Electronic Journal of Probability

Vol. 11 (2006), Paper no. 27, pages 686–699.

Journal URL
http://www.math.washington.edu/~ejpecp/

Spatial smoothness of the stationary solutions of the 3D Navier–Stokes equations

Cyril ODASSO

Ecole Normale Supérieure de Cachan, antenne de Bretagne,
Avenue Robert Schuman, Campus de Ker Lann, 35170 Bruz (FRANCE).
and
IRMAR, UMR 6625 du CNRS, Campus de Beaulieu, 35042 Rennes cedex (FRANCE)

Abstract

We consider stationary solutions of the three dimensional Navier–Stokes equations (NS3D) with periodic boundary conditions and driven by an external force which might have a deterministic and a random part. The random part of the force is white in time and very smooth in space. We investigate smoothness properties in space of the stationary solutions.

Classical technics for studying smoothness of stochastic PDEs do not seem to apply since global existence of strong solutions is not known. We use the Kolmogorov operator and Galerkin approximations. We first assume that the noise has spatial regularity of order p in the L^2 based Sobolev spaces, in other words that its paths are in H^p. Then we prove that at each fixed time the law of the stationary solutions is supported by H^{p+1}.

Then, using a totally different technic, we prove that if the noise has Gevrey regularity then at each fixed time, the law of a stationary solution is supported by a Gevrey space. Some informations on the Kolmogorov dissipation scale are deduced

Key words: Stochastic three-dimensional Navier-Stokes equations, invariant measure, Gevrey spaces, Kolmogorov operator, Kolmogorov dissipation scale

AMS 2000 Subject Classification: Primary 35Q30 ; 76D05.

Submitted to EJP on January 23 2006, final version accepted June 7 2006.

Introduction

We are concerned with the stochastic Navier–Stokes equations in dimension 3 (NS3D) with periodic boundary conditions and zero mean value. These equations describe the time evolution of an incompressible fluid and are given by

$$\begin{cases} dX + \nu(-\Delta)X\,dt + (X,\nabla)X\,dt + \nabla p\,dt = \phi(X)dW + g(X)dt, \\ (\text{div } X)\,(t,\xi) = 0, \qquad \text{for } \xi \in D, \ \ t>0, \\ \int_D X(t,\xi)d\xi = 0, \qquad \text{for } t>0, \\ X(0,\xi) = x_0(\xi), \qquad \text{for } \xi \in D, \end{cases} \tag{0.1}$$

where $D = (0, 2\pi)^3$. We have denoted by $X(t, \xi)$ the velocity and by $p(t, \xi)$ the pressure at time t and at the point $\xi \in D$, also ν denotes the viscosity. The external force acting on the fluid is the sum of a random force of white noise type $\phi(X)dW$ and a deterministic one $g(X)dt$.

As it is well known, in the deterministic case, global existence of weak (in the PDE sense) solutions and uniqueness of strong solutions hold for the Navier-Stokes equations. In space dimension two, weak solutions are strong and global existence and uniqueness follows. Such a result is an open problem in dimension three (see [19] for a survey on these questions).

In the stochastic case, the situation is similar. However due to the lack of uniqueness, we have to work with global weak (in the PDE sense) solutions of the martingale problem (see [7] for a survey on the stochastic case). Roughly speaking, this means that in (0.1), we take X, p and W for unknown.

As is usual in the context of the incompressible Navier-Stokes equation, we get rid of the pressure thanks to the Leray projector. Let us denote by (X, W) a weak (in the PDE sense) stationary solution of the martingale problem (0.1) and by μ the the law of $X(t)$, which is an invariant measure if we can prove that (0.1) defines a Markov evolution. In this article, we establish that μ admits a finite moment in spaces of smooth functions provided the external force is sufficiently smooth. We think that this is an interesting question to study. First, it can be seen that if we were able to prove that μ has a moment of sufficiently high order in a well chosen Sobolev norm (order 4 in H^1 or 2 in H^2 for instance) then this would imply global existence of strong solutions for μ almost every initial data.

Moreover, this result is an important ingredient if one tries to follow the method of [2] to construct a Markov transition semi-group in $H^p(D)$ under suitable conditions on ϕ and g. Since even uniqueness in law is not known for NS3D, such a result might be important.

We first prove that if the external force is in $H^{p-1}(D)$ and the noise term has paths in $H^p(D)$ then μ admits a finite moment in the Sobolev space $H^{p+1}(D)$

Note that analogous results are well-known for the two dimensional Navier–Stokes equations (NS2D). Actually a stronger result is true for NS2D. Namely, for any square integrable x_0, the unique solution of NS2D is continuous from $(0, \infty)$ into $H^p(D)$ and is square integrable from (t_0, t_1) into $H^{p+1}(D)$. It follows that μ admits moments of any orders in $H^p(D)$ and a moment of order 2 in $H^{p+1}(D)$. This stronger result is linked to the global existence of strong solutions for NS2D.

This kind of idea cannot be used for NS3D and we use a generalization of an idea used in [2] for the case $p = 1$. The method is based on the use of the Kolmogorov operator applied to suitable Lyapunov functional. These functionals have already been used in the deterministic case in [18], chapter 4.

Using a totally different method, we establish also that the invariant measure μ admits a moment in a Gevrey class of functions provided the external force has the same regularity. Gevrey regularity has been studied in the deterministic case in [10] and [11]. Our method is based on tools developed in [10]. In [14], [17] these tools have been used to obtain an exponential moment for the invariant measure in Gevrey norms in the two dimensional case. The arguments used in [14], [17] do not generalize to the three dimensional case since there strong existence and uniqueness is used. The three dimensional case NS3D requires substantial adaptations. We develop a framework which gives a control on a Gevrey norm by using a control of the $H^1(D)$–norm of X only at fixed time.

Actually, in this way, we are able to generalize to NS3D the results of [14], [17]. However, we do not have exponential moments. We deduce that the Kolmogorov dissipation scale is larger than $\nu^{6+\delta}$. This is certainly not optimal since it is expected that the scale is of order $\nu^{\frac{3}{4}}$. Note that our result is rigorous and does not use any heuristic argument.

1 Notations

For $m \in \mathbb{N}$, we denote by $\mathbb{H}^m_{\text{per}}(D)$ the space of functions which are restrictions of periodic functions in $H^m_{loc}(D)^3$ and whose average is zero on D. We set

$$H = \left\{ X \in \mathbb{H}^0_{\text{per}}(D) \mid \text{ div } X = 0 \text{ on } D \right\},$$

and

$$V = H \cap \mathbb{H}^1_{\text{per}}(D).$$

Let π be the orthogonal projection in $L^2(D)^3$ onto the space H. We set

$$A = \pi\left(-\Delta\right), \quad D(A) = V \cap \mathbb{H}^2_{\text{per}}(D) \text{ and } B(u) = \pi\left((u, \nabla)u\right).$$

It is convenient to endow $\mathbb{H}^m_{\text{per}}(D)$ with the inner product $((\cdot,\cdot))_m = (A^{\frac{m}{2}}\cdot, A^{\frac{m}{2}}\cdot)_{L^2(D)^3}$. The corresponding norm is denoted by $\|\cdot\|_m$. It is classical that this defines a norm which is equivalent to the usual one. For $m = 0$ we write $|\cdot| = \|\cdot\|_0$ and for $m = 1$ we write $\|\cdot\| = \|\cdot\|_1$. Note that, since we work with functions whose average is zero on $(0, 2\pi)^3$, we have the following Poincaré type inequality

$$\|x\|_{m_1} \le \|x\|_{m_2}, \ m_1 \le m_2, \ x \in \mathbb{H}^{m_2}_{\text{per}}(D).$$

We also use the spaces $L^p(D)^3$ endowed with their usual norm denoted by $|\cdot|_p$. Moreover, given two Hilbert spaces K_1 and K_2, $\mathcal{L}_2(K_1; K_2)$ is the space of Hilbert-Schmidt operators from K_1 to K_2 .

The noise is described by a cylindrical Wiener process W defined on a Hilbert space U and a mapping ϕ defined on H with values in $\mathcal{L}_2(U; H)$. We also consider a deterministic forcing term described by a mapping g from H into H. More precise assumptions on ϕ and g are made below.

Now, we can write problem (0.1) in the form

$$\left\{\begin{array}{rcl} dX+\nu AXdt+B(X)dt & = & \phi(X)dW+g(X)dt, \\ X(0) & = & x_0. \end{array}\right. \tag{1.1}$$

In all the paper, we consider a H−valued stationary solution (X,W) of the martingale problem (1.1). Existence of such a solution has been proved in [8]. We denote by μ the law of $X(t)$. We do not consider any stationary solutions but only those which are limit in distribution of stationary solutions of Galerkin approximations of (1.1). More precisely, for any $N\in\mathbb{N}$, we denote by P_N the projection of A onto the vector space spanned by the first N eigenvalues and consider the following approximation of (1.1)

$$\left\{\begin{array}{rcl} dX_N+\nu AX_Ndt+P_NB(X_N)dt & = & P_N\phi(X_N)dW+P_Ng(X_N)dt, \\ X_N(0) & = & P_Nx_0. \end{array}\right. \tag{1.2}$$

It can be easily shown that (1.2) has a stationary solution X_N. Proceeding as in [9], we can see that their laws are tight in suitable functional spaces, and, up to a subsequence, (X_N,W) converges in law to a stationary solution (X,W) of (1.1). Actually the convergence holds in $C(0,t;D(A^{-s}))\cap L^2(0,t;D(A^{\frac{1}{2}-s}))$ for any $t,s>0$. We only consider stationary solutions constructed in that way.

To obtain an estimate for stationary solutions of (1.1) (limit of Galerkin approximations), we proceed as follows. We first prove the desired estimate for every stationary solutions of (1.2) and then we take the limit.

The reason why our results are only stated for solutions limit of Galerkin approximations comes from the fact that it is not known if computations applied to solutions of Galerkin approximations can be applied directly on solutions (X,W) of the three-dimensional Navier-Stokes equations.

Some of our results describe properties of μ in Gevrey type spaces. These spaces contain functions with exponentially decaying Fourier coefficients. According to the setting given in [10], we set for any $(\alpha,\beta)\in\mathbb{R}_*^+\times(0,1]$

$$\left\{\begin{array}{rcl} \|x\|^2_{G(\alpha,\beta)} & = & \left|A^{\frac{1}{2}}e^{\alpha A^{\frac{\beta}{2}}}x\right|^2=\sum_{k\in Z^3}|k|^2e^{2\alpha|k|^\beta}|\hat{x}(k)|^2, \\ G(\alpha,\beta) & = & \left\{x\in H\ \middle|\ \|x\|_{G(\alpha,\beta)}<\infty\right\}, \end{array}\right.$$

where $(\hat{x}(k))_{k\in\mathbb{Z}^3}$ are the Fourier coefficients of $x\in H$. Moreover, for any $(x,y)\in G(\alpha,\beta)^2$, we set

$$(x,y)_{G(\alpha,\beta)}=\left(A^{\frac{1}{2}}e^{\alpha A^{\frac{\beta}{2}}}x,A^{\frac{1}{2}}e^{\alpha A^{\frac{\beta}{2}}}y\right)=\sum_{k\in Z^3}|k|^2e^{2\alpha|k|^\beta}\Re e\left(\hat{x}(k)\overline{\hat{y}(k)}\right).$$

Clearly, $\left(G(\alpha,\beta),(\cdot,\cdot)_{G(\alpha,\beta)}\right)$ is a Hilbert space.

We are not interested in large viscosities and in all the article it is assumed that $\nu\le 1$. We will use various constants which may depend on some parameter such as $p,\nu,\ldots$ When this dependance is important, we make it explicit.

2 $\mathbb{H}^p_{\mathrm{per}}(D)$–regularity

Let $p \in \mathbb{N}$. We now make the following smoothness assumptions on the forcing terms.

Hypothesis 2.1 *The mapping ϕ (resp. g) takes values in $\mathcal{L}_2(U; H \cap \mathbb{H}^p_{per}(D))$ (resp. $H \cap \mathbb{H}^{p-1}_{per}(D)$) and $\phi : H \to \mathcal{L}_2(U; H \cap \mathbb{H}^p_{per}(D))$ and $g : H \to H \cap \mathbb{H}^{p-1}_{per}(D)$ are bounded.*

We set, when Hypothesis 2.1 holds,

$$B_p = \sup_H \left(\|\phi\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\mathrm{per}}(D))} + \|g\|^2_{p-1} \right).$$

It is also convenient to define

$$\bar{B}_p = \sup_H \|\phi\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\mathrm{per}}(D))} + \frac{1}{\nu} \sup_H \|g\|^2_{p-1}.$$

The aim of this section is to establish the following result.

Theorem 2.2 *Let μ be the invariant law of a stationary solution X of the three dimensional Navier-Stokes equations that is limit of stationary solutions of Galerkin approximations. Assume that Hypothesis 2.1 holds for some $p \geq 1$. For any $\nu \leq 1$, there exists $c_{p,\nu}$ depending on p, ν and B_p such that*

$$\int_H \|x\|^{\frac{2}{2p+1}}_{p+1} \, d\mu(x) \leq c_{p,\nu}.$$

Let us make few comments.

Note that it would be very important to obtain an estimate on $\int_H \|x\|^{\delta_p}_{p+1} \, d\mu(x)$ with $p\delta_p > 3$. Indeed, by Agmon inequality , we have

$$\int_H |x|^2_\infty \, d\mu(x) \leq c \int_H |x|^{2-3/p} \|x\|^{\frac{3}{p}}_p \, d\mu(x)$$

and this would give an estimate on the left hand side. Since uniqueness is easily shown to hold for solutions in $L^2(0,T; L^\infty(D)^3)$, a classical argument could be used to deduce that for μ almost every initial data there exists a unique global weak solution. Combining with the result in [6], this would partially solve Leray's conjecture.

Consider the case $g = 0$, $U = H$ and $\phi = A^{-s-\frac{3}{4}}$. Then Hypothesis 2.1 holds for any $p < s$ and the unique invariant measure of the three dimensional linear stochastic Stokes equations in H is in $\mathbb{H}^{r+1}_{\mathrm{per}}(D)$ with probability zero if $r > s$. Therefore it seems that $\|\cdot\|_{p+1}$ is the strongest norm we can control under Hypothesis 2.1.

Remark that in the two dimensional case a much stronger result holds. Indeed, standard arguments imply that under Hypothesis 2.1 we have for any invariant measure μ and any $q \in \mathbb{N}^*$

$$\int_H \|x\|^{2q}_p \, d\mu(x) < \infty, \quad \int_H \|x\|^2_{p+1} \, d\mu(x) < \infty.$$

In the proof, we use ideas developped in [18]. Similar but more refined techniques have been used in [11] to derive interesting properties on the decay of the Fourier spectrum of smooth solutions of the deterministic Navier-Stokes equations. Using such techniques does not seem to yield great improvement of our result. Indeed, trying to do so, we have been able to improve the estimate of Theorem 2.2 as follows

$$\nu \int_H \|x\|_{p+1}^{\frac{c_*}{p}} \, d\mu(x) \leq 2\bar{B}_p + c2^p(1+\bar{B}_0),$$

where c and c_* are positive constants and c_* is close to 1.02. We have not been able to derive very interesting results from this improved estimate and therefore have preferred to give the simpler one which follows from easier arguments.

Proof: The proof is rather standard. We do not give the details.

Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of invariant measures of stationary solutions $(X_N)_N$ of (1.2) such that there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ such that X_{N_k} converges to X in law. It follows that $(\mu_{N_k})_{k\in\mathbb{N}}$ converges to μ (considered as probability measures on $D(A^{-1})$).

We denote by L_N the Kolmogorov operator associated to the Galerkin approximation (1.2) of the stochastic Navier-Stokes equations

$$L_N f(x) = \frac{1}{2} tr\left((P_N\phi)(x)(P_N\phi)^*(x)D^2 f(x)\right) - (\nu Ax + B(x) - g(x), Df(x)),$$

for any $f \in C^2(P_N H;\mathbb{R})$ and $x \in P_N H$.

The proof of Theorem 2.2 is based on the fact that, for any $N \in \mathbb{N}$, we have

$$\int_{P_N H} L_N f(x) d\mu_N(x) = 0, \tag{2.1}$$

provided $f \in C^2(P_N H;\mathbb{R})$ verifies

$$\begin{cases} i) & \int_{P_N H} |f(x)| \, d\mu_N(x) < \infty, \\ ii) & \int_{P_N H} |L_N f(x)| \, d\mu_N(x) < \infty, \\ iii) & \int_{P_N H} |(P_N\phi)^*(x) Df(x)|^2 \, d\mu_N(x) < \infty. \end{cases} \tag{2.2}$$

It follows from [7], Chapter 1.2, Corollary 1.12 that for any $p \in \mathbb{N}$

$$\int_{P_N H} |x|^{2p} \, d\mu_N(x) = \mathbb{E}\left(|X_N(0)|^{2p}\right) \leq 2C_{p,\nu} < \infty, \tag{2.3}$$

and

$$\nu \int_{P_N H} \|x\|^2 \, d\mu_N(x) \leq \bar{B}_0. \tag{2.4}$$

The result in [7] is given for $g = 0$ but the generalization is easy.

Thanks to (2.3), we use (2.1) with

$$f = \frac{1}{\left(1 + \|\cdot\|_p^2\right)^{\varepsilon_p}}, \quad \varepsilon_p = \frac{1}{2p-1}.$$

We obtain after lengthy but easy computations and some estimates on the nonlinear term borrowed from [18], chapter 4 (in particular equation (4.8)) that there exists c_p such that

$$R_p \leq 2\bar{B}_p + c_p\bar{B}_0 + 1, \tag{2.5}$$

with

$$R_p = \nu \int_{P_N H} \frac{1 + \|x\|_{p+1}^2}{\left(1 + \|x\|_p^2\right)^{1+\varepsilon_p}} d\mu_N(x).$$

Then arguing as in [18], chapter 4, we set

$$M_p = \nu \int_{P_N H} \left(1 + \|x\|_p^2\right)^{1/2p-1} d\mu_N(x),$$

and deduce

$$M_{p+1} \leq R_p^{1/2p+1} M_p^{2p/2p+1},$$

which yields

$$\int_{P_N H} \|x\|_{p+1}^{\frac{2}{2p+1}} \, d\mu_N(x) \leq c_{p,\nu}. \tag{2.6}$$

It is then standard to deduce the results thanks to (2.6), the subsequence $(N_k)_k$, Fatou Lemma and lower semicontinuity of $\|\cdot\|_p$.

3 Gevrey regularity

3.1 Statement of the result

We now state and prove the main result of this paper. It states that if the external force is bounded in a Gevrey class of functions, then μ have support in another Gevrey class of function.

The main assumption in this section is the following

Hypothesis 3.1 *There exists $(\alpha, \beta) \in \mathbb{R}_+^* \times (0,1]$ such that the mappings $g : H \to G(\alpha, \beta)$ and $\phi : H \to \mathcal{L}_2(U; G(\alpha, \beta))$ are bounded.*

We set

$$B_0' = \sup_{x\in H} \|\phi(x)\|^2_{\mathcal{L}_2(U;G(\alpha,\beta))} + \sup_{x\in H} \|g(x)\|^2_{G(\alpha,\beta)}.$$

The aim of this section is to establish the following results proved in the following subsections.

Theorem 3.2 *Let μ be the invariant law of a stationary solution X of the three dimensional Navier-Stokes equations that is limit of stationary solutions of Galerkin approximations. Assume that Hypothesis 3.1 holds. There exist a family of constants $(K_\gamma)_{\gamma\in(0,1)}$ only depending on (α, β, B_0') and a family $(\alpha_\nu)_{\nu\in(0,1)}$ of measurable mappings $H \to (0, \alpha)$ such that for any $\nu \in (0, 1)$*

$$\int_H \|x\|^{2\gamma}_{G(\nu\alpha_\nu(x),\beta)} \, d\mu(x) \leq K_\gamma (1 + \bar{B}_0)^2 \nu^{-\frac{7}{2}}, \tag{3.1}$$

$$\int_H (\alpha_\nu(x))^{-\frac{\gamma}{2}} \, d\mu(x) \leq K_\gamma (1 + \bar{B}_0) \nu^{-\frac{5}{2}}, \tag{3.2}$$

for any $\gamma \in (0, 1)$.

This result gives some informations on the Kolmogorov dissipation scale. Indeed, it follows from (3.1), (3.2) that

$$|\hat{x}(k)| \le \|x\|_{G(\nu\alpha_\nu(x),\beta)} |k|^{-1} e^{-\nu\alpha_\nu(x)|k|^\beta},$$

where $(\hat{x}(k))_{k\in\mathbb{Z}^3}$ are the Fourier coefficients of x.

Hence, if Hypothesis 3.1 holds with $\beta = 1$ and $g = 0$, then $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg (\nu\alpha_\nu(x))^{-1}$. By (3.2), for any $\delta > 0$

$$\frac{1}{\alpha_\nu(x)} \le c_{\delta,\nu}(x)\nu^{-5(1+\delta)} \quad \text{with} \quad \int c_{\delta,\nu}(x)^{\frac{1}{2(1+\delta)}}\,\mu(dx) \le \Theta_\delta < \infty,$$

and Θ_δ not depending on ν. It follows that $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg \nu^{-(6+5\delta)}$. This indicates that the 3D–Kolmogorov dissipation scale is larger than $\nu^{6+5\delta}$. Note that by physical arguments it is expected that the 3D–Kolmogorov dissipation scale is of order of $\nu^{\frac{3}{4}}$.

Making analogous computations, we obtain that, for $g \ne 0$ and $\beta \in (0, 1]$, the 3D–Kolmogorov dissipation scale is larger than $\nu^{\frac{8}{\beta}+\delta}$ for $\delta > 0$.

In [14], [17], analogous results to Theorem 3.2 are proved for the 2D–Navier-Stokes equations. Moreover, it is deduced in [14] that the 2D–Kolmogorov dissipation scale is larger than $\nu^{\frac{25}{2}}$. In [1], it is established that the 2D–Kolmogorov dissipation scale is larger than $\nu^{\frac{1}{2}}$, which is the physically expected result.

We can also control a moment of a fixed Gevrey norm.

Corollary 3.3 *Under the same assumptions, there exists a family $(C_{\gamma,\alpha',\beta',\nu})_{\gamma,\alpha',\beta',\nu}$ only depending on $(\alpha, \beta, B_0', \nu)$ such that*

$$\int \left(\ln^+ \|x\|^2_{G(\alpha',\beta')}\right)^\gamma d\mu(x) \le C_{\gamma,\alpha',\beta',\nu}, \tag{3.3}$$

where $\ln^+ r = \max\{0, \ln r\}$ and provided $\alpha' > 0$ and $\beta', \gamma > 0$ verify

$$\beta' < \beta \quad \text{and} \quad 2\gamma < \frac{\beta}{\beta'} - 1.$$

3.2 Estimate of blow-up time in Gevrey spaces

Before proving Theorem 3.2, we establish the following result which implies that the time of blow-up of the solution in Gevrey spaces admits a negative moment.

Lemma 3.4 *Assume that Hypothesis 3.1 holds. For any N, any stationary solution X_N of (1.2) and any $\nu \in (0, 1)$, there exist K only depending on (α, β, B_0') and a stopping time $\tau^N > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in(0,\tau^N)} \|X_N(t)\|^2_{G(\nu t,\beta)}\right) \le \frac{4}{\nu}(\bar{B}_0 + 1), \tag{3.4}$$

$$\mathbb{P}\left(\tau^N < t\right) \le K(\bar{B}_0 + 1)t^{\frac{1}{2}}\nu^{-\frac{5}{2}}. \tag{3.5}$$

This result is a refinement of the result developed by Foias and Temam in [10] and is strongly based on the tools developed in this latter article. We denote by μ_N the invariant law associated to X_N. Let us set

$$\tau^N = \inf\left\{ t \geq 0 \,\middle|\, 1 + \|X_N(t)\|^2_{G(\nu t,\beta)} > 4\left(\|X_N(0)\|^2 + 1\right)\right\}. \tag{3.6}$$

Clearly

$$\mathbb{E}\left(\sup_{t\in(0,\tau^N)} \|X_N(t)\|^2_{G(\nu t,\beta)}\right) \leq 4\mathbb{E}\left(\|X_N(0)\|^2 + 1\right)$$

and (3.4) follows from (2.4). It remains to prove (3.5).

We apply Itô formula to $\|X_N(t)\|^2_{G(\nu t,\beta)}$ for $t \in (0,\alpha)$

$$\begin{aligned} d\|X_N(t)\|^2_{G(\nu t,\beta)} + 2\nu \left\|A^{\frac12} X_N(t)\right\|^2_{G(\nu t,\beta)} dt = \nu \left\|A^{\frac{\beta}{2}} X_N(t)\right\|^2_{G(\nu t,\beta)} dt \\ + dM(t) + I(t)dt, \end{aligned} \tag{3.7}$$

where

$$\begin{cases} I(t) = 2I_g(t) + 2I_B(t) + I_\phi(t), & I_B(t) = -\left(X_N(t), B(X_N(t))\right)_{G(\nu t,\beta)}, \\ I_\phi(t) = \|P_N\phi(X_N(t))\|^2_{\mathcal{L}_2(U;G(\nu t,\beta))}, & I_g(t) = \left(g(X_N(t)), X_N(t)\right)_{G(\nu t,\beta)}, \\ M(t) = 2\int_0^t \left(X_N(s), \phi(X_N(s))dW(s)\right)_{G(\nu t,\beta)}. \end{cases}$$

The following inequality is proved in [10] for $\beta \leq 1$

$$2I_B(t) \leq \nu \left\|A^{\frac12} X_N(t)\right\|^2_{G(\nu t,\beta)} + \frac{c}{\nu^3}\|X_N(t)\|^6_{G(\nu t,\beta)}. \tag{3.8}$$

By Hypothesis 3.1 we have

$$I_\phi(t) + 2I_g(t) \leq \|X_N(t)\|^6_{G(\nu t,\beta)} + B_0' + 1. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain since $\beta, \nu \leq 1$

$$d\|X_N(t)\|^2_{G(\nu t,\beta)} \leq dM(t) + \frac{c}{\nu^3}\|X_N(t)\|^6_{G(\nu t,\beta)} dt + (B_0'+1)dt. \tag{3.10}$$

Applying Ito formula to $\left(1 + \|X_N(t)\|^2_{G(\nu t,\beta)}\right)^{-2}$, we then deduce from (3.10) and from Hypothesis 3.1 that for any $t \in (0,\alpha)$ and any $\nu \leq 1$

$$-d\left(1 + \|X_N(t)\|^2_{G(\nu t,\beta)}\right)^{-2} \leq d\mathcal{M}(t) + C_0\nu^{-3}dt, \tag{3.11}$$

where $C_0 = c(B_0'+1)$ and

$$\mathcal{M}(t) = 4\int_0^t \left(1 + \|X_N(s)\|^2_{G(\nu t,\beta)}\right)^{-3} \left(X_N(s)\,,\, \phi(X_N(s))dW(s)\right)_{G(\nu t,\beta)}.$$

Setting

$$\begin{cases} \tau_0^N &= \inf\left\{t \in (0,\alpha) \,\middle|\, \mathcal{M}(t) > \frac{1}{4\left(1+\|X_N(0)\|^2\right)^2}\right\}, \\ \tau_1^N &= \tau_0^N \wedge \left(\frac{\nu^3}{4C_0\left(1+\|X_N(0)\|^2\right)^2}\right), \end{cases}$$

we obtain by integration of (3.11) on $[0,t]$ for $t \in (0, \tau_1^N)$

$$1 + \|X_N(t)\|^2_{G(\nu t,\beta)} \leq 4\left(1 + \|X_N(0)\|^2\right).$$

We deduce that $\tau^N \geq \tau_1^N$ and

$$\mathbb{P}\left(\tau^N < t\right) \leq \mathbb{P}\left(\tau_0^N < t\right) + \mathbb{P}\left(\left(1 + \|X_N(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right). \tag{3.12}$$

Since μ is the law of $X_N(0)$, we have

$$\mathbb{P}\left(\left(1 + \|X_N(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right) = \mu_N\left(x \in H,\, 1 + \|x\|^2 \geq \frac{\nu^{\frac{3}{2}}}{(4C_0 t)^{\frac{1}{2}}}\right).$$

Applying Chebyshev inequality, we deduce from (2.4)

$$\begin{aligned} &\mathbb{P}\left(\left(1 + \|X_N(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right) \\ &\leq 2\nu^{-\frac{3}{2}}\left(C_0 t\right)^{\frac{1}{2}} \int_H (1 + \|x\|^2) d\mu_N(x) \\ &\leq 2\nu^{-\frac{5}{2}}\left(1 + \bar{B}_0\right)\left(C_0 t\right)^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

Moreover

$$\begin{aligned} \mathbb{P}\left(\tau_0^N < t\right) &= \mathbb{P}\left(4\left(\left(1 + \|X_N(0)\|^2\right)^2 \sup_{s\in[0,t\wedge\tau_0^N]} \mathcal{M}(s)\right) > 1\right) \\ &\leq 4\mathbb{E}\left(\left(1 + \|X_N(0)\|^2\right)^2 \sup_{s\in[0,t\wedge\tau_0^N]} \mathcal{M}(s)\right). \end{aligned}$$

Taking conditional expectation with respect to the σ–algebra $\mathcal{F}_0$ generated by $X_N(0)$ inside the expectation, it follows

$$\mathbb{P}\left(\tau_0^N < t\right) \leq 4\mathbb{E}\left(\left(1 + \|X_N(0)\|^2\right)^2 \mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_0^N]} \mathcal{M}(s) \,\middle|\, \mathcal{F}_0\right)\right).$$

By Burkholder-Davis-Gundy inequality (see Theorem 3.28 page 166 in [13]) we obtain

$$\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_0^N]} \mathcal{M}(s) \,\middle|\, \mathcal{F}_0\right) \leq c\mathbb{E}\left(\langle\mathcal{M}\rangle^{\frac{1}{2}}\left(t \wedge \tau_0^N\right) \,\middle|\, \mathcal{F}_0\right),$$

and

$$\mathbb{P}\left(\tau_0^N < t\right) \leq 4\mathbb{E}\left(\left(1 + \|X_N(0)\|^2\right)^2 \mathbb{E}\left(\langle\mathcal{M}\rangle^{\frac{1}{2}}\left(t \wedge \tau_0^N\right) \,\middle|\, \mathcal{F}_0\right)\right). \tag{3.14}$$

We have

$$\langle \mathcal{M} \rangle (t) = 4 \int_0^t \left(1 + \|X_N(s)\|^2_{G(\nu t,\beta)}\right)^{-6} \left| \left(A^{\frac{1}{2}} e^{\nu t A^{\frac{\beta}{2}}} \phi(X_N(s))\right)^* \left(A^{\frac{1}{2}} e^{\nu t A^{\frac{\beta}{2}}} X_N(s)\right) \right|_U^2 ds.$$

Therefore

$$\langle \mathcal{M} \rangle (t) \le 4 \int_0^t \left(1 + \|X_N(s)\|^2_{G(\nu t,\beta)}\right)^{-6} \|\phi(X_N(s))\|^2_{\mathcal{L}(U;G(\nu t,\beta))} \|X_N(s)\|^2_{G(\nu t,\beta)} \, ds.$$

It follows that

$$\langle \mathcal{M} \rangle (t \wedge \tau_0^N) \le \frac{B_0' t}{4^3 \left(1 + \|X_N(0)\|^2\right)^4}$$

Hence we infer from (3.14) and from $\|\cdot\| \le \|\cdot\|_{G(\nu t,\beta)}$ that

$$\mathbb{P}\left(\tau_0^N \le t\right) \le \sqrt{B_0' t}. \tag{3.15}$$

Combining (3.12), (3.13) and (3.15), we deduce (3.5).

3.3 Proof of Theorem 3.2

Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of invariant measures of stationary solutions $(X_N)_N$ of (1.2) such that there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ such that X_{N_k} converges to X in law. It follows that $(\mu_{N_k})_{k\in\mathbb{N}}$ converges to μ (considered as probability measures on $D(A^{-1})$).

Setting

$$\alpha_\nu(x) = \inf\left\{ s \ge 0 \,\middle|\, \|x\|^2_{G(\nu s,\beta)} > \frac{4}{\nu s^{\frac{1}{2}}}\left(\bar{B}_0 + 1\right) \right\},$$

it follows that for any $\gamma \in (0,1)$

$$\int \|x\|^{2\gamma}_{G(\nu\alpha_\nu(x),\beta)} \, d\mu(x) \le \left(\frac{4}{\nu}\right)^\gamma \left(\bar{B}_0 + 1\right)^\gamma \int (\alpha_\nu(x))^{-\frac{\gamma}{2}} \, d\mu(x). \tag{3.16}$$

Hence (3.1) is consequence of (3.2). Then in order to establish Theorem 3.2, it is sufficient to prove (3.2).

Clearly

$$\mathbb{P}\left(\|X_N(t)\|^2_{G(\nu t,\beta)} > \tfrac{4}{\nu t^{\frac{1}{2}}}\left(\bar{B}_0 + 1\right)\right) \le \mathbb{P}\left(\sup_{s\in[0,\tau^N]} \|X_N(s)\|^2_{G(\nu s,\beta)} > \tfrac{4}{\nu t^{\frac{1}{2}}}\left(\bar{B}_0 + 1\right)\right) + \mathbb{P}\left(\tau^N < t\right),$$

where τ^N has been defined in section 3.2. Applying Chebyshev inequality, we infer from Lemma 3.4 and from the fact that, for any $t > 0$, μ_N is the law of $X_N(t)$

$$\mu_N(\mathcal{O}_t) = \mathbb{P}\left(X_N(t) \in \mathcal{O}_t\right) \le (K+1)(1+\bar{B}_0) t^{\frac{1}{2}} \nu^{-\frac{5}{2}}. \tag{3.17}$$

where

$$\mathcal{O}_t = \left\{ x \in D(A^{-1})\,,\; \|x\|^2_{G(\nu t,\beta)} > \frac{4}{\nu t^{\frac{1}{2}}}\left(\bar{B}_0 + 1\right) \right\}.$$

Notice that $\mathcal{O}$ is an open subset of $D(A^{-1})$. Hence, since $\mu_{N_k} \to \mu$ (considered as probability measures on $D(A^{-1})$), then we deduce from (3.17) that

$$\mu(\mathcal{O}_t) \leq \liminf_k \left(\mu_{N_k}(\mathcal{O}_t)\right) \leq (K+1)(1+\bar{B}_0)t^{\frac{1}{2}}\nu^{-\frac{5}{2}}. \tag{3.18}$$

Notice that

$$\left\{ x \in D(A^{-1})\,,\; \alpha_\nu(x) < t \right\} \subset \mathcal{O}_t,$$

which yields, by (3.18) and $\mu(H) = 1$,

$$\mu\left(x \in H\,,\; \alpha_\nu(x) \leq t\right) \leq (K+1)(1+\bar{B}_0)t^{\frac{1}{2}}\nu^{-\frac{5}{2}}. \tag{3.19}$$

It is well-known that (3.19) for any $t > 0$ implies (3.2), which yields Theorem 3.2.

3.4 Proof of Corollary 3.3

To deduce Corollary 3.3 from Theorem 3.2, it is sufficient to prove that for any $(\alpha', \alpha, \beta', \beta) \in (0,\infty)^2 \times (0,1]^2$ such that $\beta' < \beta$, we have

$$\|x\|_{G(\alpha',\beta')} \leq \exp\left(c(\beta,\beta')\left(\alpha'\right)^{\frac{\beta}{\beta-\beta'}}\left(\alpha\right)^{-\frac{\beta'}{\beta-\beta'}} \right)\|x\|_{G(\alpha,\beta)}\,. \tag{3.20}$$

Indeed, (3.20) implies that for any $\gamma \in \mathbb{R}^+_*$

$$\left(\ln^+\|x\|^2_{G(\alpha',\beta')}\right)^\gamma \leq c_\gamma\left(c(\beta,\beta') + \left(\alpha'\right)^{\frac{\gamma\beta}{\beta-\beta'}}\left(\nu\alpha_\nu(x)\right)^{-\frac{\gamma\beta'}{\beta-\beta'}} + \left(\ln^+\|x\|^2_{G(\nu\alpha_\nu(x),\beta)}\right)^\gamma \right),$$

which yields Corollary 3.3 provided Theorem 3.2 is true.
We now establish (3.20). It follows from arithmetic-geometric inequality that for any $k \in \mathbb{Z}^3$

$$\alpha'\,|k|^{\beta'} \leq c(\beta,\beta')\left(\alpha'\right)^{\frac{\beta}{\beta-\beta'}}\left(\alpha\right)^{-\frac{\beta'}{\beta-\beta'}} + \alpha\,|k|^\beta\,. \tag{3.21}$$

Recalling that

$$\|x\|^2_{G(\alpha',\beta')} = \sum_{k\in\mathbb{Z}^3}|k|^2\exp\left(2\alpha'\,|k|^{\beta'}\right)|\hat{x}(k)|^2\,,$$

we infer (3.20) from (3.21).

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