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Stochastic Volterra equations under perturbations

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Abstract

We study stochastic perturbed Volterra equations of convolution type in an infinite dimensional case. Our interest is directed towards the existence and regularity of stochastic convolutions connected to the equations considered under some kind of perturbations. We use an operator theoretical method for the representation of solutions.

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1 Introduction

Since abstract linear Volterra equations are being used in many applications, it is becoming increasingly desirable to study stochastic perturbations. To date there is little published work on the conditions that an additive noise must satisfy to produce solutions, or some kind of qualitative behavior like asymptotic stability, see Appleby [1], [3] and Appleby-Freeman [2]. On the other hand, recently it has been demonstrated that the use of one-parameter systems of bounded and linear operators can help to analyze the existence of strong solutions to stochastic Volterra equations in the infinite dimensional framework [10], [11].

In the current paper we study stochastic Volterra equations (SVEs) of the convolution type with some perturbations. Up till now, see Karczewska [8, 9] and Karczewska and Lizama [10, 11], we have used the so-called resolvent approach which is a natural generalization of the semigroup approach usually used in the study of stochastic differential equations considered in infinite dimensions. The semigroup methods are a powerful tool. Unfortunately, this method can not be used in the study of stochastic integral equations, particularly of stochastic Volterra equations considered in this paper.

There are some generalizations of the classical theory of SDEs presented e.g. in Da Prato and Zabczyk [6]. One of them is the resolvent approach mentioned above. In Karczewska [12] several problems and difficulties arising during the study of SVEs were discussed in details. Moreover, the semigroup and the resolvent case have also been compared.

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The other generalization is that developed by K. Liu in the papers [14, 15, 16, 17], where stochastic retarded differential equations (SRDEs) in Hilbert/Banach space are studied. Liu has generalized classical results in a different direction than we have done. He has studied SRDEs with some kind of retarding part, some times referred to as the Hale type operator.

In his study, Liu [14] used two kinds of families of operators connected to the retarded equations. First, he used the semigroup of operators generated by the linear part of the equation. He proved the existence and uniqueness and a variation of constants formula of mild solutions to the equations considered. This result was proved by the contraction mapping theorem. Next, Liu constructed a new strong continuous one-parameter family of bounded linear operators, called the fundamental solutions or Green's operators of the stochastic retarded equations. This family was then used to define mild, weak and strong solutions to the SRDEs considered. Liu obtained relationships between the mentioned solutions and found conditions under which mild solutions became strong ones. In the remaining part of the paper, Liu obtained a very interesting result, that is, the strong solution approximations for mild solutions of the systems under consideration. This trick has played an important role, for instance, in stability and controllability starting from A. Ichikawa [7], till now, see, e.g., Liu [13, 16] and references therein. The paper [14] was finished with Burkholder-Davis-Grundy's inequalities for stochastic convolutions connected to the retarded equations studied. In general the stochastic convolutions do not necessarily have continuous trajectories. Liu shown that continuous versions of stochastic convolutions can be found by a strong approximation procedure.

The methods developed in [14], have been used in [15, 16, 17]. In [15] a class of retarded Langevin equations on Hilbert space were studied. In [16] neutral models were considered. The paper [17] deals on some regularity results for mild solutions of a class of linear functional equations on Hilbert space. The obtained results are generalization of those given in [14].

Following the ideas from [8, 9, 10, 11], in this work we investigate strong solutions, Itô-type formula and continuity of trajectories of mild solutions for a stochastic version of the following perturbed linear Volterra equation

$$u(t)=\int_0^t [a(t-s)+(a*k)(t-s)]\, Au(s)\, ds+\int_0^t b(t-s)\, u(s)\, ds+f(t),\ t\in[0,T],\ T<\infty, \tag{1.1}$$

that is, the equation

$$X(t)=X_0+\int_0^t [a(t-\tau)+(a*k)(t-\tau)]AX(\tau)d\tau+\int_0^t b(t-\tau)X(\tau)d\tau+\int_0^t \Psi(\tau)dW(\tau). \tag{1.2}$$

Here A is a closed linear unbounded operator defined later in details and f is a continuous function on $[0,T]$. We assume that a,k,b are real valued and locally integrable functions defined on $\mathbb{R}_+$. X_0 denotes a random variable, W is a cylindrical Wiener process, and the process Ψ is W-integrable. Precise description of the equation (1.1) will be given in section 4.

For $a(t)=t$ and $b(t)\equiv 0$ the equation (1.1) is equivalent to

$$\begin{array}{rcl} \ddot{u}(t) &=& \dot{f}(t)+Au(t)+k*Au(t), \qquad t\geq 0, \\ u(0) &=& f(0), \qquad \dot{u}(0)=\dot{f}(0), \end{array} \tag{1.3}$$

where $\ddot{u}$ denotes second derivative w.r.t. time of u. This problem arises in several applied fields like viscoelasticity or heat conduction with memory, and in such applications the operator A typically is the negative Laplacian in $H = L^2(\Omega)$, or the elasticity operator, the Stokes operator, or the biharmonic Δ^2 operator, equipped with suitable boundary conditions. We note that the recent paper [21] considered stability properties of the solutions of (1.3) in Hilbert spaces, in particular the decay of polynomial or exponential type. Typically, the kernel k is nonnegative nonincreasing, and it satisfies $\int_0^\infty k(t)dt < 1$; the latter is necessary for the well-posedness of the problem, see [21].

For the case $a(t) \equiv 1$, the equation (1.1) is equivalent to

$$\begin{aligned} \dot{u}(t) &= \dot{f}(t) + Au(t) + k * Au(t) + db * u(t), \qquad t \geq 0, \\ u(0) &= f(0), \end{aligned} \tag{1.4}$$

where $\dot{u}$ and $\dot{f}$ denote first derivatives w.r.t. time of u and f, respectively. Note that [20, Theorem 1.2, p.40] states that (1.4) is well-posed if and only if A generates a C_0-semigroup in H.

The problem

$$\begin{aligned} \dot{u}(t) &= Au(t), \qquad t \leq 1, \\ \dot{u}(t) &= Au(t) + Au(t-1), \qquad t > 1, \\ u(0) &= x \end{aligned} \tag{1.5}$$

is of the form (1.1) with $b(t) \equiv 0$ and $k(t) = \delta_0(t-1)$, $a(t) \equiv 1$. From [20, Example 1.1 p.41] it is known that (1.5) is well-posed if and only if A generates an analytic C_0-semigroup in H.

Deterministic convolution type Volterra equations with $k(t) \equiv 0$ and $b(t) \equiv 0$, that is equations of the form

$$u(t) = f(t) + \int_0^t a(t-s)\, Au(s)\, ds, \qquad t \in [0,T], \tag{1.6}$$

considered in a more general context of a Banach space, have been the subject of many papers; see e.g. [20] and references therein. Stochastic versions of the equation (1.6) in a separable Hilbert space have been studied, among others, in [8], [9], [10] and [11]. However, to the best of our knowledge, the study of stochastic versions of the perturbed Volterra equation (1.1) remains an open problem.

As we have already written, our approach to SVEs and results are basing on the different tool. Our starting point is a family of so-called resolvent operators, or simply resolvents. Usually such family consists of operators generated by the linear operator and kernel function. In this paper the resolvent, denoted by $R(t),\ t \geq 0$, is generated by the foursome $(A; a, b, k)$, where A is an operator and a, b, k are kernel functions which occur in the SVE under consideration.

In the paper we define mild, weak and strong solutions to the Volterra equation studied and state some relationships between them. Particularly, we give sufficient conditions under which the mild solution becomes the strong one. We show this using the theorem on approximation of resolvents. Next, we discuss the Itô-type formula for the perturbed Volterra equation considered. This formula will be a good starting point to the study of stability of the mild solution to the Volterra equation studied. Then we provide continuity of trajectories of stochastic convolution corresponding to the particular form of the equation. We did this result by conducting the stochastic convolution to some formula easier for the study and without the convolution.

The paper is organized as follows. First, in section 2, we recall some known facts on the equation (1.1) in the setting of Banach spaces. In section 3 we provide sufficient conditions for the existence of the resolvent family for (1.1) and we give our first main theorem on approximation of resolvents. This result will be used in the remaining part of the paper where we study the stochastic equation (1.2) in Hilbert spaces. In section 4 our principal results provide sufficient conditions which guarantee existence of strong solution to stochastic perturbed Volterra equation (1.2). In section 5 we discuss the Itô-type formula and its application to the exponential stability of stochastic convolution introduced earlier. Section 6 deals with the continuity of trajectories of mild solution to the stochastic perturbed Volterra equation (1.2). Section 7 contains the proof of Theorem 6.1.

2 Preliminaries

In the paper we use an operator approach to the perturbed equation (1.1) introducing a family of operators corresponding to (1.1) called resolvent, see the definition below. We shall assume that the equation (1.1) is well-posed. The concepts of solutions, well-posedness and resolvent to (1.1) are extensions of that introduced for the equation (1.6); see [20].

In this section A is a closed linear unbounded operator defined in a Banach space B with a dense domain $D(A)$ equipped with the graph norm.

Definition 2.1. *A family $(R(t))_{t\geq 0}$ of linear bounded operators defined in the space B is called a* `resolvent` *to the equation (1.1) if the following conditions are fulfilled:*

1. *$R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0)=I$;*
2. *$R(t)$ commutes with A, that is, $R(t)(D(A))\subset D(A)$ and for all $x\in D(A),\ \ t\geq 0$, $AR(t)x=R(t)Ax$;*
3. *for all $x\in D(A),\ t\geq 0$, the following* `resolvent equation` *holds*

$$R(t)x=x+\int_0^t [a(t-s)+(a*k)(t-s)]AR(s)x\,ds+\int_0^t b(t-s)R(s)x\,ds.$$

Let us recall now some interesting results about the resolvent for the equation (1.1).

Theorem 2.2. (see, e.g. Theorem 1.2 [20])
Let $k,b\in BV_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$. Then (1.1) is well-posed iff (1.6) is well-posed. Also, the resolvent $R(t),\ t\geq 0$, of (1.1) is differentiable iff the resolvent $S(t),\ t\geq 0$, of (1.6) has this property.

Corollary 2.3. (see, e.g. Corollary 1.3 [20])
Suppose $k,b\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ and let $S(t),\ \ t\geq 0$, be the resolvent for (1.6). Then the resolvent $R(t),\ t\geq 0$, of (1.1) admits the decomposition

$$R(t)=S(t)+\tilde{S}(t),\qquad t\geq 0,$$

where $\tilde{S}(t)$ is continuous in $\mathcal{B}(B)$ for all $t\geq 0$. If $S(t),t\geq 0$, is also differentiable then $\dot{\tilde{S}}\in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{B}(D(A),B))$.

Corollary 2.4. (see, e.g. Corollary 1.4 [20])
Assume that $k,b\in BV_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$. Then (1.4) is well-posed iff A generates a C_0-semigroup in B. If this is the case, the resolvent for (1.4) is differentiable.

Corollary 2.5. (see, e.g. Corollary 1.5 [20])
Suppose $k,b\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ and $k\in BV_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$. Then (1.3) is well-posed iff A generates a cosine family in B. If this is the case, the resolvent for (1.3) is differentiable.

Corollary 2.6. (see, Example 1.1 p.41 [20])
The problem (1.3) admits a resolvent iff A generates an analytic C_0-semigroup in B.

3 Representation of the solution and qualitative properties

Concerning representation of the solution to the equation (1.1) we have the following result.

Theorem 3.1. *Assume that the equation (1.1) admits a resolvent $R(t), t > 0$. Then its solution can be represented by*

$$v(t) = \frac{d}{dt}\int_0^t R(t-s)f(s)ds = R(t)f(0) + \int_0^t R(t-s)\dot{f}(s)ds, \quad t \geq 0, \tag{3.1}$$

whenever $f \in W^{1,1}(\mathbb{R}_+; B)$.

Proof. Note that $(1{*}v)(t) = (R{*}f)(t)$ and hence from properties of the finite convolution, and definition of R, we obtain

$$\begin{aligned} 1 * (a * Av + a * k * Av) &= Aa * (1 * v) + Aa * k * (1 * v) \\ &= Aa * R * f + Aa * k * R * f \\ &= (Aa * R + Aa * k * R) * f \\ &= (R - I - b * R) * f \\ &= R * f - 1 * f - b * R * f \\ &= 1 * v - 1 * (b * v) - 1 * f \\ &= 1 * (v - b * v - f), \end{aligned}$$

that is

$$\int_0^t (a * Av)(s)ds + \int_0^t (a * k * Av)(s)ds = \int_0^t v(s)ds - \int_0^t (b * v)(s)ds - \int_0^t f(s)ds, \quad t \geq 0.$$

By differentiation with respect to t on both sides of the above identity, we obtain the desired representation. ☐

Corollary 3.2. *Assume that the equation (1.1) admits a differentiable resolvent $R(t)$, $t \geq 0$. Then the solution can be represented by*

$$v(t) = f(t) + \int_0^t \dot{R}(t-s)f(s)ds, \quad t \geq 0. \tag{3.2}$$

Next we study the approximation of $R(t)$, $t \geq 0$, by a sequence of bounded linear operators involving the unbounded operator A.

Let $\alpha(t)$ and $\beta(t)$ be the unique solutions of the scalar convolution equations

$$\alpha(t) = (\alpha * b)(t) + (a * k)(t) + a(t), \quad t \geq 0, \tag{3.3}$$

and

$$\beta(t) = 1 + (\beta * b)(t), \qquad t \geq 0, \tag{3.4}$$

respectively. Note that $\beta(0) = 1$. Taking Laplace transform, whenever exists, we obtain the following identities

$$\hat{\alpha}(\lambda) = \frac{\hat{a}(\lambda)(1 + \hat{k}(\lambda))}{1 - \hat{b}(\lambda)} \tag{3.5}$$

and

$$\hat{\beta}(\lambda) = \frac{1}{\lambda(1 - \hat{b}(\lambda))} \qquad \text{for all} \quad Re(\lambda) \quad \text{large enough.} \tag{3.6}$$

We shall assume the following hypotheses:

($\mathbf{H}_{\boldsymbol{\alpha}}$) The solution of the equation (3.3) is nonnegative, nonincreasing and convex.

($\mathbf{H}_{\boldsymbol{\beta}}$) The solution of the equation (3.4) is differentiable.

Let us note that there are several kernel functions fulfilling the above conditions.

Example 3.3. *Consider the equation (1.1) with the kernel functions* $a(t) \equiv 1$, $b(t) \equiv 0$ *and* $k(t) = \begin{cases} 2t - 2 & \text{if } t \leq 1; \\ 0 & \text{if } t > 1. \end{cases}$

Then $\alpha(t) = \begin{cases} (t-1)^2, & \text{if } t \leq 1; \\ 0, & \text{if } t > 1, \end{cases}$ *and* $\beta(t) \equiv 1$ *verify the hypotheses* **(H**$_{\boldsymbol{\alpha}}$**)** *and* **(H**$_{\boldsymbol{\beta}}$**)**.

Example 3.4. *Consider the equation (1.1) with the kernel functions* $b(t) \equiv 0, a(t) = p$, *where* $p > 0$ *and* $k(t) = -e^{-qt}$, *where* $pq \geq 1$. *Then* $\alpha(t) = \frac{p}{q}e^{-qt} + p(1 - \frac{1}{q}),\ \ q \geq 1$. *Moreover,* $\dot{\alpha}(t) < 0$ *and* $\ddot{\alpha}(t) > 0$ *implying that hypotheses* **(H**$_{\boldsymbol{\alpha}}$**)** *and* **(H**$_{\boldsymbol{\beta}}$**)** *are verified.*

Comment Under the hypothesis **(H**$_{\boldsymbol{\alpha}}$**)** it follows from [20, Proposition 3.3, p.72] that $\alpha(t)$ is 1-regular and of positive type.

Now, we are ready to formulate the main result of this section.

Theorem 3.5. *Assume that* A *is the generator of a bounded analytic semigroup on* B. *Suppose that the hypothesis* **(H**$_{\boldsymbol{\alpha}}$**)** *and* **(H**$_{\boldsymbol{\beta}}$**)** *are satisfied. Then the equation (1.1) admits a resolvent* $(R(t))_{t\geq 0}$ *on* B. *Additionally, there exist bounded operators* A_n *and corresponding resolvent families* $R_n(t)$ *satisfying* $||R_n(t)|| \leq M\beta(t)$ *for all* $t > 0,\ n \in \mathbb{N}$, *such that*

$$R_n(t)x \to R(t)x \quad \text{as} \quad n \to +\infty \tag{3.7}$$

for all $x \in B,\ t \geq 0$. *Moreover, the convergence is uniform in* t *on every compact subset of* $\mathbb{R}_+$.

Proof. Since the function $\alpha(t),\ t > 0$, is of positive type, we obtain by [20, Corollary 3.1, p.69] that $\frac{1}{\hat{\alpha}(\lambda)} \in \rho(A)$ for all $Re\ \lambda > 0$. Moreover, there is a constant $M \geq 1$ such that $H(\lambda) := (I - \hat{\alpha}(\lambda)A)^{-1}/\lambda$ satisfies

$$||H(\lambda)|| \leq \frac{M}{|\lambda|} \ \text{ for all } \ Re\ \lambda > 0.$$

From the above, and since $\alpha(t)$ is 1-regular, we get by [20, Theorem 3.1 p.73] that A generates a strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{B}(B)$ such that

$$\hat{S}(\lambda) = H(\lambda), \quad Re\ \lambda > 0.$$

Additionally, there is a constant $C \geq 1$ such that the estimate $||S(t)|| \leq C,\ t > 0$, is valid.

Let $x \in B$ and define

$$R(t)x := S(t)x + \int_0^t \dot{\beta}(t-s)S(\tau)x d\tau, \quad t > 0.$$

Then we can see that $R(t),\ t > 0$, is a resolvent for the equation (1.1).

On the other hand, since A generates a bounded analytic semigroup, the resolvent set $\rho(A)$ of the operator A contains the ray $(0, \infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{|\lambda|^k} \qquad \text{for } \lambda > 0, \qquad k \in \mathbb{N}.$$

Define

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \qquad n > 0,$$

the *Yosida approximation* of A. Then

$$||e^{tA_n}|| = e^{-nt}||e^{n^2 R(n,A)t}|| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!} ||R(n, A)^k|| \leq M\, e^{(-n + \frac{n^2}{n})t} = M.$$

Hence, for $n > 0$ we obtain

$$||e^{A_n t}|| \leq M, \quad t \geq 0. \tag{3.8}$$

Since each A_n is bounded and hence generates bounded analytic semigroups $e^{A_n t}$ verifying (3.8), it follows that there exist uniformly continuous families $(S_n(t))_{t\geq 0}$ such that $||S_n(t)|| \leq M$, see [20, Corollary 3.1 p.69 and Theorem 3.1 p.73]. Then, for each $n \in \mathbb{N}$ and $x \in B$ the formula

$$R_n(t)x = S_n(t)x + \int_0^t \dot{\beta}(t-s)S_n(\tau)x d\tau, \quad t > 0,$$

defines corresponding resolvent families such that

$$||R_n(t)|| \leq M + M\int_0^t \dot{\beta}(s)ds = M\beta(t). \tag{3.9}$$

Now, we recall the fact that $R(\mu, A_n)x \to R(\mu, A)x$ as $n \to \infty$ for all μ sufficiently large, from which it follows that

$$\hat{\beta}(\lambda)(I - \hat{\alpha}(\lambda)A_n)^{-1}x \to \hat{\beta}(\lambda)(I - \hat{\alpha}(\lambda)A)^{-1}x$$

as $n \to \infty$ for all λ sufficiently large. Then the uniform stability condition (3.9) and [18, Theorem 2.2] implies that

$$R_n(t)x \to R(t)x \quad \text{as} \quad n \to +\infty$$

for all $x \in B$, uniformly for t on every compact subset of $\mathbb{R}_+$. □

4 Stochastic convolution

Now, we are able to study the stochastic version of the equation (1.1) in a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$ denote a stochastic basis. We consider the following stochastic Volterra equation

$$\begin{aligned} X(t) = X_0 \quad &+ \quad \int_0^t [a(t-\tau) + (a*k)(t-\tau)]AX(\tau)d\tau \\ &+ \quad \int_0^t b(t-\tau)X(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad t \geq 0. \end{aligned} \tag{4.1}$$

In the equation (4.1), X_0 is a $\mathcal{F}_0$-measurable H-valued random variable and the kernel functions a, k, b are as previously. The operator A is a closed linear unbounded with the dense domain $D(A) \subset H$ equipped with the graph norm defined as follows: $|h|_{D(A)} := (|h|_H^2 + |Ah|_H^2)^{\frac{1}{2}}$ for $h \in D(A)$. Here $|\cdot|_H$ denotes the norm in the space H defined by the inner product $\langle\cdot,\cdot\rangle_H$.

W is a cylindrical Wiener process on another Hilber space U with the strictly positive covariance operator Q. Let the subspace U_0 of U be defined by $U_0 := Q^{\frac{1}{2}}(U)$, with the norm $|u|_{U_0} := |Q^{\frac{1}{2}}u|_U$ for $u \in U_0$. We denote by $L_2^0 := L_2(U_0, H)$ the space of Hilbert-Schmidt operators acting from U_0 into H.

The process Ψ belongs to the space $\mathcal{N}^2(0,T;L_2^0)$, that is the space of all operator valued stochastic processes

$$\Phi : [0,T] \times \Omega \to L_2(U_0, H) \tag{4.2}$$

such that

$$\|\Phi\|_T := \left[E\left(\int_0^T \|\Phi(t)\|^2_{L_2(U_0,H)}\, dt\right)\right]^{\frac{1}{2}} < \infty\ . \tag{4.3}$$

For all $u \in U_0$, the process $(\Phi(t)u)$, $t \in [0,T]$, is an H-valued and $\mathcal{F}_t$-adapted stochastic process.

In what follows, we prove some results on stochastic convolution corresponding to the equation (4.1). As we have already written, we shall use the resolvent approach to the equation (4.1).

For definitions of strong, weak and mild solutions to stochastic Volterra equations of convolution type without perturbation we can refer to [9] or [11]. The concepts of solutions have obvious extensions to the stochastic equation (4.1). For the reader's convenience we formulate them below.

Definition 4.1. *An H-valued predictable process $X(t),\ t \in [0,T]$, is said to be a* `strong solution` *to (4.1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0,T]$; for any $t \in [0,T]$*

$$\int_0^t |[a(t-\tau) + (a \star k)(t-\tau)]AX(\tau)|_H \, d\tau < \infty, \quad P-a.s. \tag{4.4}$$

and for any $t \in [0,T]$ the equation (4.1) holds $P-a.s.$

Let A^* denote the adjoint of the operator A with the dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$ defined as $|h|_{D(A^*)} := (|h|_H^2 + |A^*h|_H^2)^{\frac{1}{2}}$, for $h \in D(A^*)$. Then the space $(D(A^*), |\cdot|_{D(A^*)})$ is a separable Hilbert space.

Definition 4.2. *An H-valued predictable process $X(t),\ t \in [0,T]$, is said to be a* `weak solution` *to (1.1), if $P(\int_0^t |[a(t-\tau) + (a \star k)(t-\tau)]X(\tau)|_H d\tau < \infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds*

$$\begin{aligned} \langle X(t), \xi \rangle_H &= \langle X_0, \xi \rangle_H + \langle \int_0^t [a(t-\tau) + (a \star k)(t-\tau)]X(\tau)\, d\tau, A^*\xi \rangle_H \\ &+ \langle \int_0^t b(t-\tau)X(\tau) d\tau, \xi \rangle_H + \langle \int_0^t \Psi(\tau)\, dW(\tau), \xi \rangle_H, \quad P\text{−a.s.} \end{aligned} \tag{4.5}$$

In the remaining part of the paper $R(t),\ t \ge 0$, will denote the resolvent operators (the resolvent) to the equation (1.1).

Definition 4.3. *An H-valued predictable process $X(t),\ t \in [0,T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (4.1), if*

$$\mathbb{E}\left(\int_0^t ||R(t-\tau)\Psi(\tau)||_{L_2^0}^2 d\tau\right) < \infty \quad for \quad t \le T \tag{4.6}$$

and, for arbitrary $t \in [0,T]$,

$$X(t) = R(t)X_0 + \int_0^t R(t-\tau)\Psi(\tau)\, dW(\tau), \quad P-a.s. \tag{4.7}$$

Then the `stochastic convolution` corresponding to (4.1) is defined as

$$W^\Psi(t) := \int_0^t R(t-\tau)\Psi(\tau) dW(\tau), \quad t \ge 0. \tag{4.8}$$

Using the same arguments like in the papers [8, 9], we can prove the following results analogous to those obtained previously for the equation without perturbations.

Proposition 4.4. *For any process $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, the process $W^\Psi(t),\ t \ge 0$, given by (4.8) has a predictable version.*

Proposition 4.5. *Let* $\Psi \in \mathcal{N}^2(0,T;L_2^0)$*. Then the process* $W^\Psi(t)$*,* $t\in[0,T]$*, defined by (4.8) has square integrable trajectories.*

Proposition 4.6. *If* $\Psi \in \mathcal{N}^2(0,T;L_2^0)$*, then the stochastic convolution* $W^\Psi(t),\ t\in[0,T]$*, fulfills the equation (4.5).*

Corollary 4.7. *Assume that* A *is a bounded operator. If* Ψ *belongs to* $\mathcal{N}^2(0,T;L_2^0)$ *then*

$$\begin{aligned} W^\Psi(t) &= \int_0^t [a(t-\tau)+(a\star k)(t-\tau)]AW^\Psi(\tau)d\tau + \int_0^t b(t-\tau)W^\Psi(\tau)d\tau \\ &+ \int_0^t \Psi(\tau)dW(\tau), \quad P-a.s. \end{aligned} \tag{4.9}$$

Theorem 4.8. *Suppose that the assumptions of Theorem 3.5 hold. When* $\Psi,\ A\Psi \in \mathcal{N}^2(0,T;L_2^0)$*, and* $\Psi(\cdot,\cdot)(U_0)\subset D(A)$*, P-a.s., then the equation (4.1) has a strong solution. Precisely, the convolution* $W^\Psi(t),\ t\in[0,T]$*, defined by (4.8) is the strong solution to (4.1).*

Comment The above results can be read as follows. Proposition 4.6 says that the stochastic convolution $W^\Psi(t),\ t\in[0,T]$, is the weak solution to (1.1). Moreover, under some additional assumptions, by Theorem 4.8, the convolution $W^\Psi(t),\ t\in[0,T]$, is even strong solution to the equation (1.1). Because proofs of that facts are very similar to those in [8, 9], we omit them.

Remark Let us note that we have more general result. In fact, the process

$$X^\Psi(t) := W^\Psi(t) + R(t)X_0, \quad t\geq 0,$$

when $X_0\in D(A)$, satisfies the equation (4.5). It comes from the resolvent equation in Definition 2.1.

5 Itô-type formula

In this section we derive the Itô-type formula to the perturbed Volterra equation (4.1) where we take X_0 nonrandom for simplicity.

Let us provide, using standard argumentation, an analogue of the Itô formula for the stochastic Volterra equation (4.1).

Proposition 5.1. *(Analogue of the Itô formula)*
Suppose that the function $F(t,x):[0,T]\times H\to H$ *is continuous and has the following properties:*

1. $F(t,x)$ *is differentiable in* t*;*
2. $F(t,x)$ *is twice Fréchet differentiable in* x *and* $F_x(t,x)\,x\in H$*,* $F_{xx}(t,x)(x_1,x_2)\in H$*, are continuous on* $[0,T]\times H$ *for all* $x,x_1,x_2\in H$*.*

The process $X(t),\ t\geq 0$*, is given by the equation (4.1).*

Then the process $Z(t):=F(t,X(t)),\ t\in[0,T]$*, fulfills the following equation*

$$\begin{aligned} dZ(t) &= \left[F_t(t,X(t)) + \left\langle F_x(t,X(t)), \left(\int_0^t \tilde{a}(t-\tau)X(\tau)d\tau\right)'\right\rangle_H \right. \\ &\left. + \frac{1}{2}\mathrm{Tr}\,\Psi(t)Q(\Psi(t))^* F_{xx}(t,X(t))\right] dt + \langle F_x(t,X(t)), \Psi(t)dW(t)\rangle_H, \quad t\geq 0, \end{aligned} \tag{5.1}$$

where

$$\left(\int_0^t \tilde{a}(t-\tau)X(\tau)d\tau\right)' := \int_0^t [\dot{a}(t-\tau) + ((a*k)(t-\tau))']AX(\tau)\,d\tau + [a(0)+(a*k)(0)]AX(t) + \int_0^t \dot{b}(t-\tau)X(\tau)\,d\tau + b(0)X(t). \tag{5.2}$$

In (5.2) all kernel functions a, b, k *are like in the equation (4.1).*

Now, we are ready to apply the Itô-type formula (5.1).

Let $v:\ H \to \mathbb{R}$ be a function satisfying the following conditions:

$v(x)$ is twice Fréchet differentiable and $v(x), v_x(x),$ and $v_{xx}(x)$ are continuous in $\mathbb{R}^1, H$ and $L(H)$, respectively; (5.3)

$$|v(x)| + |x|\,|v_x(x)| + |x|^2\,|v_{xx}(x)| \le c|x|^p \quad \text{for some} \quad p \ge 2 \quad \text{and} \quad c > 0; \tag{5.4}$$

$$Lv(x) + \alpha v(x) \le 0 \quad \text{for all} \quad x \in D(A); \tag{5.5}$$

where α is a real number and

$$Lv(x) := \langle v_x(x), Ax + f(x)\rangle + \frac{1}{2}\mathrm{Tr}\, G(x)\, W G^*(x)\, v_{xx}(x).$$

The formula (5.1) seems to be a good starting point in the study of stability of mild solution to (4.1).

Conjecture 5.2. *Assume that the conditions (5.3), (5.4) and (5.5) hold. Then the mild solution* $X(t),\ t \ge 0$*, to the equation (4.1) satisfies the inequality*

$$\mathbb{E}\, v(X(t)) \le e^{-\alpha t} M\, v(X_0). \tag{5.6}$$

The idea of the proof bases on Ichikawa's scheme, see [7, Theorem 3.1], and on Theorem 3.5 and Proposition 5.1.

The stability of mild solution to the equation (4.1) will be studied in details in a future paper.

6 Continuity of trajectories of stochastic convolution

In this section we give sufficient conditions for the continuity of trajectories of the stochastic convolution corresponding to the equation

$$X(t) = X_0 + \int_0^t [a(t-\tau) + (a \star k)(t-\tau)]AX(\tau)d\tau + W(t), \tag{6.1}$$

$t \in [0,T]$, when the covariance operator Q of the Wiener process W is nuclear one, that is, $\mathrm{Tr}\, Q < \infty$. The stochastic convolution corresponding to the equation (6.1) has the form

$$W^R(t) := \int_0^t R(t-\tau)dW(\tau), \quad t \in [0,T], \tag{6.2}$$

where $R(t),\ t \ge 0$, denotes here the resolvent family for the equation (1.1) in the case $b(t) \equiv 0$.

The study of continuity of trajectory of stochastic convolution is rather complicated. In the semigroup case, see, e.g. [6], two methods can be used. The first, direct method, bases on the so-called factorization method developed in [5]. Unfortunately, this method cannot be used in the resolvent case. The second, indirect method, relies on results for regularity of solutions of an appropriate Cauchy problem connected to the stochastic convolution considered.

In this paper we use the second method. In the following theorem we conduct the convolution (6.2) to the formula without the stochastic integral. We substitute the convolution (6.2) by the formula (6.3) below with a Bochner type integral.

Theorem 6.1. *Assume that the operator A in the equation (6.1) is the generator of an analytic semigroup $T(t), t \in [0,T]$. Let the kernel functions $a(t)$, $k(t)$, $t \in [0,T]$, satisfy assumptions of Theorem 3.5, and, additionally, $\dot{a} \in L^1_{\mathrm{loc}}([0,T];\mathbb{R})$. Let $R(t)$, $W(t)$ and $W^S(t), t \in [0,T]$, be like above. Then the following formula holds*

$$\begin{aligned} W^R(t) &= cA\int_0^t T(t-\tau)\times \\ &\quad \times\left[\int_0^\tau \left[\dot{a}(\tau-\sigma)+[(a\star k)(t-\tau)]'\right]W^R(\sigma)d\sigma + cW(\tau)\right]d\tau + W(t), \end{aligned} \tag{6.3}$$

$t \in [0,T]$, *where $c = a(0)$ is a constant.*

Proof of Theorem 6.1 is given in section 7.

Comment One can see that the formula (6.3) is more complicated than the analogous one for the semigroup case. In fact, if $a(t) = 1$ and $k(t) \equiv 0$, the formula (6.3) reduces to

$$W^T(t) = A\int_0^t T(t-s)W(s)ds + W(t), \qquad t \in [0,T]\,,$$

where $T(t),\ t \geq 0$, denotes the semigroup generated by A.

For our convenience we will assume in the sequel that $c \equiv a(0) = 1$.

Lemma 6.2. *Suppose that the assumptions of Theorem 6.1 hold and $a(0)=1$. Define the process*

$$Y(t) := \int_0^t T(t-s)\left[\widetilde{W}(s)+W(s)\right]ds, \qquad t \in [0,T]\,, \tag{6.4}$$

where

$$\widetilde{W}(s) := \int_0^s \dot{a}(s-\sigma)\,W^R(\sigma)\,d\sigma, \qquad s \in [0,T]\,.$$

Then

$$W^R(t) = A\,Y(t) + W(t), \qquad t \in [0,T]. \tag{6.5}$$

Additionally, Y belongs to $C^1([0,T];D(A)),\ \mathbb{P}-a.s.$, and

$$\frac{dY(t)}{dt} = A\,Y(t) + \left[\widetilde{W}(t)+W(t)\right], \qquad t \in [0,T]\,. \tag{6.6}$$

Proof. From the definition (formula (6.4)) of the process Y and properties of convolution, $Y \in C^1([0,T];D(A)),\ P{-}a.s.$ Next, from the Leibniz rule and property of semigroup we obtain

$$\begin{aligned} \frac{dY(t)}{dt} &= \int_0^t \frac{dT(t-s)}{dt}\left[\widetilde{W}(s)+W(s)\right]ds + T(0)\left[\widetilde{W}(t)+W(t)\right] \\ &= A\int_0^t T(t-s)\left[\widetilde{W}(s)+W(s)\right]ds + \left[\widetilde{W}(t)+W(t)\right] \\ &= A\,Y(t) + \left[\widetilde{W}(t)+W(t)\right], \qquad t \in [0,T]. \end{aligned}$$

□

Now, we are able to use regularity results of solutions for the non homogeneous Cauchy problem, due to Da Prato and Grisvard [4], to the problem (6.5).

We will need the following space related to the so-called interpolation theory, see, e.g., [19]. It will be denoted $D_A(\alpha,\infty)$ for $\alpha\in(0,1)$.

For reader's convenience we recall the definition of that space. For any $\alpha\in(0,1)$, we set

$$|x|_{\alpha,\infty} := \sup_{t>0}\frac{|T(t)x-x|}{t^\alpha}, \qquad x\in H.$$

We denote by $D_A(\alpha,\infty)$ the Banach space of all $x\in H$ such that $|x|_{\alpha,\infty}<\infty$, endowed with the norm $|\cdot|_H+|\cdot|_{\alpha,\infty}$. By the interpolation theory, $D_A(\alpha,\infty)$ is an invariant space of $T(t),\ t>0$, and the restriction of $T(t)$ to $D_A(\alpha,\infty)$ generates a $\mathcal{C}_0$-semigroup in $D_A(\alpha,\infty)$.

Hence, we can deduce the following result from the formula (6.5).

Theorem 6.3. *Let us take the same assumptions as in Theorem 6.1. If both processes Y and W have continuous trajectories in the space $D_A(\alpha,\infty)$, then the stochastic convolution $W^R(t)$, $t\in[0,T]$, has continuous trajectories in $D_A(\alpha,\infty)$.*

7 Appendix

Proof. **of Theorem 6.1**
Because the formula (4.9) holds for any bounded operator, then it holds for the Yosida approximation A_n, $n\in\mathbb{N}$, of the operator A and in our case it has the form

$$W^{R_n}(t)=\int_0^t[a(t-\tau)+(a\star k)(t-\tau)]A_nW^{R_n}(\tau)\,d\tau+W(t),\quad t\in[0,T]. \tag{7.1}$$

In the formula (7.1), $R_n(t)$, $t\geq 0$, is the resolvent corresponding to the equation (6.1) with the operator A_n and

$$W^{R_n}(t):=\int_0^t R_n(t-\tau)\,dW(\tau)\quad t\in[0,T]\,. \tag{7.2}$$

Let us denote

$$Z_n(t):=\int_0^t[a(t-\tau)+(a\star k)(t-\tau)]W^{R_n}(\tau)\,d\tau,\quad t\in[0,T]\,. \tag{7.3}$$

Then, from the Leibniz rule

$$\begin{aligned} Z_n'(t) &= \left(\int_0^t a(t-\tau)W^{R_n}(\tau)\,d\tau\right)'+\left(\int_0^t(a\star k)(t-\tau)W^{R_n}(\tau)\,d\tau\right)' \\ &= \int_0^t \dot a(t-\tau)\,W^{R_n}(\tau)\,d\tau+a(0)\,W^{R_n}(t) \\ &\quad+\int_0^t[(a\star k)(t-\tau)]'W^{R_n}(\tau)\,d\tau,\quad t\in[0,T]\,. \end{aligned} \tag{7.4}$$

From (7.1) and (7.3), we can write

$$W^{R_n}(t)=A_n\,Z_n(t)+W(t),\quad t\in[0,T]\,.$$

Let us denote

$$\widetilde{W}^{R_n}(t):=\int_0^t[\dot a(t-\tau)+[(a\star k)(t-\tau)]']\,W^{R_n}(\tau)\,d\tau\,.$$

From (7.4), if $a(0) \neq 0$, we have

$$W^{R_n}(t) = \frac{1}{a(0)} \left[Z_n'(t) - \widetilde{W}^{R_n}(t) \right] .$$

So, we obtained

$$Z_n'(t) - \widetilde{W}^{R_n}(t) = a(0) \, [A_n \, Z_n(t) + W(t)], \quad t \in [0,T].$$

And next

$$Z_n'(t) = a(0) \, A_n \, Z_n(t) + \widetilde{W}^{R_n}(t) + a(0) \, W(t), \quad t \in [0,T].$$

So, we have

$$Z_n'(t) = c \, A_n \, Z_n(t) + \left[\widetilde{W}^{R_n}(t) + c \, W(t) \right], \quad \text{where} \quad c = a(0).$$

From the above formula

$$Z_n(t) = \int_0^t e^{c(t-\tau)A_n} \left[\widetilde{W}^{R_n}(\tau) + c \, W(\tau) \right] d\tau, \quad t \in [0,T] \, ,$$

because $Z_n(0) = 0$.

From the formula (7.1),

$$W^{R_n}(t) = A_n \, Z_n(t) + W(t), \quad t \in [0,T] \, , \qquad \text{or}$$

$$W^{R_n}(t) = A J_n Z_n(t) + W(t), \qquad t \in [0,T],$$

where $J_n := nR(n,A)$.

So, we obtain

$$W^{R_n}(t) = A J_n \int_0^t e^{c(t-\tau)A_n} \left[\widetilde{W}^{R_n}(\tau) + c \, W(\tau) \right] d\tau + W(t), \quad t \in [0,T].$$

Basing on Theorem 3.5, properties of the Yosida approximation A_n of the operator A, and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} J_n \, x = x, \quad \text{for any} \quad x \in H;$$

$$\lim_{n\to\infty} A_n \, x = A \, x, \quad \text{for any} \quad x \in D(A);$$

$$\lim_{n\to\infty} e^{tA_n} \, x = T(t) \, x, \quad \text{for any} \quad x \in H;$$

$$\text{and} \quad \lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E} \left| W^{R_n}(t) - W^R(t) \right|_H^2 = 0.$$

Because the operator A is closed, we can conclude that the integral $\int_0^t T(t-\tau) \left[\widetilde{W}^R(\tau) + c \, W(\tau) \right] d\tau$ belongs to the domain $D(A)$.

Hence, passing to the limit with $n \to +\infty$, we obtain

$$W^R(t) = cA \int_0^t T(t-\tau) \left[\widetilde{W}^R(\tau) + c \, W(\tau) \right] d\tau + W(t), \quad t \in [0,T],$$

where

$$\widetilde{W}^R(\tau) = \int_0^\tau [\dot{a}(\tau-\sigma) + [(a \star k)(\tau-\sigma)]'] W^R(\sigma) d\sigma \, , \quad \tau \in [0,T] \, .$$

In the case $a(0) = 0$, from the formula (7.1), passing to the limit we would have only

$$W^R(t) = \int_0^t [a(t-\tau) + (a \star k)(t-\tau)] A W^R(\tau) d\tau + W(t), \quad t \in [0,T] \, .$$

□

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