

## Corrections to "On Lévy processes conditioned to stay positive"

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### Abstract

We correct two errors of omission in our paper, [2].

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We would like to correct two errors of omission in our paper, [2]. The first occurs in equation (2.4), where we overlooked the possibility that the downgoing ladder time process has a positive drift. This happens if and only if 0 is not regular for  $(0, \infty)$ . If this drift is denoted by  $\eta$ , the correct version of (2.4) is

$$\begin{aligned} & \mathbb{P}_x(\tau_{(-\infty, 0)} > \mathbf{e}/\varepsilon) = \mathbb{P}(\underline{X}_{\mathbf{e}/\varepsilon} \geq -x) \\ & = \mathbb{E} \left( \int_{[0, \infty)} e^{-\varepsilon t} \mathbb{1}_{\{\underline{X}_t \geq -x\}} d\underline{L}_t \right) [\eta\varepsilon + \underline{n}(\mathbf{e}/\varepsilon < \zeta)], \end{aligned} \quad (1)$$

and the correct version of (2.5) is:

$$h(x) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_x(\tau_{(-\infty, 0)} > \mathbf{e}/\varepsilon)}{\eta\varepsilon + \underline{n}(\mathbf{e}/\varepsilon < \zeta)}. \quad (2)$$

However this makes no essential difference to the proof of the following Lemma 1: we just need to replace  $\underline{n}(\mathbf{e}_\varepsilon < \zeta)$  by  $\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)$  four times, and  $\underline{n}(\zeta)$  by  $\eta + \underline{n}(\zeta)$  in (2.6). The details can be seen in section 8.2 of [3]. We should also mention that (1) can be found in [1]: see equation (8), p 174.

The second omission is that we failed to give any proof of

**Corollary 1.** *Assume that 0 is regular upwards. For any  $t > 0$  and for any  $\mathcal{F}_t$ -measurable, continuous and bounded functional  $F$ ,*

$$\underline{n}(F, t < \zeta) = k \lim_{x \rightarrow 0} \mathbb{E}_x^\uparrow(h(X_t)^{-1}F).$$

The clear implication from our paper is that this follows immediately from our main result, Theorem 2, but this overlooks the singularity at zero of the function  $1/h(x)$ . Since this Corollary has been cited in a number of recent papers, we give here a full proof of it.

*Proof.* From (3.2) and Theorem 2 of [2] we see that, for any fixed  $\delta > 0, t > 0$ ,

$$\underline{n}(F, t < \zeta, X_t > \delta) = k \lim_{x \rightarrow 0} \mathbb{E}_x^\uparrow(h(X_t)^{-1}F, X_t > \delta),$$

and in particular, taking  $F \equiv 1$ ,

$$\begin{aligned} \underline{n}(t < \zeta, X_t > \delta) &= k \lim_{x \rightarrow 0} \mathbb{E}_x^\uparrow(h(X_t)^{-1}, X_t > \delta) \\ &= k \lim_{x \rightarrow 0} \mathbb{P}_x(X_t > \delta, \tau_{(-\infty, 0)} > t)/h(x). \end{aligned}$$

Suppose we can show that

$$\underline{n}(t < \zeta) = k \lim_{x \rightarrow 0} \mathbb{P}_x(\tau_{(-\infty, 0)} > t)/h(x). \quad (3)$$

Then, by subtraction,

$$\begin{aligned} \underline{n}(t < \zeta, X_t \leq \delta) &= k \lim_{x \rightarrow 0} \mathbb{P}_x(X_t \leq \delta, \tau_{(-\infty, 0)} > t)/h(x) \\ &= k \lim_{x \rightarrow 0} \mathbb{E}_x^\uparrow(h(X_t)^{-1}, X_t \leq \delta). \end{aligned}$$

Since  $\underline{n}(t < \zeta, X_t = 0) = 0$ , if  $K$  is an upper bound for  $F$ , we also have

$$\lim_{\delta \rightarrow 0} \underline{n}(F, t < \zeta, X_t \leq \delta) \leq K \lim_{\delta \rightarrow 0} \underline{n}(t < \zeta, X_t \leq \delta) = 0,$$

and the required conclusion follows.

To prove (3) we start with (1), and, since we are assuming that 0 is regular upwards, the drift  $\eta$  in the downwards ladder time process is zero, so we can write it as

$$\int_0^\infty e^{-\varepsilon t} \mathbb{P}_x(\tau_{(-\infty, 0)} > t) dt = h^{(\varepsilon)}(x) \int_0^\infty e^{-\varepsilon t} \underline{n}(\zeta > t) dt,$$

where  $h^{(\varepsilon)}(x) = \mathbb{E} \left( \int_0^\infty e^{-\varepsilon t} \mathbb{1}_{\underline{X}_t \geq -x} dL_t \right)$ . We know  $0 \leq h^{(\varepsilon)}(x) \leq h(x)$ , so

$$\int_0^\infty e^{-\varepsilon t} \mathbb{P}_x(\tau_{(-\infty, 0)} > t) dt \leq h(x) \int_0^\infty e^{-\varepsilon t} \underline{n}(\zeta > t) dt.$$

But we also have

$$\begin{aligned} \liminf_{x \downarrow 0} \frac{\mathbb{P}_x(\tau_{(-\infty, 0)} > t)}{h(x)} &\geq \lim_{\delta \downarrow 0} \lim_{x \downarrow 0} \frac{\mathbb{P}_x(\tau_{(-\infty, 0)} > t, X_t > \delta)}{h(x)} \\ &= \lim_{\delta \downarrow 0} \underline{n}(\zeta > t, X_t > \delta) = \underline{n}(\zeta > t). \end{aligned}$$

Together, these prove that

$$\lim_{x \downarrow 0} \int_0^\infty e^{-\varepsilon t} \frac{\mathbb{P}_x(\tau_{(-\infty, 0)} > t) dt}{h(x)} = \int_0^\infty e^{-\varepsilon t} \underline{n}(\zeta > t) dt.$$

Thus the measure with density  $\mathbb{P}_x(\tau_{(-\infty, 0)} > t)/h(x)$  converges weakly to the measure with the continuous density  $\underline{n}(\zeta > t)$ . But if  $0 < c < t$  are fixed we have

$$\begin{aligned} \lim_{x \rightarrow 0} \mathbb{P}_x(\tau_{(-\infty, 0)} > t)/h(x) &\geq c^{-1} \lim_{x \rightarrow 0} \int_t^{t+c} \mathbb{P}_x(\tau_{(-\infty, 0)} > s) ds/h(x) \\ &= c^{-1} \int_t^{t+c} \underline{n}(\zeta > s) ds \geq \underline{n}(\zeta > t+c), \\ \lim_{x \rightarrow 0} \mathbb{P}_x(\tau_{(-\infty, 0)} > t)/h(x) &\leq c^{-1} \lim_{x \rightarrow 0} \int_{t-c}^t \mathbb{P}_x(\tau_{(-\infty, 0)} > s) ds/h(x) \\ &= c^{-1} \int_{t-c}^t \underline{n}(\zeta > s) ds \leq \underline{n}(\zeta > t-c), \end{aligned}$$

and letting  $c \downarrow 0$  we conclude that (3) holds, and hence the Corollary.  $\square$

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## References

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