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Optimization of joint p-variations of Brownian semimartingales*

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Abstract

We study the optimization of the joint $(p^Y, p^Z)-$variations of two continuous semimartingales (Y, Z) driven by the same Itô process X. The p-variations are defined on random grids made of finitely many stopping times. We establish an explicit asymptotic lower bound for our criterion, valid in rather great generality on the grids, and we exhibit minimizing sequences of hitting time form. The asymptotics is such that the spatial increments of X and the number of grid points are suitably converging to 0 and $+\infty$ respectively.

Keywords: p-variation; almost-sure convergence; optimal stopping times.
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1 Introduction

Setting and objectives. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a one-dimensional Brownian motion B, with usual conditions on the filtration. Let $T > 0$ be a fixed time horizon. We consider a scalar Itô semimartingale X with dynamics

$$dX_t = b_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}B_t, \tag{1.1}$$

and two general continuous semimartingales Y and Z both driven by X,

$$\mathrm{d}Y_t = \mathrm{d}A_t^Y + \sigma_t^Y \mathrm{d}X_t, \qquad \mathrm{d}Z_t = \mathrm{d}A_t^Z + \sigma_t^Z \mathrm{d}X_t, \tag{1.2}$$

where A^Y and A^Z are continuous finite variation processes. In this setting, X should be viewed as a common control for Y and Z. Consider the weighted p^Y- and p^Z-variations of Y and Z, defined by

$$\begin{cases} \mathcal{Y}_T^n := \sum_{\tau_{i-1}^n < T} w_{\tau_{i-1}^n}^Y |Y_{\tau_i^n} - Y_{\tau_{i-1}^n}|^{p^Y}, & p^Y \in (0,2), \\ \mathcal{Z}_T^n := \sum_{\tau_{i-1}^n < T} w_{\tau_{i-1}^n}^Z |Z_{\tau_i^n} - Z_{\tau_{i-1}^n}|^{p^Z}, & p^Z \in (2,+\infty), \end{cases} \tag{1.3}$$

where $\mathcal{T}^n = \{\tau_0^n = 0 < \tau_1^n < \cdots < \tau_{N_T^n}^n = T\}$ is a random grid, namely a finite sequence of stopping times, and w^Y, w^Z are non-negative stochastic processes. The integer n is a

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convergence parameter explained later: essentially, as $n \to +\infty$, the $[0,T]$-partition $\mathcal{T}^n$ gets more and more dense.
Our aim is to minimize, asymptotically as $n \to +\infty$, the product

$$(\mathcal{Y}_T^n)^{1/p}(\mathcal{Z}_T^n)^{1/q} \tag{1.4}$$

for appropriate positive exponents p and q, over a large class of sequences of stopping times. By a simple rescaling, we are reduced to the case $1/p + 1/q = 1$: indeed, if $1/p + 1/q = 1/r$ with $r > 0$, by setting $p_r = p/r$ and $q_r = q/r$ satisfying $1/p_r + 1/q_r = 1$, it is equivalent to study the asymptotic behavior of $[(\mathcal{Y}_T^n)^{1/p_r}(\mathcal{Z}_T^n)^{1/q_r}]^{1/r}$. In that case, Theorem 2.1 should read as $\liminf_{n\to+\infty}(\mathcal{Y}_T^n)^{1/p}(\mathcal{Z}_T^n)^{1/q} \geq (L_T)^{1/r}$ for $p = rp^*$ and $q = rq^*$. Other results can be similarly adapted. Hence, from now on, we assume the case of conjugate exponents p and q, i.e. $1/p + 1/q = 1$. Since $p^Z > 2$, $\mathcal{Z}_T^n$ converges in probability to 0 as $n \to +\infty$ (see [8, 10, 7]), thus $\mathcal{Z}_T^n$ can be interpreted as an error or a functional performance. On the contrary, due to $p^Y < 2$, under mild conditions $\mathcal{Y}_T^n$ converges in probability to $+\infty$ and thus, $\mathcal{Y}_T^n$ can be viewed as a functional cost.

Applications. For instance, for a given n, $\mathcal{T}^n$ is a sequence of decisions, and the objective is to achieve the best performance (measured by $\mathcal{Z}_T^n$) with the minimal cost (i.e. $\mathcal{Y}_T^n$) among a set of admissible decisions: therefore, solving asymptotically the above optimization problem helps to exhibit an approximative optimal solution for fixed n.
Besides, it turns out that the minimizing sequence achieving the lower bound is related to hitting times for the control process X: thus, simply by observing X enables to find the best trade-off between cost and performance related to Y and Z.
Lastly, as explained in [4], the above minimization problem also allows to optimize a more general criterion of the form $\mathcal{C}(\mathcal{Y}_T^n, \mathcal{Z}_T^n)$, where the function $\mathcal{C}$ is increasing w.r.t. both variables.
In the limit case $p^Y = 0$ (discarded by our assumptions) with $w^Y \equiv 1$ (so that $\mathcal{Y}_T^n = N_T^n$), the problem is interpreted as the optimal discretization of Z with minimal number of discretization times, see for instance [5, 6].
If we denote by L_T the asymptotic lower bound of (1.4), our work provides a general non trivial lower-bound relation between p-variations of Y and Z,

$$\mathcal{Z}_T^n \geq \frac{L_T^q}{(\mathcal{Y}_T^n)^{q/p}},$$

which is valid asymptotically as $n \to +\infty$.

Litterature background and our contributions. So far, we have been voluntarly vague about the sense of the limit: actually it is either in *a.s.* sense either in probability, depending on the chosen asymptotics. We now define the set $\mathfrak{T}_{\rho_N}^{\text{adm.}}$ of admissible sequences of random grids, emphasizing the role of the control process X: it depends on a given deterministic positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0. The set $\mathfrak{T}_{\rho_N}^{\text{adm.}}$ is parameterized by a deterministic parameter ρ_N satisfying [1]

$$1 \leq \rho_N < (1 + \frac{\theta_X}{2}) \wedge \frac{4}{3}. \tag{1.5}$$

The above restriction on ρ_N is not easily explainable using heuristics. It comes from technical considerations detailed in [6] in order to control the convergence of discretized quantities built on the random grids related to the process X. We use some of these results, such as Propositions 1.3 and 1.4 for instance.

[1] the parameter $\theta_X \in (0,1]$ is related to the smoothness of σ^X, defined later in $(\mathbf{H}_i)$.

Definition 1.1. *We denote by* $\mathfrak{T}$ *the set of sequences of random grids* $\mathcal{T} = \{\mathcal{T}^n : n \geq 0\}$*, i.e.* $\mathcal{T}^n = \{\tau_0^n = 0 < \tau_1^n < \cdots < \tau_{N_T^n}^n = T\}$ *is built with a finite sequence of increasing stopping times on* $[0,T]$*. We say that* $\mathcal{T} \in \mathfrak{T}$ *is an admissible sequence (and we denote it by* $\mathcal{T} \in \mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$*) if in addition the assumptions* **($\mathbf{A}_X$)** *and* **($\mathbf{A}_N$)** *are fulfilled.*

($\mathbf{A}_X$) (spatial control) *For a finite r.v.* $C > 0$, $\sup_{1 \leq i \leq N_T^n} \sup_{\tau_{i-1}^n \leq t \leq \tau_i^n} |X_t - X_{\tau_{i-1}^n}| \leq C\varepsilon_n$ *for any* n, a.s. .

($\mathbf{A}_N$) (number of stopping times) *For a finite r.v.* $C > 0$, $N_T^n \leq C\varepsilon_n^{-2\rho_N}$ *for any* n, a.s. .

The larger ρ_N, the larger the set $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ considered for the minimization of (1.4). In [6], it is proved that $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ is quite large, since it contains most usual deterministic partitions (provided that $\rho_N > 1$) and exit times of various random sets with radius ε_n, i.e. all the usual stopping times we wish to consider in applications. For example, the general class of hitting times of the form

$$\tau_0^n := 0, \quad \tau_i^n := \inf\left\{t > \tau_{i-1}^n : X_t - X_{\tau_{i-1}^n} \notin]-\varepsilon_n f^{(1)}_{\tau_{i-1}^n}, \varepsilon_n f^{(2)}_{\tau_{i-1}^n}[\right\} \wedge T, \tag{1.6}$$

for two continuous adapted positive processes $(f_t^{(1)})_{t\geq 0}$ and $(f_t^{(2)})_{t\geq 0}$, belongs to the admissible sequences $\mathfrak{T}_1^{\mathrm{adm.}} \subset \mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$, provided that $\sum_{n=0}^{+\infty} \varepsilon_n^2 < +\infty$; see [6] for details.

The main results are stated in Section 2 and proved in Section 3. Our Theorem 2.1 states that $\liminf_{n\to+\infty} (\mathcal{Y}_T^n)^{1/p^*} (\mathcal{Z}_T^n)^{1/q^*}$ has *a.s.* a non-degenerate lower bound over the set $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$, for suitable p^* and q^*. For other (p,q), the limit of (1.4) is degenerate (Theorem 2.2). In addition, in Theorems 2.3 and 2.4, we show the existence of an admissible sequence of random grids of the hitting time form attaining the *a.s.* lower bound. All these sharp *a.s.* results are obtained under the stringent condition $\sum_{n=0}^{+\infty} \varepsilon_n^2 < +\infty$. Our arguments are inspired by *a.s.* asymptotic results from [6] (with some reinforcement). Then, assuming only $\varepsilon_n \to_{n\to+\infty} 0$, we get a lower bound in probability with a convergence in probability of the optimal sequence, see Theorem 2.5. However, it is worth noting that, here, obtaining directly this type of convergence is a tough task because our set $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ of admissible sequences of random grids is firstly far too large and secondly, it is not described in a way to apply standard results (like those of [7, Section 2.2]). Actually, one of the breakthrough of our work is the use of *a.s.* arguments to prove probability statements via a subsequence principle (see proof of Theorem 2.5); it may seem odd at first glance, actually almost sure convergence results are in this broad framework very efficient, practical and bespoke tools. Specifically, we can obtain *a.s.* uniform estimates of the increments $|M_{\tau_i^n} - M_{\tau_{i-1}^n}|$ between two dates and *a.s.* convergence of quadratic quantities like $\sum_{\tau_{i-1}^n < t} w_{\tau_{i-1}^n} |M_{\tau_i^n \wedge t} - M_{\tau_{i-1}^n}|^2$ to $\int_0^t w_s \mathrm{d}\langle M\rangle_s$ under the additional assumption $\sum_{n=0}^{+\infty} \varepsilon_n^2 < +\infty$, for any local martingale M and any continuous process w.

To our best knowledge, the first author dealing with this kind of optimization criterion is Fukasawa in [5], but rather with an expectation viewpoint. Extension to jump processes has recently been done in [9]. We refer to the introduction of [6] for the advantages of the current *a.s.* approach.

- We discard the case $p^Y = 0$ from our current work, it requires a quite different analysis which has been partly done in [6]. On the other hand, for the first time, in this article we consider general p-variations of Y and Z.
- The pure quadratic variation cases ($p^Y = 2$ or $p^Z = 2$) are uninteresting regarding the optimization of $\mathcal{T}$, since $\mathcal{Y}_T^n$ or $\mathcal{Z}_T^n$ in (1.3) then converges to a limit independent of $\mathcal{T}$ (see Proposition 1.5).

Notations

- C stands for a finite positive random variable, which will change from line to line, independent of n.
- Let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ be two non null sequences of random variables. We write $\alpha_n = O(\beta_n)$ (resp. $o(\beta_n)$), if $\sup_{n\geq 0}(|\beta_n^{-1}\alpha_n|) < +\infty$ *a.s.* (resp. $|\beta_n^{-1}\alpha_n| \xrightarrow[n\to+\infty]{a.s.} 0$).
- For any càdlàg process U, we define $|U|_* := \sup_{0\leq t\leq T}|U_t|$: we say that U is finite *a.s.* if $|U|_* < +\infty$ *a.s.* . In addition, we set $|\Delta U|_* := \sup_{1\leq i\leq N_T^n}\sup_{\tau_{i-1}^n\leq t\leq\tau_i^n}|\Delta U_t|$, where $\Delta U_t := U_t - U_{\varphi(t-)}$ and $\varphi(t) := \max\{\tau_j^n; \tau_j^n \leq t\}$. In particular, $\Delta U_{\tau_i^n} := U_{\tau_i^n} - U_{\tau_{i-1}^n}$.

Standing assumptions on the processes X, Y, Z, w^Y, w^Z

(H$_i$) The so-called control process X is defined in (1.1): its coefficients b^X and σ^X are scalar continuous adapted processes, with σ^X satisfying the ellipticity condition $\sigma^X > 0$ *a.s.* (in the sense $\sigma_t^X > 0$ for all $t\in[0,T]$ *a.s.*) and the smoothness assumption $|\sigma_t^X - \sigma_s^X| \leq C|t-s|^{\theta_X/2}$ for any $0\leq s,t\leq T$, with $\theta_X > 0$.
To simplify, we directly assume $b^X \equiv 0$, since b^X can be removed by a change of measure (under mild assumptions on the integrability of b^X/σ^X) and we study convergences in *a.s.* sense or in probability (valid under any equivalent measure).

(H$_{ii}$) The coefficients σ^Y, σ^Z of Y and Z defined in (1.2) are scalar continuous adapted processes, where $\sigma^Y > 0$, $\sigma^Z > 0$, $|\sigma_t^Y - \sigma_s^Y| \leq C|t-s|^{\theta_Y/2}$ and $|\sigma_t^Z - \sigma_s^Z| \leq C|t-s|^{\theta_Z/2}$ for any $0\leq s,t\leq T$, with $\theta_Y, \theta_Z > 0$. The finite variation processes A^Y and A^Z are *a.s.* $\frac{1}{2}^-$-Hölder continuous, i.e. for any $\rho > 0$ $\sup_{0\leq s\neq t\leq T}\frac{|A_t^Y - A_s^Y| + |A_t^Z - A_s^Z|}{|t-s|^{1/2-\rho}} < +\infty$ *a.s.* .

(H$_{iii}$) The weights w^Y and w^Z are non-negative continuous adapted stochastic processes.

Remark 1.2. *The smoothness assumptions in (**H**$_i$-**H**$_{ii}$) are little demanding. For instance, regarding X, if $\sigma_t^X = \sigma(t, X_t)$ with $|\sigma(t,x) - \sigma(s,y)| \leq c_\sigma(|y-x|^2 + |t-s|)^{\theta/2}$ (for two constants $c_\sigma > 0$ and $\theta > 0$) and σ is bounded, it is an easy exercice using the Garsia-Rodemich-Rumsey lemma to prove that **(H$_i$)** is valid for any $\theta_X \in (0,\theta)$. The assumptions on $A = A^Y, A^Z$ in **(H$_{ii}$)** are automatically satisfied if $\mathrm{d}A_t = b_t\mathrm{d}t$ with a bounded b. Different examples are $A_t = \mathrm{Cste}\times t^\theta$ with $\theta \geq 1/2$, or $A_t =$Brownian motion local time at a given level.*

We now highlight nice general properties available for sequences of random grids, this is repeatedly used in this work.

Proposition 1.3 ([6, Corollary 2.2])**.** *Let ρ_N be a parameter as in* (1.5) *and $\mathcal{T} \in \mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ for a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ satisfying $\sum_{n=0}^{+\infty}\varepsilon_n^2 < +\infty$. For any $\rho > 0$,*

$$\sup_{n\geq 0}\left(\varepsilon_n^{\rho-2}\sup_{1\leq i\leq N_T^n}\Delta\tau_i^n\right) < +\infty \quad \text{a.s.}.$$

Proposition 1.4 ([6, Corollary 2.3])**.** *Let $\mathcal{T} \in \mathfrak{T}$ be a sequence of random grids satisfying (**A**$_X$) for a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ satisfying $\sum_{n=0}^{+\infty}\varepsilon_n^2 < +\infty$. Let $((M_t^n)_{0\leq t\leq T})_{n\geq 0}$ be a sequence of scalar continuous local martingales such that $\langle M^n\rangle_t = \int_0^t \alpha_r^n\mathrm{d}r$ for a non-negative measurable adapted α^n satisfying the following inequality: there exists a non-negative* a.s. *finite random variable C_α and a parameter $\theta \geq 0$ such that*

$$0 \leq \alpha_r^n \leq C_\alpha|\Delta r|^\theta, \quad \forall\, 0\leq r\leq T, \forall n\geq 0, \quad \text{a.s.}.$$

Then, the following estimates hold, for any $\rho > 0$:

i) Under $(\mathbf{A}_X)$, $\sup_{n\geq 0}\big(\varepsilon_n^{\rho-\frac{1+\theta}{2}} \sup_{1\leq i\leq N_T^n}\sup_{\tau_{i-1}^n\leq t\leq \tau_i^n}|\Delta M_t^n|\big)<+\infty$, a.s. .

ii) Assuming additionally $(\mathbf{A}_N)$ *with a parameter* ρ_N *as in* (1.5) *(i.e.* $\mathcal{T}\in\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$*),* $\sup_{n\geq 0}\big(\varepsilon_n^{\rho-(1+\theta)} \sup_{1\leq i\leq N_T^n}\sup_{\tau_{i-1}^n\leq t\leq \tau_i^n}|\Delta M_t^n|\big)<+\infty$, a.s. .

The interest in the p-variation of stochastic processes was initiated by Lévy's result [8] on the quadratic variation of Brownian motion along dyadic grids:

$$\lim_{n\to+\infty}\sum_{i=0}^{2^n-1}\big(B_{i/2^n}-B_{(i+1)/2^n}\big)^2=1,\quad \textit{a.s.}.$$

Generalizations of this result to different grids and martingales lead to complications (see [2, 3, 10]). In our setting, we obtain the *a.s.* convergence of weighted quadratic variations under mild conditions; the next result is proved in Appendix.

Proposition 1.5. *Let* $\mathcal{T}\in\mathfrak{T}$ *be a sequence of random grids satisfying* $(\mathbf{A}_X)$ *for a sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *satisfying* $\sum_{n=0}^{+\infty}\varepsilon_n^2<+\infty$*. Let* $(H_t^n)_{0\leq t\leq T}$ *and* H *be scalar continuous adapted processes such that* $\sup_{t\in[0,T]}|H_t^n-H_t|\xrightarrow[n\to+\infty]{a.s.}0$*, and let* $(M_t)_{0\leq t\leq T}$ *be a scalar continuous local martingale such that* $\langle M\rangle_t=\int_0^t\alpha_r\mathrm{d}r$ *with* $\sup_{0\leq t\leq T}|\alpha_t|<+\infty$ a.s. . *Then*

$$\sum_{\tau_{i-1}^n<T}H_{\tau_{i-1}^n}^n(\Delta M_{\tau_i^n})^2\xrightarrow[n\to+\infty]{a.s.}\int_0^T H_t\mathrm{d}\langle M\rangle_t.$$

2 Main results

2.1 Almost sure convergence

Our first result gives a lower bound for a generic criterion in the *a.s.* sense.

Theorem 2.1. *Assume* $\sum_{n=0}^{+\infty}\varepsilon_n^2<+\infty$*. Set* $p^*=\frac{p^Z-p^Y}{p^Z-2}$ *and* $q^*=\frac{p^Z-p^Y}{2-p^Y}$*. Let* $\mathcal{T}$ *be an admissible sequence of random grids in* $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ *with, in addition to* (1.5), [2]

$$\rho_N<1+\frac{1}{2}\begin{cases}\dfrac{p^Y}{p^*-p^Y}\wedge\dfrac{\theta_Y p^Y}{p^*}\wedge\dfrac{1}{q^*-1}\wedge\dfrac{\theta_Z}{q^*} & \textit{if } p^Y\in(0,1),\\[2mm] \dfrac{1}{q^*-1}\wedge\dfrac{1}{p^*-1}\wedge\dfrac{\theta_Y}{p^*}\wedge\dfrac{\theta_Z}{q^*} & \textit{if } p^Y\in[1,2).\end{cases}$$

Then, setting $\mathcal{L}_t:=(w_t^Y(\sigma_t^Y)^{p^Y})^{1/p^*}(w_t^Z(\sigma_t^Z)^{p^Z})^{1/q^*}$*, we have*

$$\liminf_{n\to+\infty}(\mathcal{Y}_T^n)^{1/p^*}(\mathcal{Z}_T^n)^{1/q^*}\geq\int_0^T\mathcal{L}_t\mathrm{d}\langle X\rangle_t:=L_T,\quad\text{a.s.}.$$

Moreover, the above conjugate exponents p^* and q^* are in some sense optimal.

Theorem 2.2. *Assume* $\sum_{n=0}^{+\infty}\varepsilon_n^2<+\infty$*, our standing assumptions with* $X=Y=Z$ *and* $\omega^Y\omega^Z\neq 0$ a.s. *. Then, for any conjugate* p *and* q*,*

- *if* $p<p^*$ *and* $q>q^*$*, the* $\liminf$ *of* $(\mathcal{Y}_T^n)^{1/p}(\mathcal{Z}_T^n)^{1/q}$ *is* $+\infty$ a.s. *for any* $\mathcal{T}\in\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$ *with* ρ_N *as in* (1.5)*;*
- *if* $p>p^*$ *and* $q<q^*$*, the* $\liminf$ *of* $(\mathcal{Y}_T^n)^{1/p}(\mathcal{Z}_T^n)^{1/q}$ *is* a.s. *0 for all the sequences of random grids defined by* (1.6)*, where* $f^{(1)}$ *and* $f^{(2)}$ *are continuous adapted positive processes (which yield admissible sequences of random grids in* $\mathfrak{T}_1^{\mathrm{adm.}}$*).*

[2] for better analysing the inequality on ρ_N, observe that $p^Y<1\Rightarrow 1/p^Y+1/p^Z>1\Leftrightarrow p^*-p^Y>0$.

Observe interestingly that the lower bound L_T does not depend on X, but rather on the Itô decompositions of Y and Z w.r.t. the Brownian motion: indeed, denoting by $\bar\sigma_t^Y = \sigma_t^Y\sigma_t^X$ and $\bar\sigma_t^Z = \sigma_t^Z\sigma_t^X$ the diffusion coefficients of Y and Z w.r.t. B, using $p^Y/p^* + p^Z/q^* = 2$ we obtain $L_T = \int_0^T (w_t^Y(\bar\sigma_t^Y)^{p^Y})^{1/p^*}(w_t^Z(\bar\sigma_t^Z)^{p^Z})^{1/q^*}\mathrm{d}t$. Actually the independence of L_T w.r.t. the characteristics of X is not that surprising since we optimize the p-variations of Y and Z: the fundamental point is that, for any given intermediate process X (both controlling Y and Z, under rather general conditions), one can achieve the lower bound via a sequence of random grids built on this control X (see below Theorems 2.3 and 2.4). For applications where (Y,Z) is a system controlled by X, this is quite relevant; see for instance in stochastic finance [6], where X stands for a tradable asset and (Y,Z) is related to the portfolio invested in the asset.

We now provide an optimal admissible sequence of random grids such that our criterion converges *a.s.* to the above lower bound. Let $\chi(.)$ be a smooth function such that $\mathbf{1}_{]-\infty,1/2]} \le \chi(.) \le \mathbf{1}_{]-\infty,1]}$ and for $\mu \in (0,1]$, set $\chi_\mu(x) = \chi(x/\mu)$.

Theorem 2.3. *Assume the assumptions of Theorem 2.1 and let $\mu \in (0,1]$. For any $n \in \mathbb{N}$, define the random grids $\mathcal{T}_\mu^n$ by $\tau_0^n := 0$ and*

$$\tau_i^n := \inf\Bigg\{ t \ge \tau_{i-1}^n : |X_t - X_{\tau_{i-1}^n}| > \varepsilon_n \left(\frac{w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y} + \mu\chi_\mu\big(w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y}\big)}{w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z} + \mu\chi_\mu\big(w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z}\big)}\right)^{\frac{1}{p^Z-p^Y}} \Bigg\} \wedge T.$$

Then, $\mathcal{T}_\mu = \{\mathcal{T}_\mu^n : n \in \mathbb{N}\}$ belongs to $\mathfrak{T}_1^{\mathrm{adm.}}$ and is asymptotically μ-optimal in the following sense:

$$\limsup_{n\to+\infty} \left| (\mathcal{Y}_T^n)^{1/p^*}(\mathcal{Z}_T^n)^{1/q^*} - \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t \right| \le C\mu^{\frac{1}{(p^*\vee q^*)^2}} \left(\int_0^T \Big\{ \chi_\mu(w_t^Z|\sigma_t^Z|^{p^Z}) + \chi_\mu(w_t^Y|\sigma_t^Y|^{p^Y}) \Big\} \mathrm{d}\langle X\rangle_t \right)^{\frac{1}{p^*\vee q^*}} \underset{\mu\to 0}{\overset{a.s.}{\longrightarrow}} 0.$$

In particular, on the event $\{\forall t \in [0,T] : w_t^Z|\sigma_t^Z|^{p^Z} \wedge w_t^Y|\sigma_t^Y|^{p^Y} \ge \mu\}$, $(\mathcal{Y}_T^n)^{1/p^*}(\mathcal{Z}_T^n)^{1/q^*}$ converges a.s. *to $\int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t$.*

Theorem 2.4. *Assume the assumptions of Theorem 2.1 with in addition $\rho_N > 1$. Let ρ_μ satisfies $\rho_\mu \in]0,(\rho_N-1)/2]$ and $\rho_\mu < 2/(p^Z-2)$ and set $\mu_n = \varepsilon_n^{\rho_\mu(p^Z-p^Y)}$. For any $n \in \mathbb{N}$, define the sequence of stopping times $\mathcal{T}_{\rho_\mu}^{*,n}$ by $\tau_0^n := 0$ and*

$$\tau_i^n := \inf\Bigg\{ t \ge \tau_{i-1}^n : |X_t - X_{\tau_{i-1}^n}| > \varepsilon_n^{1+\rho_\mu} \left(\frac{w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y} + \mu_n\chi\big(w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y}\big)}{w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z} + \mu_n\chi\big(w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z}\big)}\right)^{\frac{1}{p^Z-p^Y}} \Bigg\} \wedge T. \tag{2.1}$$

Then, $\mathcal{T}^*_{\rho_\mu} = \{\mathcal{T}^{*,n}_{\rho_\mu} : n \ge 0\}$ belongs to $\mathfrak{T}^{\mathrm{adm.}}_{\rho_N}$ and is asymptotically optimal:

$$(\mathcal{Y}_T^n)^{1/p^*}(\mathcal{Z}_T^n)^{1/q^*} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t.$$

This is an improvement to the type of results proved in [6], where only μ-optimality is established. With the current arguments, we could also derive optimal sequences in [6]. Actually, the optimal sequence $\mathcal{T}^*_{\rho_\mu}$ is not in $\mathfrak{T}_1^{\mathrm{adm.}}$ (see the proof of Theorem 2.4): for $\rho_N = 1$, so far we can prove only the existence of a μ-optimal sequence (Theorem 2.3).

2.2 Convergence in probability

Theorem 2.5. *Assume only that $\varepsilon_n \to_{n\to+\infty} 0$.*
1) Consider the notation and definition of Theorem 2.1; for any admissible sequence of random grids $\mathcal{T} \in \mathfrak{T}^{\text{adm.}}_{\rho_N}$, we have

$$\forall\delta > 0, \qquad \lim_{n\to+\infty} \mathbb{P}\Big((\mathcal{Y}^n_T)^{1/p^*}(\mathcal{Z}^n_T)^{1/q^*} \geq \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t - \delta\Big) = 1.$$

2) For the admissible sequence $\mathcal{T}^*_{\rho_\mu}$ defined in Theorem 2.4, we have

$$\forall\delta > 0, \quad \lim_{n\to+\infty} \mathbb{P}\Big(\Big|(\mathcal{Y}^n_T)^{1/p^*}(\mathcal{Z}^n_T)^{1/q^*} - \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t\Big| \leq \delta\Big) = 1.$$

Proof. To go from *a.s.* results to convergence in probability results, we use the subsequence principle of [1, Theorem 20.5], stated as follows.

- $\mathcal{X}_n \underset{n\to+\infty}{\overset{\text{in prob.}}{\longrightarrow}} \mathcal{X}$ if, and only if, for any subsequence $(\mathcal{X}_{\sigma(n)})_{n\in\mathbb{N}}$ of $(\mathcal{X}_n)_{n\in\mathbb{N}}$, we can extract another subsequence $(\mathcal{X}_{\sigma\circ\sigma'(n)})_{n\in\mathbb{N}}$ such that $\mathcal{X}_{\sigma\circ\sigma'(n)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \mathcal{X}$.

Then, for any $\mathcal{T} \in \mathfrak{T}^{\text{adm.}}_{\rho_N}$, set $\mathcal{X}_n = \min\big(0, (\mathcal{Y}^n_T)^{1/p^*}(\mathcal{Z}^n_T)^{1/q^*} - \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t\big)$ and consider an arbitrary subsequence $(\mathcal{X}_{\sigma(n)})_n$: take another subsequence $(\varepsilon_{\sigma\circ\sigma'(n)})_{n\geq 0}$ such that $\sum_{n=0}^{+\infty} \varepsilon^2_{\sigma\circ\sigma'(n)} < +\infty$, then apply Theorem 2.1 to show that $\mathcal{X}_{\sigma\circ\sigma'(n)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 0$. Thus, we conclude that $\mathcal{X}_n \underset{n\to+\infty}{\overset{\text{in prob.}}{\longrightarrow}} 0$, which proves our first statement.

The argument is similar for $\mathcal{T}^*_{\rho_\mu}$ and yields our second statement. □

3 Proofs of *a.s.* convergence results

3.1 Proof of Theorem 2.1

We split the proof into three steps: decomposition of the criterion (1.4), lower bound for the main contribution, justification of neglected terms.

Step 1: decomposition of $\mathcal{Y}^n_T$ and of $\mathcal{Z}^n_T$. We follow a standard approach which consists in approximating the increments of the semimartingales Y and Z by the increments of their local martingale components and showing that the residual terms (i.e. the increments of their finite variation parts) tend to 0 quickly enough :

$$\mathcal{Y}^n_T := \sum_{\tau^n_{i-1}<T} w^Y_{\tau^n_{i-1}} |\Delta Y_{\tau^n_i}|^{p^Y} = \sum_{\tau^n_{i-1}<T} w^Y_{\tau^n_{i-1}} |\sigma^Y_{\tau^n_{i-1}} \Delta X_{\tau^n_i}|^{p^Y} + \mathcal{E}^n_{Y,1,T}, \tag{3.1}$$

$$\mathcal{Z}^n_T := \sum_{\tau^n_{i-1}<T} w^Z_{\tau^n_{i-1}} |\Delta Z_{\tau^n_i}|^{p^Z} = \sum_{\tau^n_{i-1}<T} w^Z_{\tau^n_{i-1}} |\sigma^Z_{\tau^n_{i-1}} \Delta X_{\tau^n_i}|^{p^Z} + \mathcal{E}^n_{Z,1,T}, \tag{3.2}$$

where

$$\mathcal{E}^n_{Y,1,T} := \sum_{\tau^n_{i-1}<T} w^Y_{\tau^n_{i-1}} \Big(|\Delta Y_{\tau^n_i}|^{p^Y} - |\sigma^Y_{\tau^n_{i-1}} \Delta X_{\tau^n_i}|^{p^Y}\Big),$$
$$\mathcal{E}^n_{Z,1,T} := \sum_{\tau^n_{i-1}<T} w^Z_{\tau^n_{i-1}} \Big(|\Delta Z_{\tau^n_i}|^{p^Z} - |\sigma^Z_{\tau^n_{i-1}} \Delta X_{\tau^n_i}|^{p^Z}\Big).$$

Step 2: lower bound for the main term. The aim of this step is to provide a simple proof of the lower bound stated in Theorem 2.1. The Hölder inequality immediately yields a lower bound for the product of the two main terms of (3.1) and (3.2), that is

$$\Big(\sum_{\tau_{i-1}^n<T} w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Y}\Big)^{1/p}\Big(\sum_{\tau_{i-1}^n<T} w^Z_{\tau^n_{i-1}}|\sigma^Z_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Z}\Big)^{1/q}$$
$$\geq \sum_{\tau_{i-1}^n<T}(w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y})^{1/p}(w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z})^{1/q}|\Delta X_{\tau^n_i}|^{p^Y/p+p^Z/q}. \tag{3.3}$$

Then, for $p^Y/p^*+p^Z/q^*=2$, we obtain a non trivial lower bound. Since we restrict to conjugate exponents, this corresponds to $p^*=\frac{p^Z-p^Y}{p^Z-2}$ and $q^*=\frac{p^Z-p^Y}{2-p^Y}$ and then,

$$\Big(\sum_{\tau_{i-1}^n<T} w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Y}\Big)^{1/p^*}\Big(\sum_{\tau_{i-1}^n<T} w^Z_{\tau^n_{i-1}}|\sigma^Z_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Z}\Big)^{1/q^*}$$
$$\geq \sum_{\tau_{i-1}^n<T}\mathcal{L}_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2 \xrightarrow[n\to+\infty]{a.s.} \int_0^T \mathcal{L}_t \mathrm{d}\langle X\rangle_t.$$

The last convergence follows from Proposition 1.5.

Step 3: the renormalized errors $\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\mathcal{E}^n_{Y,1,T}$ and $\varepsilon_n^{(p^Y-2\rho_N)(q^*-1)}\mathcal{E}^n_{Z,1,T}$ converge to 0 *a.s.* . If we admit the above convergences, then in view of (3.1), (3.2) and Step 2, we easily complete the proof of Theorem 2.1. Indeed , the cross products where $\mathcal{E}^n_{Y,1,T}$ and $\mathcal{E}^n_{Z,1,T}$ are involved, are easily estimated combining the above results with

$$|b^p-a^p|\leq|b-a|^p \text{ for } (a,b)\in\mathbb{R}^2_+ \text{ and } p\in]0,1], \tag{3.4}$$

($\mathbf{A}_X$) and **($\mathbf{A}_N$)**. Details are left to the reader.
• *Proof of* $\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\mathcal{E}^n_{Y,1,T} \xrightarrow[n\to+\infty]{a.s.} 0$. First, define $\mathrm{d}M^Y_t:=\sigma^Y_t\mathrm{d}X_t$ so that $\mathrm{d}Y_t=\mathrm{d}A^Y_t+\mathrm{d}M^Y_t$. We distinguish two cases.
▷ **Case** $p^Y\geq 1$. Use Taylor's theorem applied to the function $x\mapsto x^{p^Y}$ to get

$$\begin{aligned}
&\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}|\mathcal{E}^n_{Y,1,T}|\\
&\leq \varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\Big\{\sum_{\tau_{i-1}^n<T} w^Y_{\tau^n_{i-1}}\Big||\Delta M^Y_{\tau^n_i}|^{p^Y}-|\sigma^Y_{\tau^n_i}\Delta X_{\tau^n_i}|^{p^Y}\Big|\\
&\qquad\qquad + \sum_{\tau_{i-1}^n<T} w^Y_{\tau^n_{i-1}}\Big||\Delta Y_{\tau^n_i}|^{p^Y}-|\Delta M^Y_{\tau^n_i}|^{p^Y}\Big|\Big\}\\
&\leq \varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}|w^Y|_* p^Y\Big\{N^n_T\Big|\int_{\varphi(.)}^{.}\Delta\sigma^Y_s\mathrm{d}X_s\Big|_*\Big(|\Delta M^Y|_*\vee|\sigma^Y_{\varphi(.)}\Delta X|_*\Big)^{p^Y-1}\\
&\qquad\qquad + \sum_{\tau_{i-1}^n<T}|\Delta A^Y_{\tau^n_i}|\left(|\Delta Y|_*\vee|\Delta M^Y|_*\right)^{p^Y-1}\Big\}.
\end{aligned}$$

Apply Proposition 1.4 with $\theta=\theta_Y$ to bound $|\int_{\varphi(.)}^{.}\Delta\sigma^Y_s\mathrm{d}X_s\Big|_*$ by $C_\rho\varepsilon_n^{1+\theta_Y-\rho}$ for any $\rho>0$, Proposition 1.4 with $\theta=0$ to bound $|M^Y|_*\vee|\sigma^Y_{\varphi(.)}\Delta X|_*$ by $C_\rho\varepsilon_n^{1-\rho}$ for any $\rho>0$: combining this with **($\mathbf{A}_N$)** it yields that the first term in the bracket is bounded by $C_\rho\varepsilon_n^{-2\rho_N+1+\theta_Y+p^Y-1-\rho}$ for any $\rho>0$ (where the values of ρ and C_ρ have changed). The second term in the bracket is bounded $C_\rho\varepsilon_n^{p^Y-1-\rho}$ for any $\rho>0$: this is proved using similar arguments, the finite variation property of A^Y and Proposition 1.3 combined with $(\mathbf{H}_{ii})$. Since $-2\rho_N+1+\theta_Y\le 0$, the second term bound is smaller than the first term one; observe here that, whenever needed, we could relax the $\frac{1}{2}^-$-Hölder property of A^Y to a smaller Hölder exponent depending on ρ_N,θ_Y,p^Y, without changing the conclusion. To summarize, we have proved (for any $\rho>0$)

$$\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}|\mathcal{E}^n_{Y,1,T}|\le O\left(\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)+p^Y-2\rho_N+\theta_Y-\rho}\right).$$

It converges *a.s.* to 0 as soon as

$$\begin{aligned}0&<(p^Z-2\rho_N)(p^*-1)+p^Y-2\rho_N+\theta_Y\\&=(p^Z-2)(p^*-1)+p^Y-2+2(1-\rho_N)(p^*-1)+\theta_Y+2(1-\rho_N)=2(1-\rho_N)p^*+\theta_Y,\end{aligned}$$

which holds by taking $\rho_N<1+\frac{\theta_Y}{2p^*}$.

▷ **Case** $p^Y\in(0,1)$. Using (3.4) we have

$$\begin{aligned}\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}|\mathcal{E}^n_{Y,1,T}|&\le\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\sum_{\tau^n_{i-1}<T}w^Y_{\tau^n_{i-1}}\left|\Delta Y_{\tau^n_i}-\sigma^Y_{\tau^n_i}\Delta X_{\tau^n_i}\right|^{p^Y}\\&=\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\sum_{\tau^n_{i-1}<T}w^Y_{\tau^n_{i-1}}\left|\int_{\tau^n_{i-1}}^{\tau^n_i}\Delta\sigma^Y_t\,\mathrm{d}X_t+\Delta A^Y_{\tau^n_i}\right|^{p^Y}\\&\le\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}|w^Y|_*\Big(N^n_T\Big|\int_{\varphi(.)}^{.}\Delta\sigma^Y_s\,\mathrm{d}X_s\Big|_*^{p^Y}+\sum_{\tau^n_{i-1}<T}|\Delta A^Y_{\tau^n_i}|^{p^Y}\Big).\end{aligned}$$

In view of Proposition 1.4, the first term is $O(\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)-2\rho_N+(1+\theta_Y)p^Y-\rho})$, for any $\rho>0$, thus it converges to 0 provided that $0<(p^Z-2\rho_N)(p^*-1)-2\rho_N+(1+\theta_Y)p^Y=2(1-\rho_N)(p^*-1)+2(1-\rho_N)+\theta_Yp^Y=2(1-\rho_N)p^*+\theta_Yp^Y$, which holds under our assumptions. For the second term, we use Hölder's inequality

$$\begin{aligned}\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\sum_{\tau^n_{i-1}<T}|\Delta A^Y_{\tau^n_i}|^{p^Y}&\le\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)}\Big(\sum_{\tau^n_{i-1}<T}|\Delta A^Y_{\tau^n_i}|\Big)^{p^Y}(N^n_T)^{1-p^Y}\\&=O(\varepsilon_n^{(p^Z-2\rho_N)(p^*-1)-2\rho_N(1-p^Y)}).\end{aligned}$$

It converges to 0 since the exponent of ε_n is equal to $2(1-\rho_N)(p^*-1)-2\rho_N(1-p^Y)+2-p^Y=2(1-\rho_N)(p^*-p^Y)+p^Y>0$. Observe that in this case, we do not need the Hölder property of A^Y, we only use the finite variation property.

• *Proof of* $\varepsilon_n^{(p^Y-2\rho_N)(q^*-1)}\mathcal{E}^n_{Z,1,T}\xrightarrow[n\to+\infty]{a.s.}0$. The computations are identical to those for $\mathcal{E}^n_{Y,1,T}$, when $p^Y\ge1$, we skip details. □

3.2 Proof of Theorem 2.2

▷ **Case** $p<p^*$ **and** $q>q^*$, i.e. $p^Y/p+p^Z/q<2$. Using a lower bound as for (3.3) specialized to the assumption $X=Y=Z$, we deduce

$$\begin{aligned}(\mathcal{Y}^n_T)^{1/p}(\mathcal{Z}^n_T)^{1/q}&\ge\sum_{\tau^n_{i-1}<T}(w^Y_{\tau^n_{i-1}})^{1/p}(w^Z_{\tau^n_{i-1}})^{1/q}|\Delta X_{\tau^n_i}|^{p^Y/p+p^Z/q}\\&\ge|\Delta X|_*^{p^Y/p+p^Z/q-2}\sum_{\tau^n_{i-1}<T}(w^Y_{\tau^n_{i-1}})^{1/p}(w^Z_{\tau^n_{i-1}})^{1/q}|\Delta X_{\tau^n_i}|^2.\end{aligned}$$

On the one hand, $|\Delta X|_*^{p^Y/p+p^Z/q-2}\xrightarrow[n\to+\infty]{a.s.}+\infty$ because of $(\mathbf{A}_X)$; on the other hand, the above sum converges *a.s.* to a positive r.v. thanks to Proposition 1.5 and $w^Yw^Z\ne0$. We are done.

$\triangleright$ **Case** $p > p^*$ **and** $q < q^*$, i.e. $p^Y/p + p^Z/q > 2$. The sequence $\mathcal{T}^n_{f^{(1)},f^{(2)}}$ of increasing stopping times defined in (1.6) is such that $\mathcal{T}_{f^{(1)},f^{(2)}} = \{\mathcal{T}^n_{f^{(1)},f^{(2)}} : n \in \mathbb{N}\}$ belongs to $\mathfrak{T}_1^{\text{adm.}}$, see [6, proof of Proposition 2.4]. Then

$$\Big(\sum_{\tau^n_{i-1}<T} w^Y_{\tau^n_{i-1}} |\Delta X_{\tau^n_i}|^{p^Y}\Big)^{1/p}\Big(\sum_{\tau^n_{i-1}<T} w^Z_{\tau^n_{i-1}} |\Delta X_{\tau^n_i}|^{p^Z}\Big)^{1/q} = O(\varepsilon_n^{p^Y/p+p^Z/q-2}) \xrightarrow[n\to+\infty]{a.s.} 0.$$

Hence, for each element $\mathcal{T}_{f^{(1)},f^{(2)}}$, the limit of $(\mathcal{Y}^n_T)^{1/p}(\mathcal{Z}^n_T)^{1/q}$ equals 0. $\square$

3.3 Proof of Theorem 2.3

We repeatedly use the inequality $x + \mu \geq x + \mu\chi_\mu(x) \geq \mu/2, \forall x \geq 0$.
Firstly, we check the admissibility of $\mathcal{T}_\mu$: the verification of the assumption **(A$_X$)** is immediate thanks to $\mu > 0$. Clearly $\mathcal{T}^n_\mu$ is a sequence of increasing stopping times. Regarding the assumption **(A$_N$)**, we point out that

$$\begin{aligned}\varepsilon_n^2 N^n_T &= \varepsilon_n^2 + \sum_{1\leq i\leq N^n_T-1}\left[\frac{w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y} + \mu\chi_\mu(w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y})}{w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z} + \mu\chi_\mu(w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z})}\right]^{\frac{-2}{p^Z-p^Y}} (\Delta X_{\tau^n_{i-1}})^2 \\ &\xrightarrow[n\to+\infty]{a.s.} \int_0^T \left[\frac{w^Y_t(\sigma^Y_t)^{p^Y} + \mu\chi_\mu(w^Y_t(\sigma^Y_t)^{p^Y})}{w^Z_t(\sigma^Z_t)^{p^Z} + \mu\chi_\mu(w^Z_t(\sigma^Z_t)^{p^Z})}\right]^{\frac{-2}{p^Z-p^Y}} \mathrm{d}\langle X\rangle_t\end{aligned}$$

using Proposition 1.5, available under **(A$_X$)** only. This proves that $N^n_T = O(\varepsilon_n^{-2})$ and finally, $\mathcal{T}_\mu \in \mathfrak{T}_1^{\text{adm.}}$.
Secondly, let us show the μ-optimality. Define for $t \geq 0$,

$$\mathcal{L}^\mu_t := \left(w^Y_t(\sigma^Y_t)^{p^Y} + \mu\chi_\mu(w^Y_t(\sigma^Y_t)^{p^Y})\right)^{1/p^*}\left(w^Z_t(\sigma^Z_t)^{p^Z} + \mu\chi_\mu(w^Z_t(\sigma^Z_t)^{p^Z})\right)^{1/q^*}.$$

Starting from the decompositions (3.1)-(3.2), write

$$\varepsilon_n^{2-p^Y}\mathcal{Y}^n_T = \sum_{1\leq i\leq N^n_T-1} \mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2 + \varepsilon_n^{2-p^Y}\left(\mathcal{E}^n_{Y,1,T} + \mathcal{E}^n_{Y,2,T} + \mathcal{E}^n_{Y,3,T}\right), \tag{3.5}$$

$$\mathcal{E}^n_{Y,2,T} := \sum_{1\leq i\leq N^n_T-1}\Big[w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Y} - \varepsilon_n^{p^Y-2}\mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2\Big],$$

$$\mathcal{E}^n_{Y,3,T} := w^Y_{N^n_T-1}|\sigma^Y_{N^n_T-1}\Delta X_T|^{p^Y},$$

$$\varepsilon_n^{2-p^Z}\mathcal{Z}^n_T = \sum_{1\leq i\leq N^n_T-1} \mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2 + \varepsilon_n^{2-p^Z}\left(\mathcal{E}^n_{Z,1,T} + \mathcal{E}^n_{Z,2,T} + \mathcal{E}^n_{Z,3,T}\right), \tag{3.6}$$

$$\mathcal{E}^n_{Z,2,T} := \sum_{1\leq i\leq N^n_T-1}\Big[w^Z_{\tau^n_{i-1}}|\sigma^Z_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Z} - \varepsilon_n^{p^Z-2}\mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2\Big],$$

$$\mathcal{E}^n_{Z,3,T} := w^Z_{N^n_T-1}|\sigma^Z_{N^n_T-1}\Delta X_T|^{p^Z}.$$

We now aim at establishing *a.s.* boundedness of $\varepsilon_n^{2-p^Y}\mathcal{Y}^n_T$ and $\varepsilon_n^{2-p^Z}\mathcal{Z}^n_T$. So far, we know that $\sum_{1\leq i\leq N^n_T-1}\mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2 \xrightarrow[n\to+\infty]{a.s.} \int_0^T \mathcal{L}^\mu_t \mathrm{d}\langle X\rangle_t$ (see Proposition 1.5). Furthermore, we have already established (see proof of Theorem 2.1, Step 3) that $\mathcal{E}^n_{Y,1,T} = o(\varepsilon_n^{-(p^Z-2\rho_N)(p^*-1)}) = o(\varepsilon_n^{p^Y-2})$ (using $\rho_N \geq 1$) and $\mathcal{E}^n_{Z,1,T} = o(\varepsilon_n^{-(p^Y-2\rho_N)(q^*-1)}) = o(\varepsilon_n^{p^Z-2})$ for any admissible sequence of random grids.
Moreover, $\mathcal{E}^n_{Y,3,T} = O(\varepsilon_n^{p^Y}) = o(\varepsilon_n^{p^Y-2})$ and $\mathcal{E}^n_{Z,3,T} = O(\varepsilon_n^{p^Z}) = o(\varepsilon_n^{p^Z-2})$.

Finally, regarding $\mathcal{E}^n_{Y,2,T}$ and $\mathcal{E}^n_{Z,2,T}$, we obtain that $|\varepsilon_n^{2-p^Y}\mathcal{E}^n_{Y,2,T}|$ is bounded by

$$\begin{aligned}
&\varepsilon_n^{2-p^Y}\sum_{1\le i\le N^n_T-1}\left|w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Y}-\varepsilon_n^{p^Y-2}\mathcal{L}^\mu_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2\right|\\
&=\sum_{1\le i\le N^n_T-1}\mu\chi_\mu(w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}|^{p^Y})\left(\frac{w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z}+\mu\chi_\mu(w^Z_{\tau^n_{i-1}}(\sigma^Z_{\tau^n_{i-1}})^{p^Z})}{w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y}+\mu\chi_\mu(w^Y_{\tau^n_{i-1}}(\sigma^Y_{\tau^n_{i-1}})^{p^Y})}\right)^{1/q^*}(\Delta X_{\tau^n_i})^2\\
&\le C\mu^{1/p^*}\sum_{1\le i\le N^n_T-1}\chi_\mu(w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}|^{p^Y})(\Delta X_{\tau^n_i})^2.
\end{aligned}$$

Moreover and similarly, $|\varepsilon_n^{2-p^Z}\mathcal{E}^n_{Z,2,T}|\le C\mu^{1/q^*}\sum_{1\le i\le N^n_T-1}\chi_\mu(w^Z_{\tau^n_{i-1}}|\sigma^Z_{\tau^n_{i-1}}|^{p^Z})(\Delta X_{\tau^n_i})^2$. Thus,

$$\begin{aligned}
\limsup_{n\to+\infty}|\varepsilon_n^{2-p^Y}\mathcal{E}^n_{Y,2,T}|&\le C\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t,\quad \textit{a.s.},\\
\limsup_{n\to+\infty}|\varepsilon_n^{2-p^Z}\mathcal{E}^n_{Z,2,T}|&\le C\mu^{1/q^*}\int_0^T\chi_\mu(w^Z_t|\sigma^Z_t|^{p^Z})\mathrm{d}\langle X\rangle_t,\quad \textit{a.s.}.
\end{aligned}$$

Let us summarize: setting $L^\mu_T:=\int_0^T\mathcal{L}^\mu_t\mathrm{d}\langle X\rangle_t$, $L_T:=\int_0^T\mathcal{L}_t\mathrm{d}\langle X\rangle_t$, we have established

$$\begin{aligned}
\limsup_{n\to+\infty}\left|\varepsilon_n^{2-p^Y}\mathcal{Y}^n_T-L^\mu_T\right|&\le C\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t,\\
\limsup_{n\to+\infty}\left|\varepsilon_n^{2-p^Z}\mathcal{Z}^n_T-L^\mu_T\right|&\le C\mu^{1/q^*}\int_0^T\chi_\mu(w^Z_t|\sigma^Z_t|^{p^Z})\mathrm{d}\langle X\rangle_t.
\end{aligned}$$

Then, using the inequality (3.4) , by simple computations we conclude that

$$\begin{aligned}
&\limsup_{n\to+\infty}\left|(\mathcal{Y}^n_T)^{1/p^*}(\mathcal{Z}^n_T)^{1/q^*}-L_T\right|\\
&\le\limsup_{n\to+\infty}\left|\varepsilon_n^{2-p^Z}\mathcal{Z}^n_T-L_T\right|^{1/q^*}\limsup_{n\to+\infty}\left(\varepsilon_n^{2-p^Y}\mathcal{Y}^n_T\right)^{1/p^*}\\
&\qquad+L_T^{1/q^*}\limsup_{n\to+\infty}\left|\varepsilon_n^{2-p^Y}\mathcal{Y}^n_T-L_T\right|^{1/p^*}\\
&\le C\Big[L^\mu_T-L_T+\mu^{1/q^*}\int_0^T\chi_\mu(w^Z_t|\sigma^Z_t|^{p^Z})\mathrm{d}\langle X\rangle_t\Big]^{1/q^*}\\
&\qquad\times\Big[L^\mu_T+\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t\Big]^{1/p^*}\\
&\quad+CL_T^{1/q^*}\Big[L^\mu_T-L_T+\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t\Big]^{1/p^*}\\
&\le C\Big[\mu^{1/q^*}\int_0^T\chi_\mu(w^Z_t|\sigma^Z_t|^{p^Z})\mathrm{d}\langle X\rangle_t+\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t\Big]^{1/p^*}\\
&\quad+C\Big[\mu^{1/q^*}\int_0^T\chi_\mu(w^Z_t|\sigma^Z_t|^{p^Z})\mathrm{d}\langle X\rangle_t+\mu^{1/p^*}\int_0^T\chi_\mu(w^Y_t|\sigma^Y_t|^{p^Y})\mathrm{d}\langle X\rangle_t\Big]^{1/q^*}.
\end{aligned}$$

Theorem 2.3 is proved. □

3.4 Proof of Theorem 2.4

▷ **Admissibility of the sequence** (2.1)**.** We set $\mathcal{R}_n(t):=\frac{w^Y_t(\sigma^Y_t)^{p^Y}+\mu_n\chi\left(w^Y_t(\sigma^Y_t)^{p^Y}\right)}{w^Z_t(\sigma^Z_t)^{p^Z}+\mu_n\chi\left(w^Z_t(\sigma^Z_t)^{p^Z}\right)}$ and we repeatedly use the *a.s.* inequality (for a r.v. $C\ge1$)

$$\frac{1}{C}(\mu_n\wedge1)\le\mathcal{R}_n(t)\le C(\mu_n\wedge1)^{-1},\quad\forall n\ge0,\ \forall t\in[0,T],\tag{3.7}$$

which comes from the property of χ and the boundedness of $w^Y, w^Z, \sigma^Y, \sigma^Z$. Then

$$\sup_{n\geq 0}\Big(\varepsilon_n^{-1}\sup_{1\leq i\leq N_T^n}\sup_{t\in(\tau_{i-1}^n,\tau_i^n]}|\Delta X_t|\Big)\leq \sup_{n\geq 0}\Big(\varepsilon_n^{\rho_\mu}\sup_{1\leq i\leq N_T^n}\big(\mathcal{R}_n(\tau_{i-1}^n)\big)^{\frac{1}{p^Z-p^Y}}\Big)<+\infty,\ \textit{a.s.},$$

which validates the assumption $(\mathbf{A}_X)$. Moreover, writing $N_T^n = 1+\sum_{1\leq i\leq N_T^n-1}1$ and using (3.7), we obtain *a.s.*

$$\begin{aligned}\varepsilon_n^{2\rho_N}N_T^n &\leq \varepsilon_n^{2\rho_N}+\varepsilon_n^{2(\rho_N-(1+\rho_\mu))}\sup_{1\leq i\leq N_T^n}\Big(\mathcal{R}_n(\tau_{i-1}^n)\Big)^{\frac{-2}{p^Z-p^Y}}\sum_{1\leq i\leq N_T^n-1}|\Delta X_{\tau_i^n}|^2\\ &\leq C\Big(\varepsilon_n^{2\rho_N}+\varepsilon_n^{2(\rho_N-(1+\rho_\mu))-2\rho_\mu}\sum_{1\leq i\leq N_T^n-1}|\Delta X_{\tau_i^n}|^2\Big).\end{aligned}$$

Then, thanks to convergence of Proposition 1.5, we obtain $\varepsilon_n^{2\rho_N}N_T^n\leq C\varepsilon_n^{2(\rho_N-(1+2\rho_\mu))}$ with a larger random variable C. Since $\rho_N\geq 1+2\rho_\mu$, the assumption $(\mathbf{A}_N)$ is fulfilled, thus $\mathcal{T}_{\rho_\mu}^*$ belongs to $\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$.

$\triangleright$ **Asymptotic optimality.** Let $\eta := \rho_\mu\frac{(2-p^Y)(p^Z-2)}{p^Z-p^Y}$. Using a similar decomposition to (3.5)-(3.6), write

$$\begin{aligned}\widetilde{\mathcal{L}}_t^\mu &:= \Big(w_t^Y(\sigma_t^Y)^{p^Y}+\mu\chi(w_t^Y(\sigma_t^Y)^{p^Y})\Big)^{1/p^*}\Big(w_t^Z(\sigma_t^Z)^{p^Z}+\mu\chi(w_t^Z(\sigma_t^Z)^{p^Z})\Big)^{1/q^*},\\ \varepsilon_n^{2-p^Y+\eta p^*}\mathcal{Y}_T^n &= \sum_{1\leq i\leq N_T^n-1}\widetilde{\mathcal{L}}^{\mu_n}_{\tau_{i-1}^n}(\Delta X_{\tau_i^n})^2+\varepsilon_n^{2-p^Y+\eta p^*}\Big(\mathcal{E}_{Y,1,T}^n+\widetilde{\mathcal{E}}_{Y,2,T}^n+\mathcal{E}_{Y,3,T}^n\Big),\\ \widetilde{\mathcal{E}}_{Y,2,T}^n &:= \sum_{1\leq i\leq N_T^n-1}\Big[w^Y_{\tau_{i-1}^n}|\sigma^Y_{\tau_{i-1}^n}\Delta X_{\tau_i^n}|^{p^Y}-\varepsilon_n^{p^Y-2-\eta p^*}\widetilde{\mathcal{L}}^{\mu_n}_{\tau_{i-1}^n}(\Delta X_{\tau_i^n})^2\Big],\\ \varepsilon_n^{2-p^Z-\eta q^*}\mathcal{Z}_T^n &= \sum_{1\leq i\leq N_T^n-1}\widetilde{\mathcal{L}}^{\mu_n}_{\tau_{i-1}^n}(\Delta X_{\tau_i^n})^2+\varepsilon_n^{2-p^Z-\eta q^*}\Big(\mathcal{E}_{Z,1,T}^n+\widetilde{\mathcal{E}}_{Z,2,T}^n+\mathcal{E}_{Z,3,T}^n\Big),\\ \widetilde{\mathcal{E}}_{Z,2,T}^n &:= \sum_{1\leq i\leq N_T^n-1}\Big[w^Z_{\tau_{i-1}^n}|\sigma^Z_{\tau_{i-1}^n}\Delta X_{\tau_i^n}|^{p^Z}-\varepsilon_n^{p^Z-2+\eta q^*}\widetilde{\mathcal{L}}^{\mu_n}_{\tau_{i-1}^n}(\Delta X_{\tau_i^n})^2\Big].\end{aligned}$$

• *Terms $\mathcal{E}_{Y,1,T}^n$ and $\mathcal{E}_{Z,1,T}^n$.* Because $\mathcal{T}_{\rho_\mu}^*\in\mathfrak{T}_{\rho_N}^{\mathrm{adm.}}$, the convergences to 0 given in Step 3 of the proof of Theorem 2.1 remain true: we argue that they imply the convergences to 0 of our renormalized terms. Indeed, on the one hand, observe that $(p^Z-2\rho_N)(p^*-1)\leq(p^Z-2)(p^*-1)=2-p^Y<2-p^Y+\eta p^*$. On the other hand, simple computations show that $(p^Y-2\rho_N)(q^*-1)\leq 2-p^Z-\eta q^*$ if and only if $\rho_\mu(2-p^Y)\leq 2(\rho_N-1)$, which holds true in view of our assumption $\rho_\mu\leq(\rho_N-1)/2$. Consequently, $\varepsilon_n^{2-p^Y+\eta p^*}\mathcal{E}_{Y,1,T}^n$ and $\varepsilon_n^{2-p^Z-\eta q^*}\mathcal{E}_{Z,1,T}^n$ both converge to 0 *a.s.* .

• *Terms $\mathcal{E}_{Y,3,T}^n$ and $\mathcal{E}_{Z,3,T}^n$.* Easily, $\varepsilon_n^{2-p^Y+\eta p^*}\mathcal{E}_{Y,3,T}^n=O(\varepsilon_n^{2+\eta p^*})=o(1)$ and $\varepsilon_n^{2-p^Z-\eta q^*}\mathcal{E}_{Z,3,T}^n=O(\varepsilon_n^{2-\eta q^*})=o(1)$ since $2-\eta q^*>0\iff\rho_\mu<2/(p^Z-2)$ which we assume.

• *Terms $\widetilde{\mathcal{E}}_{Y,2,T}^n$ and $\widetilde{\mathcal{E}}_{Z,2,T}^n$.* The touchy point concerns these terms. Writing

$$\begin{aligned}w^Y_{\tau_{i-1}^n}|\sigma^Y_{\tau_{i-1}^n}\Delta X_{\tau_i^n}|^{p^Y}=\Big(\big(w^Y_{\tau_{i-1}^n}(\sigma^Y_{\tau_{i-1}^n})^{p^Y}+\mu_n\chi(w^Y_{\tau_{i-1}^n}(\sigma^Y_{\tau_{i-1}^n})^{p^Y})\big)|\Delta X_{\tau_i^n}|^{p^Y-2}\\ -\mu_n\chi(w^Y_{\tau_{i-1}^n}(\sigma^Y_{\tau_{i-1}^n})^{p^Y}))|\Delta X_{\tau_i^n}|^{p^Y-2}\Big)(\Delta X_{\tau_i^n})^2,\end{aligned}$$

and remarking that $2-p^Y+\eta p^*+(1+\rho_\mu)(p^Y-2)=0$ by definition of η, we obtain that

$|\varepsilon_n^{2-p^Y+\eta p^*}\widetilde{\mathcal{E}}^n_{Y,2,T}|$ is bounded by

$$\begin{aligned}
&\varepsilon_n^{2-p^Y+\eta p^*}\sum_{1\le i\le N_T^n-1}\Big|w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}\Delta X_{\tau^n_i}|^{p^Y}-\varepsilon_n^{p^Y-2-\eta p^*}\widetilde{\mathcal{L}}^{\mu_n}_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2\Big|\\
&=\sum_{1\le i\le N_T^n-1}\mu_n\chi(w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}|^{p^Y})\big(\mathcal{R}_n(\tau^n_{i-1})\big)^{-1/q^*}(\Delta X_{\tau^n_i})^2\\
&\le C\mu_n^{1/p^*}\sum_{1\le i\le N_T^n-1}\chi(w^Y_{\tau^n_{i-1}}|\sigma^Y_{\tau^n_{i-1}}|^{p^Y})(\Delta X_{\tau^n_i})^2=O(\mu_n^{1/p^*})\xrightarrow[n\to+\infty]{a.s.}0.
\end{aligned}$$

Similarly, we can obtain $|\varepsilon_n^{2-p^Z-\eta q^*}\widetilde{\mathcal{E}}^n_{Z,2,T}|=O(\mu_n^{1/q^*})\xrightarrow[n\to+\infty]{a.s.}0$.

• *End of proof.* The above justifies that $\varepsilon_n^{2-p^Y+\eta q^*}\mathcal{Y}^n_T$ and $\varepsilon_n^{2-p^Z-\eta q^*}\mathcal{Z}^n_T$ both converge *a.s.* to $\lim_{n\to+\infty}\sum_{1\le i\le N_T^n-1}\widetilde{\mathcal{L}}^{\mu_n}_{\tau^n_{i-1}}(\Delta X_{\tau^n_i})^2=L_T$ owing to Proposition 1.5. The proof is done. □

A Proof of Proposition 1.5

As mentioned in the introduction, we use profusely results on almost sure convergence. This is an easy variant of [6, Proposition 2.3], we give details for the sake of completeness. The proof necessitates the following result.

Proposition A.1 ([6, Corollary 2.1]). *Let $p>0$ and let $\{(M_t^n)_{0\le t\le T}: n\ge 0\}$ be a sequence of scalar continuous local martingales vanishing at zero. Then,*

$$\sum_{n\ge 0}\langle M^n\rangle_T^{p/2}<+\infty\ \text{a.s.}\quad\Longleftrightarrow\quad\sum_{n\ge 0}\sup_{0\le t\le T}|M_t^n|^p<+\infty\ \text{a.s.}.$$

We are now in a position to prove Proposition 1.5. Itô's lemma yields

$$\sum_{\tau^n_{i-1}<T}H^n_{\tau^n_{i-1}}(\Delta M_{\tau^n_i})^2=2\int_0^T H^n_{\varphi(t)}\Delta M_t\mathrm{d}M_t+\int_0^T H^n_{\varphi(t)}\mathrm{d}\langle M\rangle_t.$$

The second term in the above r.h.s. readily converges *a.s.* to $\int_0^T H_t\mathrm{d}\langle M\rangle_t$. Thus, it remains to show that the stochastic integral w.r.t. $\mathrm{d}M_t$ converges *a.s.* to 0. Owing to Proposition A.1, it is enough to study the series of quadratic variations, i.e. to show that $\sum_{n\ge0}\Big[\int_0^T(\Delta M_t)^2(H^n_{\varphi(t)})^2\mathrm{d}\langle M\rangle_t\Big]^3<+\infty$ *a.s.*, and since α and H^n are *a.s.* bounded, uniformly in n, it is sufficient to show

$$\sum_{n\ge0}\Big[\int_0^T|\Delta M_t|^2\mathrm{d}t\Big]^3<+\infty\ \textit{a.s.}. \tag{A.1}$$

Clearly $\big[\int_0^T|\Delta M_t|^2\mathrm{d}t\big]^3$ is bounded by $T^3|\Delta M|_*^6\le C\varepsilon_n^2$ owing to Proposition 1.4 (item i)) for $\theta=0$ and $\rho=\frac{1}{6}$. The convergence (A.1) is proved and we are done. □

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