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Integrability of exit times and ballisticity for random walks in Dirichlet environment*

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Abstract

We consider random walks in Dirichlet random environment. Since the Dirichlet distribution is not uniformly elliptic, the annealed integrability of the exit time out of a given finite subset is a non-trivial question. In this paper we provide a simple and explicit equivalent condition for the integrability of Green functions and exit times on any finite directed graph. The proof relies on a quotienting procedure allowing for an induction argument on the cardinality of the graph. This integrability problem arises in the definition of Kalikow auxiliary random walk. Using a particular case of our condition, we prove a refined version of the ballisticity criterion given by Enriquez and Sabot in [EnSa06].

Key words: Random walks in random environment, Dirichlet distribution, exit time, reinforced random walks, quotient graph, ballisticity.

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1 Introduction

Since their introduction in the 70's, models of random walks in random environment have mostly been studied in the one dimensional case. Using specific features of this setting, like the reversibility of the Markov chain, Solomon [So75] set a first milestone by proving simple explicit necessary and sufficient conditions for transience, and a law of large numbers. In contrast, the multidimensional situation is still poorly understood. A first general transience criterion was provided by Kalikow [Ka81], which Sznitman and Zerner [SzZe99] later proved to imply ballisticity as well. Under an additional uniform ellipticity hypothesis, Sznitman ([Sz01], [Sz04]) could weaken this ballisticity criterion, but not much progress was made since then about the delicate question of sharpening transience or ballisticity criterions.

Another approach consists in deriving explicit conditions in more specific random environments. Among them, Dirichlet environments, first studied by Enriquez and Sabot in [EnSa06], appear as a natural choice because of their connection with oriented edge linearly reinforced random walks (cf. [EnSa02], and [Pe07] for a review on reinforced processes). Another interest in this case comes from the existence of algebraic relations involving Green functions. These relations allowed Enriquez and Sabot to show that Kalikow's criterion is satisfied under some simple condition, thus proving ballistic behaviour, and to give estimates of the limiting velocity.

Defining Kalikow's criterion raises the problem of integrability of Green functions on finite subsets. While this property is very easily verified for a uniformly elliptic environment, it is no longer the case in the Dirichlet situation. In [EnSa06], the condition on the environment allowed for a quick proof, and the general case remained unanswered.

The main aim of this article is to state and prove a simple necessary and sufficient condition of integrability of these Green functions in Dirichlet environment on general directed graphs. Integrability conditions for exit times are then easily deduced. The "sufficiency" part of the proof is the more delicate. It procedes by induction on the size of the graph by going through an interesting quotienting procedure.

This sharpening of the integrability criterion, along with an additional trick, allows us to prove a refined version of Enriquez and Sabot's ballisticity criterion. The condition of non integrability may also prove useful in further analysis of random walks in Dirichlet environment. Indeed, finite subsets with non integrable exit times play the role of "strong traps" for the walk. As a simple example, one can prove that the existence of such a subset implies a null limiting velocity.

Next section introduces the notations, states the results and various corollaries. Section 3 contains the proofs of the main result and corollary. Finally, Section 4 proves the generalization of Enriquez and Sabot's criterion.

2 Definitions and statement of the results

2.1 Dirichlet distribution

Let us first recall the definition of the usual Dirichlet distribution. Let I be a finite set. The set of probability distributions on I is denoted by $\mathrm{Prob}(I)$:

$$\mathrm{Prob}(I) = \Big\{(p_i)_{i\in I} \in \mathbb{R}_+^I \;\Big|\; \textstyle\sum_{i\in I} p_i = 1\Big\}.$$

Given a family $(\alpha_i)_{i\in I}$ of positive real numbers, the **Dirichlet distribution** of parameter $(\alpha_i)_{i\in I}$ is the probability distribution $\mathscr{D}((\alpha_i)_{i\in I})$ on $\mathrm{Prob}(I)$ of density:

$$(x_i)_{i\in I} \mapsto \frac{\Gamma(\sum_{i\in I}\alpha_i)}{\prod_{i\in I}\Gamma(\alpha_i)}\prod_{i\in I} x_i^{\alpha_i-1}$$

with respect to the Lebesgue measure $\prod_{i\neq i_0} dx_i$ (where i_0 is any element of I) on the simplex $\mathrm{Prob}(I)$. We will recall a few properties of this distribution on page 437.

2.2 Definition of the model

In order to later deal with multiple edges, we define a **directed graph** as a quadruplet $G=(V,E,$ $head, tail)$ where V and E are two sets whose elements are respectively called the **vertices** and **edges** of G, endowed with two maps $head: e\mapsto \overline{e}$ and $tail: e\mapsto \underline{e}$ from E to V. An edge $e\in E$ is thought of as an oriented link from $\underline{e}$ (*tail*) to $\overline{e}$ (*head*), and the usual definitions apply. Thus, a vertex x is **connected** to a vertex y in G if there is an oriented path from x to y, i.e. a sequence $e_1,\dots,e_n$ of edges with $\underline{e_1}=x$, $\overline{e_k}=\underline{e_{k+1}}$ for $k=1,\dots,n-1$, and $\overline{e_n}=y$. For brevity, we usually only write $G=(V,E)$, the tail and head of an edge e being always denoted by $\underline{e}$ and $\overline{e}$.

Unless explicitely stated otherwise (i.e. before the quotienting procedure page 441), graphs are however supposed not to have multiple edges, so that the notation (x,y) for the edge from x to y makes sense.

In the following, we consider *finite* directed graphs $G=(V\cup\{\partial\},E)$ (without multiple edges) possessing a cemetery vertex ∂. In this setting, we always assume that the set of edges is such that:

(i) ∂ is a dead end: no edge in E exits this vertex;

(ii) every vertex is connected to ∂ through a path in E.

Let $G=(V\cup\{\partial\},E)$ be such a graph. For all $x\in V$, let $\mathscr{P}_x$ designate the set of probability distributions on the set of edges originating at x:

$$\mathscr{P}_x=\left\{(p_e)_{e\in E,\underline{e}=x}\in\mathbb{R}_+^{\{e\in E\,|\,\underline{e}=x\}}\;\middle|\;\textstyle\sum_{e\in E,\underline{e}=x}p_e=1\right\}.$$

Then the set of **environments** is

$$\Omega=\prod_{x\in V}\mathscr{P}_x\subset\mathbb{R}^E.$$

We will denote by $\omega=(\omega_e)_{e\in E}$ the canonical random variable on Ω, and we usually write $\omega(x,y)$ instead of $\omega_{(x,y)}$.

Given a family $\vec{\alpha}=(\alpha_e)_{e\in E}$ of positive weights indexed by the set of edges of G, the **Dirichlet distribution on environments** of parameter $\vec{\alpha}$ is the product measure on $\Omega=\prod_{x\in V}\mathscr{P}_x$ of Dirichlet distributions on each of the $\mathscr{P}_x$, $x\in V$:

$$\mathbb{P}=\mathbb{P}^{(\vec{\alpha})}=\bigotimes_{x\in V}\mathscr{D}((\alpha_e)_{e\in E,\ \underline{e}=x}).$$

Note that this distribution does not satisfy the usual uniform ellipticity condition: there is no positive constant bounding $\mathbb{P}$-almost surely the transition probabilities ω_e from below.

In the case of $\mathbb{Z}^d$, we always consider translation invariant distributions of environments, hence the parameters are identical at each vertex and we only need to be given a $2d$-uplet $(\alpha_e)_{e\in\mathscr{V}}$ where $\mathscr{V} = \left\{e \in \mathbb{Z}^d \,\middle|\, |e| = 1\right\}$. This is the usual case of i.i.d. Dirichlet environment.

The canonical process on V will be denoted by $(X_n)_{n\geq 0}$, and the canonical shift on $V^{\mathbb{N}}$ by Θ: for $p \in \mathbb{N}$, $\Theta_p((X_n)_{n\in\mathbb{N}}) = (X_{p+n})_{n\in\mathbb{N}}$.

For any environment $\omega \in \Omega$, and any vertex $x \in V$, the **quenched law** in the environment ω starting at $x \in V$ is the distribution $P_{x,\omega}$ of the Markov chain starting at x with transition probabilities given by ω and stopped when it hits ∂. Thus, for every $o \in V$, $(x, y) \in E$, $\omega \in \Omega$, $n \in \mathbb{N}$:

$$P_{o,\omega}(X_{n+1} = y | X_n = x) = \omega(x, y)$$

and

$$P_{o,\omega}(X_{n+1} = \partial | X_n = \partial) = 1.$$

The **annealed law** starting at $x \in V$ is then the following averaged distribution on random walks on G:

$$P_x(\cdot) = \int P_{x,\omega}(\cdot)\mathbb{P}(d\omega) = \mathbb{E}[P_{x,\omega}(\cdot)].$$

We will need the following stopping times: for $A \subset E$, $T_A = \inf\{n \geq 1 \,|\, (X_{n-1}, X_n) \notin A\}$, for $U \subset V$, $T_U = \inf\{n \geq 0 \,|\, X_n \notin U\}$ and, for any vertex x, $H_x = \inf\{n \geq 0 \,|\, X_n = x\}$ and $\widetilde{H}_x = \inf\{n \geq 1 \,|\, X_n = x\}$.

If the random variable N_y denotes the number of visits of $(X_n)_{n\geq 0}$ at site y (before it hits ∂), then the **Green function** G^ω of the random walk in the environment ω is given by:

$$\text{for all } x, y \in V,\ G^\omega(x, y) = E_{x,\omega}[N_y] = \sum_{n\geq 0} P_{x,\omega}(X_n = y).$$

Due to the assumption (ii), $G^\omega(x, y)$ is $\mathbb{P}$-almost surely finite for all $x, y \in V$. The question we are concerned with is the integrability of these functions under $\mathbb{P}$, depending on the value of $\vec{\alpha}$.

2.3 Integrability conditions

The main quantity involved in our conditions is the sum of the coefficients α_e over the edges e exiting some set. For every subset A of E, define:

$$\begin{aligned}
\underline{A} &= \left\{\underline{e} \,\middle|\, e \in A\right\} \subset V, \\
\overline{A} &= \{\overline{e} \,|\, e \in A\} \subset V \cup \{\partial\}, \\
\underline{\overline{A}} &= \left\{\underline{e} \,\middle|\, e \in A\right\} \cup \{\overline{e} \,|\, e \in A\} \subset V \cup \{\partial\}, \\
\partial_E A &= \left\{e \in E \setminus A \,|\, \underline{e} \in \underline{A}\right\} \subset E,
\end{aligned}$$

and the sum of the coefficients of the edges "exiting A":

$$\beta_A = \sum_{e\in\partial_E A} \alpha_e.$$

A is said to be **strongly connected** if, for all $x, y \in \overline{A}$, x is **connected** to y in A, i.e. there is an (oriented) path from x to y through edges in A. If A is strongly connected, then $\underline{A} = \overline{A}$.

THEOREM 1. – Let $G = (V \cup \{\partial\}, E)$ be a finite directed graph and $\vec{\alpha} = (\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Let $o \in V$. For every $s > 0$, the following statements are equivalent:

(i) $\mathbb{E}[G^\omega(o,o)^s] < \infty$;

(ii) for every strongly connected subset A of E such that $o \in \underline{A}$, $\beta_A > s$.

Undirected graphs are directed graphs where edges come in pair: if $(x, y) \in E$, then $(y, x) \in E$ as well. In this case, the previous result translates into a statement on subsets of V. For any $S \subset V$, we denote by β_S the sum of the coefficients of the edges "exiting S":

$$\beta_S = \sum_{\underline{e}\in S,\ \overline{e}\notin S} \alpha_e.$$

Suppose there is no loop in G (i.e. no edge both exiting from and heading to the same vertex). For any strongly connected subset A of E, if we let $S = \underline{A}$, then S is connected, $|S| \geq 2$ and $\beta_S \leq \beta_A$. Conversely, provided the graph is undirected, a connected subset S of V of cardinality at least 2 satisfies $\beta_S = \beta_A$ where $A = \{e \in E | \underline{e} \in S, \overline{e} \in S\}$, which is strongly connected. This remark yields:

THEOREM 2. – Let $G = (V \cup \{\partial\}, E)$ be a finite undirected graph without loop and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Let $o \in V$. For every $s > 0$, the following statements are equivalent:

(i) $\mathbb{E}[G^\omega(o,o)^s] < \infty$;

(ii) every connected subset S of V such that $\{o\} \subsetneq S$, $\beta_S > s$.

In particular, we get the case of i.i.d. environments in $\mathbb{Z}^d$. Given a subset U of $\mathbb{Z}^d$, let us introduce the Green function G_U^ω of the random walk on $\mathbb{Z}^d$, in environment ω, killed when exiting U. Identifying the complement of U with a cemetery point ∂ allows to apply the previous theorem to G_U^ω. Among the connected subsets S of vertices of $\mathbb{Z}^d$ such that $\{o\} \subsetneq S$, the ones minimizing the "exit sum" β_S are made of the two endpoints of an edge. The result may therefore be stated as:

THEOREM 3. – Let $\vec{\alpha} = (\alpha_e)_{e\in\mathcal{V}}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the translation invariant Dirichlet distribution on environments on $\mathbb{Z}^d$ associated with $\vec{\alpha}$. Let U be a finite subset of $\mathbb{Z}^d$ containing o. Let $\Sigma = \sum_{e\in\mathcal{V}} \alpha_e$. Then for every $s > 0$, the following assertions are equivalent:

(i) $\mathbb{E}[G_U^\omega(o,o)^s] < \infty$;

(ii) for every edge $e = (o, x)$ with $x \in U$, $2\Sigma - \alpha_e - \alpha_{-e} > s$.

Assuming the hypothesis of Theorem 1 to be satisfied relatively to all vertices instead of only one provides information about exit times:

COROLLARY 4. – Let $G = (V \cup \{\partial\}, E)$ be a finite strongly connected directed graph and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. For every $s > 0$, the following properties are equivalent:

(i) for every vertex x, $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$;

(ii) for every vertex x, $\mathbb{E}[G^\omega(x,x)^s] < \infty$;

(iii) every non-empty strongly connected subset A of E satisfies $\beta_A > s$;

(iv) there is a vertex x such that $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$.

And in the undirected case:

Corollary 5. – Let $G = (V \cup \{\partial\}, E)$ be a finite connected undirected graph without loop, and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. For every $s > 0$, the following properties are equivalent:

(i) for every vertex x, $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$;

(ii) for every vertex x, $\mathbb{E}[G^\omega(x,x)^s] < \infty$;

(iii) every connected subset S of V of cardinality ≥ 2 satisfies $\beta_S > s$;

(iv) there is a vertex x such that $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$.

2.4 Ballisticity criterion

We now consider the case of random walks in i.i.d. Dirichlet environment on $\mathbb{Z}^d$, $d \geq 1$.

Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{Z}^d$, and $\mathscr{V} = \left\{e \in \mathbb{Z}^d \,\middle|\, |e| = 1\right\}$. Let $(\alpha_e)_{e\in\mathscr{V}}$ be positive numbers. We will write either α_i or α_{e_i}, and α_{-i} or α_{-e_i}, $i = 1, \ldots, d$.

Enriquez and Sabot proved that the random walk in Dirichlet environment has a ballistic behaviour as soon as $\max_{1\leq i\leq d} |\alpha_i - \alpha_{-i}| > 1$. Our improvement replaces ℓ^∞-norm by ℓ^1-norm:

Theorem 6. – If $\sum_{i=1}^{d} |\alpha_i - \alpha_{-i}| > 1$, then there exists $v \neq 0$ such that, P_0-a.s., $\dfrac{X_n}{n} \to_n v$, and the following bound holds:

$$\left| v - \frac{\Sigma}{\Sigma - 1} d_m \right|_1 \leq \frac{1}{\Sigma - 1},$$

where $\Sigma = \sum_{e\in\mathscr{V}} \alpha_e$, $d_m = \sum_{i=1}^{d} \frac{\alpha_i - \alpha_{-i}}{\Sigma} e_i$ is the drift in the averaged environment, and $|X|_1 = \sum_{i=1}^{d} |X \cdot e_i|$ for any $X \in \mathbb{R}^d$.

3 Proof of the main result

Let us first give a few comments about the proof of Theorem 1. Proving this integrability condition amounts to bounding the tail probability $\mathbb{P}(G^\omega(o,o) > t)$ from below and above.

In order to get the lower bound, we consider an event consisting of environments with small transition probabilities out of a given subset containing o. This forces the mean exit time out of this subset to be large. However, getting a large number of returns to the starting vertex o requires an additional trick: one needs to control from below the probability of some paths leading back to o.

The important yet basic remark here is that, at each vertex, there is at least one exiting edge with transition probability greater than the inverse number of neighbours at that vertex. By restricting the probability space to an event where, at each vertex, this (random) edge is fixed, we are able to compensate for the non uniform ellipticity of $\mathbb{P}$.

The upper bound is more elaborate. Since $G^{\omega}(o,o) = 1/P_{o,\omega}(H_{\partial} < \widetilde{H}_o)$, we need lower bounds on the probability to reach ∂ without coming back to o. If $\mathbb{P}$ were uniformly elliptic, it would suffice to consider a single path from o to ∂. In the present case, we construct a random subset $C(\omega)$ of E containing o where a weaker ellipticity property holds anyway: vertices in $\underline{C(\omega)}$ can be easily connected to each other through paths inside $C(\omega)$ (cf. Proposition 8). The probability, from o, to reach ∂ without coming back to o is greater than the probability, from o, to exit $C(\omega)$ without coming back to o and then to reach ∂ without coming back to $C(\omega)$. The uniformity property of $C(\omega)$ allows to bound $P_{o,\omega}(T_{C(\omega)} < \widetilde{H}_o)$ by a simpler quantity, and to relate the probability of reaching ∂ without coming back to $C(\omega)$ to the probability, *in a quotient graph* $\widetilde{G}$, of reaching ∂ from a vertex $\widetilde{o}$ (corresponding to $C(\omega)$) without coming back to $\widetilde{o}$. We thus get a lower bound on $P_{o,\omega}(H_{\partial} < \widetilde{H}_o)$ involving the same probability relative to a quotient graph. This allows us to perform an induction on the size of the graph. Actually, the environment we get on the quotient graph is not exactly Dirichlet, and we first need to show (cf. Lemma 9) that its density can be compared to that of a Dirichlet environment.

3.1 Properties of Dirichlet distributions

Notice that if (p_1, p_2) is a random variable with distribution $\mathscr{D}(\alpha, \beta)$ under P, then p_1 is a Beta variable of parameter (α, β), and has the following tail probability:

$$P(p_1 \le \varepsilon) \underset{\varepsilon \to 0}{\sim} C\varepsilon^{\alpha}, \tag{$*$}$$

where $C = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}$.

Let us now recall two useful properties that are simple consequences of the representation of a Dirichlet random variable as a normalized vector of independent gamma random variables (cf. for instance [Wi63]). Let $(p_i)_{i \in I}$ be a random variable distributed according to $\mathscr{D}((\alpha_i)_{i \in I})$. Then:

(Associativity) Let $I_1, \dots, I_n$ be a partition of I. The random variable $\left(\sum_{i \in I_k} p_i\right)_{k \in \{1,\dots,n\}}$ on $\mathrm{Prob}(\{1, \dots, n\})$ follows the Dirichlet distribution $\mathscr{D}((\sum_{i \in I_k} \alpha_i)_{1 \le k \le n})$.

(Restriction) Let J be a nonempty subset of I. The random variable $\left(\frac{p_i}{\sum_{j \in J} p_j}\right)_{i \in J}$ on $\mathrm{Prob}(J)$ follows the Dirichlet distribution $\mathscr{D}((\alpha_i)_{i \in J})$ and is independent of $\sum_{j \in J} p_j$ (which follows a Beta distribution $B(\sum_{j \in J} \alpha_j, \sum_{j \notin J} \alpha_j)$ due to the associativity property).

Thanks to the associativity property, the asymptotic estimate $(*)$ holds as well (with a different C) for the marginal p_1 of a Dirichlet random variable $(p_1, \dots, p_n)$ with parameters $(\alpha, \alpha_2, \dots, \alpha_n)$.

3.2 First implication (lower bound)

Let $s > 0$. Suppose there exists a strongly connected subset A of E such that $o \in \underline{A}$ and $\beta_A \le s$. We shall prove the stronger statement that $\mathbb{E}[G_A^\omega(o,o)^s] = \infty$ where G_A^ω is the Green function of the random walk in the environment ω killed when exiting A.

Let $\varepsilon > 0$. Define the event $\mathscr{E}_\varepsilon = \{\forall x \in \underline{A},\ \sum_{e\in\partial_E A,\ \underline{e}=x} \omega_e \le \varepsilon\}$. On $\mathscr{E}_\varepsilon$, one has:

$$E_{o,\omega}[T_A] \ge \frac{1}{\varepsilon}. \tag{1}$$

Indeed, by the Markov property, for all $n \in \mathbb{N}^*$,

$$P_{o,\omega}(T_A > n) = E_{o,\omega}[\mathbf{1}_{\{T_A>n-1\}} P_{X_{n-1},\omega}(T_A > 1)] \ge P_{o,\omega}(T_A > n-1) \min_{x\in\overline{A}} P_{x,\omega}(T_A > 1)$$

and if $\omega \in \mathscr{E}_\varepsilon$ then, for all $x \in \overline{A} = \underline{A}$, $P_{x,\omega}(T_A > 1) \ge 1-\varepsilon$, hence, by recurrence:

$$P_{o,\omega}(T_A > n) \ge P_{o,\omega}(T_A > 0)(1-\varepsilon)^n = (1-\varepsilon)^n,$$

from which inequality (1) results after summation over $n \in \mathbb{N}$.

As a consequence,

$$\mathbb{E}[E_{o,\omega}[T_A]^s] = \int_0^\infty \mathbb{P}\left(E_{o,\omega}[T_A]^s \ge t\right) dt \ge \int_0^\infty \mathbb{P}\left(\mathscr{E}_{\frac{1}{t^{1/s}}}\right) dt.$$

Notice now that, due to the associativity property of section 3.1, for every $x \in \underline{A}$, the random variable $\sum_{e\in\partial_E A,\ \underline{e}=x} \omega_e$ follows a Beta distribution with parameters $(\sum_{e\in E\setminus A,\ \underline{e}=x} \alpha_e, \sum_{e\in A,\ \underline{e}=x} \alpha_e)$, so that the tail probability $(*)$ together with the spatial independence gives:

$$\mathbb{P}(\mathscr{E}_\varepsilon) \underset{\varepsilon\to 0}{\sim} C\varepsilon^{\beta_A},$$

where C is a positive constant. Hence $\mathbb{P}\left(\mathscr{E}_{\frac{1}{t^{1/s}}}\right) \underset{t\to\infty}{\sim} Ct^{-\frac{\beta_A}{s}}$, and the assumption $\beta_A \le s$ leads to

$$\mathbb{E}[E_{o,\omega}[T_A]^s] = \infty.$$

Dividing T_A into the time spent at each point of $\underline{A}$, one has:

$$E_{o,\omega}[T_A]^s = \left(\sum_{x\in\underline{A}} G_A^\omega(o,x)\right)^s \le \left(|\underline{A}| \max_{x\in\underline{A}} G_A^\omega(o,x)\right)^s = |\underline{A}|^s \max_{x\in\underline{A}} G_A^\omega(o,x)^s \le |\underline{A}|^s \sum_{x\in\underline{A}} G_A^\omega(o,x)^s, \tag{2}$$

so that there is a vertex $x \in \underline{A}$ such that $\mathbb{E}[G_A^\omega(o,x)^s] = \infty$.

Getting the result on $G_A^\omega(o,o)$ requires to refine this proof. To that aim, we shall introduce an event $\mathscr{F}$ of positive probability on which, from every vertex of $\underline{A}$, there exists a path toward o whose transition probability is bounded from below uniformly on $\mathscr{F}$.

Let $\omega \in \Omega$. Denote by $\vec{G}(\omega)$ the set of the edges $e^* \in E$ such that $\omega_{e^*} = \max\{\omega_e | e \in E, \underline{e} = \underline{e^*}\}$. If $e^* \in \vec{G}(\omega)$, then (by a simple pigeonhole argument):

$$\omega_{e^*} \ge \frac{1}{n_{\underline{e^*}}} \ge \frac{1}{|E|},$$

where n_x is the number of neighbours of a vertex x. In particular, there is a positive constant κ depending only on G such that, if x is connected to y through a (simple) path π in $\vec{G}(\omega)$ then $P_{x,\omega}(\pi) \geq \kappa$. Note that for $\mathbb{P}$-almost every ω in Ω, there is exactly one maximizing edge e^* exiting each vertex.

Since A is a strongly connected subset of E, it possesses at least one spanning tree T oriented toward o. Let us denote by $\mathscr{F}$ the event $\{\vec{G}(\omega) = T\}$: if $\omega \in \mathscr{F}$, then every vertex of $\underline{A}$ is connected to o in $\vec{G}(\omega)$. One still has:

$$\mathbb{P}(\mathscr{E}_\varepsilon \cap \mathscr{F}) \geq \mathbb{P}(\mathscr{E}_\varepsilon \cap \{\forall e \in T, \omega_e > 1/2\}) \underset{\varepsilon \to 0}{\sim} C' \varepsilon^{\beta_A},$$

where C' is a positive constant (depending on the choice of T). Indeed, using the associativity property and the spatial independence, this asymptotic equivalence reduces to the fact that if (p_1, p_2) has distribution $\mathscr{D}(\alpha, \beta)$ or if (p_1, p_2, p_3) has distribution $\mathscr{D}(\alpha, \beta, \gamma)$, then there is $c > 0$ such that $P(p_1 \leq \varepsilon, p_2 > 1/2) \underset{\varepsilon \to 0}{\sim} c\varepsilon^\alpha$. In the first case, this is exactly $(*)$, and in the case of 3 variables,

$$\begin{aligned} P(p_1 \leq \varepsilon, p_2 > 1/2) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\varepsilon \int_{1/2}^1 x_1^{\alpha-1} x_2^{\beta-1} (1-x_1-x_2)^{\gamma-1} dx_2 dx_1 \\ &\underset{\varepsilon \to 0}{\sim} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\varepsilon x_1^{\alpha-1} dx_1 \int_{1/2}^1 x_2^{\beta-1} (1-x_2)^{\gamma-1} dx_2 = c\varepsilon^\alpha. \end{aligned}$$

Then, as previously:

$$\mathbb{E}[E_{o,\omega}[T_A]^s, \mathscr{F}] = \int_0^\infty \mathbb{P}(E_{o,\omega}[T_A]^s \geq t, \mathscr{F}) dt \geq \int_0^\infty \mathbb{P}\left(\mathscr{E}_{\frac{1}{t^{1/s}}} \cap \mathscr{F}\right) dt = +\infty,$$

and subsequently there exists $x \in \underline{A}$ such that $\mathbb{E}[G_A^\omega(o,x)^s, \mathscr{F}] = \infty$. Now, there is an integer l and a real number $\kappa > 0$ such that, for every $\omega \in \mathscr{F}$, $P_{x,\omega}(X_l = o) \geq \kappa$. Thus, thanks to the Markov property:

$$\begin{aligned} G_A^\omega(o,x) &= \sum_{k \geq 0} P_{o,\omega}(X_k = x, T_A > k) \\ &\leq \sum_{k \geq 0} \frac{1}{\kappa} P_{o,\omega}(X_{k+l} = o, T_A > k + l) \\ &\leq \frac{1}{\kappa} G_A^\omega(o,o). \end{aligned}$$

Therefore we get:

$$\mathbb{E}[G_A^\omega(o,x)^s, \mathscr{F}] \leq \frac{1}{\kappa^s} \mathbb{E}[G_A^\omega(o,o)^s, \mathscr{F}],$$

and finally $\mathbb{E}[G_A^\omega(o,o)^s] = \infty$.

3.3 Converse implication (upper bound)

Scheme of the proof The proof of the upper bound procedes by induction on the number of edges of the graph. More precisely, we prove the following property by induction on $n \geq 1$:

Proposition 7. – Induction hypothesis

Let $n \in \mathbb{N}^*$. Let $G = (V \cup \{\partial\}, E)$ be a directed graph possessing at most n edges and such that every vertex is connected to ∂, and $(\alpha_e)_{e\in E}$ be positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Then, for every vertex $o \in V$, there exist real numbers $C, r > 0$ such that, for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon) \le C\varepsilon^\beta(-\ln\varepsilon)^r,$$

where $\beta = \min\left\{\beta_A \,\middle|\, A \text{ is a strongly connected subset of } E \text{ and } o \in \underline{A}\right\}$.

The implication (ii)⇒(i) in Theorem 1 results from this proposition using the integrability of $t \mapsto \frac{(\ln t)^r}{t^\beta}$ in the neighbourhood of $+\infty$ as soon as $\beta > 1$, and from the following Markov chain identity:

$$G^\omega(o,o) = \frac{1}{P_{o,\omega}(H_\partial < \widetilde{H}_o)}.$$

Let us initialize the induction. If a graph $G = (V \cup \{\partial\}, E)$ with a vertex o is such that $|E| = 1$ and o is connected to ∂, the only edge links o to ∂, so that $P_{o,\omega}(H_\partial < \widetilde{H}_o) = 1$, and the property is true (β is infinite here).

Let $n \in \mathbb{N}^*$. We suppose the induction hypothesis to be true at rank n. Let $G = (V \cup \{\partial\}, E)$ be a directed graph with $n+1$ edges, o be a vertex and $(\alpha_e)_{e\in E}$ be positive parameters. As usual, $\mathbb{P}$ is the corresponding Dirichlet distribution. In the next paragraph, we introduce the random subset $C(\omega)$ of E we will then be interested to quotient G by.

Construction of $C(\omega)$ Let $\omega \in \Omega$. We define inductively a finite sequence $e_1 = (x_1, y_1), \ldots, e_m = (x_m, y_m)$ of edges in the following way: letting $y_0 = o$, if $e_1, \ldots, e_{k-1}$ have been defined, then e_k is the edge in E which maximizes the exit distribution out of $C_k = \{e_1, \ldots, e_{k-1}\}$ starting at y_{k-1}, that is:

$$e \mapsto P_{y_{k-1},\omega}((X_{T_{C_k}-1}, X_{T_{C_k}}) = e).$$

The integer m is the least index ≥ 1 such that $y_m \in \{o, \partial\}$. In words, the edge e_k is, among the edges exiting the set $C_k(\omega)$ of already visited edges, the one maximizing the probability for a random walk starting at y_{k-1} to exit $C_k(\omega)$ through it; and the construction ends as soon as an edge e_k heads at o or ∂. The assumption that each vertex is connected to ∂ guarantees the existence of an exit edge out of $C_k(\omega)$ for $k \le m$, and the finiteness of G ensures that m exists: the procedure ends. We set:

$$C(\omega) = C_{m+1} = \{e_1, \ldots, e_m\}.$$

Note that the maximizing edges, and thus $C(\omega)$, are in fact well defined only for ω out of a Lebesgue negligible subset of Ω.

Notice that $C_1 = \varnothing$, hence $T_{C_1} = 1$ and e_1 is the edge maximizing $e \mapsto \omega_e$ among the edges starting at o. Hence e_1 is the edge of $\vec{G}(\omega)$ starting at o (cf. the proof of the first implication). More generally, for $1 \le k \le m$, if $y_{k-1} \notin \{x_1, \ldots, x_{k-1}\}$, then e_k is the edge of $\vec{G}(\omega)$ starting at y_{k-1}.

The main property of $C(\omega)$ is the following:

Proposition 8. – There exists a constant $c > 0$ such that, for every $\omega \in \Omega$ such that $C(\omega)$ is well defined, for all $x \in \overline{C(\omega)} \setminus \{o\}$,

$$P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_{C(\omega)}) \ge c. \tag{3}$$

Proof. Let ω be such that $C(\omega)$ is well defined. For $k = 1,\dots,m$, due to the choice of e_k as a maximizer over E (or $\partial_E C_k$), we have:

$$P_{y_{k-1},\omega}((X_{T_{C_k}-1}, X_{T_{C_k}}) = e_k) \geq \frac{1}{|E|} = \kappa.$$

For every k such that $y_k \neq o$ (that is for $k = 1,\dots,m-1$ and possibly $k = m$), we deduce that:

$$P_{y_{k-1},\omega}(H_{y_k} < \widetilde{H}_o \wedge T_{C(\omega)}) \geq P_{y_{k-1},\omega}(X_{T_{C_k}} = y_k) \geq \kappa.$$

Then, by the Markov property, for any $x \in \overline{C(\omega)} = \{y_1,\dots,y_m\}$, if $x \neq o$,

$$P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_{C(\omega)}) \geq \kappa^m \geq \kappa^{|E|} = c,$$

as expected. □

The support of the distribution of $\omega \mapsto C(\omega)$ writes as a disjoint union $\mathscr{C} = \mathscr{C}_o \cup \mathscr{C}_\partial$ depending on whether o or ∂ belongs to $\overline{C(\omega)}$. For any $C \in \mathscr{C}$, we define the event

$$\mathscr{E}_C = \{C(\omega) = C\}.$$

On such an event, the previous proposition gives uniform lower bounds inside C, "as if" a uniform ellipticity property held. Because $\mathscr{C}$ is finite, it will be sufficient to prove the upper bound separately on the events $\mathscr{E}_C$ for $C \in \mathscr{C}$.

If $C \in \mathscr{C}_\partial$, then $\partial \in \overline{C}$ and the proposition above provides $c > 0$ such that, on $\mathscr{E}_C$, $P_{o,\omega}(H_\partial < \widetilde{H}_o) \geq c$ hence, for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon,\ \mathscr{E}_C) = 0.$$

In the following we will therefore work on $\mathscr{E}_C$ where $C \in \mathscr{C}_o$. In this case, C is a strongly connected subset of E. Indeed, $y_0 = y_m = o$ and, due to the construction method, y_k is connected in $C(\omega)$ to y_{k+1} for $k = 0,\dots,m-1$.

Quotienting procedure We intend to introduce a quotient of G by contracting the edges of C, and to relate its Green function to that of G. Let us first give a general definition:

Definition. – If A is a strongly connected subset of edges of a graph $G = (V, E, head, tail)$, the quotient graph of G obtained by contracting A to $\widetilde{a}$ is the graph $\widetilde{G}$ deduced from G by deleting the edges of A, replacing all the vertices of $\underline{A}$ by one new vertex $\widetilde{a}$, and modifying the endpoints of the edges of $E \setminus A$ accordingly. Thus the set of edges of $\widetilde{G}$ is naturally in bijection with $E \setminus A$ and can be thought of as a subset of E.

In other words, $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{head}, \widetilde{tail})$ where $\widetilde{V} = (V \setminus \underline{A}) \cup \{\widetilde{a}\}$ ($\widetilde{a}$ being a new vertex), $\widetilde{E} = E \setminus A$ and, if π denotes the projection from V to $\widetilde{V}$ (i.e. $\pi_{|V\setminus\underline{A}} = \mathrm{id}$ and $\pi(x) = \widetilde{a}$ if $x \in \underline{A}$), $\widetilde{head} = \pi \circ head$ and $\widetilde{tail} = \pi \circ tail$.

Notice that this quotient may well introduce multiple edges.

In our case, we consider the quotient graph $\widetilde{G} = (\widetilde{V} = (V \setminus \underline{C}) \cup \{\partial, \widetilde{o}\}, \widetilde{E} = E \setminus C, \widetilde{head}, \widetilde{tail})$ obtained by contracting C to a new vertex $\widetilde{o}$.

Starting from $\omega \in \Omega$, let us define the *quotient environment* $\widetilde{\omega} \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is the analog of Ω for $\widetilde{G}$. For every edge $e \in \widetilde{E}(\subset E)$, if $e \notin \partial_E C$ (i.e. if $\widetilde{tail}(e) \neq \widetilde{o}$), then $\widetilde{\omega}_e = \omega_e$, and if $e \in \partial_E C$, then $\widetilde{\omega}_e = \frac{\omega_e}{\Sigma}$, where:

$$\Sigma = \sum_{e \in \partial_E C} \omega_e.$$

This environment allows to bound the Green function of G using that of $\widetilde{G}$ in a convenient way. Notice that, from o, one way for the walk to reach ∂ without coming back to o consists in exiting C without coming back to o and then reaching ∂ without coming back to $\underline{C}$. Thus we have, for $\omega \in \mathscr{E}_C$:

$$\begin{aligned} P_{o,\omega}(H_\partial < \widetilde{H}_o) &\geq \sum_{x \in \underline{C}} P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_C, H_\partial < H_x + \widetilde{H}_{\underline{C}} \circ \Theta_{H_x}) \\ &= \sum_{x \in \underline{C}} P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_C) P_{x,\omega}(H_\partial < \widetilde{H}_{\underline{C}}) \\ &\overset{(3)}{\geq} c \sum_{x \in \underline{C}} P_{x,\omega}(H_\partial < \widetilde{H}_{\underline{C}}) \\ &= c\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \end{aligned}$$

where the first equality is an application of the Markov property at time H_x, and the last equality comes from the definition of the quotient: both quantities correspond to the same set of paths viewed in G and in $\widetilde{G}$, and, for all $x \in \underline{C}$, $P_{x,\omega}$-almost every path belonging to the event $\{H_\partial < \widetilde{H}_{\underline{C}}\}$ contains exactly one edge exiting from $\underline{C}$ so that the normalization by Σ appears exactly once by quotienting.

Finally, for $c' = 1/c > 0$, we have:

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon,\ \mathscr{E}_C) \leq \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq c'\varepsilon,\ \mathscr{E}_C). \quad (4)$$

Back to Dirichlet environment It is important to remark that, under $\mathbb{P}$, $\widetilde{\omega}$ does not follow a Dirichlet distribution because of the normalization (and neither is it independent of Σ). We can however reduce to the Dirichlet situation and thus procede to induction. This is the aim of the following lemma, inspired by the restriction property of Section 3.1. Because its proof, while simple, is a bit tedious to write, we defer it until the Appendix page 449.

LEMMA 9. – Let $(p_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (p_i^{(r)})_{1 \leq i \leq n_r}$ be independent Dirichlet random variables with respective parameters $(\alpha_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (\alpha_i^{(r)})_{1 \leq i \leq n_r}$. Let $m_1, \ldots, m_r$ be integers such that $1 \leq m_1 < n_1, \ldots, 1 \leq m_r < n_r$, and let $\Sigma = \sum_{j=1}^r \sum_{i=1}^{m_j} p_i^{(j)}$ and $\beta = \sum_{j=1}^r \sum_{i=1}^{m_j} \alpha_i^{(j)}$. There exists positive constants C, C' such that, for any positive measurable function $f : \mathbb{R} \times \mathbb{R}^{\sum_j m_j} \to \mathbb{R}$,

$$E\left[f\left(\Sigma, \frac{p_1^{(1)}}{\Sigma}, \ldots, \frac{p_{m_1}^{(1)}}{\Sigma}, \ldots, \frac{p_1^{(r)}}{\Sigma}, \ldots, \frac{p_{m_r}^{(r)}}{\Sigma}\right)\right] \leq C \cdot \widetilde{E}\left[f\left(\widetilde{\Sigma}, \widetilde{p}_1^{(1)}, \ldots, \widetilde{p}_{m_1}^{(1)}, \ldots, \widetilde{p}_1^{(r)}, \ldots, \widetilde{p}_{m_r}^{(r)}\right)\right],$$

where, under the probability $\widetilde{P}$, $(\widetilde{p}_1^{(1)}, \ldots, \widetilde{p}_{m_1}^{(1)}, \ldots, \widetilde{p}_1^{(r)}, \ldots, \widetilde{p}_{m_r}^{(r)})$ is sampled from a Dirichlet distribution of parameter $(\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}, \ldots, \alpha_1^{(r)}, \ldots, \alpha_{m_r}^{(r)})$, $\widetilde{\Sigma}$ is bounded and satisfies $\widetilde{P}(\widetilde{\Sigma} < \varepsilon) \leq C'\varepsilon^\beta$ for every $\varepsilon > 0$, and these two random variables are independent.

We apply this lemma to ω by normalizing the transition probabilities of the edges exiting C. Using (4) and Lemma 9, we thus get:

$$\begin{aligned}
\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon,\ \mathscr{E}_C) &\le \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \le c'\varepsilon,\ \mathscr{E}_C) \\
&\le \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \le c'\varepsilon) \\
&\le C \cdot \widetilde{\mathbb{P}}(\widetilde{\Sigma} \cdot P_{\widetilde{o},\omega}(H_\partial < \widetilde{H}_{\widetilde{o}}) \le c'\varepsilon), \qquad (5)
\end{aligned}$$

where $\widetilde{\mathbb{P}}$ is the Dirichlet distribution of parameter $(\alpha_e)_{e\in\widetilde{E}}$ on $\widetilde{\Omega}$, ω is the canonical random variable on $\widetilde{\Omega}$ (which can be seen as a restriction of the canonical r.v. on Ω) and, under $\widetilde{\mathbb{P}}$, $\widetilde{\Sigma}$ is a positive bounded random variable independent of ω and such that, for all $\varepsilon > 0$, $\widetilde{\mathbb{P}}(\widetilde{\Sigma} \le \varepsilon) \le C'\varepsilon^{\beta_C}$.

Induction Equation (5) relates the same quantities in G and $\widetilde{G}$, allowing to finally complete the induction argument.

Because the induction hypothesis deals with graphs with simple edges, it may be necessary to reduce the graph $\widetilde{G}$ to this case. Another possibility would have been to define the random walk as a sequence of edges, thus allowing to define T_C (where $C \subset E$) for graphs with multiple edges. In the Dirichlet case, the reduction is however painless since it leads to another closely related Dirichlet distribution.

Note indeed first that, for any $\widetilde{\omega} \in \widetilde{\Omega}$, the quenched laws $P_{x,\widetilde{\omega}}$ (where $x \in \widetilde{V}$) are not altered if we replace multiple (oriented) edges by simple ones with transition probabilities equal to the sum of those of the deleted edges. Due to the associativity property of section 3.1, the distribution under $\widetilde{\mathbb{P}}$ of this new environment is the Dirichlet distribution with weights equal to the sum of the weights of the deleted edges. Hence the annealed laws on $\widetilde{G}$ and on the new simplified graph $\widetilde{G}'$ with these weights are the same and, for the problems we are concerned with, we may use $\widetilde{G}'$ instead of $\widetilde{G}$.

The edges in C do not appear in $\widetilde{G}$ anymore. In particular, $\widetilde{G}$ (and a fortiori $\widetilde{G}'$) has strictly less than n edges. In order to apply the induction hypothesis, we need to check that each vertex is connected to ∂. This results directly from the same property for G. We apply the induction hypothesis to the graph $\widetilde{G}'$ and to $\widetilde{o}$. It states that, for small $\varepsilon > 0$:

$$\widetilde{\mathbb{P}}(P_{\widetilde{o},\omega}(H_\partial < \widetilde{H}_{\widetilde{o}}) \le \varepsilon) \le C''\varepsilon^{\widetilde{\beta}}(-\ln\varepsilon)^r, \qquad (6)$$

where $C'' > 0, r > 0$ and $\widetilde{\beta}$ is the exponent "β" from the statement of the induction hypothesis corresponding to the graph $\widetilde{G}'$ (it is in fact equal to the "β" for $\widetilde{G}$). As for the left-hand side of (6), it may equivalently refer to the graph $\widetilde{G}$ or to $\widetilde{G}'$, as explained above.

Considering (5) and (6), it appears that the following lemma allows to carry out the induction.

LEMMA 10. – If X and Y are independent positive bounded random variables such that, for some real numbers $\alpha_X, \alpha_Y, r > 0$:

- there exists $C > 0$ such that $P(X < \varepsilon) \le C\varepsilon^{\alpha_X}$ for all $\varepsilon > 0$ (or equivalently for small ε);
- there exists $C' > 0$ such that $P(Y < \varepsilon) \le C'\varepsilon^{\alpha_Y}(-\ln\varepsilon)^r$ for small $\varepsilon > 0$,

then there exists a constant $C'' > 0$ such that, for small $\varepsilon > 0$:

$$P(XY \le \varepsilon) \le C''\varepsilon^{\alpha_X \wedge \alpha_Y}(-\ln\varepsilon)^{r+1}$$

(and $r+1$ can be replaced by r if $\alpha_X \neq \alpha_Y$).

Proof. We denote by M_X and M_Y (deterministic) upper bounds of X and Y. We have, for $\varepsilon > 0$:

$$P(XY \le \varepsilon) = P\left(Y \le \frac{\varepsilon}{M_X}\right) + P\left(XY \le \varepsilon, Y > \frac{\varepsilon}{M_X}\right).$$

Let $\varepsilon_0 > 0$ be such that the upper bound in the statement for Y is true as soon as $\varepsilon < \varepsilon_0$. Then, for $0 < \varepsilon < \varepsilon_0$, we compute:

$$\begin{aligned}
P(XY \le \varepsilon, Y > \frac{\varepsilon}{M_X}) &= \int_{\frac{\varepsilon}{M_X}}^{M_Y} P\left(X \le \frac{\varepsilon}{y}\right) P(Y \in dy) \\
&\le C\int_{\frac{\varepsilon}{M_X}}^{M_Y} \left(\frac{\varepsilon}{y}\right)^{\alpha_X} P(Y \in dy) \\
&= C\varepsilon^{\alpha_X} E\left[\mathbf{1}_{(Y \ge \frac{\varepsilon}{M_X})}\frac{1}{Y^{\alpha_X}}\right] \\
&= C\varepsilon^{\alpha_X}\left(\int_{\frac{\varepsilon}{M_X}}^{M_Y} P(\frac{\varepsilon}{M_X} \le Y \le x)\frac{\alpha_X dx}{x^{\alpha_X+1}} + \frac{P\left(Y \ge \frac{\varepsilon}{M_X}\right)}{{M_Y}^{\alpha_X}}\right) \\
&\le C\varepsilon^{\alpha_X}\left(\alpha_X C' \int_{\frac{\varepsilon}{M_X}}^{\varepsilon_0} x^{\alpha_Y}(-\ln x)^r \frac{dx}{x^{\alpha_X+1}} + \frac{1}{{M_Y}^{\alpha_X}}\right) \\
&\le C\varepsilon^{\alpha_X}\left(\alpha_X C' \int_{\frac{\varepsilon}{M_X}}^{\varepsilon_0} x^{\alpha_Y-\alpha_X-1}dx(-\ln\frac{\varepsilon}{M_X})^r + \frac{1}{{M_Y}^{\alpha_X}}\right) \\
&\le C''\varepsilon^{\alpha_X \wedge \alpha_Y}(-\ln\varepsilon)^{r+1}.
\end{aligned}$$

Indeed, if $\alpha_Y > \alpha_X$, the integral converges as $\varepsilon \to 0$; if $\alpha_Y = \alpha_X$, it is equivalent to $-\ln\varepsilon$; if $\alpha_Y < \alpha_X$, the equivalent becomes $\frac{1}{\varepsilon^{\alpha_X-\alpha_Y}}$. And the formula is checked in every case (note that $-\ln\varepsilon > 1$ for small ε). □

Using (6), Lemma 10 and (5), we get constants $c, r > 0$ such that, for small $\varepsilon > 0$:

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon, \mathscr{E}_C) \le c\varepsilon^{\beta_C \wedge \widetilde{\beta}}(-\ln\varepsilon)^{r+1}. \tag{7}$$

Let us prove that $\widetilde{\beta} \ge \beta$, where β is the exponent defined in the induction hypothesis relative to G and o (remember that $\widetilde{\beta}$ is the same exponent, relative to $\widetilde{G}'$ (or $\widetilde{G}$) and $\widetilde{o}$). Let $\widetilde{A}$ be a strongly connected subset of $\widetilde{E}$ such that $\widetilde{o} \in \underline{\widetilde{A}}$. Set $A = \widetilde{A} \cup C \subset E$. In view of the definition of $\widetilde{E}$, every edge exiting $\widetilde{A}$ corresponds to an edge exiting A and vice-versa (the only edges to be deleted by the quotienting procedure are those of C). Thus, recalling that the weights of the edges are preserved in the quotient (cf. Lemma 9), $\beta_{\widetilde{A}} = \beta_A$. Moreover, $o \in \underline{A}$, and A is strongly connected (so are $\widetilde{A}$ and C, and $\widetilde{o} \in \widetilde{A}$, $o \in C$), so that $\beta_A \ge \beta$. As a consequence, $\widetilde{\beta} \ge \beta$, as announced.

Then $\beta_C \wedge \widetilde{\beta} \ge \beta_C \wedge \beta = \beta$ because C is strongly connected and $o \in \underline{C}$. Hence (7) becomes: for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon, \mathscr{E}_C) \le c\varepsilon^{\beta}(-\ln\varepsilon)^{r+1}.$$

Summing on all events $\mathscr{E}_C$, $C \in \mathscr{C}$, this concludes the induction.

Remark This proof (with both implications) gives the following more precise result: there exist $c, C, r > 0$ such that, for large enough t,

$$c\frac{1}{t^{\min_A \beta_A}} \le \mathbb{P}(G^\omega(o,o) > t) \le C\frac{(\ln t)^r}{t^{\min_A \beta_A}},$$

where the minimum is taken over all strongly connected subsets A of E such that $o \in \underline{A}$.

3.4 Proof of the corollary

We prove Corollary 4. Let s be a positive real number. The equivalence of (i) and (ii) results from the inequalities below: for every $\omega \in \Omega$, $x \in V$,

$$G^\omega(x,x)^s = E_{x,\omega}[N_x]^s \le E_{x,\omega}[T_V]^s = \left(\sum_{y\in V} P_{x,\omega}(H_y < H_\partial) G^\omega(y,y)\right)^s \le |V|^s \sum_{y\in V} G^\omega(y,y)^s,$$

where the second inequality is obtained by bounding the probability by 1 and proceding as in equation (2). Theorem 1 provides the equivalence of (ii) and (iii). The fact that (i) implies (iv) is trivial.

Let us suppose that (iii) is not satisfied: there is a strongly connected subset A of E such that $\beta_A \le s$. Let o be a vertex. If $o \in \underline{A}$, then $\mathbb{E}[E_{o,\omega}[T_V]^s] \ge \mathbb{E}[G^\omega(o,o)^s] = \infty$ thanks to Theorem 1; and if $o \notin \underline{A}$, there exists (thanks to strong connexity) a path π from o to some vertex $x \in \underline{A}$ which remains outside $\underline{A}$ (before x), and we recall that Theorem 1 proves $\mathbb{E}[G_A^\omega(x,x)^s] = \infty$ hence, thanks to spatial independence of the environment:

$$\mathbb{E}[E_{o,\omega}[T_V]^s] \ge \mathbb{E}[G^\omega(o,x)^s] \ge \mathbb{E}[P_{o,\omega}(\pi)^s G_A^\omega(x,x)^s] = \mathbb{E}[P_{o,\omega}(\pi)^s] \times \mathbb{E}[G_A^\omega(x,x)^s] = \infty,$$

so that in both cases, $\mathbb{E}[E_{o,\omega}[T_V]^s] = \infty$. Thus, (iv) is not true. So (iv) implies (iii), and we are done.

Remark Under most general hypotheses, (i) and (ii) are still equivalent (same proof). The equivalence of (i) and (iv) can be shown to hold as well in the following general setting:

PROPOSITION 11. – Let $G = (V \cup \{\partial\}, E)$ be a finite strongly connected graph endowed with a probability measure $\mathbb{P}$ on the set of its environments satisfying:

- the transition probabilities $\omega(x,\cdot)$, $x \in V$, are independent under $\mathbb{P}$;
- for all $e \in E$, $\mathbb{P}(\omega_e > 0) > 0$.

If there exists $x \in V$ such that $E_x[T_V] = +\infty$, then for all $y \in V$, $E_y[T_V] = +\infty$.

PROOF. Suppose $x \in V$ satisfies $E_x[T_V] = +\infty$. We denote by A a subset of E satisfying $E_x[T_A] = +\infty$, and being *minimal* (with respect to inclusion) among the subsets of E sharing this property. Since E is finite, the existence of such an A is straightforward.

Let $y \in \underline{A}$: there is an $e \in A$ such that $\underline{e} = y$. Let us prove $E_y[T_A] = +\infty$. We have, by minimality of A, $E_x[T_{A\setminus\{e\}}] < \infty$. Let $H_e = \inf\{n \ge 1 \,|\, (X_{n-1}, X_n) = e\}$. Then:

$$\begin{aligned} E_x[T_A] &= E_x[T_A, H_e < T_A] + E_x[T_A, H_e > T_A] \\ &\le E_x[T_A, H_e < T_A] + E_x[T_{A\setminus\{e\}}], \end{aligned}$$

hence $E_x[T_A, H_e < T_A] = +\infty$. Thus, using the Markov property:

$$\begin{aligned} +\infty &= E_x[T_A - T_{A\setminus\{e\}} + 1, H_e < T_A] = E_x[T_A - (H_e - 1), H_e < T_A] \\ &\leq E_x[T_A - (H_e - 1), H_e - 1 < T_A] = E_x[T_A \circ \Theta_{H_e - 1}, H_e - 1 < T_A] \\ &= \mathbb{E}[E_{x,\omega}[E_{X_{H_e-1},\omega}[T_A], H_e - 1 < T_A]] = \mathbb{E}[E_{\underline{e},\omega}[T_A] P_{x,\omega}(H_e - 1 < T_A)] \\ &\leq E_{\underline{e}}[T_A], \end{aligned}$$

which gives $E_y[T_A] = +\infty$ as announced.

Let $z \in V$. If $z \in \underline{A}$, we have of course $E_z[T_V] \geq E_z[T_A] = +\infty$. Suppose $z \in V \setminus \underline{A}$. By strong connexity of G, one can find a simple path $e_1, \cdots, e_n$ from z to a point $y = \overline{e_n} \in \underline{A}$ such that $\underline{e_1}, \ldots, \underline{e_n} \notin \underline{A}$ (by taking any simple path from z to any point in $\underline{A}$ and stopping it just before it enters $\underline{A}$ for the first time). Then, by the Markov property and using independence between the vertices in the environment:

$$\begin{aligned} E_z[T_V] &\geq E_z[T_V, X_i = \overline{e_i} \text{ for } i = 1, \ldots, n] \\ &= \mathbb{E}[\omega_{e_1} \cdots \omega_{e_n} E_{y,\omega}[T_V + n]] \\ &\geq \mathbb{E}[\omega_{e_1} \cdots \omega_{e_n} E_{y,\omega}[T_A + n] \\ &= \mathbb{E}[\omega_{e_1}] \cdots \mathbb{E}[\omega_{e_n}](E_y[T_A] + n) \end{aligned}$$

hence $E_z[T_V] = +\infty$ because the first factors are positive and the last one is infinite *via* the first part of the proof. This concludes the proof of Proposition 11. □

4 Proof of the ballisticity criterion

We now consider random walks in i.i.d. Dirichlet environment on $\mathbb{Z}^d$, $d \geq 1$. Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{Z}^d$, and $\mathcal{V} = \left\{ e \in \mathbb{Z}^d \,\middle|\, |e| = 1 \right\}$. Let $(\alpha_e)_{e \in \mathcal{V}}$ be positive numbers. We will write either α_i or α_{e_i}, and α_{-i} or α_{-e_i}, $i = 1, \ldots, d$. Let us recall the statement of Theorem 6:

THEOREM – If $\sum_{i=1}^{d} |\alpha_i - \alpha_{-i}| > 1$, then there exists $v \neq 0$ such that, P_0-a.s., $\frac{X_n}{n} \to_n v$, and the following bound holds:

$$\left| v - \frac{\Sigma}{\Sigma - 1} d_m \right|_1 \leq \frac{1}{\Sigma - 1},$$

where $\Sigma = \sum_{e \in \mathcal{V}} \alpha_e$ and $d_m = \sum_{i=1}^{d} \frac{\alpha_i - \alpha_{-i}}{\Sigma} e_i$ is the drift under the averaged environment.

PROOF. This proof relies on properties and techniques of [EnSa06]. Our improvement is twofold: first, thanks to the previous sections, we are able to define the Kalikow random walk under weaker conditions, namely those of the statement; second, we get a finer bound on the drift of this random walk.

Let us recall a definition. Given a finite subset U of $\mathbb{Z}^d$ and a point $z_0 \in U$ such that $\mathbb{E}[G_U^\omega(z_0, z_0)] < \infty$, the **Kalikow auxiliary random walk** related to U and z_0 is the Markov chain on $U \cup \partial_V U$ (where $\partial_V U$ is the set of the vertices neighbouring U) given by the following transition probabilities:

$$\text{for all } z \in U \text{ and } e \in \mathcal{V},\ \widehat{\omega}_{U,z_0}(z, z+e) = \frac{\mathbb{E}[G_U^\omega(z_0, z)\omega(z, z+e)]}{\mathbb{E}[G_U^\omega(z_0, z)]}$$

and $\widehat{\omega}_{U,z_0}(z,z) = 1$ if $z \in \partial_V U$. For the sake of making formal computations rigorous, Enriquez and Sabot first consider the **generalized Kalikow random walk**. Given an additional parameter $\delta \in (0,1)$, the new transition probabilities $\widehat{\omega}_{U,z_0,\delta}(z, z+e)$ are defined like the previous ones except that, in place of $G_U^{\omega}(z_0, z)$, we use the Green function of the random walk under the environment ω killed at rate δ and at the boundary of U:

$$G_{U,\delta}^{\omega}(z_0, z) = E_{z_0,\omega}\left[\sum_{k=0}^{T_U} \delta^k \mathbf{1}_{(X_k = z)}\right]$$

(and we don't need any assumption on $\mathbb{P}$ anymore).

The following identity (equation (2) of [EnSa06]) was a consequence of an integration by part formula: for all finite $U \subset \mathbb{Z}^d$, $z \in U$, $e \in \mathscr{V}$, $\delta \in (0,1)$,

$$\widehat{\omega}_{U,z_0,\delta}(z, z+e) = \frac{1}{\Sigma - 1}\left(\alpha_e - \frac{\mathbb{E}[G_{U,\delta}^{\omega}(z_0, z) p_{\omega,\delta}(z, z+e)]}{\mathbb{E}[G_{U,\delta}^{\omega}(z_0, z)]]}\right)$$

where $p_{\omega,\delta}(z, z+e)) = \omega(z, z+e)(G_{U,\delta}^{\omega}(z,z) - \delta G_{U,\delta}^{\omega}(z+e, z))$. The Markov property for the killed random walk shows that, for all z, the components of $(p_{\omega,\delta}(z, z+e))_{e\in\mathscr{V}}$ are positive and sum up to 1: this is a probability measure. Besides, after a short computation, it can be rewritten as:

$$p_{\omega,\delta}(z, z+e) = P_{z,\omega}(X_1 = z+e | H_\partial < \widetilde{H}_z),$$

which highlights its probabilistic interpretation. This remark allows us to refine the estimates of [EnSa06]. The drift of the generalized Kalikow random walk at z is:

$$\widehat{d}_{U,z_0,\delta}(z) = \frac{1}{\Sigma - 1}\left(\sum_{i=1}^{d}(\alpha_i - \alpha_{-i})e_i - \widetilde{d}\right) = \frac{1}{\Sigma - 1}(\Sigma d_m - \widetilde{d}), \tag{8}$$

where $\widetilde{d}$ (depending on all parameters) is the expected value of the following probability measure:

$$\frac{\mathbb{E}[G_{U,\delta}^{\omega}(z_0, z) p_{\omega,\delta}(z, z+\cdot)]}{\mathbb{E}[G_{U,\delta}^{\omega}(z_0, z)]}.$$

This measure is supported by $\mathscr{V}$, hence $\widetilde{d}$ belongs to the convex hull of $\mathscr{V}$, which is the closed $|\cdot|_1$-unit ball $\mathrm{B}_{|\cdot|_1}$:

$$|\widetilde{d}|_1 \le 1.$$

On the other hand, the assumption gives $\Sigma d_m \notin \mathrm{B}_{|\cdot|_1}$, and the convexity of $\mathrm{B}_{|\cdot|_1}$ provides $l \in \mathbb{R}_d \setminus \{0\}$ and $c > 0$ (depending only on the parameters $(\alpha_e)_{e\in\mathscr{V}}$) such that, for all $X \in \mathrm{B}_{|\cdot|_1}$,

$$\Sigma d_m \cdot l > c > X \cdot l.$$

Therefore, noting that our assumption implies $\Sigma > 1$, we have, for every finite subset U of $\mathbb{Z}^d$, every $z_0, z \in U$ and $\delta \in (0,1)$:

$$\widehat{d}_{U,z_0,\delta}(z) \cdot l = \frac{1}{\Sigma - 1}(\Sigma d_m \cdot l - \widetilde{d} \cdot l) \ge \frac{\Sigma d_m \cdot l - c}{\Sigma - 1} > 0.$$

It is time to remark that Theorem 3 applies under our condition: the hypothesis implies $\Sigma > 1$ so that, for all i, $2\Sigma - \alpha_i - \alpha_{-i} > 1$. This guarantees the integrability of $G_U^\omega(z_0, z)$ and allows us to make δ converge to 1 in the last inequality (monotone convergence theorem applies because $G_{U,\delta}^\omega$ increases to G_U^ω as δ increases to 1). We get a uniform lower bound concerning the drift of Kalikow random walk:

$$\widehat{d}_{U,z_0}(z) \cdot l \geq \frac{\Sigma d_m \cdot l - c}{\Sigma - 1} > 0.$$

In other words, Kalikow's criterion is satisfied for finite subsets U of $\mathbb{Z}^d$. As underlined in [EnSa06], this is sufficient to apply Sznitman and Zerner's law of large numbers ([SzZe99]), hence there is a deterministic $v \neq 0$ such that, $\mathbb{P}$-almost surely,

$$\frac{X_n}{n} \underset{n}{\longrightarrow} v.$$

Finally, identity (8) gives:

$$\left| \widehat{d}_{U,z_0,\delta}(z) - \frac{\Sigma}{\Sigma - 1} d_m \right|_1 = \frac{1}{\Sigma - 1} |\widetilde{d}|_1 \leq \frac{1}{\Sigma - 1},$$

from which the stated bound on v results using Proposition 3.2 of [Sa04]: v is an accumulation point of the convex hull of $\left\{ \widehat{d}_{U,z_0,\delta}(z) \,\middle|\, U \text{ finite}, z_0, z \in U \right\}$ when δ tends to 1, and any point of these convex hulls lies in the desired (closed, convex) ℓ^1-ball. □

5 Concluding remarks and computer simulations

In the case of $\mathbb{Z}^d$, we have provided a criterion for non-zero limiting velocity. One may prove the following criterion as well, thanks to Theorem 3 (for a proof, please refer to the Appendix):

Proposition 12. – If there exists $i \in \{1, \ldots, d\}$ such that $\alpha_i + \alpha_{-i} \geq 2\Sigma - 1$, then:

$$P_0\text{-a.s.}, \ \frac{X_n}{n} \underset{n}{\longrightarrow} 0.$$

The question remains open whether one of these criterions is sharp. Actually, computer simulations let us think that neither is. We were especially able to find parameters such that exit times of all finite subsets are integrable and the random walk has seemingly zero speed (more precisely, X_n looks to be on the order of n^κ for some $0 < \kappa < 1$). Figure 1 shows some results obtained with $(\alpha_1, \alpha_{-1}, \alpha_2, \alpha_{-2}) = (0.5, 0.2, 0.1, 0.1)$. We performed 10^3 numerical simulations of trajectories of random walks up to time $n_{\max} = 10^6$ and compared the averaged values of $y_n = X_n \cdot e_1$ with $C_\alpha n^\alpha$, where C_α is chosen so as to make curves coincide at $n = n_{\max}$. The first graph shows the average of y_n and the second one the maximum over $n \in \{10^5 + 1, \ldots, 10^6\}$ of the relative error $\left|1 - \frac{y_n}{C_\alpha n^\alpha}\right|$, as α varies. The minimizing α is 0.9, corresponding to a uniform relative error of .0044. However we could not yet prove that such an intermediary regime happens.

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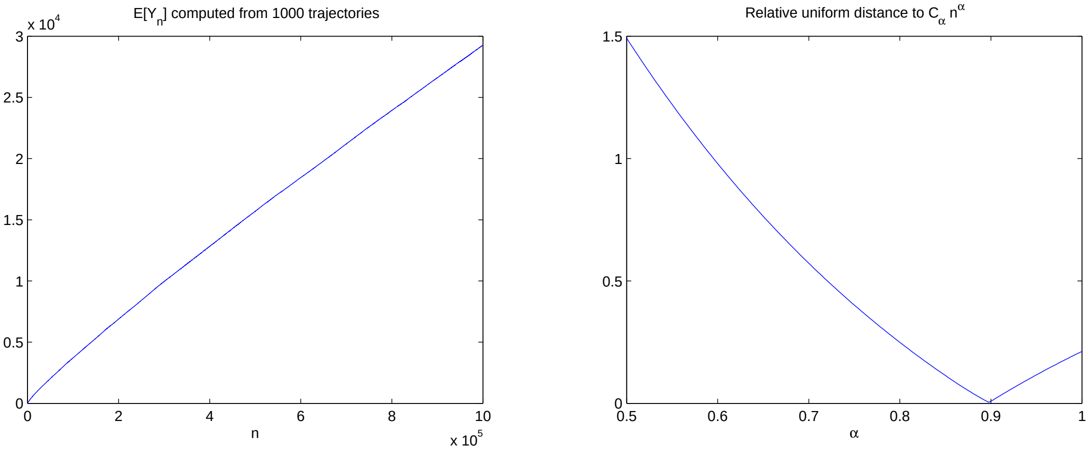

Figure 1: These plots refer to computer simulation: averages are taken over 10^3 trajectories up to time 10^6 (see section 5)

Appendix

Proof of Lemma 9

For readibility and tractability reasons, we will prove this lemma in the case of only two Dirichlet random variables $(p_1, \ldots, p_{k+1})$ and $(p'_1, \ldots, p'_{l+1})$. The proof of the general case would go exactly along the same lines. The associativity property allows as well to reduce to the case when we normalize all components but one: $m_1 = k$ and $m_2 = l$ (by replacing the other marginals by their sum).

We set $\gamma = \alpha_{k+1}$ and $\gamma' = \alpha'_{l+1}$. Writing the index $(\cdot)_i$ instead of $(\cdot)_{1\le i\le k}$ and the same way with j and l, the left-hand side of the statement in this simplified setting equals:

$$\int_{\{\sum_i x_i \le 1,\ \sum_j y_j \le 1\}} f\left(\sum_i x_i + \sum_j y_j, \left(\frac{x_i}{\sum_i x_i + \sum_j y_j}\right)_i, \left(\frac{y_j}{\sum_i x_i + \sum_j y_j}\right)_j\right) \phi((x_i)_i, (y_j)_j) \prod_i dx_i \prod_j dy_j,$$

where for some positive c_0, $\phi((x_i)_i, (y_j)_j) = c_0 \left(\prod_i x_i^{\alpha_i - 1}\right) (1 - \sum_i x_i)^\gamma \left(\prod_j y_j^{\alpha'_j - 1}\right) (1 - \sum_j y_j)^{\gamma'}$. We successively procede to the following changes of variable: $x_1 \mapsto u = \sum_i x_i + \sum_j y_j$, then $x_i \mapsto \widetilde{x}_i = \frac{x_i}{u}$ for every $i \neq 1$, and $y_j \mapsto \widetilde{y}_j = \frac{y_j}{u}$ for every j. The previous integral becomes:

$$\int_{\left\{\begin{array}{c}\sum_{i\neq 1} \widetilde{x}_i + \sum_j \widetilde{y}_j \le 1, \\ 1 - \frac{1}{u} \le \sum_j \widetilde{y}_j \le \frac{1}{u}\end{array}\right\}} f\left(u, 1 - \sum_{i\neq 1} \widetilde{x}_i + \sum_j \widetilde{y}_j, (\widetilde{x}_i)_{i\neq 1}, (\widetilde{y}_j)\right) \psi(u, (\widetilde{x}_i)_{i\neq 1}, (\widetilde{y}_j)_j)\, du \prod_{i\neq 1} d\widetilde{x}_i \prod_j d\widetilde{y}_j,$$

Bounding from above by 1 the last two factors of ψ where u appears, we get that the last quantity is less than:

$$\int_{\left\{\begin{array}{c}\sum_{i\neq 1}\widetilde{x}_i+\sum_j \widetilde{y}_j\leq 1,\\ u\leq 2\end{array}\right\}} f\left(u,1-\sum_{i\neq 1}\widetilde{x}_i+\sum_j \widetilde{y}_j,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)\right)\theta(u,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)_j)\,du\prod_{i\neq 1}d\widetilde{x}_i\prod_j d\widetilde{y}_j,$$

where $\theta(u,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)_j)=c_0\left(1-\sum_{i\neq 1}\widetilde{x}_i-\sum_j \widetilde{y}_j\right)^{\alpha_1-1}\prod_{i\neq 1}\widetilde{x}_i^{\alpha_i-1}\prod_j \widetilde{y}_j^{\alpha'_j-1}$.

This rewrites, for some positive c_1, as: $c_1\widetilde{E}\left[f(\widetilde{\Sigma},\widetilde{p}_1,\ldots,\widetilde{p}_k,\widetilde{p'}_1,\ldots,\widetilde{p'}_l)\right]$, with the notations of the statement. We have here $\widetilde{P}(\widetilde{\Sigma}<\varepsilon)=c\int_0^\varepsilon u^{\sum_i\alpha_i+\sum_j\alpha'_j-1}du=c'\varepsilon^{\sum_i\alpha_i+\sum_j\alpha'_j}$.

Proof of Proposition 12

Without loss of generality, we may assume $\alpha_1+\alpha_{-1}\geq 2\Sigma-1$. As a consequence of Theorem 3, the time spent by the random walk inside the edge $(0,e_1)$ is non-integrable under P_0.

We prove that, for every $e\in\mathcal{V}$, P_0-a.s., $\frac{T_k^e}{k}\to_k+\infty$, where :

$$T_k^e=\inf\{n\geq 0|X_n\cdot e\geq 2k\}.$$

This implies that P_0-a.s., $\limsup_n\frac{X_n\cdot e}{n}\leq 0$ for all $e\in\mathcal{V}$, and thus the proposition. Let $e\in\mathcal{V}$. We introduce the exit times :

$$\tau_0=\inf\{n\geq 0|X_n\notin\{X_0,X_0+e_1\}\}$$

(with a minus sign instead of the plus if $e=-e_1$) and, for $k\geq 1$,

$$\tau_k=\tau_0\circ\Theta_{T_k^e},$$

with the convention that $\tau_k=\infty$ if $T_k^e=\infty$. The only dependence between the times τ_k, $k\in\mathbb{N}$, comes from the fact that $\tau_k=\infty$ implies $\tau_l=\infty$ for all $l\geq k$. The "2" in the definition of T_k^e causes indeed the τ_k's to depend on disjoint parts of the environment, namely slabs $\{x\in\mathbb{Z}^d|x\cdot e\in\{2k,2k+1\}\}$. For $t_0,\ldots,t_k\in\mathbb{N}$, one has, using the Markov property at time T_k^e, the independence and the translation invariance of $\mathbb{P}$:

$$\begin{aligned}P_0(\tau_0=t_0,\ldots,\tau_k=t_k) &= P_0(\tau_0=t_0,\ldots,\tau_{k-1}=t_{k-1},\tau_k=t_k,T_k^e<\infty)\\ &\leq P_0(\tau_0=t_0,\ldots,\tau_{k-1}=t_{k-1})P_0(\tau_0=t_k)\\ &\leq \cdots\leq P_0(\tau_0=t_0)\cdots P_0(\tau_0=t_{k-1})P_0(\tau_0=t_k)\\ &= P(\widehat{\tau}_0=t_0,\ldots,\widehat{\tau}_k=t_k),\end{aligned}$$

where, under P, the random variables $\widehat{\tau}_k$, $k\in\mathbb{N}$, are independent and have the same distribution as τ_0 (and hence are finite P-a.s.). From this we deduce that, for all $A\subset\mathbb{N}^{\mathbb{N}}$, $P_0((\tau_k)_k\in A)\leq P((\widehat{\tau}_k)_k\in A)$. In particular,

$$P_0\left(\liminf_k\frac{\tau_0+\cdots+\tau_{k-1}}{k}<\infty\right)\leq P\left(\liminf_k\frac{\widehat{\tau}_0+\cdots+\widehat{\tau}_{k-1}}{k}<\infty\right)=0,$$

where the equality results from the law of large number for i.i.d. random variables (recall $E[\widehat{\tau}_k]=E_0[\tau_0]=\infty$). Finally, $T_k^e\geq\tau_0+\cdots+\tau_{k-1}$, so that $\liminf_k\frac{T_k^e}{k}=\infty$ P_0-a.s., as wanted.

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