Electron. Commun. Probab. **19** (2014), no. 52, 1–8.
DOI: 10.1214/ECP.v19-3490
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Disjoint crossings, positive speed and deviation estimates for first passage percolation

Ghurumuruhan Ganesan*

Abstract

Consider bond percolation on the square lattice $\mathbb{Z}^2$ where each edge is independently open with probability p. For some positive constants $p_0 \in (0,1), \epsilon_1$ and ϵ_2, the following holds: if $p > p_0$, then with probability at least $1-\frac{\epsilon_1}{n^4}$ there are at least $\frac{\epsilon_2 n}{\log n}$ disjoint open left-right crossings in $B_n := [0,n]^2$ each having length at most $2n$, for all $n \geq 2$. Using the proof of the above, we obtain positive speed for first passage percolation with independent and identically distributed edge passage times $\{t(e_i)\}_i$ satisfying $\mathbb{E}(\log t(e_1))^+ < \infty$; namely, $\limsup_n \frac{T_{pl}(0,n)}{n} \leq Q$ a.s. for some constant $Q < \infty$, where $T_{pl}(0,n)$ denotes the minimum passage time from the point $(0,0)$ to the line $x = n$ taken over all paths contained in B_n. Finally, we also obtain deviation corresponding estimates for nonidentical passage times satisfying $\inf_i \mathbb{P}(t(e_i) = 0) > \frac{1}{2}$.

Keywords: First passage percolation, zero passage times, deviation estimates.
AMS MSC 2010: 60K35.
Submitted to ECP on May 7, 2014, final version accepted on August 7, 2014.

1 Introduction

Consider bond percolation in $\mathbb{Z}^2$ where each bond is independently open with probability p. A bond that is not open is said to be closed. For integer $M \geq 1$ and $\epsilon > 0$, let $R_{n,\epsilon}(M)$ denote the rectangle $[0,n]\times[0,\lceil (M\log n)^{1+\epsilon}\rceil]$. Here $\lceil x \rceil$ represents the smallest integer greater than x. All constants mentioned henceforth are independent of n. A path $\Pi = (e_1, ..., e_t)$ of edges contained in $R_{n,\epsilon}(M)$ is said to be an open left-right crossing if every e_i is open and e_1 intersects the left side of $R_{n,\epsilon}(M)$ and e_t touches the right side of $R_{n,\epsilon}(M)$.

Proposition 1.1. *Fix $\delta > 0$ and $\epsilon > 0$.*
(i) If $p > \frac{1}{2}$, there are positive constants $C_1 = C_1(p,\delta,\epsilon)$ so that with probability at least $1-\frac{C_1}{n^\delta}$, there exists an open left-right crossing of $R_{n,\epsilon}(1)$, for all $n \geq 1$.
(ii) If $p > \frac{2}{3}$, there are positive constants $M_2 = M_2(p,\delta)$ and $C_2 = C_2(p,\delta)$ so that with probability at least $1-\frac{C_2}{n^\delta}$, there exists an open left-right crossing of $R_{n,0}(M_2)$, for all $n \geq 1$.
(iii) If $p > p_0 := \left(1-3^{-8}\right)^{\frac{1}{2}}$, there are positive constants $M_3 = M_3(p,\delta)$ and $C_3 = C_3(p,\delta)$ so that with probability at least $1-\frac{C_3}{n^\delta}$, there exists an open left-right crossing of $R_{n,0}(M_3)$ containing at most $2n$ edges, for all $n \geq 1$.

*EPFL, Lausanne. E-mail: gganesan82@gmail.com

The proof of (i) uses the exponential decay theorem for subcritical percolation (Chapter 4, Bollobas and Riordan (2006)). For (ii), we use a contour argument without resorting to the exponential decay theorem. Finally, we use oriented paths to control the length of the crossings in (iii). For further analysis regarding critical values for oriented percolation, we refer to Durrett (1984) and references therein.

The following result is a consequence of Proposition 1.1. For $n \geq 1$, let $B_n := [0,n]^2$.

Theorem 1.2. *Fix $\delta > 0$ and $\epsilon > 0$ and let p_0 be as in Proposition 1.1.*
(i) If $p > \frac{1}{2}$, there are positive constants $\gamma_1 = \gamma_1(p,\delta,\epsilon)$ and $C_1 = C_1(p,\delta,\epsilon)$ so that with probability at least $1 - \frac{C_1}{n^\delta}$, there are at least $\frac{\gamma_1 n}{(\log n)^{1+\epsilon}}$ disjoint open left-right crossings of B_n, for all $n \geq 2$.
(ii) If $p > \frac{2}{3}$, there are positive constants $\gamma_2 = \gamma_2(p,\delta)$ and $C_2 = C_2(p,\delta)$ so that with probability at least $1 - \frac{C_2}{n^\delta}$, there are at least $\frac{\gamma_2 n}{\log n}$ open left-right crossings of B_n, for all $n \geq 2$.
(iii) If $p > p_0$, there are positive constants $\gamma_3 = \gamma_3(p,\delta)$ and $C_3 = C_3(p,\delta)$ so that with probability at least $1 - \frac{C_3}{n^\delta}$, there are at least $\frac{\gamma_3 n}{\log n}$ open left-right crossings of B_n, each containing at most $2n$ edges, for all $n \geq 2$.

As an application of Proposition 1.1, we obtain positive speed and deviation estimates for first passage percolation.

1.1 First passage percolation

Consider the square lattice $\mathbb{Z}^2$ with edges $\{e_i\}_{i\geq 1}$. The passage times $\{t(e_i)\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) having the same distribution as a random variable X. For a path π containing k edges $g_1,...,g_k$, let $T(\pi) := \sum_{i=1}^{k} t(g_i)$ denote its passage time. Let $T_{ll}(0,n) = \min_\pi T(\pi)$ be the minimum passage time from the line $x = 0$ to the line $x = n$ where the minimum is taken over all paths π contained in B_n. Similarly, $T_{pl}(0,n)$ and $T_{pp}(0,n) = \min_\pi T(\pi)$ be, respectively, the minimum passage time from $(0,0)$ to the line $x = n$ and from $(0,0)$ to $(n,0)$, again over all paths contained in B_n. Clearly,

$$T_{ll}(0,n) \leq T_{pl}(0,n) \leq T_{pp}(0,n) \ \ a.s.$$

We have the following result.

Theorem 1.3. *If $\mathbb{P}(X < \infty) = 1$, there exists a finite constant $Q_1 \geq 0$ so that:*
(i) $\limsup_n \frac{T_{ll}(0,n)}{n} \leq Q_1$ a.s.
If $\mathbb{E}(\log X)^+ < \infty$, there exists a finite constant $Q_2 \geq 0$ so that:
(ii) $\limsup_n \frac{T_{pl}(0,n)}{n} \leq Q_2$ a.s.
(iii) For every $\epsilon > 0$, we have

$$\mathbb{P}\left(\frac{T_{pp}(0,n)}{n} > Q_2 + \epsilon\right) \longrightarrow 0$$

as $n \to \infty$.
If $\mathbb{E}X < \infty$, then:
(iv) $\limsup_n \frac{T_{pp}(0,n)}{n} \leq \mathbb{E}X \ \ a.s.$

If we think of $\frac{n}{T_{pp}(0,n)}, \frac{n}{T_{pl}(0,n)}$ or $\frac{n}{T_{ll}(0,n)}$ as the corresponding speed of first passage percolation, then (i)-(iv) of the above result implies positive speed even if expected passage time is not finite.

Suppose now the passage time is zero with large probability. It is then intuitive to expect infinite speed and we have the following result.

Proposition 1.4. *Suppose one of the following two conditions hold:*
(i) $\mathbb{P}(X = 0) > \frac{1}{2}$ and $\mathbb{E}Y^{1+\eta} < \infty$ for some $\eta > 0$, where $Y = (\log X)^+$.

(ii) $\mathbb{P}(X = 0) > \frac{2}{3}$ *and* $\mathbb{E}(\log X)^+ < \infty$.
We have that $\frac{T_{pl}(0,n)}{n} \longrightarrow 0$ *a.s. as* $n \to \infty$ *and* $\frac{T_{pp}(0,n)}{n} \longrightarrow 0$ *in probability as* $n \to \infty$.

Finally, we obtain deviation estimates for first passage percolation with independent passage times but not necessarily identically distributed.

Theorem 1.5. *Suppose* $\inf_i \mathbb{P}(t(e_i) = 0) > \frac{1}{2}$ *and* $\inf_i \mathbb{P}(t(e_i) < \infty) = 1$.
(i) We have $\frac{T_{ll}(0,n)}{n} \longrightarrow 0$ *a.s. as* $n \to \infty$.
(ii) If $\sup_i \mathbb{E}t(e_i) < \infty$, *then*

$$\mathbb{E}T_{pp}(0,n) \leq C_1(\log n)^2 \tag{1.1}$$

for all $n \geq 1$ *and for some positive constants* β_1 *and* C_1. *In particular, we have* $\frac{T_{pp}(0,n)}{n} \longrightarrow 0$ *in probability as* $n \to \infty$.
(iii) If $\sup_i \mathbb{E}t(e_i)^K < \infty$ *for some* $K > 1$, *we have*

$$\mathbb{P}\left(T_{pp}(0,n) \geq n^{\beta_2}\right) \leq \frac{C_2}{n^{\beta_3}} \tag{1.2}$$

for all $n \geq 1$ *and for some positive constants* $\beta_2 < \min\left(1, \frac{1}{K-1}\right), \beta_3 > 1$ *and* C_2. *In particular, we have* $\frac{T_{pp}(0,n)}{n} \longrightarrow 0$ *a.s. as* $n \to \infty$.
(iv) If $\sup_i \mathbb{E}e^{st(e_i)} < \infty$ *for some* $s > 0$, *we have*

$$\mathbb{P}\left(T_{pp}(0,n) \geq \beta_4(\log n)^3\right) \leq \frac{C_3}{n^{\beta_5}} \tag{1.3}$$

for all $n \geq 1$ *and for some positive constants* $\beta_4, \beta_5 > 1$ *and* C_3.

The proof of the above theorem uses Proposition 1.1(i) which in turn uses the exponential decay theorem. If suppose $\inf_i \mathbb{P}(t(e_i) = 0) > \frac{2}{3}$, the above result holds and is also proved using Proposition 1.1(ii).

The paper is organized as follows: In Section 2, we prove Theorems 1.3 and 1.5 assuming Proposition 1.1. In Section 3, we prove Proposition 1.1.

2 Proof of Theorems 1.3 and 1.5

Proof of Theorem 1.3: We prove (iv) first. If $\mathbb{E}X < \infty$, then the result follows from strong law of large numbers since

$$\frac{T_b(0,n)}{n} \leq \frac{1}{n}\sum_{i=1}^{n} t(f_i).$$

This proves (iv). We now prove (i)-(iii).

(i) Since $\mathbb{P}(X < \infty) = 1$, we choose N large so that $\mathbb{P}(X \leq N) > p_0$. For $M \geq 1$ let $R'_{n,0}(M) = [0,n] \times [1, M\log n + 1]$ be the shifted rectangle. Also for $i \geq 1$, let f_i denote the edge from $(i-1, 0)$ to $(i, 0)$. We set an edge e in $R'_{n,0}(M)$ to be open if its passage time $t(e) \leq N$. Set $\delta = 2$ in Proposition 1.1 and let A_n denote the event that the rectangle $R'_{n,0}(M)$ contains an open left-right crossing containing less than $2n$ edges, where $M = M_3$ is as in Proposition 1.1(iii). Since we only need to hit the line $x = n$ from the line $x = 0$, we have that

$$\frac{T_{ll}(0,n)}{n} \leq 2N + J_n 1\!\!1(A_n^c) \tag{2.1}$$

where $J_n = \frac{1}{n}\sum_{i=1}^{n} t(f_i)$. Since $\mathbb{P}(A_n^c) \leq \frac{C_1}{n^2}$ for some positive constant C_1, we have that

$$\sum_{n \geq 1} \mathbb{P}(A_n^c) < \infty.$$

Thus by Borel-Cantelli Lemma, we have that $\mathbb{P}(\liminf_n A_n) = 1$ and this implies that

$$J_n \mathbb{1}(A_n^c) \longrightarrow 0 \text{ a.s.} \tag{2.2}$$

as $n \to \infty$. This proves (i) with $Q_1 = 2N$.

To prove (ii)-(iii), we assume henceforth that $X < \infty$ a.s. and $\mathbb{E}X = \infty$. Choose N sufficiently large so that $\mathbb{P}(X \leq N) > p_0$. Set the edge e to be open if $t(e) \leq N$. Clearly each edge is independently open with probability at least p_0.

We prove (iii) first and obtain (ii) as a Corollary.

(iii) Let $\{h_{1,i}\}_{1 \leq i \leq M \log n+1}$ be the set of edges forming the left vertical side of $R'_{n,0}(M)$ and including the edge from $(0,0)$ to $(0,1)$. Similarly let $\{h_{n,i}\}_{1 \leq i \leq M \log n+1}$ be the set of edges forming the right vertical side of $R'_{n,0}(M)$ and including the edge from $(n,0)$ to $(n,1)$. We then have that

$$\begin{aligned} T_{pp}(0,n) &= T_{pp}(0,n)\mathbb{1}(A_n) + T_{pp}(0,n)\mathbb{1}(A_n^c) \\ &\leq \left(\sum_{j=1,n} \sum_{i=1}^{M \log n+1} t(h_{j,i}) + 2Nn \right) \mathbb{1}(A_n) + \left(\sum_{i=1}^{n} t(f_i) \right) \mathbb{1}(A_n^c). \end{aligned} \tag{2.3}$$

Thus

$$\frac{T_{pp}(0,n)}{n} \leq 2N + I_{1,n} + I_{2,n} + J_n \mathbb{1}(A_n^c) \tag{2.4}$$

where

$$I_{1,n} = \frac{1}{n} \sum_{i=1}^{M \log n+1} t(h_{1,i}), \quad I_{2,n} = \frac{1}{n} \sum_{i=1}^{M \log n+1} t(h_{n,i})$$

and J_n is as in (2.1). From (2.2), the third term goes to zero a.s. as $n \to \infty$. For the first two terms, we apply Feller's theorem (Theorem 8.9, Durrett (2001)) with $a_n = \exp\left(\frac{n-1}{M}\right)$. Indeed for $a_m \leq n < a_{m+1}$, we have

$$\frac{1}{n} \sum_{i=1}^{M \log n+1} t(h_{1,i}) \leq \frac{1}{a_m} \sum_{i=1}^{m+1} t(h_{1,i}) = \frac{e^{1/M}}{a_{m+1}} \sum_{i=1}^{m+1} t(h_{1,i})$$

so that

$$\limsup_n I_{1,n} \leq e^{1/M} \limsup_m \frac{1}{a_m} \sum_{i=1}^{m} t(h_{1,i}). \tag{2.5}$$

Since

$$\mathbb{P}\left(t(h_{1,n}) > a_n\right) = \mathbb{P}\left(\log X > \frac{n-1}{M}\right) = \mathbb{P}\left((\log X)^+ > \frac{n-1}{M}\right)$$

for $n \geq 2$, we have that

$$\sum_n \mathbb{P}\left(t(h_{1,n}) > a_n\right) = \sum_n \mathbb{P}\left(M(\log X)^+ + 1 > n\right) \leq M\mathbb{E}(\log X)^+ + 2.$$

Since $\mathbb{E}(\log X)^+ < \infty$, we have from Feller's theorem that the right hand side of (2.5) is zero a.s. This implies that

$$I_{1,n} \longrightarrow 0 \text{ a.s.} \tag{2.6}$$

as $n \to \infty$. Since $I_{1,n}$ has the same distribution as $I_{2,n}$ we have that

$$I_{2,n} \longrightarrow 0 \text{ in probability.}$$

From (2.4), the result (iii) then follows with $Q_2 = 2N$.

(ii) The analysis is the same as above except that we obtain

$$\frac{T_{pl}(0,n)}{n} \leq 2N + I_{1,n} + J_n 1\!\!1(A_n^c) \tag{2.7}$$

instead of (2.5), since we only need to hit the line $x = n$. From (2.6) and (2.2), the result then follows. □

Proof of Proposition 1.4: We prove (ii) using Proposition 1.1(ii). An analogous analysis holds for (i) using Proposition 1.1(i). Fix $\delta = 2$ in Proposition 1.1(ii) and let A_n denote the event that the rectangle $R'_{n,0}(M)$ defined in proof of Theorem 1.3 contains a left-right crossing containing only edges with zero passage time. Here $M = M_2$ is as in Proposition 1.1(ii). Arguing as in the paragraph preceding (2.4), we get that

$$\frac{T_{pp}(0,n)}{n} \leq I_{1,n} + I_{2,n} + J_n 1\!\!1(A_n^c)$$

where $I_{1,n}, I_{2,n}$ and J_n are as in (2.4). The result follows by an analogous analysis following (2.4). □

Proof of Theorem 1.5: We consider the shifted rectangle $R'_{n,1}(M) = [1,n]\times[1,(\log n)^2+1]$. Let A_n denote the event that $R'_{n,1}(M)$ has a left-right crossing consisting of edges with zero passage times. Following an analogous analysis as preceding (2.3) we obtain

$$T_{pp}(0,n) \leq \left(\sum_{j=1,n}\sum_{i=1}^{(\log n)^2+1} t(h_{j,i})\right)1\!\!1(A_n) + \left(\sum_{i=1}^{n} t(f_i)\right)1\!\!1(A_n^c). \tag{2.8}$$

(i) follows by an analogous analysis as Proposition 1.4.

(ii) For the estimate on $\mathbb{E}T_{pp}(0,n)$, we obtain from (2.8) that

$$\begin{aligned}
\mathbb{E}T_{pp}(0,n) &\leq (2(\log n)^2+2)\mathbb{E}t(h_{1,1}) + \mathbb{E}\left(\sum_{i=1}^{n} t(f_i)1\!\!1(A_n^c)\right)\\
&= (2(\log n)^2+2)\mathbb{E}t(h_{1,1}) + \mathbb{E}\left(\sum_{i=1}^{n} t(f_i)\right)\mathbb{P}(A_n^c)\\
&\leq (2(\log n)^2+2)\mathbb{E}t(h_{1,1}) + n\mathbb{E}t(f_1)\frac{C_1}{n}
\end{aligned}$$

for some constant $C_1 > 0$. The second equality is because A_n^c is independent of $\{t(f_i)\}_i$. This proves (1.1) in (ii). The convergence in probability follows since $\frac{1}{n}\mathbb{E}T_{pp}(0,n) \longrightarrow 0$ as $n \to \infty$.

(iii) Let $\delta > 0$ be fixed. From (2.3) and from the estimate on $\mathbb{P}(A_n^c)$ in Proposition 1.1 we then have for $x > 0$ that

$$\begin{aligned}
&\mathbb{P}\left(T_{pp}(0,n) > 2x\right)\\
&\leq \mathbb{P}\left(\sum_{j=1,n}\sum_{i=1}^{(\log n)^2+1} t(h_{j,i}) > 2x\right) + \frac{C}{n^\delta}\\
&\leq \mathbb{P}\left(\bigcup_{j=1,n}\bigcup_{i=1}^{(\log n)^2+1}\left\{t(h_{j,i}) > \frac{x}{(\log n)^2+1}\right\}\right) + \frac{C}{n^\delta}.\\
&\leq \sum_{j,i}\mathbb{P}\left(t(h_{j,i}) > x((\log n)^2+1)^{-1}\right) + \frac{C}{n^\delta}.
\end{aligned} \tag{2.9}$$

We now set $\delta = 2$ and $x = ((\log n)^2 + 1)n^\theta$ in (2.9). Here we choose $\theta > 0$ so that $\max(1, K-1) < \theta^{-1} < K$. Thus from (2.9), we get

$$\begin{aligned}
&\mathbb{P}\left(T_{pp}(0,n) > (2(\log n)^2 + 2)n^\theta\right) \\
&\leq \sum_{i,j} \mathbb{P}\left(t(h_{i,j}) > n^\theta\right) + \frac{C}{n^\delta} \\
&\leq (2(\log n)^2 + 2)\frac{1}{n^{\theta K}} \sup_i \mathbb{E}t(e_i)^K + \frac{C}{n^\delta}.
\end{aligned} \tag{2.10}$$

This proves (1.3) in (iii). Since $\theta K > 1$ and $\theta < 1$, we obtain from (2.10) and Borel-Cantelli Lemma that $\frac{T_{pp}(0,n)}{n} \longrightarrow 0$ a.s. as $n \to \infty$.

(iv) We set $\delta = 2$ and $x = 2\delta_1(\log n)^3$ in (2.9). From (2.9), we get

$$\begin{aligned}
&\mathbb{P}\left(T_{pp}(0,n) > 4\delta_1(\log n)^3\right) \\
&\leq \sum_{i,j} \mathbb{P}\left(t(h_{i,j}) > 2\delta_1(\log n)^3((\log n)^2+1)^{-1}\right) + \frac{C}{n^\delta} \\
&\leq \sum_{i,j} \mathbb{P}\left(t(h_{i,j}) > \delta_1 \log n\right) + \frac{C}{n^\delta} \\
&\leq \sum_{i,j} e^{-s\delta_1 \log n}\mathbb{E}e^{st(h_{i,j})} + \frac{C}{n^\delta} \\
&\leq (2(\log n)^2 + 2)e^{-s\delta_1 \log n} \sup_i \mathbb{E}e^{st(e_i)} + \frac{C}{n^\delta},
\end{aligned}$$

where the second estimate follows from Markov inequality. Choosing δ_1 large, proves (iv). $\square$

3 Proof of Proposition 1.1

Proof of Proposition 1.1(i): Fix a midpoint x of an edge in the bottom side of $R_{n,\epsilon}(M)$. From exponential decay theorem (Chapter 4, Bollobas and Riordan (2006)), we know that a closed dual top bottom crossing of $R_{n,\epsilon}(M)$ intersecting x occurs with probability at most $\exp\left(-C_1\frac{(\log n)^{1+\epsilon}}{(\log\log n)^2}\right)$ for some constant $C_1 > 0$. This is seen by considering the $(\log n)^{1+\epsilon} \times (\log n)^{1+\epsilon}$ box $B'_n(x)$ centred at x. If there is a closed dual top bottom crossing, then there is a closed dual path intersecting x and hitting the boundary of $B'_n(x)$. Since there are at most n choices for x, we have that a closed dual top bottom crossing occurs with probability at most

$$n\exp\left(-C_1\frac{(\log n)^{1+\epsilon}}{(\log\log n)^2}\right) \leq n\exp\left(-(\delta+2)\log n\right)$$

for all n sufficiently large. Since a closed dual top-bottom crossing or an open left-right crossing of $R_{n,\epsilon}(M)$ always must occur (Chapter 1, Bollobas and Riordan (2006)), we have the result.

Proof of Proposition 1.1(ii): We use a counting argument and again the fact that either an open left-right crossing occurs or a closed dual top-bottom crossing occurs but not both. As above, we fix a midpoint x of an edge at the bottom side of $R_{n,0}(M)$ and suppose that there is a dual top-bottom crossing of $R_{n,0}(M)$ with length $k \geq M\log n$, intersecting x. There are at most 4.3^{k-1} dual paths intersecting x and each is closed with probability $(1-p)^k$. Since there are at most n choices for x, we have that a closed dual top bottom crossing occurs with probability at most

$$n\sum_{k\geq M\log n} 4.3^{k-1}(1-p)^k \leq \frac{1}{n^{\delta+1}}$$

provided M is large. The sum above is convergent since $1-p<\frac{1}{3}$. Fixing such an M proves the result.

Proof of Proposition 1.1(iii): Fix $p_0>0$ and $p>p_0$. We use comparison with oriented percolation. We draw oriented arrows from (i,j) to $(i+1,j-1)$ and from (i,j) to $(i+1,j+1)$. To draw arrows from (i,j) to $(i+1,j-1)$, we let S_1 and S_2 be the bonds from (i,j) to $(i,j-1)$ and from $(i,j-1)$ to $(i+1,j-1)$, respectively. Let E_i denote the event that S_i is open. We draw arrow from (i,j) to $(i+1,j-1)$ if $E_1\cap E_2$ holds. An analogous procedure is used for drawing oriented arrows from (i,j) to $(i+1,j+1)$.

We have that $\mathbb{P}_p(E_1\cap E_2)=p^2\geq p_0^2$. We start from the left side of $R_{n,0}(M)$ and continue this oriented percolation process iteratively. Let $\mathbb{P}_{or}$ denote the corresponding probability measure and let LR_n denote the event that there exists an oriented left-right crossing of $R_{n,0}(M)$. If LR_n occurs, there is an open left-right crossing of $R_{n,0}(M)$ containing at most $2n$ edges in the original bond percolation. Using a contour argument as in Durrett (1984), we have that

$$\mathbb{P}_{or}(LR_n)\geq 1-\frac{1}{n^{\delta+1}}$$

provided p_0 and M are large. Fixing such a p_0 and M establishes the result.

To obtain the estimate on $\mathbb{P}_{or}(LR_n)$, we let $\mathcal{C}$ denote the collection of all oriented paths starting from the left side E_{left} of $R_{n,0}(M)$. Recall that we grow the cluster from E_{left} and for every vertex $x\in\mathcal{C}$, there is an oriented path from E_{left} to x. As in Durrett (1984), we place a square S''_x on each vertex $x\in\mathbb{Z}^2$ of $\mathcal{C}$. If $x=(i,j)$, then S''_x has endvertices $(i,j-1)$, $(i+1,j)$, $(i,j+1)$ and $(i-1,j)$. The edges of S''_x are oriented in such a way that the square S''_x forms a clockwise oriented contour around x. If two oriented edges in opposite directions coincide, they "cancel" each other and we draw nothing. There is an outermost contour Π of $\cup_{x\in\mathcal{C}}S''_x$ that is oriented clockwise and encloses E_{left}. Oriented arrows with at least one end-vertex in $\mathcal{C}$ and crossing Π are called boundary arrows and we say such arrows were terminated in the cluster growing process.

Suppose that there is no oriented left-right crossing of $R_{n,0}(M)$. Let z_u and z_f denote the rightmost points of intersection of Π and the top and bottom edge of $R_{n,0}(M)$, respectively. Let Π_1 denote the part of Π from z_u to z_f. The path Π_1 is contained in $R_{n,0}(M)$. We write $LR_n^c=\bigcup_{1\leq j\leq n-1}A_j\cap LR_n^c$, where A_j denotes the event that Π_1 cuts the horizontal segment $[j-1,j]\times\{MK_n\}$ of the top edge of R_T^{int}. Suppose that $A_j\cap LR_n^c$ occurs and suppose that Π_1 contains $k\geq M\log n$ oriented edges.

We count up and down arrows as in Durrett (1984) and obtain a subset Π_2 of Π_1 consisting of at least $\frac{k}{8}$ edges, each cutting a boundary arrow that was independently terminated. Since the number of choices of Π_1 is at most 4.3^{k-1} and each boundary arrow was terminated with probability at most $1-p_0^2$, we obtain that

$$\mathbb{P}_{or}(A_j\cap LR_n^c)\leq 4\sum_{k\geq M\log n}3^{k-1}\left(1-p_0^2\right)^{k/8}\leq e^{-\alpha M\log n}$$

for all n sufficiently large and for some constant $\alpha>0$, provided
$p_0>(1-3^{-8})^{1/2}$. Fixing such an p_0, we get that

$$\mathbb{P}_{or}(LR_n^c)=\sum_{1\leq j\leq n-1}\mathbb{P}_{or}(A_j\cap LR_n^c)\leq ne^{-\alpha M\log n}\leq\frac{1}{n^{\delta+1}},$$

provided M is large. □

Acknowledgements

I thank Professors Rahul Roy, Thomas Mountford and the referee for crucial comments that led to an improvement of the paper. I also thank Professors Rahul Roy and

Thomas Mountford for my postdoctoral fellowship.

References

[1] Bollobas and Riordan: Percolation. *Cambridge Univ. Press*, (2006). MR-2283880

[2] R. Durrett: Oriented Percolation in Two Dimensions. *Annals Prob.*, **12**, (1984), 999–1040. MR-0757768

[3] R. Durrett: Probability: Theory and Examples. *Cambridge Univ. Press*, (2001). MR-1609153

[4] H. Kesten: On the Speed of convergence in first-passage percolation. *Annals Appl. Prob.*, **3**, (1993), 296–338. MR-1221154

[5] R. T. Smythe and J. C. Wierman: First-Passage Percolation on the Square Lattice. *Springer-Verlag*, (2008). MR-0513421