsilon^2x)) \cdot \nabla.$$

Various heat kernel estimates for operators of the above type have been studied in literature.

The following uniform interior lower bound estimate follows immediately from [14, Lemma 3.3] and the application of Chapman-Kolmogorov equation to extend the range of t . When a and b are smooth, one could also apply [1, Theorem 8]. (A sharp two-sided Dirichlet heat kernel estimate is available from [10, Theorem 2.1], but we do not need it in this paper. Though it is assumed in [14] that the dimension $k \geq 3$, the interior lower bound estimate [14, Lemma 3.3] in fact holds for any $k \geq 1$; see [10].)

Theorem 3.1. *Let $B = B(x_0, R)$ be an open ball in \mathbb{R}^k . For each given $\varepsilon_0 > 0$, $t > 0$ and $0 < \gamma < 1$, there is a constant $c > 0$ so that $p_{\widehat{\mathcal{L}}_\varepsilon, B}^\dagger(t, x, y) \geq c$ for every $x, y \in B(x_0, \gamma R)$ and $\varepsilon \in (0, \varepsilon_0]$.*

We note that Lemma 3.3 of [14] requires that a is Hölder continuous and $|\widehat{b}_\varepsilon|$ is in Kato class K_{k+1} , where $\widehat{b}_\varepsilon(\cdot) = b(\varepsilon^2 \cdot) - \varepsilon^2\widehat{b}(\varepsilon^2 \cdot)$. It is immediate from the definition of K_{k+1} that any bounded measurable function is in K_{k+1} . From Condition 2.3(i) and (ii), we now see that the coefficients $a(\varepsilon^2 \cdot)$ and \widehat{b}_ε are uniformly bounded and so they are in Kato class K_{k+1} with uniform bounded Kato norms. Furthermore, from the proof of the lemma it can be seen that the constant c depends on the coefficients of the diffusion only through the constants $\underline{\sigma}$, γ_1 for the upper bound of b and γ_2 in Condition 2.3 and the constant ε_0 (which imply the uniform ellipticity and uniform Hölder continuity of the diffusion matrices $a(\varepsilon^2x)$ and the uniform boundedness of the drift coefficients \widehat{b}_ε for $\varepsilon \in (0, \varepsilon_0]$). This ensures that the constant c in Theorem 3.1 can be chosen independent of $\varepsilon \in (0, \varepsilon_0]$.

Theorem 3.1 will be a key ingredient in the proof of Theorem 2.7. Note that if for some $x_0 \in G^\circ$ and $r > 0$, $\mathbb{B}_r(x_0) \subset G^\circ$, then

$$p_\varepsilon(t, y, z) \geq p_{\mathcal{L}_{\varepsilon, \varepsilon^2}(\mathbb{B}_r(x_0))^\circ}(t, y, z), \text{ for all } t > 0, y, z \in \varepsilon^2(\mathbb{B}_r(x_0))^\circ.$$

This elementary inequality will allow us to obtain lower bounds on transition densities of constrained diffusions using the Dirichlet heat kernel estimates of Theorem 3.1.

3.1 Proof of Theorem 2.7.

Fix $t_1, M, \varepsilon_0 > 0$ and $M_1 \in (0, M)$. Also, fix $x_0 \in G^\circ$ such that $|x_0| < M_1$. Choose $r_2 > 0$ such that $\mathbb{B}_{r_2}(x_0) \subset G^\circ \cap \{x : |x| < M_1\}$. Fix $0 < r_0 < r_1 < r_2$. Also, fix $t_2, t_3 > 0$ such that $t_2 + t_3 = t_1$. We will now show that Condition 2.4 holds with $E = \mathbb{B}_{r_0}(x_0)$. Let $\varphi : G \rightarrow [0, 1]$ be a continuous function such that $\varphi(x) = 1$ for all $x \in \mathbb{B}_{r_0}(x_0)$ and $\varphi(x) = 0$ for all $x \in (\mathbb{B}_{r_1}(x_0))^c$. We first show that

$$\text{For some } \kappa_0 > 0, \int_{\mathbb{R}^k} \varphi(z/\varepsilon^2) p_\varepsilon(t_2 \varepsilon^2, x, z) dz \geq \kappa_0 \text{ for all } x \in \mathbb{B}_{M\varepsilon^2} \text{ and } \varepsilon \in (0, \varepsilon_0] \tag{3.2}$$

For this, note that

$$\int_{\mathbb{R}^k} \varphi(z/\varepsilon^2) p_\varepsilon(t_2 \varepsilon^2, x, z) dz = \mathbb{E}_x^\varepsilon \left[\varphi \left(\frac{Z(t_2 \varepsilon^2)}{\varepsilon^2} \right) \right]$$

and, \mathbb{P}_x^ε a.s.,

$$Z(t) = \Gamma \left(x + \int_0^\cdot b(Z(s)) ds + \varepsilon \int_0^\cdot \sigma(Z(s)) dW(s) \right) (t).$$

Therefore, letting $Z^\varepsilon(t) = \frac{Z(t_2 \varepsilon^2)}{\varepsilon^2}$ and using the scaling property of the Skorohod map noted in (2.2), we have

$$Z^\varepsilon(t) = \Gamma \left(\frac{x}{\varepsilon^2} + \int_0^\cdot b(\varepsilon^2 Z^\varepsilon(s)) ds + \int_0^\cdot \sigma(\varepsilon^2 Z^\varepsilon(s)) dW^\varepsilon(s) \right) (t),$$

\mathbb{P}_x^ε a.s., where $W^\varepsilon(t) = \frac{1}{\varepsilon} W(t\varepsilon^2)$. Suppose now that (3.2) fails. Then there exists a sequence $\varepsilon_n \rightarrow 0$ and $\{x_n\} \subset G$ such that $x_n \in \mathbb{B}_{M\varepsilon_n^2}$ and

$$\lim_{n \rightarrow \infty} \mathbb{E}_{x_n}^{\varepsilon_n} [\varphi(Z^{\varepsilon_n}(t_2))] = 0.$$

Without loss of generality we can assume that x_n/ε_n^2 converges to some $\bar{x} \in \mathbb{B}_M$. A standard weak convergence argument now shows that

$$0 = \lim_{n \rightarrow \infty} \mathbb{E}_{x_n}^{\varepsilon_n} [\varphi(Z^{\varepsilon_n}(t_2))] = \tilde{E} \left[\varphi(\tilde{X}(t_2)) \right], \tag{3.3}$$

where \tilde{X} is a G valued continuous stochastic process given on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, supporting a k dimensional Brownian motion \tilde{W} , such that

$$\tilde{X}(t) = \Gamma \left(\bar{x} + b(0)\iota(\cdot) + \sigma(0)\tilde{W}(\cdot) \right) (t), t \geq 0.$$

From Condition 2.3 (iii) it follows that (cf. Lemma 5.7[5])

$$\tilde{E} \left[\varphi(\tilde{X}(t_2)) \right] \geq \tilde{P}(\tilde{X}(t_2) \in \mathbb{B}_{r_0}(\bar{x})) > 0.$$

This contradicts (3.3) and therefore (3.2) follows.

Note that if U is a diffusion process on \mathbb{R}^k having transition density function $p(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R}^k , then $Y(\cdot) = \varepsilon^{-2}U(\varepsilon^2 \cdot)$ has transition density function $p^Y(t, x, y) = \varepsilon^{2k}p(\varepsilon^2 t, \varepsilon^2 x, \varepsilon^2 y)$.

Let \mathcal{L}_ε be as introduced in (1.2). Its associated diffusion is given by

$$dU(t) = \varepsilon\sigma(U(t))dW(t) + b(U(t))dt.$$

Using Brownian scaling, we see that $Y(t) = \varepsilon^{-2}U(\varepsilon^2t)$ satisfies

$$dY(t) = \sigma(\varepsilon^2Y(t))dW(t) + b(\varepsilon^2Y(t))dt,$$

and thus has infinitesimal generator $\widehat{\mathcal{L}}_\varepsilon$ introduced above Theorem 3.1, namely,

$$\widehat{\mathcal{L}}_\varepsilon f(x) = \frac{1}{2}\text{Tr}(\sigma(\varepsilon^2x)D^2f(x)\sigma(\varepsilon^2x)^{tr}) + b(\varepsilon^2x) \cdot \nabla f(x), \quad f \in C_b^2(\mathbb{R}^k).$$

Next, let $p_\varepsilon^\dagger = p_{\mathcal{L}_\varepsilon, B_\varepsilon}^\dagger$, where $B_\varepsilon = \varepsilon^2(\mathbb{B}_{r_2}(x_0))^\circ$. Then by the above scaling relation, we conclude that

$$\varepsilon^{2k}p_\varepsilon^\dagger(\varepsilon^2t, \varepsilon^2x, \varepsilon^2y) = p_{\widehat{\mathcal{L}}_\varepsilon, (\mathbb{B}_{r_2}(x_0))^\circ}^\dagger(t, x, y)$$

for every $t > 0$ and $x, y \in (\mathbb{B}_{r_2}(x_0))^\circ$.

Applying Theorem 3.1 with $\gamma = r_1/r_2 \in (0, 1)$, we have that there is a constant $\kappa_1 > 0$ so that for every $\varepsilon \in (0, \varepsilon_0]$ and for $y \in \varepsilon^2\mathbb{B}_{r_1}(x_0)$ and $z \in \varepsilon^2\mathbb{B}_{r_1}(x_0)$,

$$\varepsilon^{2k}p_\varepsilon^\dagger(\varepsilon^2t_3, y, z) = p_{\widehat{\mathcal{L}}_\varepsilon, (\mathbb{B}_{r_2}(x_0))^\circ}^\dagger(t_3, \varepsilon^{-2}y, \varepsilon^{-2}z) \geq \kappa_1. \quad (3.4)$$

Finally, take $E = \mathbb{B}_{r_1}(x_0)$. Then for every for $x \in \mathbb{B}_{M\varepsilon^2}$ and $z \in \varepsilon^2E$,

$$\begin{aligned} \varepsilon^{2k}p_\varepsilon(t_1\varepsilon^2, x, z) &\geq \varepsilon^{2k} \int_{\varepsilon^2\mathbb{B}_{r_1}(x_0)} p_\varepsilon(t_2\varepsilon^2, x, y)p_\varepsilon(t_3\varepsilon^2, y, z)dy \\ &\geq \left(\inf_{y, z \in \varepsilon^2\mathbb{B}_{r_1}(x_0)} \varepsilon^{2k}p_\varepsilon^\dagger(t_3\varepsilon^2, y, z) \right) \int_{\varepsilon^2\mathbb{B}_{r_1}(x_0)} p_\varepsilon(t_2\varepsilon^2, x, y)dy \\ &\geq \kappa_1 \int_{\mathbb{R}^k} \varphi\left(\frac{y}{\varepsilon^2}\right)p_\varepsilon(t_2\varepsilon^2, x, y)dy \\ &\geq \kappa_1 \cdot \kappa_0, \end{aligned}$$

where the third inequality uses (3.4) and the fourth follows from (3.2). The result now follows on taking $\kappa = \kappa_1 \cdot \kappa_0$. \square

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