Electronic Journal of Probability

Vol. 3 (1998) Paper no. 13, pages 1–19.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol3/paper13.abs.html

Laplace Asymptotic Expansions for Gaussian Functional Integrals

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Abstract

We obtain a Laplace asymptotic expansion, in orders of λ, of

$$\mathbb{E}_x^\rho \left\{ G(\lambda x) e^{\{-\lambda^{-2} F(\lambda x)\}} \right\}$$

the expectation being with respect to a Gaussian process. We extend a result of Pincus [9] and build upon the previous work of Davies and Truman [1, 2, 3, 4]. Our methods differ from those of Ellis and Rosen [6, 7, 8] in that we use the supremum norm to simplify the application of the result.
A.M.S. Classification: **Primary** 60G15, **Secondary** 41A60

Submitted to EJP on January 24, 1998. Final version accepted on September 21, 1998.
Keywords: Gaussian processes, asymptotic expansions, functional integrals.

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Introduction

There is a considerable literature on the development and use of Laplace asymptotic expansions in areas related to mathematical physics. The papers of Schilder [10] and Pincus [9] dealt with Wiener integrals and Gaussian functional integrals respectively. Schilder derived the full structure of the asymptotic expansion and considered examples in the solution of functional equations and the calculus of variations whilst Pincus derived the leading order behaviour for the asymptotic expansion and had applications to Hammerstein integral equations. In the papers of Davies and Truman [1, 2, 3, 4] the Laplace asymptotic expansion of a Conditional Wiener integral (the underlying process being the Brownian Bridge) was developed to arbitrarily high orders and applications were made to obtaining generalized Mehler kernel formulæ (for Hamiltonians including magnetic fields) and to the Bender-Wu formula (concerned with the behaviour of perturbation series for ground state energies of the anharmonic oscillator). One notable application of this work was in Davies and Truman [5] where the existence of the Meissner-Ochsenfeld effect was proven for an ideal charged Boson gas.

Ellis and Rosen [6, 7, 8] have developed Laplace asymptotic expansions for Gaussian functional integrals working with the L^2 norm throughout. This gives, perhaps, a cleaner approach to the estimates required in the arguments but in our view the supremum norm is easier to work with especially when one considers applications. It was this approach that was used in the initial extension of Schilder's [10] work and we continued to use it in our subsequent work.

The seminal paper of Schilder has been, and continues to be, of topical interest. In more recent years Azencott and Doss [18] have used asymptotic expansions to study the Schrödinger equation, whilst Azencott [19, 20] has used asymptotic expansions to study the density of diffusions for small time and both sequential and parallel annealing. Ben Arous (and co-workers) [21, 22, 23, 24] have developed and utilised Laplace asymptotic techniques to study functional integrals with respect to possibly degenerate diffusions, Strassen's functional law of the iterated logarithm and the asymptotics of solutions to non-homogeneous versions of the KPP equation. Kusuoka and Stroock [25, 26] have developed asymptotic expansions of certain Wiener functionals with degenerate extrema for processes on abstract Wiener space whilst Rossignol [27] has used the Newton polyhedron to study the case of Laplace integrals on Wiener space with an isolated degenerate minimum. A generalization of the expansion formula of Ben Arous has been developed by Takanobu and Watanabe [28] in which one can handle Wiener functionals which are smooth in the sense of Malliavin but not necessarily smooth in the sense of Frechet.

Some preliminary notation and the statement of the main result follow in the next section. The subsequent section contains the necessary lemmas (and proofs in some cases) to substantiate the theorem. Those lemmas due to Pincus are included for clarity and his proofs are noted as such. The final section contains the proof of the theorem. The proof is an amalgam and extension of the methods of Pincus [9] and Schilder [10] influenced by our previous work on Conditional Wiener integrals [1, 2, 3, 4].

The Laplace Asymptotic Expansion

Let $\rho(\sigma,\tau)$, $0 \le \sigma,\ \tau \le t$, denote a continuous, symmetric, positive-definite kernel. If

$$\int_0^t \int_0^t \rho(\sigma,\tau)^2 \, d\sigma \, d\tau < \infty$$

then we may define the Hilbert-Schmidt operator A by

$$(Ax)(\sigma) = \int_0^t \rho(\sigma,\tau)x(\tau)\, d\tau, \qquad x \in L^2[0,t].$$

Note that A is a compact, self-adjoint, positive-definite operator on $L^2[0,t]$. We call $\rho(\sigma,\tau)$ a covariance function if $\rho(\sigma,\tau) = \rho(\tau,\sigma)$ and if for any finite set $0 < \tau_1 < \tau_2 \cdots < \tau_n < t$ the matrix $[\rho(\tau_i,\tau_j]$ is non-negative definite. Let $\mathbb{E}_x^\rho$ denote expectation with respect to the mean zero Gaussian process with covariance function $\rho(\sigma,\tau)$ and sample paths $x \in C[0,t]$, the space of continuous functions on $[0,t]$. Let $C_\rho[0,t] \subset C[0,t]$ denote the paths of the process.

Given that A has positive eigenvalues let $\{\rho_i\}_{i=1}^\infty$ be the reciprocal eigenvalues in order of increasing magnitude. We will make use of the notation

$$(x,x) = \|x\|_2^2 = \int_0^t x^2(\tau)\, d\tau, \qquad x \in L^2[0,t],$$

and

$$\|x\|_\infty = \sup_{0 \le \tau \le t} |x(\tau)|, \qquad x \in C[0,t].$$

Theorem. *Let $\rho(\sigma,\tau)$, $0 \le \sigma,\ \tau \le t$, be a continuous, symmetric, positive-definite kernel for which there is a Gaussian process generated by $\rho(\sigma,\tau)$ having continuous sample paths $x(\tau)$, $0 \le \tau \le t$. Let $F(x)$ and $G(x)$ be real valued continuous functionals defined on $C_\rho[0,t]$ and suppose that the functional $H(x) = \frac{1}{2}(A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x) + F(x)$ attains its unique global minimum of b at $x^* \in D(A^{-\frac{1}{2}}) \subset C_\rho[0,t]$. If $F(x)$ and $G(x)$ satisfy the conditions below, then*

$$e^{b\lambda^{-2}} \mathbb{E}_x^\rho \left\{ G(\lambda x) e^{\{-\lambda^{-2}F(\lambda x)\}} \right\} = \Gamma_0 + \lambda\Gamma_1 + \cdots + \lambda^{n-3}\Gamma_{n-3} + O(\lambda^{n-2})$$

as $\lambda \to 0$, where the Γ_i are integrals dependent only on the functionals $F(x)$, $G(x)$ and their Frechet derivatives evaluated at x^*.

1. *$F(x)$ is measurable.*
2. *$F(x) \ge -\frac{1}{2}c_1(x,x) - c_2$ where $c_1 < \rho_1$ and c_2 is real.*
3. *$|F(x) - F(y)| \le K(x-y, x-y)^{\frac{\alpha}{2}}$ for $\|x - x^*\|_\infty \le 2R$, $\|y - x^*\|_\infty \le 2R$ and $0 < \alpha < 1$ where R is defined by*

$$R = \max\{1, (4c_2/\gamma)^{\frac{1}{2}}, (2+\sqrt{2})(c_2\rho_1 M/(\rho_1 - c_1))^{\frac{1}{2}}\}.$$

4. *$F(x)$ has $n \ge 3$ continuous Frechet derivatives in a ball of radius δ centred at x^* in $C_\rho[0,t]$, $\delta > 0$. We further assume that the Frechet derivatives D^jF satisfy $D^jF(x^*+\eta)(x,x,\ldots,x) = O(\|x\|_\infty^j)$ if $\|\eta\|_\infty < \delta$.*

5. *For some* $\epsilon > 0$, *for* $\|\eta\|_\infty < \delta$, $\mathbb{E}^\rho_x\{\exp\{-(1+\epsilon)D^2F(x^*+\eta)x^2/2\}\}$ *is uniformly bounded.*

6. $G(x)$ *is measurable and is continuous at* x^*.

7. $|G(x)| \leq c_4 \exp\{c_3\|x\|^2_\infty\}$, $c_3,\ c_4 > 0$.

8. $G(x)$ *has* $n-2$ *continuous Frechet derivatives in a ball of radius* δ *centred at* x^* *in* $C_\rho[0,t]$, $\delta > 0$.

We will only prove the theorem in the case of $b = 0$ since we can deduce the corresponding result for $b \neq 0$ by use of the substitution $F(x) \to F(x) - b$. This Theorem is the analogue of Schilder's Theorem C [10] and the proof follows essentially the same route as taken by both Schilder [10] and Pincus [9].

Lemmas

Lemma 1. *If the symmetric, positive-definite kernel* $\rho(\sigma,\tau)$, $0 \leq \sigma,\ \tau \leq t$ *satisfies*

$$|\rho(\sigma,\tau) - \rho(\sigma',\tau)| \leq K|\sigma - \sigma'|^\alpha,$$

where $K > 0$ *and* $0 \leq \alpha \leq 1$ *then there is a Gaussian process generated by* $\rho(\sigma,\tau)$, *with continuous sample paths* $x(\tau)$, $0 \leq \tau \leq t$, *such that if* $a > \xi > 0$

$$\mathit{Prob}\{\|x\|_\infty \geq a\} \leq c\exp\{-\gamma a^2\},$$

where c *depends only on* ξ *and* $\gamma = 9\left(\frac{2^{4/\ln 2}(\alpha \ln 2)^4}{2\pi t^\alpha 4^4 K}\right)$.

Proof. See either Simon [11] or Prohorov [13]. □

The following three lemmas are concerned with the properties of the operators A and $A^{-\frac{1}{2}}$.

Lemma 2. $A^{-\frac{1}{2}}$ *is a Hilbert-Schmidt operator with a kernel* $K(\sigma,\tau)$ *and is a completely continuous mapping of* $L^2[0,t]$ *into* $C[0,t]$.

Proof. See Dunford and Schwarz [14]. □

Lemma 3.

a) $\|A^{-\frac{1}{2}}x\|^2_\infty \leq M(x,x)$, $\qquad M = \sup_{0\leq\sigma\leq t}\rho(\sigma,\sigma)$.

b) $(Ax,x) \leq (x,x)/\rho_1$ *where* $1/\rho_1$ *is the largest eigenvalue of the operator* A.

c) $(Ax,Ax) \leq (Ax,x)/\rho_1$.

d) $\|Ax\|^2_\infty \leq M(x,x)/\rho_1$.

Proof. See Dunford and Schwarz [14]. □

Let $D(A^{-1})$ denote the domain of the operator A^{-1}. Define the Hilbert space $L^2_A[0,t]$ as the Cauchy completion of the space $D(A^{-1})$ under the norm $(A^{-1}x,x)^{\frac{1}{2}}$.

Lemma 4. *The domain of $A^{-\frac{1}{2}}$, $D(A^{-\frac{1}{2}}) = L^2_A$.*

Proof. See Mikhlin [15]. □

Now let $[x, y]_A$ denote the inner product on L^2_A. We see from Lemma 4 that $[x, x]_A = (A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x)$. In terms of L^2_A Lemma 2 can be taken to mean that every bounded set in L^2_A is precompact in C.

The following lemmas deal with the functional $H(x) = \frac{1}{2}(A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x) + F(x)$ and its properties.

Lemma 5. *Let $F(x)$ be a real valued continuous functional on $C[0, t]$ satisfying*

$$F(x) \geq -\frac{1}{2}c_1(x, x) - c_2,$$

where $c_1 < \rho_1$, $c_2 \in \mathbb{R}$. It then follows that there exists at least one point $x^* \in D(A^{-\frac{1}{2}})$ at which $H(x)$ assumes its global minimum over $C[0, t]$.

Proof. (Pincus) Let B be the set of points at which $H(x)$ attains its global minimum, and let $\{x_n\}$ be a minimising sequence of $H(x)$. We then have $B \subset D(A^{-\frac{1}{2}})$ and that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges uniformly to a point $x^* \in B$. By Lemma 3, (b), we have

$$\frac{1}{2}(A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x) + F(x) \geq \frac{1}{2}(\rho_1 - c_1)(x, x) - c_2 \geq -c_2.$$

Thus we have $H(x)$ bounded below for all x. Without loss of generality we may assume that the global minimum of $H(x)$ is zero. Clearly, when $(A^{-\frac{1}{2}}x_n, A^{-\frac{1}{2}}x_n) \to \infty$ we will have $H(x_n) \to \infty$ as $n \to \infty$. From this we see that the sequence $\{(A^{-\frac{1}{2}}x_n, A^{-\frac{1}{2}}x_n) = [x_n, x_n]_A\}$ is bounded. By what immediately follows the proof of Lemma 4 we see that $\{x_n\}$ contains a subsequence $\{x_{n_i}\}$ which forms a Cauchy sequence in $C[0, t]$. Let $\lim_{i\to\infty} x_{n_i} = y \in C[0, t]$. We now proceed to show that $y \in D(A^{-\frac{1}{2}})$. $\{x_{n_i}\}$ is a bounded set in L^2_A and so is weakly precompact. Since any Hilbert space is weakly complete we see that there exists a subsequence of $\{x_{n_i}\}$ which converges weakly in L^2_A to a point $u \in L^2_A$. We also denote this subsequence as $\{x_{n_i}\}$ to retain clarity. By the definition of weak convergence we have

$$[z, x_{n_i}]_A \to [z, u]_A$$

as $i \to \infty$ for all $z \in L^2_A$. In particular, when $z \in D(A^{-1})$ we have

$$[z, x_{n_i} - u]_A = (A^{-1}z, x_{n_i} - u) \to 0$$

as $i \to \infty$. Since $\{x_{n_i}\}$ converges uniformly to y it follows that if $z \in D(A^{-1})$ then

$$|(A^{-1}z, x_{n_i} - y)|^2 \leq (A^{-1}z, A^{-1}z)(x_{n_i} - y, x_{n_i} - y) \to 0$$

as $i \to \infty$. Therefore, we have

$$\lim_{i\to\infty}[z, x_{n_i}]_A = (A^{-1}z, u) = (A^{-1}z, y)$$

for all $z \in D(A^{-1})$ which implies that $u = y \in D(A^{-\frac{1}{2}})$. In any normed linear space the norm is weakly lower semi-continuous, i.e $x_n \to x$ weakly implies

$$|x| \leq \liminf_n |x_n|, \qquad (|x| = \text{norm } x).$$

Applying this to L^2_A we write $H(x) = \frac{1}{2}[x,x]_A + F(x)$ and obtain

$$0 = \lim_{i\to\infty} H(x_{n_i}) = \liminf_i (\frac{1}{2}[x_{n_i}, x_{n_i}]_A + F(x_{n_i})) \geq \frac{1}{2}[y,y]_A + F(y) \geq 0.$$

Therefore, $\frac{1}{2}(A^{-\frac{1}{2}}y, A^{-\frac{1}{2}}y) + F(y) = 0$ which implies $y = x^*$ for some $x^* \in B$. □

Lemma 6. *Let $F(x)$ satisfy the conditions of Lemma 5. Of those x satisfying $\|x - x^*\|_\infty \leq R$ let x^* be the only point of B satisfying $H(x)$=0. It then follows that given $\delta > 0$ there exists a $\theta(\delta) > 0$ such that $\delta < \|x - x^*\|_\infty \leq R$ implies $H(x) \geq \theta(\delta)$.*

Proof. (Pincus) From Lemma 5 we have that every minimizing sequence $\{x_n\}$ that converges in $C[0,t]$ and satisfies $\|x - x^*\|_\infty \leq R$ must converge to x^*. Suppose that there exists a $\delta > 0$ such that $\delta < \|x - x^*\|_\infty \leq R$ implies $H(x) \leq \theta$ for all $\theta > 0$. This implies that there exists a minimising sequence $\{x_n\}$ such that $\delta \leq \|x - x^*\|_\infty \leq R$. The first statement of the proof shows us that the above is a contradiction. □

Lemma 7. *If $H(x)$ has a global minimum at x^*, $H(x^*) = 0$, and $F(x)$ is a continuous, real valued functional on $C[0,t]$ then*

$$\inf_{x\in L^2} [\frac{1}{2}(Ax, x) + F(Ax)] = 0.$$

Proof. Setting $y = Ax$, we have for $x \in L^2$,

$$\frac{1}{2}(Ax, x) + F(Ax) = \frac{1}{2}(A^{-1}y, y) + F(y),$$

given $y \in D(A^{-1})$. Since $D(A^{-1})$ is dense in $L^2_A = D(A^{-\frac{1}{2}})$, and convergence in L^2_A implies uniform convergence, we have the lemma. □

We now state and prove six lemmas which give us two transformations for the functional integral and specific bounds to ensure its existence.

Lemma 8. *Let $G(x)$ be a real valued, continuous functional on $C[0,t]$, then*

$$\mathbb{E}^\rho_x \{G(x)\} = \mathbb{E}^\rho_x \left\{G(x+y)e^{\{-[y,y]_A/2 - [y,x]_A\}}\right\}$$

where if one side of the equality exists then so does the other and they are equal.

Proof. See Kuo [16]. □

Lemma 9. *Given $\rho(\sigma,\tau)$ continuous, $G(x)$ an integrable functional and $d > 0$, then*

$$\mathbb{E}^\rho_x \{G(x)\} = D_\rho(-d)\mathbb{E}^\rho_x \left\{G(x + dAx)e^{\{-d^2(Ax,x)/2 - d(x,x)\}}\right\}$$

where $D_\rho(\ )$ is the Fredholm determinant of $\rho(\sigma,\tau)$. As in Lemma 8, if one side of the inequality exists then so does the other and they are equal.

Proof. Refer to Varberg [17] for the proof. □

Lemma 10. *Let $D_\rho(\ )$ be the Fredholm determinant of $\rho(\sigma,\tau)$ then we have*

$$D_\rho(-d) \le K_\beta \exp\{(1+2\rho_1)t\beta^2 d^2/2\rho_1\}, \qquad d \ge 1,$$

where $\beta > 0$. K_β depends on β and may be explicitly determined.

Proof. From Lemma 9 we have,

$$\mathbb{E}_x^\rho\{1\} = D_\rho(-d)\mathbb{E}_x^\rho\left\{e^{\{-d^2(Ax,x)/2-d(x,x)\}}\right\}, \qquad d > 0.$$

Lemma 3, (b), enables us to write

$$\begin{aligned} 1 &\ge D_\rho(-d)\mathbb{E}_x^\rho\left\{e^{\{-(d^2/2\rho_1)(x,x)-d(x,x)\}}\right\} \\ &= D_\rho(-d)\mathbb{E}_x^\rho\left\{e^{\{-\eta(x,x)\}}\right\} \end{aligned}$$

where $\eta = d + d^2/2\rho_1 > 0$. Using $(x,x) \le t\|x\|_\infty^2$ we have

$$1 \ge D_\rho(-d)\mathbb{E}_x^\rho\left\{e^{\{-\eta t\|x\|_\infty^2\}}\right\}.$$

Let $J = \{x : \|x\|_\infty < \beta\}$, then

$$1 \ge D_\rho(-d)\mathbb{E}_{x\in J}^\rho\left\{e^{\{-\eta t\beta^2\}}\right\}$$

where $\mathbb{E}_{x\in J}^\rho\{G(x)\} = \mathbb{E}_x^\rho\{X_J(x)G(x)\}$, $X_J(x)$ being the characteristic function of the set J. Thus,

$$1 \ge D_\rho(-d)e^{\{-\eta t\beta^2\}}\mathbb{E}_{x\in J}^\rho\{1\}$$

and so

$$D_\rho(-d) \le K_\beta \exp\{(1+2\rho_1)t\beta^2 d^2/2\rho_1\},$$

where $K_\beta^{-1} = \mathbb{E}_{x\in J}^\rho\{1\}$, for $d \ge 1$. Note that K_β is a monotonic decreasing function of β bounded below at infinity by 1. □

Lemma 11. *Suppose $F(x)$ and $G(x)$ are real valued measurable functions defined on $C[0,t]$ satisfying*

$$\begin{aligned} |G(x)| &\le c_4 \exp\{c_3\|x\|_\infty^2\} \\ F(x) &\ge -\frac{1}{2}c_1(x,x) - c_2 \end{aligned}$$

where $c_3,\ c_4 > 0$, $c_1 < \rho_1$. If

$$0 < \lambda < \min\{1,\ (\rho_1 - c_1)/(\rho_1 c_1 + 2Mc_3),\ \gamma/(2c_3 + 4c_3\sqrt{Mt/\rho_1})\}$$

then

$$\begin{aligned} &\mathbb{E}_x^\rho\left\{|G(\lambda x)|e^{\{-\lambda^{-2}F(\lambda x)\}}\right\} \\ &\qquad = D_\rho(-\lambda^{-1})\mathbb{E}_x^\rho\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\} \end{aligned}$$

is finite.

Proof. Lemma 9 with $d = \lambda^{-1}$ gives the equality of the two functional integrals and using the given conditions on $F(x)$ and $G(x)$ we may write

$$\begin{aligned}
&\mathbb{E}_x^\rho \left\{ |G(\lambda x + Ax)| e^{\{-\lambda^{-2}\{(Ax/2+\lambda x, x)+F(\lambda x+Ax)\}\}} \right\} \\
&\quad \le c_4 \mathbb{E}_x^\rho \left\{ \exp\left\{ c_3 \lambda^2 \|x\|_\infty^2 + 2c_3 \lambda \|x\|_\infty \|Ax\|_\infty + c_3 \|Ax\|_\infty^2 \right\} \right. \\
&\quad \left. \exp\left\{ -\lambda^{-2}\{(Ax/2 + \lambda x, x) - c_1(Ax + \lambda x, Ax + \lambda x)/2 - c_2\}\right\} \right\}
\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}
&(Ax/2 + \lambda x, x) - c_1(Ax + \lambda x, Ax + \lambda x)/2 - \lambda^2 c_3 \|Ax\|_\infty^2 \\
&= (Ax, x)/2 - c_1(Ax, Ax)/2 + \lambda[(x, x) - c_1(x, Ax)] - \lambda^2[c_1(x, x)/2 + c_3\|Ax\|_\infty^2] \\
&\ge (1 - c_1/\rho_1)(Ax, x)/2 + \lambda(1 - c_1/\rho_1)(x, x) - \lambda^2[c_1/2 + Mc_3/\rho_1](x, x) \\
&= (1 - c_1/\rho_1)(Ax, x)/2 + \lambda[(1 - c_1/\rho_1) - \lambda(c_1/2 + Mc_3/\rho_1)](x, x) \\
&\ge (1 - c_1/\rho_1)(Ax, x)/2 + \lambda(1 - c_1/\rho_1)(x, x)/2, \qquad \text{by choice of } \lambda \\
&\ge 0
\end{aligned}$$

since $(Ax, x) \ge \rho_1(Ax, Ax) \ge 0$. Using the above and Lemma 3 again we have

$$\begin{aligned}
&\mathbb{E}_x^\rho \left\{ |G(\lambda x + Ax)| e^{\{-\lambda^{-2}\{(Ax/2+\lambda x, x)+F(\lambda x+Ax)\}\}} \right\} \\
&\quad \le c_4 \exp\{c_2 \lambda^{-2}\} \mathbb{E}_x^\rho \left\{ \exp\left\{ c_3\lambda^2 \|x\|_\infty^2 + 2c_3 \lambda \sqrt{Mt/\rho_1} \|x\|_\infty^2 \right\} \right\} \\
&\quad \le c_4 \exp\{c_2 \lambda^{-2}\} \mathbb{E}_x^\rho \left\{ \exp\{(\gamma/2)\|x\|_\infty^2\} \right\}, \qquad \text{by choice of } \lambda.
\end{aligned}$$

Setting $f(u) = \text{Prob}\{\|x\|_\infty < u\}$ the integral above may be written as

$$\int_0^\infty e^{\gamma u^2/2} \, df(u)$$

which is finite by virtue of Lemma 1. We have the existence of $D_\rho(-\lambda^{-1})$ by Lemma 10 and so

$$\mathbb{E}_x^\rho \left\{ |G(\lambda x)| e^{\{-\lambda^{-2} F(\lambda x)\}} \right\}$$

is finite. □

Lemma 12. *If the covariance function $\rho(\sigma, \tau)$ is continuous with $0 < \alpha \le 1$, $K > 0$ and $0 < \lambda \le 1$ then*

$$\begin{aligned}
&\mathbb{E}_x^\rho \left\{ \exp\{K(x, x)^{\alpha/2}/\lambda^{2-\alpha} - (x, x)/\lambda\} \right\} \\
&\quad \le 2\exp\{K^{2/(2-\alpha)}/\lambda^{2-\alpha/2}\}.
\end{aligned}$$

Proof. (Pincus) Let $g(u) = \text{Prob}\{\|x\|_2 < u\}$, then

$$\begin{aligned}
&\mathbb{E}_x^\rho \left\{ \exp\{K(x, x)^{\alpha/2}/\lambda^{2-\alpha} - (x, x)/\lambda\} \right\} \\
&\quad = \int_0^\infty \exp\{Ku^\alpha/\lambda^{2-\alpha} - u^2/\lambda\} \, dg(u) \\
&\quad \le \int_0^{\frac{K^{1/(2-\alpha)}}{\lambda^{(1-\alpha)/(2-\alpha)}}} \exp\{Ku^\alpha/\lambda^{2-\alpha} - u^2/\lambda\} \, dg(u) + 1
\end{aligned}$$

since $\exp\{Ku^\alpha/\lambda^{2-\alpha} - u^2/\lambda\} \geq 1$ for u in the latter range of integration and $\int_0^\infty dg(u) = 1$. The supremum of the exponent above will be less than $Ku^\alpha/\lambda^{2-\alpha}$ evaluated at the largest possible value of u, and so

$$\mathbb{E}_x^\rho\left\{\exp\{K(x,x)^{\alpha/2}/\lambda^{2-\alpha} - (x,x)/\lambda\}\right\} \leq \exp\{K^{2/(2-\alpha)}/\lambda^{(4-3\alpha)/(2-\alpha)}\} + 1.$$

Since $(4-3\alpha)/(2-\alpha) \leq 2-\alpha/2$ and the fact that the exponent is always greater than 1, the lemma is proven. □

Lemma 13. *Let $F(x)$ and $G(x)$ be real valued, continuous functionals on $C[0,t]$ such that the following conditions are satisfied.*

1. $F(x) \geq -\frac{1}{2}c_1(x,x) - c_2$ *where $c_1 < \rho_1$ and c_2 is real.*
2. $|G(x)| \leq c_4 \exp\{c_3\|x\|_\infty^2\}$, $c_3,\ c_4 > 0$.
3. *There exists an $x^* \in C_\rho[0,t]$ such that the functional $H(x) = (A^{-\frac{1}{2}}x, A^{-\frac{1}{2}}x)/2 + F(x)$ attains its global minimum of zero at x^* over $C[0,t]$ and $G(x)$ is continuous at x^*.*
4. $|F(x) - F(y)| \leq K(x-y, x-y)^{\frac{\alpha}{2}}$ *for $\|x - x^*\|_\infty \leq 2R$, $\|y - x^*\|_\infty \leq 2R$ and $0 < \alpha < 1$ where*

   $$R = \max\{1, (4c_2/\gamma)^{\frac{1}{2}}, (2+\sqrt{2})(c_2\rho_1 M/(\rho_1 - c_1))^{\frac{1}{2}}\}.$$

   Furthermore, suppose that x^* is the only point in the sphere $\{x \in C[0,t] : \|x - x^*\|_\infty \leq R\}$ at which $H(x)$ attains its global minimum of zero.
5. *Both $F(x)$ and $G(x)$ are measurable.*

Then for $\delta > 0$ sufficiently small and the set J_1, defined by

$$J_1 = \{x \in C[0,t] : \|\lambda x\|_\infty \leq \delta/2, \|Ax - x^*\|_\infty \leq \delta/2\}$$

we have

$$\begin{aligned} D_\rho(-\lambda^{-1})\mathbb{E}^\rho_{x\in J_1^c}&\left\{G(\lambda x + Ax)e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\} \\ &= O(\exp\{-\xi\lambda^{-2}\}) \end{aligned} \tag{1}$$

$\xi > 0$, for sufficiently small λ, J_1^c being the complement of J_1.

This Lemma lies at the heart of the proof of the Theorem. It highlights the connection with the standard large deviation estimates (Stroock [12]) but our interest in a fully constructive result leads to the proof being somewhat involved.

Proof. Choose a δ such that $\delta < \min\{1, R\}$ and $\theta(\delta) < 1$ where θ is as defined in Lemma 6 and choose $\lambda > 0$ such that

$$\begin{aligned}\lambda < \min\{&1, (1-c_1/\rho_1)/(c_1 + 2Mc_3/\rho_1), (1/cv_1 - 1/\rho_1), (\gamma/4c_3)^{\frac{1}{2}}, \\ &\gamma/2c_3(1 + 2\sqrt{Mt/\rho_1}), [\min\{\gamma\delta^2/32, \theta(\delta/2)/4\}/K_1^{2/(2-\alpha)}]^{2/\alpha}\}.\end{aligned}$$

Let J_1 be as defined in the hypothesis of the lemma and split J_1^c into the four sets

$$\begin{aligned} J_2 &= \{x : \delta/2 < \|\lambda x\|_\infty \leq R, \|Ax - x^*\|_\infty \leq R\} \\ J_3 &= \{x : \|\lambda x\|_\infty \leq \delta/2, \delta/2 < \|Ax - x^*\|_\infty \leq R\} \\ J_4 &= \{x : R < \|\lambda x\|_\infty, \|Ax - x^*\|_\infty \leq R\} \\ J_5 &= \{x : R < \|Ax - x^*\|_\infty\} \end{aligned}$$

From Lemma 10 we have that

$$D_\rho(-\lambda^{-1}) \leq K_\beta \exp\{(1 + 2\rho_1)t\beta^2/2\rho_1\lambda^2\}$$

and we may vary $\beta > 0$ as we desire. We will consider the integral as given in the hypothesis over the above sets.

Let E_2 be given by

$$\begin{aligned} E_2 &= \left|\mathbb{E}^\rho_{x\in J_2}\left\{G(\lambda x + Ax)e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\}\right| \\ &\leq \mathbb{E}^\rho_{x\in J_2}\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\} \\ &= \mathbb{E}^\rho_{x\in J_2}\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{(Ax/2,x)+F(Ax)+F(\lambda x+Ax)-F(Ax)+(\lambda x,x)\}\}}\right\} \end{aligned}$$

Now recall from Lemma 7 that $\inf_{x\in L^2}\{(Ax/2, x) + F(Ax)\} = 0$ and so

$$E_2 \leq \mathbb{E}^\rho_{x\in J_2}\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{F(\lambda x+Ax)-F(Ax)+\lambda(x,x)\}\}}\right\}.$$

By condition (2)

$$\begin{aligned} |G(\lambda x + Ax)| &\leq c_4 \exp\{c_3\|\lambda x + Ax\|^2_\infty\} \\ &\leq c_4 \exp\{c_3\|\lambda x\|^2_\infty + 2c_3\|\lambda x\|_\infty\|Ax\|_\infty + c_3\|Ax\|^2_\infty\}. \end{aligned}$$

Given that $x \in J_2$ we have $\|\lambda x\|_\infty$ and $\|Ax\|_\infty$ both bounded and so we may choose $K_2 \in \mathbb{R}^+$, an absolute constant, such that

$$|G(\lambda x + Ax)| \leq K_2.$$

Also for $x \in J_2$ we have $\|Ax + \lambda x - x^*\|_\infty \leq 2R$ and $\|Ax - x^*\|_\infty \leq 2R$ giving

$$|F(Ax + \lambda x) - F(Ax)| \leq K_1(\lambda x, \lambda x)^{\frac{\alpha}{2}}$$

and so we obtain

$$\begin{aligned} E_2 &\leq K_2\mathbb{E}^\rho_{x\in J_2}\left\{\exp\{K_1(x, x)^{\frac{\alpha}{2}}/\lambda^{2-\alpha} - (x, x)/\lambda\}\right\} \\ &\leq K_2\left[\mathbb{E}^\rho_x\left\{\exp\{2K_1(x, x)^{\frac{\alpha}{2}}/\lambda^{2-\alpha} - 2(x, x)/\lambda\}\right\}\right]^{\frac{1}{2}}\left[\mathbb{E}^\rho_x\{X_{J_2}(x)\}\right]^{\frac{1}{2}} \end{aligned}$$

by use of the Cauchy-Schwarz inequality. Now apply the result of Lemma 1

$$\mathbb{E}^\rho_x\{X_{J_2}(x)\} \leq c \exp\{-\gamma\delta^2/4\lambda^2\}$$

and Lemma 12 to get

$$E_2 \leq \sqrt{2}K_2 \exp\{-\lambda^{-2}\{\gamma\delta^2/8 - \lambda^{\alpha/2}K_1^{2/(2-\alpha)}\}\} \qquad (2)$$

Now consider the integral in equation (1) over the set J_3. Let E_3 be given as

$$\begin{aligned} E_3 &= \left|\mathbb{E}^{\rho}_{x\in J_3}\left\{G(\lambda x + Ax)e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\}\right| \\ &\le \mathbb{E}^{\rho}_{x\in J_3}\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\} \\ &\le K_2\mathbb{E}^{\rho}_{x\in J_3}\left\{e^{\{-\lambda^{-2}\{(Ax/2,x)+F(Ax)\}\}}e^{\{K_1(x,x)^{\frac{\alpha}{2}}/\lambda^{2-\alpha}-(x,x)/\lambda\}}\right\} \end{aligned}$$

using two of our previous arguments. Letting $y = Ax$ we have $(Ax/2, x) + F(Ax) = H(y)$ and $x \in J_3$ implies $\delta/2 < \|y - x^*\|_\infty \le R$. Thus by Lemma 6,

$$(Ax/2, x) + F(Ax) > \theta(\delta/2) > 0, \qquad x \in J_3.$$

Therefore,

$$\begin{aligned} E_3 &\le K_2 \exp\{-\lambda^{-2}\theta(\delta/2)\}\mathbb{E}^{\rho}_{x\in J_3}\left\{\exp\{K_1(x,x)^{\frac{\alpha}{2}}/\lambda^{2-\alpha} - (x,x)/\lambda\}\right\} \\ &\le 2K_2 \exp\{-\lambda^{-2}\{\theta(\delta/2) - \lambda^{\alpha/2}K_1^{2/(2-\alpha)}\}\} \end{aligned} \tag{3}$$

by lemma 12.

Next define E_4 by

$$\begin{aligned} E_4 &= \left|\mathbb{E}^{\rho}_{x\in J_4}\left\{G(\lambda x + Ax)e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\}\right| \\ &\le \mathbb{E}^{\rho}_{x\in J_4}\left\{|G(\lambda x + Ax)|e^{\{-\lambda^{-2}\{(Ax/2+\lambda x,x)+F(\lambda x+Ax)\}\}}\right\}. \end{aligned}$$

By condition (2) of the hypothesis of the lemma

$$\begin{aligned} |G(\lambda x + Ax)| &\le c_4 \exp\left\{c_3\|\lambda x + Ax\|^2_\infty\right\} \\ &\le c_4 \exp\left\{c_3\|\lambda x\|^2_\infty + 2\lambda c_3\|x\|_\infty\|Ax\|_\infty + c_3\|Ax\|^2_\infty\right\}. \end{aligned}$$

Since $x \in J_4$, $\|Ax - x^*\|_\infty \le R$ implying $\|Ax\|_\infty \le R + \|x^*\|_\infty$. Thus,

$$\begin{aligned} E_4 \le c_4 e^{\left\{c_3(R+\|x^*\|_\infty)^2\right\}}\mathbb{E}^{\rho}_{x\in J_4}&\left\{\exp\left\{c_3\|\lambda x\|^2_\infty + 2\lambda c_3\|x\|_\infty(R + \|x^*\|_\infty)\right\}\right. \\ &\left.\exp\{-\lambda^{-2}\{(Ax/2 + \lambda x, x) + F(\lambda x + Ax)\}\}\right\} \end{aligned}$$

By condition (1) of our hypothesis and Lemma 3

$$\begin{aligned} &(Ax/2 + \lambda x, x) + F(\lambda x + Ax) \\ &\quad\ge (Ax, x)/2 + \lambda(x, x) - c_1(Ax + \lambda x, Ax + \lambda x)/2 - c_2 \\ &\quad= (Ax, x)/2 + \lambda(x, x) - c_1\lambda^2(x, x)/2 - c_1\lambda(x, Ax) - c_1(Ax, Ax)/2 - c_2 \\ &\quad\ge (1 - c_1/\rho_1)(Ax, x)/2 + \lambda(1 - c_1/\rho_1 - c_1\lambda/2)(x, x) - c_2 \\ &\quad\ge (1 - c_1/\rho_1)[(Ax, x) + \lambda(x, x)]/2 - c_2, \qquad \text{by choice of } \lambda \\ &\quad\ge -c_2. \end{aligned}$$

Thus,

$$\begin{aligned} E_4 &\le c_4 \exp\left\{c_3(R + \|x^*\|_\infty)^2\right\} \exp\{c_2\lambda^{-2}\} \\ &\qquad \mathbb{E}^{\rho}_{x\in J_4}\left\{\exp\left\{c_3\|\lambda x\|^2_\infty + 2\lambda c_3\|x\|_\infty(R + \|x^*\|_\infty)\right\}\right\} \\ &\le K_3 \exp\{c_2\lambda^{-2}\}\left[\mathbb{E}^{\rho}_x\{X_{J_4}(x)\}\right]^{\frac{1}{2}} \\ &\qquad \left[\mathbb{E}^{\rho}_x\left\{\exp\left\{2c_3\|\lambda x\|^2_\infty + 4\lambda c_3\|x\|_\infty(R + \|x^*\|_\infty)\right\}\right\}\right]^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. Using Lemma 1 and the bound of λ we may now write

$$E_4 \leq K_4 \exp\left\{-\lambda^{-2}\{\gamma R^2/2 - c_2\}\right\}. \tag{4}$$

We finally consider E_5,

$$\begin{aligned} E_5 &= \left|\mathbb{E}^{\rho}_{x\in J_5}\left\{G(\lambda x + Ax)\exp\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\}\right\}\right| \\ &\leq \mathbb{E}^{\rho}_{x\in J_5}\left\{|G(\lambda x + Ax)|\exp\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\}\right\} \\ &\leq c_4\mathbb{E}^{\rho}_{x\in J_5}\left\{\exp\left\{c_3\|\lambda x\|^2_\infty + 2\lambda c_3\|x\|_\infty\|Ax\|_\infty + c_3\|Ax\|^2_\infty\right\}\right. \\ &\qquad \left.\exp\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\}\right\} \\ &\leq c_4\mathbb{E}^{\rho}_{x\in J_5}\left\{\exp\left\{c_3\lambda^2\|x\|^2_\infty + 2\lambda c_3\sqrt{Mt/\rho_1}\|x\|^2_\infty\right\}\right. \\ &\qquad \left.\exp\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax) - c_3\lambda^2\|Ax\|^2_\infty\}\}\right\} \\ &\leq c_4\exp\{c_2\lambda^{-2}\}\mathbb{E}^{\rho}_{x\in J_5}\left\{\exp\left\{(c_3\lambda^2 + 2\lambda c_3\sqrt{Mt/\rho_1})\|x\|^2_\infty\right\}\right. \\ &\qquad \left.\exp\{-\lambda^{-2}\{(Ax/2+\lambda x, x) - (\lambda x + Ax, \lambda x + Ax)/2 - c_3\lambda^2\|Ax\|^2_\infty\}\}\right\}. \end{aligned}$$

From the proof of Lemma 11 we have

$$\begin{aligned} &(Ax/2+\lambda x, x) - (\lambda x + Ax, \lambda x + Ax)/2 - c_3\lambda^2\|Ax\|^2_\infty \\ &\qquad \geq (1 - c_1/\rho_1)(Ax, x)/2 \\ &\qquad \geq 0 \end{aligned}$$

and so we have

$$\begin{aligned} E_5 \leq c_4\exp\{c_2\lambda^{-2}\}\mathbb{E}^{\rho}_{x\in J_5}&\left\{\exp\left\{(c_3\lambda^2 + 2\lambda c_3\sqrt{Mt/\rho_1})\|x\|^2_\infty\right\}\right. \\ &\qquad \left.\exp\left\{-\lambda^{-2}(1 - c_1/\rho_1)(Ax, x)/2\right\}\right\}. \end{aligned}$$

If $x \in J_5$ then $\|Ax - x^*\|_\infty > R$, and from Lemma 3 we have

$$\begin{aligned} R &\leq \|Ax - x^*\|_\infty \\ &= \|A^{\frac{1}{2}}(A^{\frac{1}{2}}x - A^{-\frac{1}{2}}x^*)\|_\infty \\ &\leq M^{\frac{1}{2}}(A^{\frac{1}{2}}x - A^{-\frac{1}{2}}x^*, A^{\frac{1}{2}}x - A^{-\frac{1}{2}}x^*)^{\frac{1}{2}}. \end{aligned}$$

Now

$$\begin{aligned} (Ax, x)^{\frac{1}{2}} &= (A^{\frac{1}{2}}x, A^{\frac{1}{2}}x)^{\frac{1}{2}} \\ &\geq R/\sqrt{M} - (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*)^{\frac{1}{2}}. \end{aligned}$$

From the given conditions on $H(x)$ we have

$$0 = H(x^*) = (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*)/2 + F(x^*) \geq (\rho_1 - c_1)(x^*, x^*)/2 - c_2$$

and also

$$\begin{aligned} (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*) &= -2F(x^*) \\ &\leq c_1(x^*, x^*) + 2c_2 \\ &\leq 2c_1c_2/(\rho_1 - c_1) + 2c_2 \\ &= 2c_2\rho_1/(\rho_1 - c_1). \end{aligned}$$

We now have $(Ax,x)^{\frac{1}{2}} \geq R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1 - c_1)}$ and so

$$E_5 \leq c_4 \exp\left\{-\lambda^{-2}\left\{(1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/2 - c_2\right\}\right\}$$
$$\mathbb{E}^{\rho}_{x\in J_5}\left\{\exp\left\{(c_3\lambda^2 + 2\lambda c_3\sqrt{Mt/\rho_1})\|x\|^2_\infty\right\}\right\}.$$

Since

$$\begin{aligned}
&\mathbb{E}^{\rho}_{x\in J_5}\left\{\exp\left\{(c_3\lambda^2 + 2\lambda c_3\sqrt{Mt/\rho_1})\|x\|^2_\infty\right\}\right\} \\
&\quad \leq \mathbb{E}^{\rho}_{x}\left\{\exp\left\{(c_3\lambda^2 + 2\lambda c_3\sqrt{Mt/\rho_1})\|x\|^2_\infty\right\}\right\} \\
&\quad \leq K_5/c_4
\end{aligned}$$

for some choice of $K_5 \in \mathbb{R}^+$ by choice of λ we finally obtain

$$E_5 \leq K_5 \exp\left\{-\lambda^{-2}\left\{(1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/2 - c_2\right\}\right\}. \tag{5}$$

Recalling equation (1) we see that

$$\begin{aligned}
&|D_\rho(-\lambda^{-1})\mathbb{E}^{\rho}_{x\in J_1^c}\left\{G(\lambda x + Ax)\exp\left\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\right\}\right\}| \\
&\leq D_\rho(-\lambda^{-1})\mathbb{E}^{\rho}_{x\in J_1^c}\left\{|G(\lambda x + Ax)|\exp\left\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\right\}\right\} \\
&\leq D_\rho(-\lambda^{-1})\left[E_2 + E_3 + E_4 + E_5\right].
\end{aligned}$$

We have the following equalities to consider.

$$\begin{aligned}
&D_\rho(-\lambda^{-1})E_2 \leq \sqrt{2}K_\beta K_2 \exp\left\{-\lambda^{-2}\{\gamma\delta^2/8 - \lambda^{\alpha/2}K_1^{2/(2-\alpha)} - (1+2\rho_1)t\beta^2/2\rho_1\}\right\}, \\
&D_\rho(-\lambda^{-1})E_3 \leq 2K_\beta K_2 \exp\left\{-\lambda^{-2}\{\theta(\delta/2) - \lambda^{\alpha/2}K_1^{2/(2-\alpha)} - (1+2\rho_1)t\beta^2/2\rho_1\}\right\}, \\
&D_\rho(-\lambda^{-1})E_4 \leq K_\beta K_4 \exp\left\{-\lambda^{-2}\{\gamma R^2/2 - c_2 - (1+2\rho_1)t\beta^2/2\rho_1\}\right\}, \\
&D_\rho(-\lambda^{-1})E_5 \leq K_\beta K_5 \exp\Big\{-\lambda^{-2}\Big\{(1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/2 \\
&\qquad\qquad - c_2 - (1+2\rho_1)t\beta^2/2\rho_1\Big\}\Big\}.
\end{aligned}$$

We choose $\beta > 0$ such that

$$\begin{aligned}
\beta^2 = (2\rho_1/t(1+2\rho_1))\min\{&\gamma\delta^2/32, \theta(\delta/2)/4, (\gamma R^2/2 - c_2)/2, \\
&((1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/2 - c_2)/2\}
\end{aligned}$$

and we have

$$\lambda \leq [\min\{\gamma\delta^2/32, \theta(\delta/2)/4\}/K_1^{2/(2-\alpha)}]^{2/\alpha}$$

giving

$$\begin{aligned}
D_\rho(-\lambda^{-1})E_2 &\leq \sqrt{2}K_\beta K_2 \exp\left\{-\lambda^{-2}\gamma\delta^2/16\right\} \\
D_\rho(-\lambda^{-1})E_3 &\leq 2K_\beta K_2 \exp\left\{-\lambda^{-2}\theta(\delta/2)/2\right\}
\end{aligned}$$

Our choice of R gives

$$D_\rho(-\lambda^{-1})E_4 \le K_\beta K_4 \exp\left\{-\lambda^{-2}\gamma R^2/8\right\}$$

$$D_\rho(-\lambda^{-1})E_5 \le K_\beta K_5 \exp\left\{-\lambda^{-2}(1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/8\right\}$$

If we define ξ by

$$\min\left\{1, \gamma\delta^2/16, \theta(\delta/2)/2, \gamma R^2/8, (1-c_1/\rho_1)\left[R/\sqrt{M} - \sqrt{2c_2\rho_1/(\rho_1-c_1)}\right]^2/8\right\}$$

then we have

$$\begin{aligned} D_\rho(-\lambda^{-1})\mathbb{E}^\rho_{x\in J_1^c}&\left\{G(\lambda x + Ax)\exp\left\{-\lambda^{-2}\{(Ax/2+\lambda x, x) + F(\lambda x + Ax)\}\right\}\right\} \\ &= \mathrm{O}(\exp\{-\xi\lambda^{-2}\}) \end{aligned}$$

proving the lemma. □

Proof of Theorem

Proof. First pick a δ that satisfies both the hypothesis of the Theorem and the conditions in the proof of Lemma 13. Let $X(\delta/\lambda, x^*/\lambda, x)$ be the characteristic function of the set $\{x \in C_\rho[0,t] : \|\lambda x - x^*\|_\infty \le \delta\}$. We now consider the integral

$$\begin{aligned} &\mathbb{E}^\rho_x\left\{[1 - X(\delta/\lambda, x^*/\lambda, x)]G(\lambda x)\exp\left\{-\lambda^{-2}F(\lambda x)\right\}\right\} \\ &\quad = D_\rho(-\lambda^{-1})E^\rho_x\left\{[1 - X(\delta/\lambda, x^*/\lambda, x + \lambda^{-1}Ax)]G(\lambda x + Ax)\right. \\ &\qquad\qquad \left.\exp\left\{-\lambda^{-2}\left\{(Ax/2 + \lambda x, x) + F(\lambda x + Ax)\right\}\right\}\right\} \end{aligned}$$

(the integral above is over the complement of the set $\{x : \|\lambda x + Ax - x^*\|_\infty \le \delta\}$). If $x \in J_1$, with J_1 as in Lemma 13, then

$$\begin{aligned} \|\lambda x + Ax - x^*\|_\infty &\le \|\lambda x\|_\infty + \|Ax - x^*\|_\infty \\ &\le \delta/2 + \delta/2 \\ &= \delta \end{aligned}$$

and so $x \in J_1$ implies $x \in \{x : \|\lambda x + Ax - x^*\|_\infty \le \delta\}$ and thus $\{x : X(\delta/\lambda, x^*/\lambda, x + \lambda^{-1}Ax) = 1\}^c \subset J_1^c$. From Lemma 13 and the hypothesis of the Theorem we have

$$\mathbb{E}^\rho_x\left\{[1 - X(\delta/\lambda, x^*/\lambda, x)]G(\lambda x)\exp\left\{-\lambda^{-2}F(\lambda x)\right\}\right\} = O(\exp\{-\xi\lambda^{-2}\})$$

for λ sufficiently small. We need now only consider the integral

$$E_1 = \mathbb{E}^\rho_x\left\{X(\delta/\lambda, x^*/\lambda, x)G(\lambda x)\exp\left\{-\lambda^{-2}F(\lambda x)\right\}\right\}$$

since $O(\exp\{-\xi\lambda^{-2}\}) \le O(\lambda^{n-2})$ for any n. We now use the translation of Lemma 8 with $x \to x + x^*/\lambda$ to give

$$\begin{aligned} E_1 = \mathbb{E}^\rho_x&\left\{X(\delta/\lambda, 0, x)G(\lambda x + x^*)\right. \\ &\left.\exp\left\{-\lambda^{-2}F(\lambda x + x^*) - \lambda^{-2}(A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*)/2 - \lambda^{-1}(A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x)\right\}\right\}. \end{aligned}$$

From Taylor's theorem for functionals we have

$$\begin{aligned} F(\lambda x + x^*) &= F(x^*) + \lambda DF(x^*)x + \lambda^2 D^2F(x^*)(x,x)/2 + k_3(\lambda x) \\ &= f_0(0) + \lambda f_1(0,x) + \lambda^2 f_2(0,x) + k_3(\lambda x), \quad \text{say for convenience}^1, \end{aligned}$$

where $|k_3(\lambda x)| = O(\lambda^3 \|x\|_\infty^3)$ for $\|\lambda x\|_\infty < \delta$. Therefore,

$$\begin{aligned} E_1 &= \mathbb{E}_x^\rho \Big\{ X(\delta/\lambda, 0, x) G(\lambda x + x^*) \exp\Big\{ -\lambda^{-2}[F(x^*) + (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*)/2] \\ &\qquad - \lambda^{-1}[DF(x^*)x + (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x)] - f_2(0,x) - \lambda^{-2}k_3(\lambda x) \Big\}\Big\} \\ &= \mathbb{E}_x^\rho \Big\{ X(\delta/\lambda, 0, x) G(\lambda x + x^*) \exp\Big\{ -f_2(0,x) - \lambda^{-2}k_3(\lambda x) \Big\}\Big\} \end{aligned}$$

since

$$\begin{aligned} F(x^*) + (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x^*)/2 = 0 \\ DF(x^*)x + (A^{-\frac{1}{2}}x^*, A^{-\frac{1}{2}}x) = 0 \end{aligned}$$

except possibly on a set of zero measure by definition of x^*.

The Taylor series for $\exp\{z\}$ is $\sum_{i=0}^{n-1} z^i/i! + R_n(z)$ where

$$|R_n(z)| \le \begin{cases} (z^n/n!)\exp\{z\} & z \ge 0, \\ |z|^n/n! & z < 0. \end{cases}$$

We may now write E_1 in the form

$$\begin{aligned} E_1 = \sum_{i=0}^{n-3} (1/i!) \mathbb{E}_x^\rho \Big\{ X(\delta/\lambda, 0, x) G(\lambda x + x^*) \exp\Big\{ -f_2(0,x) \Big\} [-\lambda^{-2}k_3(\lambda x)]^i \Big\} \\ + V_{n-2}(\lambda) \end{aligned}$$

Setting $B(\lambda, x)$ to be the characteristic function of the set $\{x : k_3(\lambda x) \ge 0\}$,

$$\begin{aligned} &|V_{n-2}(\lambda)| \\ &\le (1/(n-2)!) \mathbb{E}_x^\rho \Big\{ X(\delta/\lambda, 0, x) |G(\lambda x + x^*)| |\lambda^{-2}k_3(\lambda x)|^{n-2} \\ &\qquad \exp\Big\{ -f_2(0,x) \Big\} B(\lambda, x) \Big\} \\ &+ (1/(n-2)!) \mathbb{E}_x^\rho \Big\{ X(\delta/\lambda, 0, x) |G(\lambda x + x^*)| |\lambda^{-2}k_3(\lambda x)|^{n-2} \\ &\qquad \exp\Big\{ -f_2(0,x) - \lambda^{-2}k_3(\lambda x) \Big\} [1 - B(\lambda, x)] \Big\} \end{aligned}$$

From Taylor's theorem for functionals it follows that if $\|\lambda x\|_\infty \le \delta$, then

$$\lambda^2 f_2(0,x) + k_3(\lambda x) = k_2(\lambda x) = \lambda^2 f_2(\eta, x)$$

for some $\eta \in C[0,t]$ with $\|\eta\|_\infty \le \delta$ where by hypothesis $|k_3(\lambda x)| \le C_3 \lambda^3 \|x\|_\infty^3$,

[1] $f_j(\eta, x) \equiv D^j F(x^* + \eta)(x, x, \ldots, x)/j!$

C_3 being a constant. Thus

$$
\begin{aligned}
&|V_{n-2}(\lambda)| \\
&\leq (1/(n-2)!)\mathbb{E}_x^\rho\Big\{X(\delta/\lambda,0,x)|G(\lambda x+x^*)|(C_3\lambda)^{n-2}\|x\|_\infty^{3(n-2)} \\
&\qquad\qquad \exp\Big\{-f_2(0,x)\Big\}B(\lambda,x)\Big\} \\
&+(1/(n-2)!)\mathbb{E}_x^\rho\Big\{X(\delta/\lambda,0,x)|G(\lambda x+x^*)|(C_3\lambda)^{n-2}\|x\|_\infty^{3(n-2)} \\
&\qquad\qquad \exp\Big\{-f_2(\eta,x)\Big\}[1-B(\lambda,x)]\Big\}
\end{aligned}
$$

By using the Cauchy-Schwarz inequality (or Hölder's inequality) and condition (5) of the theorem we have that $V_{n-2}(\lambda)=O(\lambda^{n-2})$. We have now proved that

$$
\begin{aligned}
E_1 = &\sum_{i=0}^{n-3}(1/i!)\mathbb{E}_x^\rho\Big\{X(\delta/\lambda,0,x)G(\lambda x+x^*)\exp\Big\{-f_2(0,x)\Big\}[-\lambda^{-2}k_3(\lambda x)]^i\Big\} \\
&\qquad +O(\lambda^{n-2}).
\end{aligned}
$$

Since $G(x)$ has $n-2$ continuous Frechet derivatives in a neighbourhood of x^* we may write

$$
G(\lambda x+x^*)=\sum_{j=0}^{n-3}\lambda^j g_j(0,x)+S_{n-2}(\lambda x)
$$

where $S_{n-2}(\lambda x)=O(\|\lambda x\|_\infty^{n-2})$. Thus, using a similar argument to before we have

$$
\begin{aligned}
E_1 = &\sum_{j=0}^{n-3}\sum_{i=0}^{n-3}(1/i!)\lambda^j\mathbb{E}_x^\rho\Big\{X(\delta/\lambda,0,x)g_j(0,x)\exp\Big\{-f_2(0,x)\Big\}[-\lambda^{-2}k_3(\lambda x)]^i\Big\} \\
&\qquad +O(\lambda^{n-2}).
\end{aligned}
$$

However $k_3(\lambda x)=\lambda^3 f_3(0,x)+\cdots+\lambda^{n-1}f_{n-1}(0,x)+k_n(\lambda x)$ where $\lambda^{-2}k_n(\lambda x)=O(\lambda^{n-2}\|x\|_\infty^n)$ for $\|\lambda x\|_\infty\leq\delta$ and so by using the above, condition (5) of the theorem and Hölder's inequality

$$
\begin{aligned}
E_1 = &\sum_{j=0}^{n-3}\sum_{i=0}^{n-3}(-1)^i(1/i!)\lambda^j\mathbb{E}_x^\rho\Big\{X(\delta/\lambda,0,x)g_j(0,x)\exp\Big\{-f_2(0,x)\Big\} \\
&\qquad\qquad [\lambda f_3(0,x)+\cdots+\lambda^{n-3}f_{n-1}(0,x)]^i\Big\}+O(\lambda^{n-2}).
\end{aligned}
$$

It can be seen from the Hölder inequality, Lemma 1 and condition (5) of the theorem that

$$
\begin{aligned}
&\sum_{j=0}^{n-3}\sum_{i=0}^{n-3}(-1)^i(1/i!)\lambda^j\mathbb{E}_x^\rho\Big\{[1-X(\delta/\lambda,0,x)]g_j(0,x)\exp\Big\{-f_2(0,x)\Big\} \\
&\qquad\qquad [\lambda f_3(0,x)+\cdots+\lambda^{n-3}f_{n-1}(0,x)]^i\Big\}=O(P(\lambda)\exp\{-\eta\lambda^{-2}\}),
\end{aligned}
$$

where P is a polynomial and η is a positive constant. Replacing X by $[1-(1-X)]$ finally gives

$$E_1 = \sum_{j=0}^{n-3}\sum_{i=0}^{n-3}(-1)^i(1/i!)\lambda^j \mathbb{E}_x^\rho\Big\{g_j(0,x)\exp\Big\{-f_2(0,x)\Big\}$$

$$[\lambda f_3(0,x)+\cdots+\lambda^{n-3}f_{n-1}(0,x)]^i\Big\} + O(\lambda^{n-2}),$$

so that

$$E_1 = \Gamma_0 + \lambda\Gamma_1 + \lambda^2\Gamma_2 + \cdots + \lambda^{n-3}\Gamma_{n-3} + O(\lambda^{n-2}),$$

where the Γ_i are only dependent on the Frechet derivatives of F and G at x^* for $i = 1, 2, 3, \ldots, n-3$. This completes the proof of the theorem. □

Acknowledgements

I wish to thank Prof Aubrey Truman for bringing to my attention the papers of Schilder [10] and Pincus [9] many years ago and for his careful direction in our early joint papers. Secondly I wish to thank the referees for bringing to my attention more recent works involving the use of asymptotic expansions of Wiener functional integrals.

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