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The mass of sites visited by a random walk on an infinite graph

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Abstract

We determine the log-asymptotic decay rate of the negative exponential moments of the mass of sites visited by a random walk on an infinite graph which satisfies a two-sided sub-Gaussian estimate on its transition kernel. This provides a new method of proof of the correct decay rate for Cayley graphs of finitely generated groups with polynomial volume growth. This method also extend known results by determining this decay rate for certain graphs with fractal-like structure or with non-Alfors regular volume growth functions.

Key words: random walk, infinite graph, visited sites, asymptotic decay rates, polynomial volume growth, Cayley graph, fractal graph, Alfors regular.

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1 Introduction

The behavior of R_n, the number of sites visited up to time n by a simple random walk in $\mathbb{Z}^d$, was studied by Dvoretsky and Erdos in 1948. [9] They computed the expected value and variance of R_n and proved a strong law of large numbers. Later Donsker and Varadhan [8] showed that

$$\lim_{n\to\infty} n^{-d/d+2} \log \mathbb{E}\left[\exp\left\{-\nu R_n\right\}\right] = -c(d,\nu), \tag{1.1}$$

for an explicitly computed constant $c(d,\nu)$. The relationship observed by Varopoulos in [25; 26] (for details see [21]) between (1.1) and the behavior of the return probability of a random walk on $F \wr \mathbb{Z}^d$ (the wreath product of a finitely generated group F with $\mathbb{Z}^d$) sparked a renewed interest in understanding the extent to which (1.1) might hold for other infinite graphs. After studying these return probabilities using isoperimetric profiles Erschler [10; 11] used this relationship in reverse to show (among other things) that on the Cayley graphs of finitely generated groups whose volume growth functions are polynomial degree d,

$$\log \mathbb{E}\left[\exp\left\{-\nu R_n\right\}\right] \simeq -n^{d/d+2}. \tag{1.2}$$

(By $f \simeq g$ we imply the existance of constants c and C such that $cg(n) \le f(n) \le Cg(n)$ for large enough n.) More recently, Rau [22] has also used this technique to obtain results similar to (1.2) for random walks on supercritical percolation clusters in $\mathbb{Z}^d$.

Here we perform a direct analysis of $\mathbb{E}\left[\exp\left\{-\nu R_n\right\}\right]$ using the coarse graining techniques popularized by Sznitman in the field of random media [5; 24]. The main results are analogous to (1.2), but apply to graphs with less regular volume growth and to fractal-like graphs [3; 4]. In Section 1 we introduce notation and state the main results. Section 2 contains the proof of the asymptotic lower bound statements, and Section 3 contains the proof of the corresponding asymptotic upper bounds. In Section 4 we present several examples of graphs to which these results apply.

1.1 Preliminaries.

A time homogeneous random walk X_k on a graph Γ may be described by a transition probability function $p(x,y) = \mathbb{P}\left[X_{k+1} = y \mid X_k = x\right]$, the probability that at step k the walk at vertex x will move to vertex y. The future path of the walk depends only on the current position (the Markov property), and this implies that $p_n(x,y) = \mathbb{P}\left[X_{k+n} = y \mid X_k = x\right]$ may be described inductively by

$$p_n(x,y) = \sum_{z\in\Gamma} p_{n-1}(x,z)p(z,y). \tag{1.3}$$

A walk is called irreducible if for any vertices x and y in Γ, $p_n(x,y) > 0$ for some n. A graph is locally finite if each vertex of the graph has at most finitely many neighbors and is connected if for any vertices x and y in Γ there is some path of edges in Γ which starts at x and ends at y. The distance $d(x,y)$ between two vertices is measured by the number of edges in any shortest path between them. Although not mentioned again, all of the random walks treated here will be irreducible walks on locally-finite, connected, infinite graphs.

A random walk is reversible with respect to a measure m on the vertices of Γ if for any x and y

$$m(x)p(x,y) = m(y)p(y,x). \tag{1.4}$$

Reversibility implies that

$$\sum_{x\in\Gamma} m(x)p(x,y) = m(y), \tag{1.5}$$

or that m is an invariant or stationary measure for $p(x,y)$. If P is an operator on $L^2(\Gamma, m)$ defined by

$$Pf(x) = \sum_{y\in\Gamma} f(y)p(x,y),$$

then P is self-adjoint whenever $p(x,y)$ is reversible with respect to m. The associated Dirichlet form is written

$$\begin{aligned}\mathcal{E}(f,f) &= \langle (I-P)f, f\rangle_{L^2(\Gamma,m)} \\ &= \frac{1}{2}\sum_{x,y\in\Gamma} (f(x)-f(y))^2\, p(x,y)m(x).\end{aligned} \tag{1.6}$$

Setting $V(x,\rho) = m(B(x,\rho))$ (the measure of the closed (path length) ball about vertex x of Γ with radius ρ), we say that m satisfies volume doubling (VD) if there is a constant C_{VD} such that

$$V(x, 2\rho) \le C_{\mathrm{VD}} V(x,\rho) \tag{1.7}$$

for all $x \in \Gamma$ and $\rho > 0$. With $\alpha = \log_2 C_{\mathrm{VD}}$ and $\rho \ge \sigma$, we will more often use the alternative formulation

$$\frac{V(x,\rho)}{V(y,\sigma)} \le C_{\mathrm{VD}} \left(\frac{d(x,y)+\rho}{\sigma}\right)^{\alpha}. \tag{1.8}$$

We say that a random walk on Γ satisfies a Neumann Poincaré inequality with parameter β (PI(β)) if there exists a constant C_{PI} such that on each ball $B(z,\rho)$

$$\inf_{\xi} \sum_{x\in B(z,\rho)} (f(x)-\xi)^2\, m(x) \le C_{\mathrm{PI}}\rho^{\beta} \sum_{x,y\in B(z,\rho)} (f(x)-f(y))^2\, p(x,y)m(x). \tag{1.9}$$

The use of $f_B = \left(\sum_{x\in B(z,\rho)} f(x)m(x)\right) V(z,\rho)^{-1}$ in place of ξ to realize the infimum on the left hand side is another common way to state PI(β). An immediate consequence of PI(β) is the bound on

$$\mu(B(z,\rho)) = \inf\left\{ \frac{1}{2}\sum_{x,y\in B(z,\rho)} (f(x)-f(y))^2\, p(x,y)m(x) \,\middle|\, \sum_{x\in B(z,\rho)} f(x)^2 m(x) = 1, \sum_{x\in B(z,\rho)} f(x)m(x) = 0 \right\}$$

(the Neumann eigenvalue of $I-P$ in $B(z,\rho)$) from below by $c\rho^{-\beta}$. Our last condition is called GE(β) and is written piecewise as

$$p_n(x,y) \le \frac{Cm(y)}{V\left(x, n^{1/\beta}\right)} \exp\left(-c\left(\frac{d(x,y)^{\beta}}{n}\right)^{1/(\beta-1)}\right), \tag{1.10}$$

and

$$p_n(x,y) + p_{n+1}(x,y) \geq \frac{cm(y)}{V\left(x, n^{1/\beta}\right)} \exp\left(-C\left(\frac{d(x,y)^\beta}{n}\right)^{1/(\beta-1)}\right). \tag{1.11}$$

That GE(β) implies VD and PI(β) is shown in appendix Proposition 5.1.

1.2 Statement of Results.

Although $m(x)$ is constant for the simple random walk in $\mathbb{Z}^d$, $m(x)$ will generally vary with x in the graph context. It is therefore more appropriate to consider $m(D_n)$, where D_n is the collection of vertices visited by the random walk up to step n. Let

$$\kappa = \inf\{m(x)\,|x \in \Gamma\}, \tag{1.12}$$

$$\eta = \inf\{p(x,y)\,|d(x,y) = 1\}, \text{ and} \tag{1.13}$$

$$d = \sup\{d(x,y)\,|p(x,y) > 0\}. \tag{1.14}$$

Theorem 1.1. *Let Γ be a locally-finite connected infinite graph, and let $p(x,y)$ be the transition operator of a locally elliptic $(\eta > 0)$, bounded range $(d < \infty)$, reversible random walk on Γ. Suppose that $p(x,y)$ together with its reversible measure m satisfies* VD, PI(2), *and $\kappa > 0$. Then for any $\nu > 0$ and every vertex x for which*

$$\liminf_{n\to\infty} V(x,n)^{-1} \inf_{y \in B(x, n^2 V(x,n))} V(y,n) > 0 \tag{1.15}$$

holds, the random walk starting at x satisfies

$$\log \mathbb{E}^x\left[\exp\left\{-\nu m\left(D_{n^2 V(x,n)}\right)\right\}\right] \simeq V(x,n). \tag{1.16}$$

Theorem 1.2. *Let Γ be a locally-finite connected infinite graph, and let $p(x,y)$ be the transition operator of a locally elliptic $(\eta > 0)$, bounded range $(d < \infty)$, reversible random walk on Γ. Suppose that $p(x,y)$ together with its reversible measure m satisfies* GE(β) *and $\kappa > 0$. Then for any $\nu > 0$ and every vertex x for which*

$$\liminf_{n\to\infty} V(x,n)^{-1} \inf_{y \in B\left(x, n^\beta V(x,n)\right)} V(y,n) > 0 \tag{1.17}$$

holds, the random walk starting at x satisfies

$$\log \mathbb{E}^x\left[\exp\left\{-\nu m\left(D_{n^\beta V(x,n)}\right)\right\}\right] \simeq V(x,n). \tag{1.18}$$

Note that when $\beta = 2$ and for strongly irreducible random walks the results of [7] show that Theorem 1.2 is equivalent to Theorem 1.1. However, the equivalence of VD and PI(2) on the one hand and GE(2) on the other (again in [7]) is not at all trivial, and part of the value of Theorem 1.1 is that its proof does not use GE(2). Also note that for any graph with bounded geometry, the simple random walk satisfies $\eta > 0$, $d < \infty$, and $\kappa > 0$ automatically. Graphs with super-polynomial volume growth are not able to be treated by the techniques used here due to the heavy dependence of these results on the volume doubling condition.

These results concern only rough rates of decay of the negative exponential moments of the mass of visited sites, and not specific asymptotic values. The letters c and C which appear in several of the proofs below will be used to represent some positive constants that may change as convenient, even line by line and do not depend on the most important parameters, such as $x \in \Gamma$ and the time parameter n. Specific information regarding constants which describe the upper and lower limits implicit in (1.16) and (1.18) may be obtained via more sophisticated methods, see [12]. We do not know whether or not a graph may satisfy the results of these theorems without also satisfying their hypotheses.

2 Lower Bounds

The lower bounds in (1.16) and (1.18) follow the classical approach as in [8]. If $H_{B(x,\rho)}$ is the first time the random walk starting at x exits $B(x,\rho)$, then

$$\begin{aligned}\mathbb{E}^x\left[\exp\left\{-\nu m\left(D_{n^\beta V(x,n)}\right)\right\}\right] &\geq \mathbb{E}^x\left[\exp\left\{-\nu V(x,n)\right\};D_{n^\beta V(n)}\subset B(x,n)\right]\\ &\geq \exp\left\{-\nu V(x,n)\right\}\mathbb{P}^x\left[H_{B(x,n)}>n^\beta V(x,n)\right].\end{aligned}$$

It remains only to show that for sufficiently large n,

$$\mathbb{P}^x\left[H_{B(x,n)}>n^\beta V(x,n)\right]\geq\exp\left(-cV(x,n)\right). \tag{2.1}$$

To do this we introduce the following notation. For any set $U\subset\Gamma$, define the killed random walk (on exit from U) by the transition function

$$p^U(x,y)=\begin{cases}p(x,y) & \text{if } x,y\in U\\ 0 & \text{otherwise,}\end{cases}$$

and let P_U denote the corresponding L_2 operator. In this context, the smallest non-zero eigenvalue of $I-P_U$ satisfies

$$\begin{aligned}\lambda(U)&=\inf\left\{\frac{\langle(I-P_U)f,f\rangle_{L^2(\Gamma,m)}}{\langle f,f\rangle_{L^2(\Gamma,m)}}\middle| f\not\equiv 0,\mathrm{supp}f\subset U\right\}\\ &=\inf\left\{\frac{1}{2}\sum_{x,y\in\Gamma}(f(x)-f(y))^2p(x,y)m(x)\middle|\|f\|_{2,m}=1,\mathrm{supp}f\subset U\right\}.\end{aligned} \tag{2.2}$$

This quantity is also refered to as the *principle Dirichlet eigenvalue* of $I-P$ in U even though $\lambda(U)$ is the bottom of the Dirichlet spectrum and may not be an eigenvalue when U is infinite. Here the set U will be finite whenever $\lambda(U)$ is used, so (2.2) follows from the Rayleigh-Ritz variational characterization of eigenvalues (e.g. [16]).

Lemma 2.1. *For σ sufficiently large with respect to ρ,*

$$\mathbb{P}^x\left[H_{B(x,\rho)}>\sigma\right]\geq c\eta^\rho\left(1-\lambda\left(B(x,\rho)\right)\right)^\sigma.$$

Proof. Let $B = B(x,\rho)$. Since the random walk may be periodic, note that

$$\begin{aligned}\mathbb{P}^x\left[H_B > \sigma\right] &\geq \frac{1}{2}\mathbb{P}^x\left[H_B > \sigma\right] + \frac{1}{2}\mathbb{P}^x\left[H_B > \sigma + 1\right] \\ &= \frac{1}{2}\sum_{y\in B} p^B_\sigma(x,y) + p^B_{\sigma+1}(x,y).\end{aligned}$$

For σ sufficiently large with respect to ρ, take $\ell \in \{\rho, \rho+1\}$ such that $\sigma - \ell$ is even. Then using (1.3)

$$\begin{aligned}\sum_{y\in B(x,\rho)} p^B_\sigma(x,y) + p^B_{\sigma+1}(x,y) &= \sum_{y\in B}\sum_{z\in B}\left(p^B_\ell(x,z) + p^B_{\ell+1}(x,z)\right)p^B_{\sigma-\ell}(z,y) \\ &\geq \sum_{y\in B}\left(p^B_\ell(x,y) + p^B_{\ell+1}(x,y)\right)p^B_{\sigma-\ell}(y,y).\end{aligned}$$

But now since $\ell \geq \rho$, there must be some path (not necessarily the shortest) of length either ℓ or $\ell+1$ from x to y for any $y \in B$. The probability that the random walk traverses this path bounds the probability of moving from x to y in ℓ or $\ell+1$ steps from below by

$$p^B_\ell(x,y) + p^B_{\ell+1}(x,y) \geq \eta^{\ell+1},$$

leaving

$$\begin{aligned}\mathbb{P}^x\left[H_B > \sigma\right] &\geq \frac{1}{2}\eta^{\rho+2}\sum_{y\in B} p^B_{\sigma-\ell}(y,y) \\ &= \frac{1}{2}\eta^{\rho+2}\text{Trace}\left(P_B^{\sigma-\ell}\right).\end{aligned}$$

All the eigenvalues of $P_B^{\sigma-\ell}$ are non-negative since $\sigma - \ell$ is even, so

$$\text{Trace}\left(P_B^{\sigma-\ell}\right) \geq (1 - \lambda(B))^\sigma,$$

and a change of constants now completes the proof. □

Applying Lemma 2.1 to $\mathbb{P}^x\left[H_{B(x,n)} > n^\beta V(x,n)\right]$,

$$\mathbb{P}^x\left[H_{B(x,n)} > n^\beta V(x,n)\right] \geq c\eta^n\left(1 - \lambda\left(B(x,n)\right)\right)^{n^\beta V(x,n)}.$$

Once

$$\lambda\left(B(x,n)\right) \leq cn^{-\beta} \tag{2.3}$$

is established,

$$\mathbb{P}^x\left[H_{B(x,n)} > n^\beta V(x,n)\right] \geq c\left(\eta\right)^n\left(1 - cn^{-\beta}\right)^{n^\beta V(x,n)}.$$

Finally, since

$$V(x,n) \geq \kappa n \tag{2.4}$$

(from (1.12), as Γ is connected and infinite), taking sufficiently large n (2.1) is complete.

For Theorem 1.1 the test function $f(y) = (\rho - d(x,y))\,\mathbf{1}_{B(x,\rho)}$ satisfies

$$\begin{aligned}\langle (I-P_U)\,f,f\rangle &\le 3d^2V(x,\rho) \text{ and}\\ \langle f,f\rangle &\ge (\rho/2)^2V(x,\rho/2).\end{aligned}$$

These estimates may be used in (2.2) together with (1.7) to establish (2.3) for $\beta = 2$. (See appendix Proposition 5.3 for details.) This completes the proof of the lower bound portion of Theorem 1.1.

For Theorem 1.2 the situation is more interesting - the test function argument, and therefore volume doubling alone, does not suffice. The following lemma follows Lemma 5.15 of [15].

Lemma 2.2. *Let $B = B\left(x, An^{1/\beta}\right)$. Assume that* GE($\beta$) *holds. Then for sufficiently large A,*

$$\lambda\,(B) \le c/n.$$

Proof. Using (1.3), (1.4), and the Cauchy-Schwarz inequality, we establish both that

$$\begin{aligned}p^B_{4n}(z,y) &= \sum_\xi p^B_{3n}\,(z,\xi)\,p^B_n\,(\xi,y)\\ &= \frac{1}{m(z)}\sum_\xi p^B_{3n}\,(\xi,z)\,p^B_n\,(\xi,y)\,m(\xi)\\ &\le \frac{1}{m(z)}\left\|p^B_{3n}\,(\cdot,z)\right\|_{2,m}\left\|p^B_n\,(\cdot,y)\right\|_{2,m},\end{aligned} \tag{2.5}$$

and that

$$p^B_{2n}\,(z,z) = \frac{1}{m(z)}\left\|p^B_n(\cdot,z)\right\|^2_{2,m}. \tag{2.6}$$

By the spectral theorem

$$\begin{aligned}\left\|p^B_{3n}\,(\cdot,z)\right\|_{2,m} &= \left\|P^{2n}_B p^B_n\,(\cdot,z)\right\|_{2,m}\\ &\le (1-\lambda(B))^{2n}\left\|p^B_n\,(\cdot,z)\right\|_{2,m}.\end{aligned} \tag{2.7}$$

By Proposition 5.3, $0 \le \lambda(B) < 1$ for large enough ρ, and so (2.5), (2.6) and (2.7) now yield

$$p^B_{4n}(z,y) \le \exp\left(-2n\lambda(B)\right)\sqrt{p_n(z,z)p_n(y,y)\frac{m(y)}{m(z)}}.$$

Two applications of *GE(β)* and (1.8) give

$$\begin{aligned}p^B_{4n}(z,y) &\le \exp\left(-2n\lambda(B)\right)\frac{m(y)}{\sqrt{m\left(B\left(z,n^{1/\beta}\right)\right)m\left(B\left(y,n^{1/\beta}\right)\right)}}\\ &\le c\exp\left(-2n\lambda(B)\right)\frac{m(y)}{m\left(B\left(y,n^{1/\beta}\right)\right)}.\end{aligned} \tag{2.8}$$

On the other hand, use Cauchy-Schwartz and (2.6) again to obtain

$$\begin{aligned}
(1-\mathbb{P}[H_B \le 2n])^2 &= \left(\sum_{\xi\in B} p_{2n}^B(x,\xi)\right)^2 \\
&= \frac{1}{m(x)^2}\left(\sum_{\xi\in B} p_{2n}^B(\xi,x)\, m(\xi)\right)^2 \qquad (2.9)\\
&\le \frac{m(B)}{m(x)^2}\sum_{\xi\in B} p_{2n}^B(\xi,x)^2\, m(\xi) \qquad (2.10)\\
&= \frac{m(B)}{m(x)} p_{4n}^B(x,x)\,. \qquad (2.11)
\end{aligned}$$

For sufficiently large A, appendix Proposition 5.2 shows that

$$\inf_{\substack{x\in\Gamma\\ n\in\mathbb{N}}} (1-\mathbb{P}[H_B\le 2n]) > 0,$$

so that (2.8) and (2.11) together are equivalent to $\lambda(B) \le c/n$. □

A change of variable in Lemma 2.2 now completes the lower bound portion of Theorem 1.2.

3 Upper Bounds

A different approach to understanding the behavior of $\mathbb{E}[\exp(-\nu R_n)]$ in $\mathbb{Z}^d$ emerges from the field of random media [5]. If each vertex is taken independently with probability $1-\exp(-\nu)$ to be a trap, then each time a vertex is visited for the first time there is an $\exp(-\nu)$ chance that the walk will not encounter a trap. Let $\mathbb{P}^\nu$ be the probability measure on the configurations of traps and $\mathbb{P}^x$ the measure on the random walk paths starting at vertex x. If T denotes the time at which the random walk first encounters a vertex with a trap – the *survival time*, then by averaging over all configurations of traps we obtain

$$\mathbb{E}^x[\exp(-\nu R_n)] = \mathbb{P}^\nu \otimes \mathbb{P}^x[T>n]\,. \qquad (3.1)$$

The notation $\mathbb{P}^\nu \otimes \mathbb{P}^x$ refers to the underlying product of σ-algebras in the spaces of configurations and of paths, respectively. More precisely, if $\omega_\nu \in \Omega_\nu$ is a configuration of traps and $\omega_x \in \Omega_x$ a path of length n starting from x, then

$$\mathbb{P}^\nu \otimes \mathbb{P}^x[T>n] = \sum_{\omega_\nu\in\Omega_\nu}\left\{\sum_{\omega_x\in\Omega_x} \mathbf{1}_{\{T>n\}}(\omega_\nu,\omega_x)\,\mathbb{P}^x[\omega_x]\right\}\mathbb{P}^\nu(\omega_\nu)\,.$$

By reversing the order of summation, conditioning on the set of vertices in the path up to time n, and considering the total probability of a configuration which has no traps at these vertices, we recover (3.1). This formulation allowed Antal [1] to apply the Method of Enlargement of

Obstacles to give a second proof of (1.1). In our context, if the probability of an obstacle at vertex x is $1-\exp(-\nu m(x))$, then

$$\mathbb{E}^x\left[\exp\left(-\nu m\left(D_n\right)\right)\right] = \mathbb{P}^\nu \otimes \mathbb{P}^x\left[T > n\right].$$

Since GE(β) implies VD and PI(β) (Lemma 5.5 in the appendix) we may prove the upper bound portion of both Theorems 1.1 and 1.2 at once. The method of proof roughly follows Lecture 3 of [5]. For convenience take $n^\beta V(x,n)=\ell$, and let $B^\omega_\rho = B(x,\rho)\setminus\Upsilon_\omega$, where Υ_ω is the collection of traps for a given configuration ω. From the spectral theorem and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\mathbb{P}^x\left[T > \ell\right] &= \mathbb{P}^x\left[T \wedge H_{B(x,d\ell)} > \ell\right] \\
&= \left\langle \frac{\delta_x}{m(\cdot)}, P^\ell_{B^\omega_{d\ell}} \mathbf{1}_{B^\omega_{d\ell}} \right\rangle_{L^2(\Gamma,m)} \\
&\leq \left\| \frac{\delta_x}{m(\cdot)} \right\|_2 \left(1-\lambda\left(B^\omega_{d\ell}\right)\right)^\ell \left\|\mathbf{1}_{B^\omega_{d\ell}}\right\|_2.
\end{aligned}$$

Computing the norms and averaging over all ω,

$$\mathbb{P}^\nu \otimes \mathbb{P}^x\left[T > \ell\right] \leq \sqrt{\frac{m(B(x,d\ell))}{m(x)}}\,\mathbb{E}^\nu\left[\left(1-\lambda\left(B^\omega_{d\ell}\right)\right)^\ell\right].$$

Since by (1.8) and (2.4)

$$\lim_{n\to\infty} V(x,n)^{-1}\log\left(\sqrt{\frac{m(B(x,d\ell))}{m(x)}}\right) \leq \lim_{n\to\infty} n^{-1}\log\left(C_{\mathrm{VD}}\left(d\ell\right)^\alpha\right) = 0,$$

it suffices to show that

$$\limsup_{n\to\infty} V(x,n)^{-1}\log\mathbb{E}^\nu\left[\left(1-\lambda\left(B^\omega_\ell\right)\right)^\ell\right] \quad < \quad 0. \tag{3.2}$$

If $\lambda\left(B^\omega_\ell\right) > c_0 n^{-\beta}$ for any $c_0>0$, then

$$\left(1-\lambda\left(B^\omega_{d\ell}\right)\right)^\ell \quad \leq \quad \left(1-c_0 n^{-\beta}\right)^\ell,$$

so that

$$\begin{aligned}
&\limsup_{n\to\infty} V(x,n)^{-1}\log\mathbb{E}^\nu\left[\left(1-\lambda\left(B^\omega_{d\ell}\right)\right)^\ell \mathbf{1}_{\left\{\lambda\left(B^\omega_{d\ell}\right) > c_0 n^{-\beta}\right\}}\right] \\
&\qquad\leq \limsup_{n\to\infty} V(x,n)^{-1}\log\left(1-c_0 n^{-\beta}\right)^\ell \\
&\qquad= \limsup_{n\to\infty} \log\left(1-c_0 n^{-\beta}\right)^{n^\beta} \quad \leq -c_0.
\end{aligned}$$

Since

$$\mathbb{E}^\nu\left[\left(1-\lambda\left(B^\omega_{d\ell}\right)\right)^\ell \mathbf{1}_{\left\{\lambda\left(B^\omega_{d\ell}\right) \leq c_0 n^{-\beta}\right\}}\right] \quad \leq \quad \mathbb{P}^\nu\left[\lambda\left(B^\omega_{d\ell}\right) \leq c_0 n^{-\beta}\right],$$

(3.2) will now follow if we show that for some c_0,

$$\limsup_{n\to\infty} V(x,n)^{-1} \log \mathbb{P}^{\nu} \left[\lambda \left(B^{\omega}_{d\ell} \right) \le c_0 n^{-\beta} \right] \quad < \quad 0. \tag{3.3}$$

Let $\{K_i, k_i\}$ be an n-net on Γ, i.e. a family of balls $K_i = B(k_i, n)$ which cover Γ and are chosen so that $\{B(k_i, n/2)\}$ are pairwise disjoint. By (2.4) and (1.8) the number of net elements which lie entirely inside $B(x,\ell)$ is necessarily bounded by a polynomial $Q(n)$. Again using (1.8), the number of net elements that overlap over any given vertex is easily bounded for any $y \in B(x, 3n)$ by

$$\frac{V(x,3n)}{V(y,n/2)} \le C_{\text{VD}} \left(\frac{3n+3n}{n/2} \right)^{\alpha} = C_{\text{over}}.$$

Now using Lemma 5.4 and PI(β),

$$\begin{aligned}
&\frac{1}{2} \sum_{x,y\in\Gamma} (f(y)-f(x))^2 m(x) p(x,y) \\
&\ge C_{\text{over}}^{-1} \sum_{i=1}^{Q(n)} \left(\frac{1}{2} \sum_{x,y\in K_i} (f(y)-f(x))^2 m(x) p(x,y) \right) \\
&\ge C_{\text{over}}^{-1} \sum_{i=1}^{Q(n)} \left(c n^{-\beta} \frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)} \sum_{x\in K_i} (f(x))^2 m(x) \right) \\
&\ge c n^{-\beta} C_{\text{over}}^{-2} \sum_{x\in\Gamma} f(x)^2 m(x) \sum_{i=1}^{Q(n)} \frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)} \\
&\ge c n^{-\beta} C_{\text{over}}^{-2} \, \|f\|_{2,m}^2 \inf_{i\in[1,Q(n)]} \frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)}.
\end{aligned}$$

Therefore, (absorbing the constants)

$$\lambda \left(B^{\omega}_{\ell} \right) \ge c n^{-\beta} \inf_{i\in[1,Q(n)]} \left\{ \frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)} \right\}$$

and

$$\begin{aligned}
\mathbb{P}^{\nu} \left[\lambda \left(B^{\omega}_{\ell} \right) \le c_0 n^{-\beta} \right] &\le \mathbb{P}^{\nu} \left[\inf_{i\in[1,Q(n)]} \frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)} \le c^{-1} c_0 \right] \\
&\le Q(n) \sup_{i\in[1,Q(n)]} \mathbb{P}^{\nu} \left[\frac{m(\{\Upsilon_\omega \cap K_i\})}{m(K_i)} \le c^{-1} c_0 \right].
\end{aligned}$$

Choosing c_0 sufficiently small with respect to ν and κ and c according to Lemma 5.5,

$$\mathbb{P}^{\nu} \left[\frac{m(\Upsilon_\omega \cap B(k_i, n/2))}{m(B(k_i, n/2))} \le c^{-1} c_0 \right] \quad \le \quad \exp\left(-\frac{1}{5} m(K_i) \right).$$

Using (2.4),

$$\begin{aligned}
&\limsup_{n\to\infty} V(x,n)^{-1}\log \mathbb{P}^{\nu}\left[\lambda\left(B_{\ell}^{\omega}\right)\le c_0 n^{-\beta}\right]\\
&\le \limsup_{n\to\infty} V(x,n)^{-1}\log\left(Q(n)\sup_{i\in[1,Q(n)]}\left\{\exp\left(-\frac{1}{5}m\left(K_i\right)\right)\right\}\right)\\
&< \limsup_{n\to\infty} V(x,n)^{-1}\log Q(n)-\frac{1}{5}\liminf_{n\to\infty} V(x,n)^{-1}\inf_{i\in[1,Q(n)]} m\left(K_i\right)\\
&< -\frac{1}{5}\liminf_{n\to\infty} V(x,n)^{-1}\inf_{i\in[1,Q(n)]} m\left(K_i\right).
\end{aligned}$$

Therefore (3.3) holds for every vertex x for which

$$\liminf_{n\to\infty} V(x,n)^{-1}\inf_{y\in B(x,n^2V(x,n))} V(y,n)>0.$$

This completes the upper bound portions of both Theorem 1.1 and 1.2. □

4 Applications

4.1 Groups of polynomial volume growth

Suppose $\Gamma=(G,S)$ is the Cayley graph of the finitely generated group G under the symmetric generating set S (each element of G is a vertex, with edges (x,y) when $y=xs$ for some $s\in S$). The simple random walk on any such graph has the counting measure as its stationary measure. Erschler [10; 11] proved the following result by different methods.

Corollary 4.1. *For the simple random walk on the Cayley graph $\Gamma=(G,S)$ with polynomial volume growth of degree d,*

$$\log \mathbb{E}^{x}\left[\exp\left\{-\nu m\left(D_n\right)\right\}\right]\simeq n^{d/d+2}.$$

Proof. In [14] it is shown that when the volume growth of Γ is polynomial of degree d, then (1.10) and (1.11) hold for the simple random walk on Γ. The desired result then follows either by direct application of Theorem 1.2 or of Theorem 1.1 after first using Proposition 5.1. However, these routes utilize the intricate techniques of [14], which may be avoided by verifying the hypotheses of Theorem 1.1 independently.

First, VD follows immediately from the degree d polynomial growth of Γ, as does (1.15) or (1.17). A weak version of PI(2) may be quickly established by taking advantage of the group structure. Specifically, for any $x,xy\in B\left(x_0,\rho\right)$ with $y=s_1s_2\dots s_k$, by the Cauchy-Schwartz inequality

$$\begin{aligned}
|f(xy)-f(x)|^2&=\left|\sum_{i=1}^{k} f\left(xs_0\dots s_i\right)-f\left(xs_0\dots s_{i-1}\right)\right|^2\\
&\le k\sum_{i=1}^{k}\left|f\left(xs_0\dots s_i\right)-f\left(xs_0\dots s_{i-1}\right)\right|^2, \qquad (4.1)
\end{aligned}$$

where s_0 is the identity in the group G. For each i, the sum of the right hand side of (4.1) over all $x \in B(x_0, \rho)$ is smaller than the sum over all $xs_0 \dots s_{i-1} \in B(x_0, 2\rho)$. Thus,

$$\begin{aligned}\sum_{x \in B(x_0,\rho)} |f(xy) - f(x)|^2 &\leq k \sum_{i=1}^{k} \sum_{x \in B(x_0,2\rho)} |f(xs_i) - f(x)|^2 \\ &\leq k^2 \sum_{x \in B(x_0,2\rho), s \in S} |f(xs) - f(x)|^2 .\end{aligned}$$

Finally, since we consider the simple random walk,

$$\begin{aligned}\sum_{x \in B(x_0,\rho)} \left|f(x) - f_{B(x_0,\rho)}\right|^2 &\leq V(x_0,\rho)^{-1} \sum_{y \in B(x_0,\rho)} \sum_{x \in B(x_0,\rho)} |f(xy) - f(x)|^2 \\ &\leq 2|S|\rho^2 \sum_{x \in B(x_0,2\rho), s \in S} |f(xs) - f(x)|^2 p(x, xs). \qquad (4.2)\end{aligned}$$

By a covering argument due to [17], (4.2) is known to improve to the full PI(2) whenever VD holds. □

4.2 An example with $V(x,\rho) \not\simeq \rho^\alpha$

Theorem 1.1 applies to the following example, even though

$$V(x,\rho) \not\simeq \rho^\alpha$$

for any x or for any α, i.e. Γ is not Alfors regular. No results similar to those presented here have been shown concerning graphs of this kind. Since the random walk presented here is strongly irreducible, Theorem 1.2 might also be applied, exercising again the equivalence shown in [7] between PI(2) and VD.

For each $i \in \mathbb{N}$, let $x_i = 2^{2^{2i}}$, $y_i = 2^{2^{2i+1}}$ and $y_0 = 0$, and define for all $x \in \mathbb{Z}_+$

$$m(x) = \begin{cases} c_i & \text{if } x \in [y_{i-1}, x_i] \\ c_i x_i^{-\alpha} x^\alpha & \text{if } x \in (x_i, y_i), \end{cases}$$

where $\alpha > 1$, $c_1 = 1$ and $c_{i+1} = c_i x_i^{-\alpha} y_i^\alpha$. Use the reversibility condition

$$\begin{aligned} p(x, x+1)m(x) &= p(x+1, x)m(x+1), \\ p(0,1) = p(0,0) &= \frac{1}{2}, \text{ and} \\ p(x,x) &= 0, \text{ if } x \neq 0 \end{aligned}$$

to recursively define $p(x, x+1)$ as a random walk transition probability function on $\mathbb{Z}_+$. Since $m(x)$ is increasing, and whenever $\rho > |x|$

$$B(0,\rho) \subset B(0, |x| + \rho) = B(x,\rho) \subset B(0, 2\rho),$$

if VD is shown to hold then (1.15) will follow for all x, since

$$\inf_{y \in B(x, n^2 V(x,n))} V(y,n) \geq V(0,n).$$

This example also serves to demonstrate that (1.15) is a weaker condition than polynomial volume growth. To finish checking the hypotheses one must then prove that $p(x, x+1)$ is bounded away from one and zero for all x and check that PI(2) holds.

We will use the approximation $V(x,n) \approx \int_{x-n}^{x+n} m(t)dt$, which results in a small relative error that does not affect the results. We will also assume that $\alpha \in \mathbb{N}$, although only small modifications are needed to show that all real $\alpha \geq 1$ is a sufficient assumption. A useful computation shows that

$$\begin{aligned}\int_{\sigma}^{\sigma+\beta} \frac{c_i}{x_i^\alpha} t^\alpha dt &= \frac{c_i}{(\alpha+1)x_i^\alpha}\left((\sigma+\beta)^{\alpha+1} - \sigma^{\alpha+1}\right) \\ &= \frac{c_i}{(\alpha+1)x_i^\alpha}\sum_{j=1}^{\alpha+1}\binom{\alpha+1}{j}\sigma^{\alpha+1-j}\beta^j \\ &= c_i\left(\frac{\sigma}{x_i}\right)^\alpha \beta\left(\frac{1}{\alpha+1}\sum_{j=0}^{\alpha}\binom{\alpha+1}{j+1}\left(\frac{\beta}{\sigma}\right)^j\right). \end{aligned} \tag{4.3}$$

Note that when $\beta < \sigma$,

$$c_i\left(\frac{\sigma}{x_i}\right)^\alpha \beta \leq \int_{\sigma}^{\sigma+\beta} \frac{c_i}{x_i^\alpha} t^\alpha dt \leq C_\alpha c_i \left(\frac{\sigma}{x_i}\right)^\alpha \beta, \tag{4.4}$$

where $C_\alpha = \frac{1}{\alpha+1}\sum_{j=0}^{\alpha}\binom{\alpha+1}{j+1}$. Also, replacing β by 2β in the above,

$$c_i\left(\frac{\sigma}{x_i}\right)^\alpha 2\beta \leq \int_{\sigma}^{\sigma+2\beta} \frac{c_i}{x_i^\alpha} t^\alpha dt \leq C_{\alpha,2} c_i \left(\frac{\sigma}{x_i}\right)^\alpha 2\beta, \tag{4.5}$$

where $C_{\alpha,2} = \frac{1}{\alpha+1}\sum_{j=0}^{\alpha}\binom{\alpha+1}{j+1}2^j > 1$. One final useful computation is that

$$V(x, y_i - x) > (\alpha+1)^{-1} c_i x_i^{-\alpha} y_i^\alpha (y_i - x) = \frac{c_{i+1}(y_i - x)}{(\alpha+1)} \tag{4.6}$$

when $y_i - x \leq x$, which follows from

$$y_i^\alpha = x^\alpha \sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{y_i - x}{x}\right)^j,$$

$$V(x, y_i - x) \geq 2c_i x_i^{-\alpha}(y_i - x)x^\alpha \left(\frac{1}{\alpha+1}\sum_{j=0}^{\alpha}\binom{\alpha+1}{j+1}\left(\frac{y_i - x}{x}\right)^j\right),$$

and

$$\binom{\alpha}{j} + \binom{\alpha}{j+1} = \binom{\alpha+1}{j+1}.$$

Proposition 4.2. *VD holds for this example.*

Proof. Note that if $\rho = x$, $V(x,\rho) = V(0,2\rho)$, and $V(x,2\rho) = V(0,3\rho)$. So, if we show that VD holds whenever $\rho \leq x$ with a constant which is independent of x, then it will also hold (for a slightly different constant) for all $\rho > 0$ for $V(0,\rho)$. But, if $\rho > x$, then $V(x,\rho) > V(0,\rho)$ and $V(x,2\rho) < V(0,3\rho)$, so VD holds for all $x \in \mathbb{Z}_+$ and all $\rho > 0$ (again for a slightly different constant).

To prove VD for $\rho \leq x$, it is necessary to check several cases. It is helpful to keep in mind that if $x < x_i$, (or y_i) then $x + \rho < 2x < y_i$ (or x_{i+1}).

If $x_i < x < y_i$, then when $x + \rho < y_i$, using (4.4) and (4.5),

$$\begin{aligned} V(x,2\rho) &\leq 2\int_x^{x+2\rho} c_i x_i^{-\alpha} t^\alpha dt \\ &\leq 4\rho C_{\alpha,2} c_i \left(\frac{x}{x_i}\right)^\alpha \\ &\leq 4C_{\alpha,2} V(x,\rho). \end{aligned}$$

When $x + \rho > y_i$ by (4.6) and $c_i x_i^{-\alpha} x^\alpha \leq c_{i+1}$,

$$\begin{aligned} V(x,2\rho) &\leq 4c_{i+1}\rho \\ &= 4(\alpha+1)\left(\frac{c_{i+1}\rho}{\alpha+1}\right) \\ &\leq 4(\alpha+1)\left(\frac{c_{i+1}(y_i - x)}{\alpha+1} + c_{i+1}(\rho - (y_i - x))\right) \\ &\leq 4(\alpha+1)\left(m\left([x-\rho, y_i]\right) + m\left([y_i, x+\rho]\right)\right) \\ &\leq 4(\alpha+1)V(x,\rho). \end{aligned}$$

If $y_{i-1} < x < x_i$, then $V(x,\rho) \geq c_i\rho$. In this case if $x + 2\rho < x_i$, then

$$V(x,2\rho) \leq 4c_i\rho \leq 4V(x,\rho),$$

and if $x + 2\rho > x_i$, then by (4.5) and $C_{\alpha,2} > 1$

$$\begin{aligned} V(x,2\rho) &\leq 2\left(c_i(x_i - x) + c_i(2\rho - (x_i - x))C_{\alpha,2}\right) \\ &\leq 4c_i C_{\alpha,2}\rho. \\ &\leq 4C_{\alpha,2} V(x,\rho). \end{aligned}$$

□

Next, show that $p(x, x+1)$ is bounded away from zero and one for all x.

Proposition 4.3. *For all x,*

$$1/2^{\alpha+1} \leq p(x+1, x) \leq 1/2.$$

Proof. Since $m(x)$ is constant between y_{i-1} and x_i, and since both $y_{i-1}$and x_i are even, the recursive definition of $p(x, x+1)$ via the reversibility condition implies that for all i,

$$p(y_{i-1}, y_{i-1}+1) = p(x_i, x_i+1).$$

It follows that if $p^*$is defined by recursively by $p^*(0,0) = p^*(0,1) = 1/2$ and $p^*(x,x) = 0$ if $x \neq 0$, based on the measure

$$m^*(x) = \begin{cases} 1 & x = 0 \\ x^\alpha & x > 0, \end{cases}$$

then

$$\inf p^*(x,x-1) \leq \inf p(x,x-1) \leq \sup p(x,x-1) \leq \sup p^*(x,x-1).$$

Using the recursive definition of p^* we obtain

$$\begin{aligned} p^*(n,n-1) &= n^{-\alpha}\left(\sum_{i=1}^{n-2}(-1)^{i+1}(n-i)^\alpha + (1/2)(-1)^{n+1}\right) \\ &= n^{-\alpha}\left(\sum_{j=2}^{n-1}(-1)^{n-1-j}(j)^\alpha + (1/2)(-1)^{n+1}\right). \end{aligned}$$

When n is even, since x^α has non-decreasing derivative, by the mean value theorem

$$\begin{aligned} \sum_{j=1}^{n-2} j^\alpha(-1)^{n-2-j} &\leq \sum_{j=2}^{n-1} j^\alpha(-1)^{n-1-j} \\ &\leq \sum_{j=3}^{n} j^\alpha(-1)^{n-j}. \end{aligned}$$

So,

$$\begin{aligned} p^*(n,n-1) &\leq \frac{1}{2}n^{-\alpha}\left(\sum_{j=2}^{n-1} j^\alpha(-1)^{n-1-j} + \sum_{j=3}^{n} j^\alpha(-1)^{n-j}\right) - \frac{1}{2}n^{-\alpha} \\ &\leq \frac{1}{2}n^{-\alpha}\left(n^\alpha - 2^\alpha\right) - \frac{1}{2}n^{-\alpha} \\ &\leq \frac{1}{2}, \end{aligned}$$

and similarly

$$\begin{aligned} p^*(n,n-1) &\geq \frac{1}{2}n^{-\alpha}\left((n-1)^\alpha - 1\right) - \frac{1}{2}n^{-\alpha} \\ &\geq \frac{1}{2}\left(\frac{1}{2}\right)^\alpha. \end{aligned} \tag{4.7}$$

When n is odd, the result follows similarly. □

Before verifying PI(2) for this example, consider how this inequality might be proved on any weighted graph. Let $B = B(x_0,\rho)$, $Q(e) = p(e_-,e_+)m(e_-)$ for each edge $e = (e_-,e_+)$,and let

γ_{xy} be a shortest path from x to y. Then (as in [23]) by the Cauchy-Schwartz inequality,

$$\begin{aligned}
& m(B)\inf_{\xi}\sum_{x\in B}(f(x)-\xi)^2\, m(x) \\
&\le \sum_{x,y\in B}(f(x)-f(y))^2\, m(x)m(y) \\
&\le \sum_{x,y\in B} d(x,y)\sum_{e\in\gamma_{xy}}(f(e_-)-f(e_+))^2 m(x)m(y) \\
&= \sum_{x,y\in B} d(x,y)\sum_{e\in\gamma_{xy}}(f(e_-)-f(e_+))^2\frac{Q(e)}{Q(e)}m(x)m(y) \\
&= \sum_{e\in B}(f(e_-)-f(e_+))^2 Q(e)\sum_{\gamma_{xy}:e\in\gamma_{xy}} d(x,y)\frac{m(x)m(y)}{Q(e)}.
\end{aligned}$$

Therefore,

$$\sum_{x\in B}(f(x)-f_B)^2\, m(x) \le \frac{1}{m(B)}\max_{e\in B}\left\{\frac{1}{Q(e)}\sum_{\substack{\gamma\in\Gamma_*\\ e\in\gamma}}|\gamma|m(\gamma)\right\}\times \sum_{x,y\in B}(f(x)-f(y))^2 p(x,y)m(x),$$

where $m(\gamma_{xy})=m(x)m(y)$ and Γ_* contains exactly one shortest path γ_{xy} for each pair of vertices $x,y\in B$. PI(2) will hold for any graph for which Γ_* can be chosen in such a way that

$$\frac{1}{m(B)}\max_{e\in B}\left\{\frac{1}{Q(e)}\sum_{\substack{\gamma\in\Gamma_*\\ e\in\gamma}}|\gamma|m(\gamma)\right\}\le c\rho^2.$$

Proposition 4.4. PI(2) *holds for this example.*

Proof. For $\mathbb{Z}$, there is only one shortest path between any pair of vertices. Furthermore, for $B=B(x_0,n)$ at most n^2 paths are required to form Γ_*, so each edge may appear in at most n^2 paths. Let γ_+ indicate the last vertex of the path, and γ_- the first vertex. Since $m(x)$ is non-decreasing, when $e\in\gamma$

$$\begin{aligned}
m(\gamma) &= m(\gamma_-)m(\gamma_+) \\
&\le m(e_-)m(x_0+n),
\end{aligned}$$

which together with $p(n,n-1) \geq 2^{-\alpha-1}$ by (4.7) yields

$$\begin{aligned}
&\frac{1}{m(B)} \max_{e\in B} \left\{ \frac{1}{p(e_-,e_+)m(e_-)} \sum_{\substack{\gamma\in\Gamma_* \\ e\in\gamma}} |\gamma| m(\gamma) \right\} \\
&\leq \frac{2nm(x_0+n)}{V(x_0,n)} \max_{e\in B} \left\{ \frac{1}{p(e_-,e_+)} \sum_{\substack{\gamma\in\Gamma_* \\ e\in\gamma}} 1 \right\} \\
&\leq \frac{2nm(x_0+n)}{V(x_0,n)} \left(2^{\alpha+1}n^2\right).
\end{aligned}$$

To complete the proof it remains only to show that $V(x_0,n) \geq cnm(x_0+n)$.
If $x_i \leq x_0+n \leq y_i$, then by (4.3)

$$\begin{aligned}
V(x_0,n) &= c_i \left(\frac{x_0}{x_i}\right)^\alpha n \left(\frac{1}{\alpha+1} \sum_{j=0}^{\alpha} \binom{\alpha+1}{j+1} \left(\frac{n}{x_0}\right)^j \right) \\
&\geq \frac{n}{\alpha+1} c_i \left(\frac{x_0}{x_i}\right)^\alpha \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{n}{x_0}\right)^j \\
&= \frac{nm(x_0+n)}{\alpha+1}.
\end{aligned}$$

If $y_{i-1} \leq x_0+n \leq x_i$, when $x_0 > y_{i-1}$,

$$V(x_0,n) \geq nm(x_0+n),$$

and when $x_0 < y_{i-1}$, by (4.6)

$$\begin{aligned}
V(x_0,n) &\geq \frac{c_i(y_{i-1}-x_0)}{(\alpha+1)} + c_i(x_0+n-y_{i-1}) \\
&\geq \frac{1}{\alpha+1} nc_i \\
&= \frac{1}{\alpha+1} nm(x_0+n).
\end{aligned}$$

□

Note that this example is recurrent ($\liminf n^{-2}V(x,n) < \infty$), but is both not Ahlfors regular and not strongly recurrent in the sense mentioned below. Although somewhat more complicated, when $\alpha \in \mathbb{N}$ it is relatively straightforward to construct a non-weighted sub-graph of $\mathbb{Z}^\alpha$ for which the simple random walk mimics the behavior of the example constructed here on $\mathbb{Z}$.

4.3 The case $\beta \neq 2$, Fractal examples

Since we are considering only cases in which $\beta \neq 2$ in this section, only Theorem 1.2 may be applied to these examples. One must first verify GE(β) in order to apply Theorem 1.2 to any specific example. This has proved to be a difficult task in most venues for which it has been accomplished, so the applications presented here will be restricted to those cases where this hypothesis is already known to hold.

Historically, the first graph for which the simple random walk was known to witness GE(β) with $\beta > 2$ was the Sierpinski gasket pre-fractal graph as pictured in figure 2 of [18]. Later this example was shown [4, Propsition 5.4] to fit into a class of strongly recurrent graphs for which GE(β) holds, namely graphs based on the pre-fractal structure of finitely ramified fractals for which the resitance metric (see section 2 of [13]) is a polynomial scaling of the graph distance. The class of generalized Sierpinski carpet pre-fractal graphs (e.g. figure 1.2 of [3]) also satisfies GE(β) [3]. Since the volume growth in all of these examples is polynomial, Theorem 1.2 applies in each case to give the asymptotic rate of decay of the negative exponential moments of the number of visited sites for the simple random walk.

A similar analysis of survival time for the Brownian motion on the Sierpinski gasket appears in [19] and [20], no results of this type have previously been explored for the discrete graph context.

As shown in [4], any nearest neighbor random walk on a tree which satisfies

$$0 < \inf_{d(x,y)=1} p(x,y)m(x) < \sup_{d(x,y)=1} p(x,y)m(x) < \infty$$

and

$$V(x,n) \simeq n^{\alpha}$$

must also satisfy GE($\alpha + 1$). The Vicsek trees ([2] figure 4) provide a nice set of examples of polynomial growth trees to which Theorem 1.2 applies.

Finally, since GE(β) is known to be stable under rough isometry (Theorem 5.11 of [13]) transformation of any of the above examples by rough isometry will generate new examples for which GE(β) holds and to which Theorem 1.2 applies.

5 Appendix

The following results are included for completeness

When $\beta = 2$, the following result and its converse for strongly irreducible nearest neighbor random walks is found in [7]. Here it is used to simplify the statement of Theorem 1.2, since PI(β) and VD are used directly in the proof of that theorem. The proof below is only slightly modified from [7], and like all the results in this section, are included for the sake of completeness.

Proposition 5.1. *Suppose that* GE(β) *holds. Then* VD *and* PI(β) also hold.

Proof. Only the lower estimate is needed to prove VD, since

$$\begin{aligned}
1 &\geq \frac{1}{2} \sum_{y \in B(x,2r)} p_{r^\beta}(x,y) + p_{r^\beta+1}(x,y) \\
&\geq \sum_{y \in B(x,2r)} \frac{c}{V(x,r)} \exp\left(-C\left((2r)^\beta / r^\beta\right)^{1/(\beta-1)}\right) \\
&= c \exp\left(-2^{\beta/(\beta-1)} C\right) \frac{V(x,2r)}{V(x,r)}.
\end{aligned}$$

Let $Q = \left(P^n + P^{n+1}\right)/2$, i.e.

$$\begin{aligned}
Qf(x) &= \sum_y f(y) Q(x,y) \\
&= \frac{1}{2} \sum_y f(y) \left(p_n(x,y) + p_{n+1}(x,y)\right),
\end{aligned}$$

let $\rho = n^{1/\beta}$, and set $B = B\left(x_0, \rho\right)$. One can easily check that Q is reversible with respect to m, since $m > 0$. When $x \in B$ by (1.11) and (1.7)

$$\begin{aligned}
Q\left(f - Qf(x)\right)^2(x) &= \sum_{y \in \Gamma} \left(f(y) - Qf(x)\right)^2 Q(x,y) \\
&= \frac{1}{2} \sum_{y \in \Gamma} \left(f(y) - Qf(x)\right)^2 \left(p_n(x,y) + p_{n+1}(x,y)\right) \\
&\geq \sum_{y \in B} \frac{cm(y)}{V(x,\rho)} \left(f(y) - Qf(x)\right)^2 \\
&\geq \sum_{y \in B} \frac{cm(y)}{V\left(x_0,\rho\right)} \left(f(y) - Qf(x)\right)^2 \\
&\geq \sum_{y \in B} \frac{cm(y)}{V\left(x_0,\rho\right)} \left(f(y) - f_B\right)^2, \qquad (5.1)
\end{aligned}$$

where $f_B = \sum_{x \in B} f(x) m(x) / m(B)$ is chosen to minimize the previous expression. Having obtained the left hand side of PI(β), we now work toward the right hand side. Starting with (5.1) and using properties of Q,

$$\begin{aligned}
\sum_{y \in B} m(y)\left(f(y) - f_B\right)^2 &\leq c \sum_{x \in B} \left(Q\left(f - (Qf)(x)\right)^2\right)(x) m(x) \\
&= c \sum_{x \in \Gamma} \left(Qf^2(x) - 2\left(Qf(x)\right)^2 + Q\left(\left(Qf\right)(x)\right)^2(x)\right) m(x) \\
&\leq c \sum_{x \in \Gamma} \left(Qf^2(x) - \left(Qf(x)\right)^2\right) m(x) \\
&= c \left(\sum_{x \in \Gamma} \left(\sum_{y \in \Gamma} f^2(y) m(y)\right) Q(y,x) - \|Qf\|_2^2\right) \\
&\leq c \left(\|f\|_2^2 - \|Qf\|_2^2\right). \qquad (5.2)
\end{aligned}$$

Rewrite

$$\|f\|_2^2 - \|Qf\|_2^2 = \left(\|P^n f\|_2^2 - \|Qf\|_2^2\right) + \sum_{i=0}^{n-1} \left(\left\|P^i f\right\|_2^2 - \left\|P^{i+1} f\right\|_2^2\right).$$

We need only control each of these terms by $\mathcal{E}(f,f)$ to complete the lemma since

$$\|f\|_2^2 - \|Qf(x)\|_2^2 \leq (n+1)\,\mathcal{E}(f,f)$$

with (5.2) for large enough ρ yields

$$\sum_{y\in B} m(y)\left(f(y)-f_B\right)^2 \leq c\rho^\beta \mathcal{E}(f,f).$$

First, since $a^2 - b^2 \leq 2a(a-b)$

$$\begin{aligned}\|f\|_2^2 - \|Pf\|_2^2 &\leq 2\langle f,(I-P)f\rangle \\ &= 2\mathcal{E}(f,f).\end{aligned}$$

Now since P is a contraction,

$$\begin{aligned}\|Pf\|_2^2 - \left\|P^2 f(x)\right\|_2^2 &\leq 2\langle Pf, P(I-P)f\rangle \\ &\leq 2\langle f,(I-P)f\rangle \\ &= 2\mathcal{E}(f,f).\end{aligned}$$

Finally,

$$\begin{aligned}\|P^n f\|_2^2 - \|Qf(x)\|_2^2 &\leq 2\langle P^n f, (P^n - Q)f\rangle \\ &= 2\left\langle P^n f, \left(P^n - \frac{1}{2}\left(P^n + P^{n-1}\right)\right) f\right\rangle \\ &= \left\langle P^n f, \left(P^n - P^{n+1}\right) f\right\rangle \\ &\leq \mathcal{E}(f,f).\end{aligned}$$

□

Another consequence of GE(β) from [14] is used to obtain a stronger upper bound than Proposition 5.3 on the Dirichlet eigenvalue for the proof of the lower bound portion of Theorem 1.2.

Proposition 5.2. *Suppose that* GE(β) *holds. Then*

$$\mathbb{P}^y\left[\sup_{1\leq i\leq n} d(y,X_i) \geq \sigma\right] \leq C\exp\left(-c\left(\frac{\sigma^\beta}{n}\right)^{1/(\beta-1)}\right).$$

Proof. Note that the result is void if σ^β/n is small. Let $B^j = B\left(y, 2^j\sigma\right)$ be the ball of radius $2^j\sigma$ about y. We first prove that

$$\mathbb{P}^y\left[X_n \notin B^0\right] \leq C\exp\left(-c\left(\frac{\sigma^\beta}{n}\right)^{1/(\beta-1)}\right). \tag{5.3}$$

Indeed, $\mathbb{P}^y\left[X_n \notin B^0\right] = \sum_{x \notin B^0} p_n(y,x)$, so that by (1.10), (1.8), and a change of constants we obtain

$$\begin{aligned} & \sum_{x \notin B(y,\sigma)} p_n(y,x) \\ \leq\ & \frac{C}{V\left(y, n^{1/\beta}\right)} \sum_{x \notin B^0} m(x) \exp\left(-c\left(\frac{(d(y,x))^\beta}{n}\right)^{\frac{1}{\beta-1}}\right) \\ =\ & \frac{C}{V\left(y, n^{1/\beta}\right)} \sum_{j=1}^{\infty}\left(\sum_{x \in B^j \setminus B^{j-1}} m(x) \exp\left(-c\left(\frac{(d(y,x))^\beta}{n}\right)^{\frac{1}{\beta-1}}\right)\right) \\ \leq\ & C \sum_{j=1}^{\infty} \frac{V\left(y, 2^j \sigma\right)}{V\left(y, n^{1/\beta}\right)} \exp\left(-c\left(\frac{\left(2^{(j-1)}\sigma\right)^\beta}{n}\right)^{\frac{1}{\beta-1}}\right) \\ \leq\ & C\left(\frac{\sigma}{n^{1/\beta}}\right)^\alpha \exp\left(-c\left(\frac{\sigma^\beta}{2n}\right)^{\frac{1}{\beta-1}}\right) \sum_{j=1}^{\infty} C_{\mathrm{VD}}^j \exp\left(-c\left(\frac{\sigma^\beta}{2n}\right)^{\frac{1}{\beta-1}}\left(2^{\frac{\beta}{\beta-1}(j)} - 1\right)\right) \\ \leq\ & C \exp\left(-c\left(\frac{\sigma^\beta}{n}\right)^{\frac{1}{\beta-1}}\right). \end{aligned}$$

The necessary size of σ^β/n only depends on the size of the constants from GE(β) and C_{VD}. Let $L_\sigma = \inf\{t \,|\, d(X_0, X_t) > \sigma\}$. Intersecting the event $\{X_n \notin B(y,\sigma/2)\}$ by the event $\{L_\sigma \leq n\}$ and using the strong Markov property,

$$\begin{aligned} & \mathbb{P}^y\left[X_n \notin B(y,\sigma/2)\right] \\ \geq\ & \mathbb{P}^y\left[X_n \notin B(y,\sigma/2), L_\sigma \leq n\right] \\ \geq\ & \mathbb{P}^y\left[L_\sigma \leq n\right] - \mathbb{P}^y\left[X_n \in B(y,\sigma/2), L_\sigma \leq n\right] \\ \geq\ & \mathbb{P}^y\left[L_\sigma \leq n\right] - \mathbb{E}^y\left[\mathbb{P}^{X_{L_\sigma}}\left[X_{n-L_\sigma} \notin B\left(X_{L_\sigma}, \sigma/2\right)\right] \mathbf{1}_{\{L_\sigma \leq n\}}\right]. \end{aligned}$$

When σ^β/n is large enough, (5.3) may be applied to obtain

$$\sup_{x, k \leq n} \mathbb{P}^x\left[X_k \notin B(x,\sigma/2)\right] \leq C' < 1.$$

Combining this with (5.3) once more shows that

$$\begin{aligned} \mathbb{P}^y\left[L_\sigma \leq n\right](1 - C') \ \leq\ & \mathbb{P}^y\left[X_n \notin B(y,\sigma/2)\right] \\ \leq\ & C \exp\left(-c\left(\frac{\sigma^\beta}{n}\right)^{1/(\beta-1)}\right). \end{aligned}$$

□

Only VD is required to obtain the $c\rho^{-2}$ upper bound on $\lambda\left(B(x,\rho)\right)$ used in the proof of the lower bound portion of Theorem 1.1. The following general argument is commonly known, see for example [6].

Proposition 5.3. *Suppose* VD *holds and that* $p(x,y)=0$ *whenever* $d(x,y)>d$. *Then,*

$$\lambda\left(B\left(x_0,\rho\right)\right) \le c\rho^{-2}.$$

Proof. Let $f(x)=(\rho-d(x,x_0))\,\mathbf{1}_B(x)$. On $B\left(x_0,\rho/2\right)$ $f(x)\ge\rho/2$, so

$$\|f\|_2^2 \ge \frac{1}{4}\rho^2 V(x_0,\rho/2).$$

To estimate $\mathcal{E}(f,f)$ from above first note that

$$\begin{aligned}|f(x)-f(y)| = &\left(d\left(x_0,y\right)-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_B(y)+\\ &\left(\rho-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_{B^C}(y)+\left(\rho-d\left(x_0,x\right)\right)^2 \mathbf{1}_{B^C}(x)\mathbf{1}_B(y),\end{aligned}$$

so that

$$\begin{aligned}\mathcal{E}(f,f) \le &\sum_{x,y}\left(d\left(x_0,y\right)-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_B(y)p(x,y)m(x)+\\ &2\sum_{x,y}\left(\rho-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_{B^C}(y)p(x,y)m(x).\end{aligned}$$

Since $p(x,y)=0$ when $d(x,y)>d$,

$$\begin{aligned}&\sum_{x,y}\left(d\left(x_0,y\right)-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_B(y)p(x,y)m(x)\\ &\le \sum_{x,y} d\left(x,y\right)^2 \mathbf{1}_B(x)\mathbf{1}_B(y)p(x,y)m(x)\\ &\le d^2 V\left(x_0,\rho\right),\end{aligned}$$

and when $y\notin B$, $p(x,y)=0$ when $d\left(x,x_0\right)<\rho-d$, making

$$\begin{aligned}&2\sum_{x,y}\left(\rho-d\left(x_0,x\right)\right)^2 \mathbf{1}_B(x)\mathbf{1}_{B^C}(y)p(x,y)m(x)\\ &\le 2\sum_{x,y}\left(\rho-(\rho-d)\right)^2 \mathbf{1}_B(x)\mathbf{1}_{B^C}(y)p(x,y)m(x)\\ &\le 2d^2 V\left(x_0,\rho\right).\end{aligned}$$

Finally, applying (1.7),

$$\begin{aligned}\frac{\mathcal{E}(f,f)}{\|f\|_2^2} &\le \frac{3d^2 V\left(x_0,\rho\right)}{\frac{1}{4}\rho^2 V\left(x_0,\rho/2\right)}\\ &\le c\rho^{-2},\end{aligned}$$

from which the result follows. □

The following lemma (nearly identical to Lemma 3.3 of [5]) is used in the upper bound proof to take full advantage of the Bernoulli distribution of traps.

Lemma 5.4. *For a finite non-empty set $U \subset \Gamma$ and $A \subset U$, define*

$$\mu_A(U) = \inf_{\substack{f \in L^2(U,m) \\ \|f\|_2^2 = 1}} \left\{ \frac{1}{2} \sum_{x,y \in U} (f(y) - f(x))^2 m(x) p(x,y) \middle| \operatorname{supp} f \subset U \setminus A \right\}.$$

Then

$$\mu_A(U) \geq \mu(U) \frac{m(A \cap U)}{m(U)}.$$

Proof. Choose $f \in L^2(U,m)$ to be vanishing on A. Let $g = f - (f,\psi)\psi$ where $\psi = \mathbf{1}_U / \sqrt{m(U)}$, so that $(g,\psi) = 0$. With f vanishing on A, for any $\alpha, \beta > 0$,

$$\begin{aligned}
&\frac{1}{2} \sum_{x,y \in U} (f(y) - f(x))^2 p(x,y) m(x) + \alpha \sum_{x \in U} f^2(x) m(x) \\
&= \frac{1}{2} \sum_{x,y \in U} ((f,\psi)\psi(y) + g(y) - (f,\psi)\psi(x) + g(x))^2 p(x,y) m(x) + \\
&\qquad \sum_{x \in U} (\alpha + \beta \mathbf{1}_A(x)) f^2(x) m(x) \\
&= \frac{1}{2} \sum_{x,y \in U} (g(y) - g(x))^2 p(x,y) m(x) + \sum_{x \in U} (\alpha + \beta \mathbf{1}_A(x)) f^2(x) m(x).
\end{aligned}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned}
&(f,\psi)^2 \\
&= \left(\sum_{x \in U} \left(\frac{1}{\alpha + \beta \mathbf{1}_A(x)} \right)^{\frac{1}{2}} \psi(x) (\alpha + \beta \mathbf{1}_A(x))^{\frac{1}{2}} f(x) m(x) \right)^2 \qquad (5.4) \\
&\leq \left(\sum_{x \in U} \left(\frac{1}{\alpha + \beta \mathbf{1}_A(x)} \right) \psi(x)^2 m(x) \right) \left(\sum_{x \in U} (\alpha + \beta \mathbf{1}_A(x)) f^2(x) m(x) \right)
\end{aligned}$$

Since $(g,\psi) = 0$,

$$\frac{1}{2} \sum_{x,y \in U} (g(y) - g(x))^2 p(x,y) m(x) \geq \mu(U) \|g\|_2^2,$$

which with (5.4) gives

$$\begin{aligned}
&\frac{1}{2} \sum_{x,y \in U} (g(y) - g(x))^2 p(x,y) m(x) + \sum_{x \in U} (\alpha + \beta \mathbf{1}_A(x)) f^2(x) m(x) \\
&\geq \mu(U) \|g\|_2^2 + \left(\sum_{x \in U} \left(\frac{1}{\alpha + \beta \mathbf{1}_A(x)} \right) \psi(x)^2 m(x) \right)^{-1} (f,\psi)^2.
\end{aligned}$$

Noticing that $(f,f) = (f,\psi)^2 + (g,g)$, it follows that

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\in U}(f(y)-f(x))^2 p(x,y)\, m(x) / \|f\|_2^2 + \alpha \\
&\geq \mu(U)\frac{\|g\|_2^2}{\|f\|_2^2} + \left(\sum_{x\in U}\left(\frac{1}{\alpha+\beta\mathbf{1}_A(x)}\right)\psi(x)^2 m(x)\right)^{-1}\frac{(f,\psi)^2}{\|f\|_2^2} \qquad (5.5)\\
&\geq \min\left\{\mu(U), \left(\sum_{x\in U}\left(\frac{1}{\alpha+\beta\mathbf{1}_A(x)}\right)\psi(x)^2 m(x)\right)^{-1}\right\}.
\end{aligned}$$

As β goes to infinity,

$$\sum_{x\in U}\left(\frac{1}{\alpha+\beta\mathbf{1}_A(x)}\right)\psi(x)^2 m(x) \to \frac{m(U\setminus A)}{\alpha m(U)},$$

which with (5.5) shows that

$$\mu_A(U) \geq \min\left\{\mu(U), \alpha\frac{m(U)}{m(U\setminus A)}\right\} - \alpha.$$

Taking $\alpha = \mu(U)\frac{m(U\setminus A)}{m(U)}$, completes the proof. □

For each $x\in\Gamma$, let Y_x be an independant Bernoulli$(1-\exp(-\nu m(x)))$ random variable. Let $S_U = \sum_{x\in U} Y_x m(x)$. The following simple large deviation result is used in the upper bound.

Lemma 5.5. *Let Γ be an infinite graph. Then for $\kappa_U = \inf\{m(x)|x\in U\}$, $\lambda = \kappa_U^{-1}+\nu$, and $c = \frac{1}{2}\nu\lambda^{-1}$,*

$$\mathbb{P}[S_U < cm(U)] \leq \exp\left(-\frac{1}{5}m(U)\right).$$

Furthermore, if $\kappa = \inf_x m(x) > 0$, then the inequality holds for $c = \frac{1}{2}\nu(\kappa+\nu)^{-1}$.

Proof. By Chebyshev's inequality, for any $\lambda > 0$,

$$\begin{aligned}
&\mathbb{P}[S_U < cm(U)] \\
&\leq \exp(\lambda cm(U))\mathbb{E}[\exp(-\lambda S_U)] \\
&= \exp(\lambda cm(U))\prod_{x\in U}\mathbb{E}[\exp(-\lambda m(x)Y_x)] \\
&= \prod_{x\in U}((\exp(\lambda(c-1)m(x)))(1-\exp(-\nu m(x))) + \exp((\lambda c-\nu)m(x)))
\end{aligned}$$

Taking $\kappa = \inf\{m(x)|x\in U\}$, $\lambda = \kappa^{-1}+\nu$, and $c = \frac{1}{2}\nu\lambda^{-1}$, a bit of calculus shows that

$$\prod_{x\in U}\exp(\lambda(c-1)m(x))(1-\exp(-\nu m(x))) + \exp((\lambda c-\nu)m(x))$$
$$\leq \exp\left(-\frac{1}{5}m(U)\right),$$

from which the first statement of the lemma follows. When $\kappa = \inf_x m(x) > 0$, as in (1.12), the last statement of the lemma follows in like fashion. □

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