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THE NORM ESTIMATE OF THE DIFFERENCE BETWEEN THE KAC OPERATOR AND SCHRÖDINGER SEMIGROUP II: THE GENERAL CASE INCLUDING THE RELATIVISTIC CASE

Takashi Ichinose[1]
Department of Mathematics, Kanazawa University, Kanazawa, 920-1192, Japan
ichinose@kappa.s.kanazawa-u.ac.jp

Satoshi Takanobu[2]
Department of Mathematics, Kanazawa University, Kanazawa, 920-1192, Japan
takanobu@kappa.s.kanazawa-u.ac.jp

Abstract More thorough results than in our previous paper in Nagoya Math. J. are given on the L_p-operator norm estimates for the Kac operator $e^{-tV/2}e^{-tH_0}e^{-tV/2}$ compared with the Schrödinger semigroup $e^{-t(H_0+V)}$. The Schrödinger operators H_0+V to be treated in this paper are more general ones associated with the Lévy process, including the relativistic Schrödinger operator. The method of proof is probabilistic based on the Feynman-Kac formula. It differs from our previous work in the point of using *the Feynman-Kac formula* not directly for these operators, but instead through *subordination* from the Brownian motion, which enables us to deal with all these operators in a unified way. As an application of such estimates the Trotter product formula in the L_p-operator norm, with error bounds, for these Schrödinger semigroups is also derived.

Keywords Schrödinger operator, Schrödinger semigroup, relativistic Schrödinger operator, Trotter product formula, Lie-Trotter-Kato product formula, Feynman-Kac formula, subordination of Brownian motion, Kato's inequality

AMS subject classification 47D07, 35J10, 47F05, 60J65, 60J35

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1. Introduction

By the *Kac operator* we mean an operator of the kind $K(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2}$, where $H = H_0 + V \equiv -\Delta/2 + V(x)$ is the nonrelativistic Schrödinger operator in $L_2(\mathbb{R}^d)$ with mass 1 with scalar potential $V(x)$ bounded from below. This $K(t)$ may correspond to the transfer operator for a lattice model in statistical mechanics studied by M. Kac [Ka]. There it is one of the important problems to know asymptotic spectral properties of $K(t)$ for $t \downarrow 0$. To this end, in [H1, H2] Helffer estimated the L_2-operator norm of the difference between $K(t)$ and the Schrödinger semigroup e^{-tH} to be of order $O(t^2)$ for small $t > 0$, if $V(x)$ satisfies $|\partial^\alpha V(x)| \le C_\alpha(1 + |x|^2)^{(2-|\alpha|)_+/2}$ for every multi-index α with a constant C_α. Then such norm estimates may be applied to get spectral properties of $K(t)$ in comparison with those of H.

In [I-Tak1] and [I-Tak2] we have extended his result to the case of more general scalar potentials $V(x)$ even in the L_p-operator norm, $1 \le p \le \infty$, making a probabilistic approach based on the Feynman-Kac formula. In [I-Tak2] we have also considered this problem for both the nonrelativistic Schrödinger operator $H = H_0 + V$ and the relativistic Schrödinger operator $H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V(x)$ with light velocity 1. The L_p-operator norm of this difference is estimated to be of order $O(t^a)$ of small $t > 0$ with $a \ge 1$, though the relativistic case shows for small $t > 0$ a slightly different behavior from the nonrelativistic case. As another application of these results the Trotter product formula for the nonrelativistic and relativistic Schrödinger operators in the L_p-operator norm with error bounds is obtained. There are also related L_2 results with operator-theoretic methods, for which we refer to [D-I-Tam].

The aim of this paper is to generalize and refine the result of [I-Tak2] in the relativistic case, admitting of *more general operators* than the free relativistic Schrödinger operator $H_0^r = \sqrt{-\Delta + 1} - 1$ as well as relaxing the conditions for the potentials $V(x)$. We use the probabilistic method with Feynman-Kac formula, though observing everything in a unified way through *subordination* from the Brownian motion. In this respect the present method differs from that in [I-Tak2] used for the relativistic Schrödinger operator H^r, which made the best of the explicit expression of the integral kernel of $e^{-tH_0^r}$.

The *more general operator* we have in mind is the following operator

$$H_0^\psi \;=\; \psi(\tfrac{1}{2}(-\Delta + 1)) - \psi(\tfrac{1}{2}), \tag{1.1}$$

which will play the same role as the relativistic Schrödinger operator

$$H_0^r \;=\; \sqrt{-\Delta + 1} - 1 \tag{1.2}$$

in [I-Tak2]. Obviously, H_0^ψ is a selfadjoint operator in $L_2(\mathbb{R}^d)$. Here $\psi(\lambda)$ is a continuous increasing function on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(\infty) = \infty$ expressed as

$$\psi(\lambda) \;=\; \int_{(0,\infty)} (1 - e^{-\lambda l}) n(dl), \quad \lambda \ge 0, \tag{1.3}$$

where $n(dl)$ is a Lévy measure on $(0, \infty)$ (i.e. a measure on $(0, \infty)$ such that $\int_{(0,\infty)} l \wedge 1 n(dl) < \infty$) with $n((0, \infty)) = \infty$. It is clear that

$$\psi(\lambda + \tfrac{1}{2}) - \psi(\tfrac{1}{2}) \;=\; \int_{(0,\infty)} (1 - e^{-\lambda l}) e^{-l/2} n(dl). \tag{1.4}$$

As a special case of H_0^ψ we have for $\psi(\lambda) = (2\lambda)^\alpha$, $0 < \alpha < 1$, the operator

$$H_0^{(\alpha)} \;=\; (-\Delta + 1)^\alpha - 1, \tag{1.5}$$

which reduces to the relativistic Schrödinger operator when $\alpha = 1/2$: $H_0^{(1/2)} = H_0^r$. In this case the Lévy measure is $n(dl) = \{2^\alpha \alpha/\Gamma(1-\alpha)\} l^{-1-\alpha} dl$.

To formulate our results we are going to describe what kind of function $V(x)$ is. Let $0 < \gamma, \delta \le 1$, $0 \le \kappa \le 1$, $0 \le \mu, \nu, \rho < \infty$, $0 \le C_1, C_2, c_1, c_2 < \infty$ and $0 < c < \infty$. Let $V : \mathbb{R}^d \to [0, \infty)$ be a continuous function satisfying one of the following five conditions:

$(\mathrm{A})_0$ $\quad |V(x) - V(y)| \;\le\; C_1|x-y|^\gamma$;

$(\mathrm{A})_1$ $\quad V$ is a C^1-function such that

(i) $|\nabla V(z)| \;\le\; C_1(1 + V(z)^{1-\delta})$, $\quad$ (ii) $|\nabla V(x) - \nabla V(y)| \;\le\; C_2|x-y|^\kappa$;

$(\mathrm{A})_2$ $\quad V$ is a C^1-function such that

(i) $|\nabla V(z)| \;\le\; C_1(1 + V(z)^{1-\delta})$,

(ii) $|\nabla V(x) - \nabla V(y)|$

$$\le\; C_2\Big\{V(x)^{(1-2\delta)_+}(1 + |x-y|^\mu) + 1 + |x-y|^\nu\Big\}|x-y|;$$

$(\mathrm{V})_1$ $\quad V$ is a C^1-function such that

(i) $V(z) \;\ge\; c\langle z\rangle^\rho$, $\quad$ (ii) $|\nabla V(z)| \;\le\; c_1\langle z\rangle^{(\rho-1)_+}$;

$(\mathrm{V})_2$ $\quad V$ is a C^2-function such that

(i) $V(z) \;\ge\; c\langle z\rangle^\rho$, $\quad$ (ii) $|\nabla V(z)| \;\le\; c_1\langle z\rangle^{(\rho-1)_+}$,

(iii) $|\nabla^2 V(z)| \;\le\; c_2\langle z\rangle^{(\rho-2)_+}$.

Here $\langle z\rangle := \sqrt{1+|z|^2}$.

Conditions $(\mathrm{A})_0$, $(\mathrm{A})_1$ and $(\mathrm{A})_2$ on $V(x)$ are used in [Tak] and are more general than in [I-Tak1,2], while conditions $(\mathrm{V})_1$ and $(\mathrm{V})_2$ are used in [D-I-Tam]. But these conditions may not be best possible. A simple example of a function which has property $(\mathrm{A})_0$, $(\mathrm{A})_1$ or $(\mathrm{A})_2$ is, needless to say, $V(x) = |x|^r$ $(0 < r < \infty)$, and a slightly complicated one $V(x) = |x|^r(2 + \sin\log|x|)$, according as $0 < r \le 1$, $1 < r < 2$ or $r \ge 2$. Also $V(x) = 1 + |x_1 - x_2|^r$ $(x = (x_1, x_2, \ldots, x_d))$ satisfies $(\mathrm{A})_0$, $(\mathrm{A})_1$ or $(\mathrm{A})_2$ with the same r as above, but neither $(\mathrm{V})_1$ nor $(\mathrm{V})_2$. To the contrary $V(x) = 1 + |x|\int_0^{|x|}(1 + \sin(\theta^2))d\theta$ satisfies $(\mathrm{V})_1$, but neither $(\mathrm{V})_2$, $(\mathrm{A})_0$, $(\mathrm{A})_1$ nor $(\mathrm{A})_2$.

The operator $H_0^\psi + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, and so its unique selfadjoint extension is also denoted by the same $H_0^\psi + V$. The semigroup $e^{-t(H_0^\psi + V)}$ on $L_2(\mathbb{R}^d)$ is extended to a strongly continuous semigroup on $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$ and $C_\infty(\mathbb{R}^d)$, to be denoted by the same $e^{-t(H_0^\psi + V)}$. Here $C_\infty(\mathbb{R}^d)$ is the Banach space of the continuous functions on $\mathbb{R}^d$ vanishing at infinity. To be complete, these and further facts are proved in Appendix.

As for the Lévy measure $n(dl)$ introduced in (1.3) and (1.4), we make the following assumption:

(L) For some $\alpha \in [0,1]$, $n((\cdot,\infty))$ is regularly varying at zero with exponent $-\alpha$, i.e., there exists a slowly varying function $L(\lambda)$ at infinity such that

$$n((t,\infty)) \sim t^{-\alpha} L(\tfrac{1}{t}) \qquad \text{as } t \downarrow 0. \tag{1.6}$$

Here a positive function $L(\cdot)$ is called *slowly varying at infinity* if for any $c > 0$,

$$\lim_{\lambda \uparrow \infty} \frac{L(c\lambda)}{L(\lambda)} = 1.$$

Let $\phi^{-1}(\cdot)$ be the inverse function of $\phi(\lambda) := \psi(\lambda + 1/2) - \psi(1/2)$. (Note that ϕ is strictly increasing.) Under the above assumption, set

$$L_1(\lambda) := \begin{cases} \Gamma(1-\alpha)L(\lambda) & \text{if} \quad 0 \le \alpha < 1 \\ \int_0^{1/\lambda} n((s,\infty))ds & \text{if} \quad \alpha = 1, \end{cases}$$

$$L_2(x) := L_1(\phi^{-1}(x))^{-1/\alpha} \quad \text{if} \quad 0 < \alpha \le 1.$$

These two functions are slowly varying at infinity, and we have $\phi(\lambda) \sim \lambda^\alpha L_1(\lambda)$ as $\lambda \to \infty$ and $\phi^{-1}(x) \sim x^{1/\alpha} L_2(x)$ as $x \to \infty$, as will be seen from Fact in Section 6, so that $\int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} d\theta$ $(0 < \alpha < 1)$ is also slowly varying at infinity.

Now we state the main results of this paper, which generalize the results in [I-Tak2]. In the following $\| \cdot \|_{p \to p}$ stands for the L_p-operator norm for $1 \le p < \infty$ and the supremum norm on $C_\infty(\mathbb{R}^d)$ for $p = \infty$.

Theorem 1. *Suppose assumption* (L) *and let* $1 \le p \le \infty$. *Then the following estimates* (i), (ii) *and* (iii) *hold for small* $t > 0$.

(i) *Under* $(\mathrm{A})_0$,

$$\begin{aligned} &\|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p\to p}, \\ &\|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p\to p}, \\ &\|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p\to p} \\ &= \begin{cases} O(t^2) & \textit{if} \quad \alpha < \gamma/2 \\ O(t^2 \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \textit{if} \quad \alpha = \gamma/2 \\ O(t^{1+\gamma/2\alpha} L_2(\frac{1}{t})^{-\gamma/2}) & \textit{if} \quad \gamma/2 < \alpha. \end{cases} \end{aligned}$$

(ii) *Under* $(\mathrm{A})_1$,

$$\|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p\to p}$$

$$
= \begin{cases} O(t^{1+1\wedge 2\delta}) & \textit{if} \quad \alpha < (1+\kappa)/2 \;\; \textit{or} \;\; \kappa = 1 \\ O(t^{1+2\delta}) + O(t^2 \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \textit{if} \quad \alpha = (1+\kappa)/2 < 1 \\ O(t^{1+2\delta}) + O(t^{1+(1+\kappa)/2\alpha} L_2(\frac{1}{t})^{-(1+\kappa)/2}) & \textit{if} \quad (1+\kappa)/2 < \alpha, \end{cases}
$$

$$
\begin{aligned}
&\|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p \to p}, \\
&\|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p \to p} \\
&= \begin{cases} O(t^{1+\delta}) & \textit{if} \quad \alpha < 1/2 \\ O(t^{1+\delta} \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \textit{if} \quad \alpha = 1/2 \\ O(t^{\delta + 1/2\alpha} L_2(\frac{1}{t})^{-1/2}) & \textit{if} \quad 1/2 < \alpha. \end{cases}
\end{aligned}
$$

(iii) *Under* (A)$_2$,

$$
\|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p \to p} \;\; = \;\; O(t^{1+1\wedge 2\delta}),
$$

$$
\begin{aligned}
&\|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p \to p}, \\
&\|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p \to p} \\
&= \begin{cases} O(t^{1+\delta}) & \textit{if} \quad \alpha < 1/2 \\ O(t^{1+\delta} \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \textit{if} \quad \alpha = 1/2 \\ O(t^{\delta + 1/2\alpha} L_2(\frac{1}{t})^{-1/2}) & \textit{if} \quad 1/2 < \alpha. \end{cases}
\end{aligned}
$$

In fact, the first estimate in (iii) *holds independent of* (L).

A consequence of Theorem 1 is the following Trotter product formula in the L_p-operator norm with error bounds.

Theorem 2. *Suppose assumption* (L) *and let* $1 \le p \le \infty$. *Then the following estimates* (i), (ii), (iii) *and* (iv) *hold uniformly on each finite* t*-interval on* $[0, \infty)$.

(i) *Under* (A)$_0$,

$$
\begin{aligned}
&\|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi + V)}\|_{p \to p}, \\
&\|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi + V)}\|_{p \to p}, \\
&\|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi + V)}\|_{p \to p} \\
&= \begin{cases} O(n^{-1}) & \textit{if} \quad \alpha < \gamma/2 \\ O(n^{-1} \int_0^{n} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \textit{if} \quad \alpha = \gamma/2 \\ O(n^{-\gamma/2\alpha} L_2(n)^{-\gamma/2}) & \textit{if} \quad \gamma/2 < \alpha. \end{cases}
\end{aligned}
$$

(ii) *Under* (A)$_1$,

$$
\begin{aligned}
&\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&= \begin{cases} O(n^{-1\wedge 2\delta}) & \text{if} \quad \alpha < (1+\kappa)/2 \ \ \text{or} \ \ \kappa = 1 \\ O(n^{-2\delta}) + O(n^{-1}\int_0^n (\phi^{-1}(\theta))^{-\alpha}d\theta) & \text{if} \quad \alpha = (1+\kappa)/2 < 1 \\ O(n^{-2\delta}) + O(n^{-(1+\kappa)/2\alpha}L_2(n)^{-(1+\kappa)/2}) & \text{if} \quad (1+\kappa)/2 < \alpha. \end{cases}
\end{aligned}
$$

(iii) *Under* (A)$_2$,

$$
\begin{aligned}
&\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&= O(n^{-1\wedge 2\delta}).
\end{aligned}
$$

(iv) *Under* (V)$_i$ $(i = 1, 2)$,

$$
\begin{aligned}
&\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&= O(n^{-i/2\vee\rho}).
\end{aligned}
$$

In fact, the asymptotic estimates (iii) *and* (iv) *hold independent of* (L).

Notice here that though the estimates with small t, in Theorem 1, for $e^{-tV}e^{-tH_0^\psi}$ and $e^{-tH_0^\psi/2}e^{-tV}e^{-tH_0^\psi/2}$ are of worse order than that for $e^{-tV/2}e^{-tH_0^\psi}e^{-tV/2}$, one has, in Theorem 2, the same error bounds with large n for these three products.

Finally we give a comment on what kind of operators are to be covered by our $H_0^\psi+V$. To this end we briefly illustrate how our result reads on the Trotter product formula in the case $H_0^{(\alpha)}+V$ with $H_0^{(\alpha)} = (-\Delta+1)^\alpha - 1$, $0 < \alpha < 1$, in (1.5). In this case, we have $n((t,\infty)) = (2^\alpha/\Gamma(1-\alpha))\,t^{-\alpha}$, or $L_2(\cdot) \equiv 2^{-1}$, so that

$$
\int_0^x (\phi^{-1}(\theta))^{-\alpha}d\theta \sim 2^\alpha \log x \quad \text{as } x \to \infty.
$$

Therefore Theorem 2 says that for $1 \le p \le \infty$ and uniformly on each finite t-interval in $[0,\infty)$,

$$
\begin{aligned}
&\|(e^{-tV/2n}e^{-tH_0^{(\alpha)}/n}e^{-tV/2n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p\to p},\\
&\|(e^{-tV/n}e^{-tH_0^{(\alpha)}/n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p\to p},\\
&\|(e^{-tH_0^{(\alpha)}/2n}e^{-tV/n}e^{-tH_0^{(\alpha)}/2n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p\to p}
\end{aligned}
$$

$$= \begin{cases} O(n^{-1}) & \text{if} \quad \alpha < \gamma/2 \\ O(n^{-1}\log n) & \text{if} \quad \alpha = \gamma/2 \qquad \text{under (A)}_0, \\ O(n^{-\gamma/2\alpha}) & \text{if} \quad \gamma/2 < \alpha \end{cases}$$

$$= \begin{cases} O(n^{-1\wedge 2\delta}) & \text{if} \quad \alpha < (1+\kappa)/2 \\ O(n^{-1}\log n) & \text{if} \quad \alpha = (1+\kappa)/2 \text{ and } 1/2 \le \delta \le 1 \\ O(n^{-2\delta}) & \text{if} \quad \alpha = (1+\kappa)/2 \text{ and } 0 < \delta < 1/2 \\ O(n^{-2\delta\wedge(1+\kappa)/2\alpha}) & \text{if} \quad (1+\kappa)/2 < \alpha \end{cases} \qquad \text{under (A)}_1.$$

An important remark is the following. In the above example, the case $\alpha = 1$ is missing. This is equivalent to the nonrelativistic case $H_0 + V = -\Delta/2 + V(x)$, treated in [Tak] (cf. [I-Tak1,2]). However we may think that this case is also implicitly contained in our results, Theorems 1 and 2, for $\alpha = 1/2$. Indeed, by using $H_0^r(c) = \sqrt{-c^2\Delta + c^4} - c^2$ with light velocity c restored in place of H_0^r in (1.2), we can obtain the case $\alpha = 1/2$ so as to involve the parameter c (light velocity). Since, in the nonrelativistic limit $c \to \infty$, the relativistic Schrödinger semigroup $e^{-t(H_0^r(c)+V)}$ is strongly convergent to the nonrelativistic Schrödinger semigroup $e^{-t(H_0+V)}$ uniformly on each finite t-interval in $[0,\infty)$ (e.g. [I2]), we can reproduce the nonrelativistic result in [Tak] (cf. Remark following Theorem 2.3).

In Section 2, we state our results in more general form: we generalize Theorems 1 and 2 to Theorems 2.1 and 2.2 / 2.3 by introducing the subordinator σ_t, namely, a time-homogeneous Lévy process associated with the Lévy measure $e^{-l/2}n(dl)$. Moreover we state Theorem 2.4 on asymptotics of the moments of the process σ_t. Once we know these asymptotics, we can obtain Theorems 1 and 2 from Theorems 2.1 and 2.2 / 2.3. These four theorems are proved in Sections 3 – 6.

In Appendix, we give a full study of the semigroups $e^{-t(H_0^\psi+V)}, t \ge 0$, on $L_p(\mathbb{R}^d), 1 \le p < \infty$ and $C_\infty(\mathbb{R}^d)$ defined through the Feynman-Kac formula. We show they constitute a strongly continuous contraction semigroup there. It is also shown that its infinitesimal generator $\mathfrak{G}_p^{\psi,V}$ has $C_0^\infty(\mathbb{R}^d)$ as a core, by establishing Kato's inequality for the operator H_0^ψ. Some of these results seem to be new.

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2. General results

In this section we shall prove the theorems in a little more general setting based on probability theory. To describe it we introduce some notations and notions. For a continuous function $V : \mathbb{R}^d \to [0,\infty)$, set

$$K(t) := e^{-tV/2}e^{-tH_0^\psi}e^{-tV/2},$$

$$G(t) := e^{-tV}e^{-tH_0^\psi},$$

$$R(t) := e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2}$$

and

$$Q_K(t) := K(t) - e^{-t(H_0^\psi + V)},$$
$$Q_G(t) := G(t) - e^{-t(H_0^\psi + V)},$$
$$Q_R(t) := R(t) - e^{-t(H_0^\psi + V)}.$$

Suppose we are given the independent random objects $N(\cdot)$ and $B(\cdot)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

(i) $N(dsdl)$ is a Poisson random measure on $[0,\infty)\times(0,\infty)$ such that $\mathbb{E}[N(dsdl)] = dse^{-l/2}n(dl)$;

(ii) $(B(t))_{t\geq 0}$ is a d-dimensional Brownian motion starting at 0.

Set

$$\sigma_t := \int_0^{t+} \int_{(0,\infty)} l\, N(dsdl). \tag{2.1}$$

Then $(\sigma_t)_{t\geq 0}$ is a time-homogeneous Lévy process with increasing paths such that

$$\mathbb{E}[e^{-\lambda\sigma_t}] = e^{-t\left(\psi(\lambda + 1/2) - \psi(1/2)\right)} \tag{2.2}$$

(e.g. Note 1.7.1 in [It-MK]). Note that σ_t has moments of all order (cf. (6.1)), which is to be seen at the beginning of Section 6. We use a subordination of $B(\cdot)$ by a subordinator $\sigma_\cdot$, i.e., a process $(B(\sigma_t))_{t\geq 0}$ on $\mathbb{R}^d$. This is a Lévy process such that

$$\mathbb{E}[e^{\sqrt{-1}\langle p, B(\sigma_t)\rangle}] = e^{-t\left(\psi((|p|^2+1)/2) - \psi(1/2)\right)},$$

which corresponds to the semigroup $\{e^{-tH_0^\psi}\}_{t\geq 0}$ with generator H_0^ψ in (1.1).
We prove the following generalization of Theorems 1 and 2.

Theorem 2.1. *Let $1 \leq p \leq \infty$ and $t \geq 0$.*
(i) *Under* $(\mathrm{A})_0$,

$$\|Q_K(t)\|_{p\to p},\ \|Q_G(t)\|_{p\to p},\ \|Q_R(t)\|_{p\to p} \leq \mathrm{const}(\gamma, d)\, C_1 t\, \mathbb{E}[\sigma_t^{\gamma/2}].$$

(ii) *Under* $(\mathrm{A})_1$,

$$\|Q_K(t)\|_{p\to p} \leq \mathrm{const}(\delta,\kappa,d) \left[C_1^2(t^2 + t^{2\delta})\mathbb{E}[\sigma_t] + \sum_{j=1}^{2} (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}]\right],$$

$$\|Q_G(t)\|_{p\to p},\ \|Q_R(t)\|_{p\to p} \leq \mathrm{const}(\delta,\kappa,d) \sum_{j=1}^{2} \left\{C_1^j(t^j + t^{j\delta})\mathbb{E}[\sigma_t^{j/2}] + (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}]\right\}.$$

(iii) *Under* $(\mathrm{A})_2$,

$$\|Q_K(t)\|_{p\to p} \;\le\; \mathrm{const}(\delta,\mu,\nu,d)\,\Big[C_1^2(t^2+t^{2\delta})\mathbb{E}[\sigma_t] + \sum_{j=1}^{2}\Big\{(C_2t)^j\mathbb{E}[\sigma_t^j] \\ + (C_2t)^j\mathbb{E}[\sigma_t^{j(1+\,\nu/2)}] + (C_2t^{1\wedge 2\delta})^j\mathbb{E}[\sigma_t^j] + (C_2t^{1\wedge 2\delta})^j\mathbb{E}[\sigma_t^{j(1+\,\mu/2)}]\Big\}\Big],$$

$$\|Q_G(t)\|_{p\to p},\ \|Q_R(t)\|_{p\to p} \;\le\; \mathrm{const}(\delta,\mu,\nu,d)\sum_{j=1}^{2}\Big\{C_1^j(t^j+t^{j\delta})\mathbb{E}[\sigma_t^{j/2}] + (C_2t)^j\mathbb{E}[\sigma_t^j] \\ + (C_2t)^j\mathbb{E}[\sigma_t^{j(1+\,\nu/2)}] + (C_2t^{1\wedge 2\delta})^j\mathbb{E}[\sigma_t^j] \\ + (C_2t^{1\wedge 2\delta})^j\mathbb{E}[\sigma_t^{j(1+\,\mu/2)}]\Big\}.$$

Theorem 2.2. *Let* $1\le p\le\infty$, $t\ge 0$ *and* $n\in\mathbb{N}$.
(i) *Under* $(\mathrm{A})_0$,

$$\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}, \\ \|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}, \\ \|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p} \\ \le\; \mathrm{const}(\gamma,d)\,C_1t\,\mathbb{E}[\sigma_{t/n}^{\gamma/2}].$$

(ii) *Under* $(\mathrm{A})_1$,

$$\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p} \\ \le\; \mathrm{const}(\delta,\kappa,d)\,\Big[C_1^2\Big((\tfrac{t}{n})^2+(\tfrac{t}{n})^{2\delta}\Big)n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}(C_2\tfrac{t}{n})^jn\mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}]\Big],$$

$$\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}, \\ \|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p} \\ \le\; \mathrm{const}(\delta,\kappa,d)\,\Big[\tfrac{1}{n}\Big(C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_1t\mathbb{E}[\sigma_t^{(1+\kappa)/2}]\Big) \\ + C_1\Big(\tfrac{t}{n}+(\tfrac{t}{n})^\delta\Big)\mathbb{E}[\sigma_{t/n}^{1/2}] + C_1^2\Big((\tfrac{t}{n})^2+(\tfrac{t}{n})^{2\delta}\Big)n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}(C_2\tfrac{t}{n})^jn\mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}]\Big].$$

(iii) *Under* $(\mathrm{A})_2$,

$$\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p} \\ \le\; \mathrm{const}(\delta,\mu,\nu,d)\,\Big[C_1^2\Big((\tfrac{t}{n})^2+(\tfrac{t}{n})^{2\delta}\Big)n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}\Big\{(C_2\tfrac{t}{n})^jn\mathbb{E}[\sigma_{t/n}^j] \\ + (C_2\tfrac{t}{n})^jn\mathbb{E}[\sigma_{t/n}^{j(1+\,\nu/2)}] + (C_2(\tfrac{t}{n})^{1\wedge 2\delta})^jn\mathbb{E}[\sigma_{t/n}^j] + (C_2(\tfrac{t}{n})^{1\wedge 2\delta})^jn\mathbb{E}[\sigma_{t/n}^{j(1+\,\mu/2)}]\Big\}\Big],$$

$$
\begin{aligned}
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&\le \text{const}(\delta,\mu,\nu,d)\Big[\tfrac{1}{n}\Big(C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_2 t^{1\wedge 2\delta}(\mathbb{E}[\sigma_t]+\mathbb{E}[\sigma_t^{1+\mu/2}])\\
&\quad + C_2 t(\mathbb{E}[\sigma_t]+\mathbb{E}[\sigma_t^{1+\nu/2}])\Big) + C_1\Big(\tfrac{t}{n}+(\tfrac{t}{n})^\delta\Big)\mathbb{E}[\sigma_{t/n}^{1/2}] + C_1^2\Big((\tfrac{t}{n})^2+(\tfrac{t}{n})^{2\delta}\Big)n\mathbb{E}[\sigma_{t/n}]\\
&\quad + \sum_{j=1}^{2}\Big\{(C_2\tfrac{t}{n})^j n\mathbb{E}[\sigma_{t/n}^j] + (C_2\tfrac{t}{n})^j n\mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}] + (C_2(\tfrac{t}{n})^{1\wedge 2\delta})^j n\mathbb{E}[\sigma_{t/n}^j]\\
&\quad + (C_2(\tfrac{t}{n})^{1\wedge 2\delta})^j n\mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}]\Big\}\Big].
\end{aligned}
$$

Theorem 2.3. *Let $1 \le p \le \infty$ and $t \ge 0$.*
(i) *Under* $(\mathrm{V})_1$ *for* $n \ge 2^{2(2\vee\rho)}$,

$$
\begin{aligned}
&\|(e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&\le \text{const}(\rho,c,c_1,d)\, n^{-1/2\vee\rho}\Big[t^{2/(\rho\wedge 2)\vee 1\,-1} + (t^2+t^{2(1\wedge((\rho\wedge 2)\vee 1)/2\rho)})n\mathbb{E}[\sigma_{t/n}]\\
&\quad + \sum_{j=1}^{2}\Big((t^j+t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&\le \text{const}(\rho,c,c_1,d)\, n^{-1/2\vee\rho}\Big[t^{2/(\rho\wedge 2)\vee 1\,-1} + (t+t^{1\wedge((\rho\wedge 2)\vee 1)/2\rho})\mathbb{E}[\sigma_t^{1/2}]\\
&\quad + t^{2/2\vee\rho}\mathbb{E}[\sigma_t] + t(\mathbb{E}[\sigma_t]+\mathbb{E}[\sigma_t^{(2\vee\rho)/2}]) + (t^2+t^{2(1\wedge((\rho\wedge 2)\vee 1)/2\rho)})n\mathbb{E}[\sigma_{t/n}]\\
&\quad + \sum_{j=1}^{2}\Big\{(t^j+t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big\}\Big].
\end{aligned}
$$

(ii) *Under* $(\mathrm{V})_2$ *for* $n \ge 1$,

$$
\begin{aligned}
&\|e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&\le \text{const}(\rho,c,c_1,c_2,d)\, n^{-2/2\vee\rho}\Big[(t^2+t^{2/1\vee\rho})n\mathbb{E}[\sigma_{t/n}]\\
&\quad + \sum_{j=1}^{2}\Big((t^j+t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\|(e^{-tV/n}e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p},\\
&\|(e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p\to p}\\
&\le \text{const}(\rho,c,c_1,c_2,d)\Big[n^{-2/2\vee\rho}\Big((t+t^{1/1\vee\rho})\mathbb{E}[\sigma_t^{1/2}] + (t+t^{2/2\vee\rho})\mathbb{E}[\sigma_t] + t\mathbb{E}[\sigma_t^{(2\vee\rho)/2}]\\
&\quad + (t^2+t^{2/1\vee\rho})n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}\Big\{(t^j+t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big\}\Big)
\end{aligned}
$$

$$+ n^{-1/1\vee\rho}\mathbb{E}[\sigma_{t/n}^{1/2}](t + t^{1/1\vee\rho})\Big].$$

Remark. As noted at the end of Section 1, the *nonrelativistic* case for $H_0 + V = -\Delta/2 + V$, being equivalent to the case $\alpha = 1$ which Theorems 1 and 2 fail to cover, can be thought to be implicitly contained in the *relativistic* case, of the above three theorems, for the relativistic Schrödinger operator $H_0^r(c) \equiv \sqrt{-c^2\Delta + c^4} - c^2$ with the light velocity $c \geq 1$ restored. We have $H_0^\psi = H_0^r(c)$, where this $\psi(\lambda)$ is a c-dependent function (1.3) given by $\psi(\lambda) := \psi(\lambda; c) = \sqrt{2c^2\lambda + c^4 - c^2} - \sqrt{c^4 - c^2}$ associated with the c-dependent Lévy measure $e^{-l/2}n(dl; c) = (2\pi)^{-1/2}ce^{-c^2l/2}l^{-3/2}dl$. In this case, Theorem 2.1 and Theorems 2.2 / 2.3 hold with the corresponding c-dependent subordinator $\sigma_t(c)$, just as they stand, namely, only with $\mathbb{E}[\sigma_s^a]$ replaced by $\mathbb{E}[\sigma_s(c)^a]$ for each respective $s > 0$ and $a > 0$. Then the nonrelativistic case in question is obtained as the nonrelativistic limit $c \to \infty$ of this c-dependent relativistic case, turning out to be just Theorems 2.1 and 2.2 / 2.3 with $\mathbb{E}[\sigma_s^a]$ replaced by s^a. This is because one can show that, as $c \to \infty$, the relativistic Schrödinger semigroup $e^{-t(H_0^r(c)+V)}$ on the LHS converges strongly to the nonrelativistic Schrödinger semigroup $e^{-t(H_0+V)}$ uniformly on each finite t-interval in $[0, \infty)$ (cf. [I2]), and $\mathbb{E}[\sigma_t(c)^a]$ on the RHS tends to t^a. Then taking the most dominant contribution on the RHS for small t or large n reproduces the same nonrelativistic result as in [Tak].

Theorems 1 and 2 follow immediately from Theorems 2.1 and 2.2 / 2.3, if one knows the asymptotics for $t \downarrow 0$ of the moments of σ_t to investigate which of the terms on the RHS makes a dominant contribution for small t or large n. These asymptotics are given by the following theorem.

Theorem 2.4. *Suppose assumption* (L). *Let* $a > 0$.
(i) *If* $\alpha < a$ *or* $a \geq 1$, *then* $\int_{(0,\infty)} l^a e^{-l/2}n(dl) < \infty$ *and*

$$\mathbb{E}[\sigma_t^a] \sim t\int_{(0,\infty)} l^a e^{-l/2}n(dl) \qquad \textit{as } t \downarrow 0.$$

In fact, for $a \geq 1$ *this always holds independent of* (L).
(ii) *If* $\alpha = a$ *and* $a < 1$, *then*

$$\mathbb{E}[\sigma_t^a] \sim \frac{1}{\Gamma(1-\alpha)}\, t\int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha}d\theta \qquad \textit{as } t \downarrow 0.$$

(iii) *If* $0 < a < \alpha$, *then*

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma(1-\frac{a}{\alpha})}{\Gamma(1-a)}\, t^{a/\alpha}L_2(\tfrac{1}{t})^{-a} \qquad \textit{as } t \downarrow 0.$$

The proofs of Theorems 2.1, 2.2, 2.3 and 2.4 are given in Sections 3, 4, 5 and 6, respectively. To show Theorem 2.1, in fact, we prove estimates of the *integral kernels* of $Q_K(t)$, $Q_G(t)$ and $Q_R(t)$ by a finite positive linear combination of $t^c\mathbb{E}[|x-y|^a\sigma_t^b p(\sigma_t, x-y)]$, where $p(t, x-y)$ is the heat kernel (see (A.2)). Such estimates of the integral kernels of the three operators of difference in Theorems 2.2 / 2.3 also can be obtained (cf. [Tak]), but are omitted.

3. Proof of Theorem 2.1

It is easily seen (see (A.6)) that for $f \in C_0^\infty(\mathbb{R}^d)$

$$K(t)f(x) \;=\; \mathbb{E}\Big[\exp\Big(-\tfrac{t}{2}(V(x)+V(x+X_t))\Big)f(x+X_t)\Big], \tag{3.1}$$

$$G(t)f(x) \;=\; \mathbb{E}\Big[\exp\Big(-tV(x)\Big)f(x+X_t)\Big], \tag{3.2}$$

$$R(t)f(x) \;=\; \mathbb{E}\Big[\exp\Big(-tV(x+X_{t/2})\Big)f(x+X_t)\Big] \tag{3.3}$$

and generally

$$K(\tfrac{t}{n})^n f(x) \;=\; \mathbb{E}\Big[\exp\Big(-\tfrac{t}{2n}\sum_{k=1}^{n}(V(x+X_{(k-1)t/n})+V(x+X_{kt/n}))\Big)f(x+X_t)\Big], \tag{3.4}$$

$$G(\tfrac{t}{n})^n f(x) \;=\; \mathbb{E}\Big[\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(k-1)t/n})\Big)f(x+X_t)\Big], \tag{3.5}$$

$$R(\tfrac{t}{n})^n f(x) \;=\; \mathbb{E}\Big[\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(2k-1)t/2n})\Big)f(x+X_t)\Big]. \tag{3.6}$$

Further, for $f \in C_0^\infty(\mathbb{R}^d)$ we have (see (A.13))

$$Q_K(t)f(x) \;=\; \int_{\mathbb{R}^d} dy f(y)\mathbb{E}_\sigma\Big[\mathbb{E}_B[v_K(t,x,y;\sigma)]p(\sigma_t,x-y)\Big], \tag{3.7}$$

$$Q_G(t)f(x) \;=\; \int_{\mathbb{R}^d} dy f(y)\mathbb{E}_\sigma\Big[\mathbb{E}_B[v_G(t,x,y;\sigma)]p(\sigma_t,x-y)\Big], \tag{3.8}$$

$$Q_R(t)f(x) \;=\; \int_{\mathbb{R}^d} f(y)dy\mathbb{E}_\sigma\Big[\mathbb{E}_B[v_R(t,x,y;\sigma)]p(\sigma_t,x-y)\Big], \tag{3.9}$$

where $\mathbb{E}_\sigma$ and $\mathbb{E}_B$ are the expectations with respect to $\sigma.$ and $B.$, respectively,

$$v_K(t,x,y;\sigma) \;:=\; \exp\Big(-\tfrac{t}{2}(V(x)+V(y))\Big)-\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big), \tag{3.10}$$

$$v_G(t,x,y;\sigma) \;:=\; \exp\Big(-tV(x)\Big)-\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big), \tag{3.11}$$

$$v_R(t,x,y;\sigma) \;:=\; \exp\Big(-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\Big)-\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big), \tag{3.12}$$

and, for $\tau > 0$, $x, y \in \mathbb{R}^d$ and $0 \le \theta \le \tau$

$$\begin{aligned} B_{0,x}^{\tau,y}(\theta) &:= x+\tfrac{\theta}{\tau}(y-x)+B_0^\tau(\theta) \\ B_0^\tau(\theta) &:= B(\theta)-\tfrac{\theta}{\tau}B(\tau). \end{aligned} \tag{3.13}$$

Since

$$e^a-e^b \;=\; (a-b)e^b+(a-b)^2\int_0^1(1-\theta)e^{\theta a}e^{(1-\theta)b}d\theta, \quad a,b\in\mathbb{R},$$

we have

$$
\begin{aligned}
v_K(t,x,y;\sigma) &= w_K(t,x,y;\sigma)\exp\Big(-\tfrac{t}{2}(V(x)+V(y))\Big) \\
&\quad - w_K(t,x,y;\sigma)^2\int_0^1(1-\theta)\exp\Big(-\theta\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big) \\
&\qquad\qquad \times\exp\Big(-(1-\theta)\tfrac{t}{2}(V(x)+V(y))\Big)d\theta \\
&=: v_{K1}(t,x,y;\sigma)+v_{K2}(t,x,y;\sigma), \qquad (3.14)\\
v_G(t,x,y;\sigma) &= w_G(t,x,y;\sigma)\exp\Big(-tV(x)\Big) \\
&\quad - w_G(t,x,y;\sigma)^2\int_0^1(1-\theta)\exp\Big(-\theta\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big) \\
&\qquad\qquad \times\exp\Big(-(1-\theta)tV(x)\Big)d\theta \\
&=: v_{G1}(t,x,y;\sigma)+v_{G2}(t,x,y;\sigma), \qquad (3.15)\\
v_R(t,x,y;\sigma) &= w_R(t,x,y;\sigma)\exp\Big(-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\Big) \\
&\quad - w_R(t,x,y;\sigma)^2\int_0^1(1-\theta)\exp\Big(-\theta\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big) \\
&\qquad\qquad \times\exp\Big(-(1-\theta)tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\Big)d\theta \\
&=: v_{R1}(t,x,y;\sigma)+v_{R2}(t,x,y;\sigma), \qquad (3.16)
\end{aligned}
$$

where

$$w_K(t,x,y;\sigma) := -\tfrac{t}{2}(V(x)+V(y))+\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds, \qquad (3.17)$$

$$w_G(t,x,y;\sigma) := -tV(x)+\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds, \qquad (3.18)$$

$$w_R(t,x,y;\sigma) := -tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))+\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds. \qquad (3.19)$$

When V is further a C^1-function, since

$$V(z)-V(w) = \langle\nabla V(w),z-w\rangle+\int_0^1\langle\nabla V(w+\theta(z-w))-\nabla V(w),z-w\rangle d\theta,$$

we have

$$
\begin{aligned}
w_K(t,x,y;\sigma) &= \tfrac{1}{2}\langle\nabla V(x)-\nabla V(y),y-x\rangle\int_0^t\tfrac{\sigma_s}{\sigma_t}ds \\
&\quad + \tfrac{1}{2}\langle\nabla V(y),y-x\rangle\Big(\int_0^t\tfrac{\sigma_s}{\sigma_t}ds-\int_0^t\tfrac{\sigma_t-\sigma_s}{\sigma_t}ds\Big) \\
&\quad + \tfrac{1}{2}\Big\langle\nabla V(x)+\nabla V(y),\int_0^t B_0^{\sigma_t}(\sigma_s)ds\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\ \tfrac{1}{2}\int_0^t ds \int_0^1 \Big\langle \nabla V(x+\theta(\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s))) - \nabla V(x), \\
&\qquad \tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)\Big\rangle d\theta \\
&+\ \tfrac{1}{2}\int_0^t ds \int_0^1 \Big\langle \nabla V(y+\theta(\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y)+B_0^{\sigma_t}(\sigma_s))) - \nabla V(y), \\
&\qquad \tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y)+B_0^{\sigma_t}(\sigma_s)\Big\rangle d\theta \\
&=: \sum_{j=1}^5 w_{Kj}(t,x,y;\sigma),
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
w_G(t,x,y;\sigma) &= \Big\langle \nabla V(x), \int_0^t (\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s))ds\Big\rangle \\
&\quad + \int_0^t ds \int_0^1 \Big\langle \nabla V(x+\theta(\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s))) - \nabla V(x), \\
&\qquad \tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)\Big\rangle d\theta \\
&=: w_{G1}(t,x,y;\sigma)+w_{G2}(t,x,y;\sigma),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
w_R(t,x,y;\sigma) &= \Big\langle \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})), \int_0^t (B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))ds\Big\rangle \\
&\quad + \int_0^t ds \int_0^1 \Big\langle \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})+\theta(B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))) \\
&\qquad - \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})), B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\Big\rangle d\theta \\
&=: w_{R1}(t,x,y;\sigma)+w_{R2}(t,x,y;\sigma).
\end{aligned}
\tag{3.22}
$$

In the following we shall prove Theorem 2.1 only in Cases $(A)_2$ and $(A)_0$. The proof of Case $(A)_1$ is omitted; it is similar to that of $(A)_2$.

3.1. Case $(A)_2$

In this subsection, we suppose condition $(A)_2$ on $V(x)$.

Claim 3.1.

$$
\begin{aligned}
&\Big|\mathbb{E}_\sigma\Big[\mathbb{E}_B[v_{K1}(t,x,y;\sigma)]p(\sigma_t,x-y)\Big]\Big| \\
&\le \mathrm{const}(\delta,\mu,\nu,d)\,C_2\Big[t^{1\wedge 2\delta}\Big(\mathbb{E}_\sigma[|x-y|^2 p(\sigma_t,x-y)]+\mathbb{E}_\sigma[\sigma_t\, p(\sigma_t,x-y)] \\
&\quad +\mathbb{E}_\sigma[|x-y|^{2+\mu}p(\sigma_t,x-y)]+\mathbb{E}_\sigma[\sigma_t^{1+\mu/2}p(\sigma_t,x-y)]\Big) \\
&\quad + t\,\Big(\mathbb{E}_\sigma[|x-y|^2 p(\sigma_t,x-y)]+\mathbb{E}_\sigma[\sigma_t\, p(\sigma_t,x-y)]+\mathbb{E}_\sigma[|x-y|^{2+\nu}p(\sigma_t,x-y)] \\
&\quad +\mathbb{E}_\sigma[\sigma_t^{1+\nu/2}p(\sigma_t,x-y)]\Big)\Big].
\end{aligned}
$$

Proof. In view of (3.14) and (3.20), we set

$$v_{K1}(t,x,y;\sigma) = \sum_{j=1}^{5} w_{Kj}(t,x,y;\sigma)e^{-t\left(V(x)+V(y)\right)/2} =: \sum_{j=1}^{5} v_{K1j}(t,x,y;\sigma). \tag{3.23}$$

Clearly

$$\mathbb{E}_B[w_{K3}(t,x,y;\sigma)] = \tfrac{1}{2}\Big\langle \nabla V(x)+\nabla V(y), \int_0^t \mathbb{E}_B[B_0^{\sigma_t}(\sigma_s)]ds\Big\rangle = 0,$$

and hence $\mathbb{E}_B\Big[v_{K13}(t,x,y;\sigma)\Big] = 0$. By the fact $(\sigma_t-\sigma_{t-s})_{0\le s\le t} \overset{\mathcal{L}}{\sim} (\sigma_s)_{0\le s\le t}$,

$$\begin{aligned}
&\mathbb{E}_\sigma\Big[w_{K2}(t,x,y;\sigma)p(\sigma_t,x-y)\Big]\\
&= \tfrac{1}{2}\langle\nabla V(y), y-x\rangle\Big(\mathbb{E}_\sigma\Big[\int_0^t \tfrac{\sigma_s}{\sigma_t}ds\, p(\sigma_t,x-y)\Big] - \mathbb{E}_\sigma\Big[\int_0^t \tfrac{\sigma_t-\sigma_{t-s}}{\sigma_t-\sigma_{t-t}}ds\, p(\sigma_t-\sigma_{t-t},x-y)\Big]\Big)\\
&= 0,
\end{aligned}$$

and hence $\mathbb{E}_\sigma\Big[\mathbb{E}_B[v_{K12}(t,x,y;\sigma)]p(\sigma_t,x-y)\Big] = \mathbb{E}_\sigma\Big[v_{K12}(t,x,y;\sigma)p(\sigma_t,x-y)\Big] = 0$. By $(\mathrm{A})_2$(ii)

$$\begin{aligned}
&|v_{K11}(t,x,y;\sigma)| = |w_{K1}(t,x,y;\sigma)|e^{-t\left(V(x)+V(y)\right)/2}\\
&\le \tfrac{1}{2}|\nabla V(x)-\nabla V(y)||x-y|\,t\,e^{-t\left(V(x)+V(y)\right)/2}\\
&\le \tfrac{C_2}{2}\Big\{V(x)^{(1-2\delta)_+}(1+|x-y|^\mu)+1+|x-y|^\nu\Big\}|x-y|^2 t\,e^{-tV(x)/2}\\
&\le \tfrac{C_2}{2}\Big\{V(x)^{(1-2\delta)_+}e^{-tV(x)/2}\,t(|x-y|^2+|x-y|^{2+\mu})+t(|x-y|^2+|x-y|^{2+\nu})\Big\}\\
&\le \tfrac{C_2}{2}\Big\{(\tfrac{2(1-2\delta)_+}{e})^{(1-2\delta)_+}t^{1\wedge 2\delta}(|x-y|^2+|x-y|^{2+\mu})+t(|x-y|^2+|x-y|^{2+\nu})\Big\}.
\end{aligned} \tag{3.24}$$

Here (and hereafter) the following inequality has been (will be) used:

$$t^b e^{-t} \le (\tfrac{b}{e})^b, \quad t\ge 0,\ b\ge 0, \tag{3.25}$$

where for $b=0$ we understand $(0/e)^0 := 1$. By $(\mathrm{A})_2$(ii) and (3.25) again

$$\begin{aligned}
&|v_{K14}(t,x,y;\sigma)| = |w_{K4}(t,x,y;\sigma)|e^{-t\left(V(x)+V(y)\right)/2}\\
&\le \tfrac{1}{2}\int_0^t ds\int_0^1 |\nabla V(x+\theta(\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)))-\nabla V(x)|\\
&\qquad\qquad \times|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|d\theta\, e^{-tV(x)/2}\\
&\le \tfrac{C_2}{2}\int_0^t\Big\{V(x)^{(1-2\delta)_+}e^{-tV(x)/2}\Big(|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^2+|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^{2+\mu}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{2+\nu}\Big\} ds \\
&\le \tfrac{C_2}{2} \int_0^t \Big\{ (\tfrac{2(1-2\delta)_+}{e})^{(1-2\delta)_+} t^{-(1-2\delta)_+} \\
&\qquad \times \Big(|\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{2+\mu} \Big) \\
&\qquad + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{2+\nu}\Big\} ds. \qquad (3.26)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&|v_{K15}(t,x,y;\sigma)| \\
&\le \tfrac{C_2}{2} \int_0^t \Big\{ (\tfrac{2(1-2\delta)_+}{e})^{(1-2\delta)_+} t^{-(1-2\delta)_+} \\
&\qquad \times \Big(|\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{2+\mu} \Big) \\
&\qquad + |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{2+\nu}\Big\} ds. \qquad (3.27)
\end{aligned}
$$

Note that for $a > 0$ and $0 \le \theta \le \tau$ $(\tau > 0)$

$$
\begin{aligned}
\mathbb{E}_B\Big[|\tfrac{\theta}{\tau}z + B_0^{\tau}(\theta)|^a\Big] &\le 3^{(a-1)_+}\Big(|z|^a + 2C(a,d)\tau^{a/2}\Big), \\
\mathbb{E}_B\Big[|\tfrac{\tau-\theta}{\tau}z + B_0^{\tau}(\theta)|^a\Big] &\le 3^{(a-1)_+}\Big(|z|^a + 2C(a,d)\tau^{a/2}\Big) \qquad (3.28)
\end{aligned}
$$

where $C(a,d) := \mathbb{E}_B[|B(1)|^a] = \int_{\mathbb{R}^d} |y|^a p(1,y)dy$. Thus, taking expectation $\mathbb{E}_B$ in (3.26) and (3.27), we have

$$
\begin{aligned}
&\mathbb{E}_B\Big[|v_{K14}(t,x,y;\sigma)|\Big] + \mathbb{E}_B\Big[|v_{K15}(t,x,y;\sigma)|\Big] \\
&\le C_2\Big\{ (\tfrac{2(1-2\delta)_+}{e})^{(1-2\delta)_+} t^{1\wedge 2\delta} \\
&\qquad \times \Big(3|x-y|^2 + 6C(2,d)\sigma_t + 3^{1+\mu}|x-y|^{2+\mu} + 3^{1+\mu}2C(2+\mu,d)\sigma_t^{1+\mu/2}\Big) \\
&\qquad + t\Big(3|x-y|^2 + 6C(2,d)\sigma_t + 3^{1+\nu}|x-y|^{2+\nu} + 3^{1+\nu}2C(2+\nu,d)\sigma_t^{1+\nu/2}\Big)\Big\}.
\end{aligned}
$$

Collecting all the above into (3.23) yields the estimate in Claim 3.1 and the proof is complete. □

Claim 3.2.

$$
\begin{aligned}
&\mathbb{E}_\sigma\Big[\mathbb{E}_B[|v_{K2}(t,x,y;\sigma)|]p(\sigma_t,x-y)\Big] \\
&\le \mathrm{const}(\delta,\mu,\nu,d)\Big[C_1^2(t^2+t^{2\delta})\Big(\mathbb{E}_\sigma[|x-y|^2 p(\sigma_t,x-y)] + \mathbb{E}_\sigma[\sigma_t\, p(\sigma_t,x-y)]\Big) \\
&\quad + C_2^2 t^{2(1\wedge 2\delta)}\Big(\mathbb{E}_\sigma[|x-y|^4 p(\sigma_t,x-y)] + \mathbb{E}_\sigma[\sigma_t^2\, p(\sigma_t,x-y)] \\
&\quad + \mathbb{E}_\sigma[|x-y|^{4+2\mu} p(\sigma_t,x-y)] + \mathbb{E}_\sigma[\sigma_t^{2+\mu} p(\sigma_t,x-y)]\Big)
\end{aligned}
$$

$$
+ C_2^2 t^2 \Big(\mathbb{E}_\sigma [|x-y|^4 p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^2 \, p(\sigma_t, x-y)]
+ \mathbb{E}_\sigma [|x-y|^{4+2\nu} p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^{2+\nu} p(\sigma_t, x-y)] \Big) \Big].
$$

Proof. By $(A)_2$(i)

$$
\begin{aligned}
\Big| \sum_{j=1}^3 w_{Kj}(t,x,y;\sigma) \Big| &= \Big| \tfrac{1}{2} \Big\langle \nabla V(x), \int_0^t (\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)) ds \Big\rangle \\
&\quad + \tfrac{1}{2} \Big\langle \nabla V(y), \int_0^t (\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)) ds \Big\rangle \Big| \\
&\le \tfrac{C_1}{2} \Big\{ (1 + V(x)^{1-\delta}) \int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)| ds \\
&\quad + (1 + V(y)^{1-\delta}) \int_0^t |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)| ds \Big\}. \qquad (3.29)
\end{aligned}
$$

This estimate together with (3.26) and (3.27) gives us that

$$
\begin{aligned}
&|w_K(t,x,y;\sigma)| e^{-\theta t (V(x)+V(y))/4} \\
&\le |\sum_{j=1}^3 w_{Kj}(t,x,y;\sigma)| e^{-\theta t (V(x)+V(y))/4} + \sum_{j=4}^5 |w_{Kj}(t,x,y;\sigma)| e^{-\theta t (V(x)+V(y))/4} \\
&\le \tfrac{C_1}{2} \Big(1 + (\tfrac{4(1-\delta)}{e})^{1-\delta} \theta^{-1+\delta} t^{-1+\delta} \Big) \\
&\quad \times \int_0^t \Big(|\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)| + |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)| \Big) ds \\
&+ \ \tfrac{C_2}{2} \int_0^t ds \Big\{ \theta^{-(1-2\delta)_+} t^{-(1-2\delta)_+} (\tfrac{4(1-2\delta)_+}{e})^{(1-2\delta)_+} \\
&\qquad \times \Big(|\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{2+\mu} \\
&\qquad + |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{2+\mu} \Big) \\
&\qquad + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{2+\nu} \\
&\qquad + |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^2 + |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{2+\nu} \Big\}.
\end{aligned}
$$

By the Schwarz inequality, it follows that

$$
\begin{aligned}
&\Big(|w_K(t,x,y;\sigma)| e^{-\theta t (V(x)+V(y))/4} \Big)^2 \\
&\le \ 12 \Big[(\tfrac{C_1}{2})^2 (t + (\tfrac{4(1-\delta)}{e})^{2(1-\delta)} \theta^{-2+2\delta} t^{-1+2\delta}) \\
&\qquad \times \Big(\int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^2 ds + \int_0^t |\tfrac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^2 ds \Big) \\
&\quad + (\tfrac{C_2}{2})^2 \Big\{ (\tfrac{4(1-2\delta)_+}{e})^{2(1-2\delta)_+} \theta^{-2(1-2\delta)_+} t^{2(1 \wedge 2\delta)-1}
\end{aligned}
$$

$$\times \Big(\int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds$$

$$+ \int_0^t |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds\Big)$$

$$+ t\Big(\int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{4+2\nu} ds$$

$$+ \int_0^t |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{4+2\nu} ds\Big)\Big\}\Big].$$

Take expectation $\mathbb{E}_B$ above, and integrate in θ. Then

$$\begin{aligned}
&\mathbb{E}_B\left[|v_{K2}(t,x,y;\sigma)|\right] \\
&\le \mathbb{E}_B\Big[w_K(t,x,y;\sigma)^2 \int_0^1 \theta e^{-\theta t\left(V(x)+V(y)\right)/2} d\theta\Big] \\
&= \int_0^1 \theta \mathbb{E}_B\Big[\Big(|w_K(t,x,y;\sigma)| e^{-\theta t\left(V(x)+V(y)\right)/4}\Big)^2\Big] d\theta \\
&\le 12\Big[(\tfrac{C_1}{2})^2 3(t^2 + (\tfrac{4(1-\delta)}{e})^{2(1-\delta)} \tfrac{1}{\delta} t^{2\delta})(|x-y|^2 + 2C(2,d)\sigma_t) \\
&\quad + (\tfrac{C_2}{2})^2 \Big\{ (\tfrac{4(1-2\delta)_+}{e})^{2(1-2\delta)_+} \tfrac{1}{1\wedge 2\delta} t^{2(1\wedge 2\delta)} \\
&\qquad \times [3^3(|x-y|^4 + 2C(4,d)\sigma_t^2) + 3^{3+2\mu}(|x-y|^{4+2\mu} + 2C(4+2\mu,d)\sigma_t^{2+\mu})] \\
&\qquad + t^2[3^3(|x-y|^4 + 2C(4,d)\sigma_t^2) + 3^{3+2\nu}(|x-y|^{4+2\nu} + 2C(4+2\nu,d)\sigma_t^{2+\nu})]\Big\}\Big],
\end{aligned}$$

whence follows immediately the estimate in Claim 3.2. □

Claim 3.3.

$$\begin{aligned}
&\mathbb{E}_\sigma\Big[\mathbb{E}_B\left[|v_G(t,x,y;\sigma)|\right] p(\sigma_t, x-y)\Big],\ \mathbb{E}_\sigma\Big[\mathbb{E}_B\left[|v_R(t,x,y;\sigma)|\right] p(\sigma_t, x-y)\Big] \\
&\le \text{const}(\delta,\mu,\nu,d) \sum_{j=1}^2 \Big[C_1^j (t^j + t^{j\delta}) \Big(\mathbb{E}_\sigma\left[|x-y|^j p(\sigma_t, x-y)\right] + \mathbb{E}_\sigma\left[\sigma_t^{j/2} p(\sigma_t, x-y)\right]\Big) \\
&\quad + C_2^j t^{j(1\wedge 2\delta)} \Big(\mathbb{E}_\sigma\left[|x-y|^{2j} p(\sigma_t, x-y)\right] + \mathbb{E}_\sigma\left[\sigma_t^j p(\sigma_t, x-y)\right] \\
&\quad + \mathbb{E}_\sigma\left[|x-y|^{j(2+\mu)} p(\sigma_t, x-y)\right] + \mathbb{E}_\sigma\left[\sigma_t^{j(1+\mu/2)} p(\sigma_t, x-y)\right]\Big) \\
&\quad + C_2^j t^j \Big(\mathbb{E}_\sigma\left[|x-y|^{2j} p(\sigma_t, x-y)\right] + \mathbb{E}_\sigma\left[\sigma_t^j p(\sigma_t, x-y)\right] \\
&\quad + \mathbb{E}_\sigma\left[|x-y|^{j(2+\nu)} p(\sigma_t, x-y)\right] + \mathbb{E}_\sigma\left[\sigma_t^{j(1+\nu/2)} p(\sigma_t, x-y)\right]\Big)\Big].
\end{aligned}$$

Proof. Similarly to what is done in (3.29), (3.26) and (3.27), we have

$$\begin{aligned}
&|w_{G1}(t,x,y;\sigma)| e^{-rtV(x)} \\
&\le C_1(1 + (\tfrac{1-\delta}{e})^{1-\delta} (rt)^{-1+\delta}) \int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)| ds, \qquad (3.30)
\end{aligned}$$

$$
\begin{aligned}
&|w_{G2}(t,x,y;\sigma)|e^{-rtV(x)}\\
&\le\ C_2\Big[(\tfrac{(1-2\delta)_+}{e})^{(1-2\delta)_+}(rt)^{-(1-2\delta)_+}\\
&\qquad\times\int_0^t\Big(|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^2+|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^{2+\mu}\Big)ds\\
&\qquad+\int_0^t\Big(|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^2+|\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^{2+\nu}\Big)ds\Big], \qquad (3.31)\\
&|w_{R1}(t,x,y;\sigma)|e^{-rtV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)}\\
&\le\ C_1(1+(\tfrac{1-\delta}{e})^{1-\delta}(rt)^{-1+\delta})\int_0^t|B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|ds, \qquad (3.32)\\
&|w_{R2}(t,x,y;\sigma)|e^{-rtV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)}\\
&\le\ C_2\Big[(\tfrac{(1-2\delta)_+}{e})^{(1-2\delta)_+}(rt)^{-(1-2\delta)_+}\\
&\qquad\times\int_0^t\Big(|B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^2+|B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^{2+\mu}\Big)ds\\
&\qquad+\int_0^t\Big(|B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^2+|B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^{2+\nu}\Big)ds\Big]. \qquad (3.33)
\end{aligned}
$$

By (3.15), (3.16), (3.21) and (3.22), note that

$$
\begin{aligned}
&|v_G(t,x,y;\sigma)|\\
&\le\ |w_{G1}(t,x,y;\sigma)|e^{-tV(x)}+|w_{G2}(t,x,y;\sigma)|e^{-tV(x)}\\
&\quad+\int_0^1\theta\Big(|w_{G1}(t,x,y;\sigma)|e^{-\theta tV(x)/2}+|w_{G2}(t,x,y;\sigma)|e^{-\theta tV(x)/2}\Big)^2d\theta, \qquad (3.34)\\
&|v_R(t,x,y;\sigma)|\\
&\le\ |w_{R1}(t,x,y;\sigma)|e^{-tV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)}+|w_{R2}(t,x,y;\sigma)|e^{-tV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)}\\
&\quad+\int_0^1\theta\Big(|w_{R1}(t,x,y;\sigma)|e^{-\theta tV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)/2}\\
&\qquad\qquad+|w_{R2}(t,x,y;\sigma)|e^{-\theta tV\big(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})\big)/2}\Big)^2d\theta. \qquad (3.35)
\end{aligned}
$$

Also note that for $a>0$ and $0\le\theta_1,\theta_2\le\tau$ ($\tau>0$) (cf. (3.28))

$$
\mathbb{E}_B\Big[|B_{0,x}^{\tau,y}(\theta_1)-B_{0,x}^{\tau,y}(\theta_2)|^a\Big]\ \le\ 3^{(a-1)_+}(|x-y|^a+2C(a,d)\tau^{a/2}). \qquad (3.36)
$$

Collecting all the above yields the estimate in Claim 3.3 immediately. □

We are now in a position to prove Theorem 2.1(iii). To do so, we need the following lemma.

Lemma 3.1. *Let $1\le p\le\infty$. Then, for $a,b\ge 0$ with $C(a,d)=\int_{\mathbb{R}^d}|y|^ap(1,y)dy$,*

$$
f_{a,b}(t)\ :=\ \Big\|\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma[|\cdot-y|^a\sigma_t^bp(\sigma_t,\cdot-y)]dy\Big\|_p
$$

$$\leq\; C(a,d)\mathbb{E}_\sigma[\sigma_t^{a/2+b}]\,\|f\|_p, \quad f\in L_p(\mathbb{R}^d).$$

Proof. For $p=\infty$, the described estimate is obvious. So let $1\le p<\infty$. First we note the Minkowski inequality for integrals: If $h(x,y)$ is a measurable function on a σ-finite product measure space $(\mathcal{X}\times\mathcal{Y},\alpha(dx)\times\beta(dy))$, then

$$\Big(\int_{\mathcal{Y}}\Big(\int_{\mathcal{X}}|h(x,y)|\alpha(dx)\Big)^p\beta(dy)\Big)^{1/p}\;\le\;\int_{\mathcal{X}}\Big(\int_{\mathcal{Y}}|h(x,y)|^p\beta(dy)\Big)^{1/p}\alpha(dx).$$

Note also that for $c\ge 0$

$$\Big\|\int_{\mathbb{R}^d}|f(y)||\cdot-y|^c p(\tau,\cdot-y)dy\Big\|_p\;\le\;C(c,d)\tau^{c/2}\|f\|_p.$$

By these inequalities, the estimate is obtained as follows:

$$\begin{aligned}\Big\|\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma[|\cdot-y|^a\sigma_t^b p(\sigma_t,\cdot-y)]dy\Big\|_p \;&\le\; \mathbb{E}_\sigma\Big[\Big\|\int_{\mathbb{R}^d}|f(y)||\cdot-y|^a\sigma_t^b p(\sigma_t,\cdot-y)dy\Big\|_p\Big]\\ &\le\; C(a,d)\mathbb{E}_\sigma[\sigma_t^{a/2+b}]\,\|f\|_p.\end{aligned}$$

□

Proof of Theorem 2.1(iii). By Claims 3.1, 3.2 with (3.7)

$$\begin{aligned}\|Q_K(t)f\|_p \;\le\;& \Big\|\int_{\mathbb{R}^d}|f(y)|\,|\mathbb{E}_\sigma[\mathbb{E}_B[v_{K1}(t,\cdot,y;\sigma)]p(\sigma_t,\cdot-y)]|dy\\ &+\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma[\mathbb{E}_B[|v_{K2}(t,\cdot,y;\sigma)|]p(\sigma_t,\cdot-y)]dy\Big\|_p\\ \le\;& \mathrm{const}(\delta,\mu,\nu,d)\Big[C_1^2(t^2+t^{2\delta})(f_{2,0}(t)+f_{0,1}(t))\\ &+\sum_{j=1}^{2}\Big\{C_2^j t^{j(1\wedge 2\delta)}(f_{2j,0}(t)+f_{0,j}(t)+f_{j(2+\mu),0}(t)+f_{0,j(1+\mu/2)}(t))\\ &+C_2^j t^j(f_{2j,0}(t)+f_{0,j}(t)+f_{j(2+\nu),0}(t)+f_{0,j(1+\nu/2)}(t))\Big\}\Big].\end{aligned}$$

By Claim 3.3 with (3.8), (3.9)

$$\begin{aligned}\|Q_{\frac{G}{R}}(t)f\|_p \;\le\;& \Big\|\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma[\mathbb{E}_B[|v_{\frac{G}{R}}(t,\cdot,y;\sigma)|]p(\sigma_t,\cdot-y)]dy\Big\|_p\\ \le\;& \mathrm{const}(\delta,\mu,\nu,d)\sum_{j=1}^{2}\Big[C_1^j(t^j+t^{j\delta})(f_{j,0}(t)+f_{0,j/2}(t))\\ &+C_2^j t^{j(1\wedge 2\delta)}(f_{2j,0}(t)+f_{0,j}(t)+f_{j(2+\mu),0}(t)+f_{0,j(1+\mu/2)}(t))\\ &+C_2^j t^j(f_{2j,0}(t)+f_{0,j}(t)+f_{j(2+\nu),0}(t)+f_{0,j(1+\nu/2)}(t))\Big].\end{aligned}$$

Combining these with Lemma 3.1 we have the assertion of Theorem 2.1(iii). □

3.2. Case $(\mathbf{A})_0$

In this subsection, we suppose condition $(\mathrm{A})_0$ on $V(x)$. In this case

$$
\begin{aligned}
|v_K(t,x,y;\sigma)| &\le |w_K(t,x,y;\sigma)| \\
&\le \tfrac{C_1}{2}\int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^\gamma ds + \tfrac{C_1}{2}\int_0^t |\tfrac{\sigma_t-\sigma_s}{\sigma_t}(x-y)+B_0^{\sigma_t}(\sigma_s)|^\gamma ds, \\
|v_G(t,x,y;\sigma)| &\le |w_G(t,x,y;\sigma)| \\
&\le C_1\int_0^t |\tfrac{\sigma_s}{\sigma_t}(y-x)+B_0^{\sigma_t}(\sigma_s)|^\gamma ds, \\
|v_R(t,x,y;\sigma)| &\le |w_R(t,x,y;\sigma)| \\
&\le C_1\int_0^t |B_{0,x}^{\sigma_t,y}(\sigma_s)-B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^\gamma ds.
\end{aligned}
$$

Here taking expectation $\mathbb{E}_B$, we have by (3.28) or (3.36),

$$
\begin{aligned}
&\mathbb{E}_B[|v_K(t,x,y;\sigma)|],\ \mathbb{E}_B[|v_G(t,x,y;\sigma)|],\ \mathbb{E}_B[|v_R(t,x,y;\sigma)|] \\
&\le C_1t(|x-y|^\gamma + 2C(\gamma,d)\sigma_t^{\gamma/2})
\end{aligned}
$$

and hence, by (3.7), (3.8) and (3.9)

$$
\begin{aligned}
&|Q_K(t)f(x)|,\ |Q_G(t)f(x)|,\ |Q_R(t)f(x)| \\
&\le C_1t\Big\{\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma\Big[|x-y|^\gamma p(\sigma_t,x-y)\Big]dy \\
&\qquad + 2C(\gamma,d)\int_{\mathbb{R}^d}|f(y)|\,\mathbb{E}_\sigma\Big[\sigma_t^{\gamma/2}\,p(\sigma_t,x-y)\Big]dy\Big\}.
\end{aligned}
$$

From this and Lemma 3.1 the assertion of Theorem 2.1(i) follows immediately.

4. Proof of Theorem 2.2

For notational simplicity we set $H_0 := H_0^\psi$ and $H := H_0 + V$, in the following, so that $K(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2}$, $G(t) = e^{-tV}e^{-tH_0}$ and $R(t) = e^{-tH_0/2}e^{-tV}e^{-tH_0/2}$.

4.1. Proof of Theorem 2.2 for $K(t)$

Since $K(t)$ and e^{-sH} are contractions, we have

$$
\begin{aligned}
\|K(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p} &= \|\sum_{k=0}^{n-1} K(\tfrac{t}{n})^{n-1-k}(K(\tfrac{t}{n}) - e^{-tH/n})e^{-ktH/n}\|_{p\to p} \\
&\le \sum_{k=0}^{n-1} \|K(\tfrac{t}{n}) - e^{-tH/n}\|_{p\to p} \\
&= n\|Q_K(\tfrac{t}{n})\|_{p\to p}.
\end{aligned}
$$

Combined with the estimates for $Q_K(t)$ in Theorem 2.1, the desired bound for $K(t/n)^n - e^{-tH}$ in Case $(\mathrm{A})_0$, $(\mathrm{A})_1$ or $(\mathrm{A})_2$ is obtained immediately.

4.2. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case $(\mathrm{A})_0$

In the same way as above

$$\begin{aligned} \|G(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p} &\le n\|Q_G(\tfrac{t}{n})\|_{p\to p}, \\ \|R(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p} &\le n\|Q_R(\tfrac{t}{n})\|_{p\to p}, \end{aligned}$$

from which together with Theorem 2.1(i), the desired bounds follow immediately.

4.3. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case $(\mathrm{A})_1$ or $(\mathrm{A})_2$

In this subsection we suppose that $V(x)$ satisfies $(\mathrm{A})_1$ or $(\mathrm{A})_2$.
We first observe that for $t \ge 0$ and $n \in \mathbb{N}$

$$\begin{aligned} G(\tfrac{t}{n})^n - e^{-tH} &= e^{-tV/2n}(K(\tfrac{n-1}{n}t\tfrac{1}{n-1})^{n-1} - e^{-(n-1)tH/n})e^{-tV/2n}e^{-tH_0/n} \\ &\quad + [e^{-tV/2n}, e^{-(n-1)tH/n}]e^{-tV/2n}e^{-tH_0/n} + e^{-(n-1)tH/n}Q_G(\tfrac{t}{n}), \\ R(\tfrac{t}{n})^n - e^{-tH} &= e^{-tH_0/2n}e^{-tV/2n}(K(\tfrac{n-1}{n}t\tfrac{1}{n-1})^{n-1} - e^{-(n-1)tH/n})e^{-tV/2n}e^{-tH_0/2n} \\ &\quad + e^{-tH_0/2n}[e^{-tV/2n}, e^{-(n-1)tH/n}]e^{-tV/2n}e^{-tH_0/2n} \\ &\quad + [e^{-tH_0/2n}, e^{-(n-1)tH/n}]e^{-tV/n}e^{-tH_0/2n} + e^{-(n-1)tH/n}Q_R(\tfrac{t}{n}), \end{aligned}$$

where $[A,B] = AB - BA$. Hence

$$\begin{aligned} \|G(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p} &\le \|K(\tfrac{n-1}{n}t\tfrac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p\to p} \\ &\quad + \|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p\to p} + \|Q_G(\tfrac{t}{n})\|_{p\to p}, \qquad (4.1) \\ \|R(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p} &\le \|K(\tfrac{n-1}{n}t\tfrac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p\to p} \\ &\quad + \|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p\to p} + \|[e^{-tH_0/2n}, e^{-(n-1)tH/n}]\|_{p\to p} \\ &\quad + \|Q_R(\tfrac{t}{n})\|_{p\to p}. \qquad (4.2) \end{aligned}$$

As for the first term on the RHS of (4.1) and (4.2), we see by Theorem 2.2 which was proved in Section 4.1

$$\begin{aligned} &\|K(\tfrac{n-1}{n}t\tfrac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p\to p} \\ &\le \begin{cases} \mathrm{const}(\delta,\kappa,d)\Big[C_1^2\Big((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2\delta}\Big)(n-1)\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}(C_2\tfrac{t}{n})^j(n-1)\mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}]\Big], \\ \hfill \text{in Case } (\mathrm{A})_1, \\ \mathrm{const}(\delta,\mu,\nu,d)\Big[C_1^2\Big((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2\delta}\Big)(n-1)\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}\Big\{(C_2\tfrac{t}{n})^j\Big((n-1)\mathbb{E}[\sigma_{t/n}^j] \\ \quad + (n-1)\mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}]\Big) + (C_2(\tfrac{t}{n})^{1\wedge 2\delta})^j\Big((n-1)\mathbb{E}[\sigma_{t/n}^j] + (n-1)\mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}]\Big)\Big\}\Big], \\ \hfill \text{in Case } (\mathrm{A})_2. \end{cases} \end{aligned}$$

As for the third term on the RHS of (4.1) and the fourth term of (4.2), we see by Theorem 2.1

$$
\begin{aligned}
&\|Q_G(\tfrac{t}{n})\|_{p\to p},\ \|Q_R(\tfrac{t}{n})\|_{p\to p}\\
&\le \begin{cases}
\mathrm{const}\,(\delta,\kappa,d)\sum_{j=1}^{2}\Big\{C_1^j((\tfrac{t}{n})^j+(\tfrac{t}{n})^{j\delta})\mathbb{E}[\sigma_{t/n}^{j/2}]+(C_2\tfrac{t}{n})^j\mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}]\Big\}, & \text{in Case (A)}_1,\\[2ex]
\mathrm{const}\,(\delta,\mu,\nu,d)\sum_{j=1}^{2}\Big\{C_1^j((\tfrac{t}{n})^j+(\tfrac{t}{n})^{j\delta})\mathbb{E}[\sigma_{t/n}^{j/2}] & \\
\qquad\qquad +(C_2(\tfrac{t}{n})^{1\wedge 2\delta})^j(\mathbb{E}[\sigma_{t/n}^{j}]+\mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}]) & \\
\qquad\qquad +(C_2\tfrac{t}{n})^j(\mathbb{E}[\sigma_{t/n}^{j}]+\mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}])\Big\}, & \text{in Case (A)}_2.
\end{cases}
\end{aligned}
$$

Therefore we need to estimate the middle terms of (4.1) and (4.2).

Claim 4.1. *Let $s \ge 0$ and $t > 0$. Then*

$$
\begin{aligned}
&\|[e^{-sV},e^{-tH}]\|_{p\to p},\ \|[e^{-sH_0},e^{-tH}]\|_{p\to p}\\
&\le \begin{cases}
\mathrm{const}\,(\delta,\kappa,d)s\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}]+C_2\mathbb{E}[\sigma_t^{(1+\kappa)/2}]\Big], & \textit{in Case } \text{(A)}_1,\\[2ex]
\mathrm{const}\,(\delta,\mu,\nu,d)s\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}]+C_2t^{-(1-2\delta)_+}(\mathbb{E}[\sigma_t]+\mathbb{E}[\sigma_t^{1+\mu/2}]) & \\
\qquad\qquad +C_2(\mathbb{E}[\sigma_t]+\mathbb{E}[\sigma_t^{1+\nu/2}])\Big], & \textit{in Case } \text{(A)}_2.
\end{cases}
\end{aligned}
$$

Proof. First we estimate the L_p-operator norm of $[e^{-sV},e^{-tH}]$. We have (by (A.13)) that for $f\in C_0(\mathbb{R}^d)$

$$
\begin{aligned}
&[e^{-sV},e^{-tH}]f(x)\\
&=\int_{\mathbb{R}^d} f(y)(e^{-sV(x)}-e^{-sV(y)})\mathbb{E}\Big[\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr\Big)p(\sigma_t,x-y)\Big]dy.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&|[e^{-sV},e^{-tH}]f(x)|\\
&\le s\int_{\mathbb{R}^d}|f(y)|\mathbb{E}\Big[|V(y)-V(x)|\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr\Big)p(\sigma_t,x-y)\Big]dy. \qquad (4.3)
\end{aligned}
$$

To estimate the integrand in (4.3), note by Taylor's theorem that

$$
\begin{aligned}
V(y)-V(x) &= \int_0^t\langle\nabla V(B_{0,x}^{\sigma_t,y}(\sigma_r)),y-x\rangle\tfrac{dr}{t}\\
&\quad+\int_0^1 d\theta\int_0^t\langle\nabla V(x+\theta(y-x))-\nabla V(B_{0,x}^{\sigma_t,y}(\sigma_r)),y-x\rangle\tfrac{dr}{t}.
\end{aligned}
$$

In Case $(\mathrm{A})_1$, it follows that

$$\begin{aligned}|V(y)-V(x)| \;&\leq\; \int_0^t C_1(1+V(B_{0,x}^{\sigma_t,y}(\sigma_r))^{1-\delta})\tfrac{dr}{t}|x-y| \\ &\quad + \int_0^1 d\theta \int_0^t C_2|(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{\kappa} dr \tfrac{|x-y|}{t} \\ &\leq\; C_1\Big(1+t^{-1+\delta}(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr)^{1-\delta}\Big)|x-y| \\ &\quad + C_2\tfrac{1}{t}\int_0^1 d\theta \int_0^t |(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{\kappa} dr|x-y| \end{aligned} \tag{4.4}$$

where the last inequality is due to Jensen's inequality. In Case $(\mathrm{A})_2$

$$\begin{aligned}&|V(y)-V(x)| \\ &\leq \int_0^t C_1(1+V(B_{0,x}^{\sigma_t,y}(\sigma_r))^{1-\delta})\tfrac{dr}{t}|x-y| \\ &\quad + \int_0^1 d\theta \int_0^t C_2\Big\{V(B_{0,x}^{\sigma_t,y}(\sigma_r))^{(1-2\delta)_+}(1+|(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{\mu}) \\ &\qquad\quad +1+|(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{\nu}\Big\}|(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|\tfrac{dr}{t}|x-y| \\ &\leq C_1\Big(1+t^{-1+\delta}(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr)^{1-\delta}\Big)|x-y| \\ &\quad + C_2 t^{-(1-2\delta)_+}\Big(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr\Big)^{(1-2\delta)_+}\int_0^1\Big(\max_{0\leq\sigma\leq\sigma_t}|(\tfrac{\sigma}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma)| \\ &\qquad\quad + \max_{0\leq\sigma\leq\sigma_t}|(\tfrac{\sigma}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma)|^{1+\mu}\Big)d\theta|x-y| \\ &\quad + C_2\tfrac{1}{t}\int_0^1 d\theta\int_0^t\Big(|(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)| \\ &\qquad\quad + |(\tfrac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{1+\nu}\Big)dr|x-y|. \end{aligned} \tag{4.5}$$

By (3.25), (4.4) and (4.5) imply the desired estimate:

$$|V(y)-V(x)|\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr\Big)$$

$$
\leq \begin{cases} C_1(1+(\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ \quad + C_2\frac{1}{t}\int_0^1 d\theta \int_0^t |(\frac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^\kappa dr|x-y|, \qquad \text{in Case (A)}_1, \\ C_1(1+(\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ \quad + C_2(\frac{(1-2\delta)_+}{e})^{(1-2\delta)_+}t^{-(1-2\delta)_+} \\ \quad\quad \times \int_0^1 \Big(\max_{0\le\sigma\le\sigma_t} |(\frac{\sigma}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma)| \\ \qquad\qquad + \max_{0\le\sigma\le\sigma_t} |(\frac{\sigma}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma)|^{1+\mu}\Big)d\theta|x-y| \\ \quad + C_2\frac{1}{t}\int_0^1 d\theta\int_0^t \Big(|(\frac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)| \\ \qquad\qquad + |(\frac{\sigma_r}{\sigma_t}-\theta)(y-x)+B_0^{\sigma_t}(\sigma_r)|^{1+\nu}\Big)dr|x-y|, \qquad \text{in Case (A)}_2. \end{cases}
$$

We take expectation $\mathbb{E}_B$ in the above. This time we use the following moment estimate: For $a > 0$, $\tau > 0$, $0 \le \theta \le 1$ and $z \in \mathbb{R}^d$

$$
\begin{aligned}
\mathbb{E}_B\Big[|(\tfrac{t}{\tau}-\theta)z + B_0^\tau(t)|^a\Big] &\le 3^{(a-1)_+}(|z|^a + 2C(a,d)\tau^{a/2}), \\
\mathbb{E}_B\Big[\max_{0\le t\le\tau}|(\tfrac{t}{\tau}-\theta)z + B_0^\tau(t)|^a\Big] &\le 3^{(a-1)_+}(|z|^a + 2\widetilde{C}(a,d)\tau^{a/2})
\end{aligned} \tag{4.6}
$$

where $C(a,d) = \mathbb{E}_B[|B(1)|^a]$ and $\widetilde{C}(a,d) = \mathbb{E}_B[\max_{0\le t\le 1}|B(t)|^a]$, and thereby we have

$$
\begin{aligned}
&\mathbb{E}_B\Big[|V(y)-V(x)|\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r))dr\Big)\Big] \\
&\leq \begin{cases} C_1(1+(\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| + C_2(|x-y|^{1+\kappa} + 2C(\kappa,d)\sigma_t^{\kappa/2}|x-y|), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in Case (A)}_1, \\ C_1(1+(\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ \quad + C_2(\frac{(1-2\delta)_+}{e})^{(1-2\delta)_+}t^{-(1-2\delta)_+} \\ \quad\quad \times \Big(|x-y| + 2\widetilde{C}(1,d)\sigma_t^{1/2} + 3^\mu(|x-y|^{1+\mu} + 2\widetilde{C}(1+\mu,d)\sigma_t^{(1+\mu)/2})\Big)|x-y| \\ \quad + C_2\Big(|x-y| + 2C(1,d)\sigma_t^{1/2} + 3^\nu(|x-y|^{1+\nu} + 2C(1+\nu,d)\sigma_t^{(1+\nu)/2})\Big)|x-y|, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in Case (A)}_2. \end{cases}
\end{aligned} \tag{4.7}
$$

Hence follows the desired bound for $[e^{-sV}, e^{-tH}]$ by Lemma 3.1 with (4.3).

Next we estimate the L_p-operator norm of $[e^{-sH_0}, e^{-tH}]$.

First we suppose that $V : \mathbb{R}^d \to [0,\infty)$ is in C^∞ and all its derivatives have polynomial growth. Then it is easily verified that (cf. Claim A.2 and its Remark)

(i) $e^{-tH}(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$, in particular, $e^{-tH_0}(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$, and

(ii) $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1\le p\le\infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,V}) \cap \bigcap_{1\le p\le\infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,0})$ and $\mathfrak{G}_p^{\psi,V} = \mathfrak{G}_p^{\psi,0} - V$ on $\mathcal{S}(\mathbb{R}^d)$.

Here $\mathfrak{G}_p^{\psi,V}$ $(1 \le p < \infty)$ is the infinitesimal generator of $\{e^{-t(H_0+V)}\}$ on $L_p(\mathbb{R}^d)$ and $\mathfrak{G}_\infty^{\psi,V}$ the one on $C_\infty(\mathbb{R}^d)$. By these facts the following formula holds in $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$ and $C_\infty(\mathbb{R}^d)$:

For each $f \in \mathcal{S}(\mathbb{R}^d)$

$$[e^{-sH_0}, e^{-tH}]f = \int_0^s e^{-uH_0}[V, e^{-tH}]e^{-(s-u)H_0} f du.$$

Hence, taking L_p-norm in the above yields that for each $f \in \mathcal{S}(\mathbb{R}^d)$

$$\|[e^{-sH_0}, e^{-tH}]f\|_p \le \int_0^s \|[V, e^{-tH}]e^{-(s-u)H_0} f\|_p du. \tag{4.8}$$

Now let V satisfy $(\mathrm{A})_1$ or $(\mathrm{A})_2$. In this case V is not necessarily smooth. So, take a nonnegative $h \in C_0^\infty$ with support in $\{x \in \mathbb{R}^d; |x| \le 1\}$ and $\int_{\mathbb{R}^d} h(x)dx = 1$. Set $V^\varepsilon = V * h_\varepsilon$ with $h_\varepsilon(x) = (1/\varepsilon)^d h(x/\varepsilon)$. Then V^ε is in $C^\infty(\mathbb{R}^d \to [0,\infty))$, and satisfies condition $(\mathrm{A})_1$ or $(\mathrm{A})_2$ with the same const's as V does. Further, by $(\mathrm{A})_1$(i) or $(\mathrm{A})_2$(ii) all the derivatives of V^ε have polynomial growth. Hence, by (4.7) and Lemma 3.1 it holds that for $g \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}
&\|[V^\varepsilon, e^{-t(H_0+V^\varepsilon)}]g\|_p \\
&\le \begin{cases}
\mathrm{const}\,(\delta,\kappa,d)\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2\mathbb{E}[\sigma_t^{(1+\kappa)/2}]\Big]\|g\|_p, & \text{in Case } (\mathrm{A})_1, \\
\mathrm{const}\,(\delta,\mu,\nu,d)\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2 t^{-(1-2\delta)_+}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) & \\
\qquad + C_2(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}])\Big]\|g\|_p, & \text{in Case } (\mathrm{A})_2.
\end{cases}
\end{aligned}$$

Since (4.8) holds with $V = V^\varepsilon$, by combining this with the above we have

$$\begin{aligned}
&\|[e^{-sH_0}, e^{-t(H_0+V^\varepsilon)}]f\|_p \\
&\le \begin{cases}
\mathrm{const}\,(\delta,\kappa,d)s\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2\mathbb{E}[\sigma_t^{(1+\kappa)/2}]\Big]\|f\|_p, & \text{in Case } (\mathrm{A})_1, \\
\mathrm{const}\,(\delta,\mu,\nu,d)s\Big[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2 t^{-(1-2\delta)_+}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) & \\
\qquad + C_2(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}])\Big]\|f\|_p, & \text{in Case } (\mathrm{A})_2.
\end{cases}
\end{aligned}$$

Finally let $\varepsilon \downarrow 0$. Since $V^\varepsilon \to V$ compact uniformly, we see by the Feynman-Kac formula (A.6) that $e^{-t(H_0+V^\varepsilon)}f \to e^{-t(H_0+V)}f$ boundedly pointwise, so that $[e^{-sH_0}, e^{-t(H_0+V^\varepsilon)}]f \to [e^{-sH_0}, e^{-t(H_0+V)}]f$ pointwise. Hence the desired bound for $[e^{-sH_0}, e^{-t(H_0+V)}]$ follows immediately by the Fatou inequality. $\square$

We return to estimate $G(t/n)^n - e^{-tH}$ and $R(t/n)^n - e^{-tH}$. By Claim 4.1

$$\begin{aligned}
&\|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p\to p},\ \|[e^{-tH_0/2n}, e^{-(n-1)tH/n}]\|_{p\to p} \\
&\le \begin{cases}
\mathrm{const}\,(\delta,\kappa,d)\frac{1}{n}\Big[C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_2 t\mathbb{E}[\sigma_t^{(1+\kappa)/2}]\Big], & \text{in Case } (\mathrm{A})_1, \\
\mathrm{const}\,(\delta,\mu,\nu,d)\frac{1}{n}\Big[C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_2 t^{1\wedge 2\delta}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) & \\
\qquad + C_2 t(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}])\Big], & \text{in Case } (\mathrm{A})_2.
\end{cases}
\end{aligned}$$

Therefore, collecting all the estimates above yields the desired bounds for $G(t/n)^n - e^{-tH}$ and $R(t/n)^n - e^{-tH}$.

5. Proof of Theorem 2.3

As in the previous section, we are setting $H_0 = H_0^\psi$ and $H = H_0 + V$.

5.1. Case (V)$_2$

Condition (V)$_2$ implies (A)$_2$ with $\delta = 1 \wedge 1/\rho$, $C_1 = c_1 c^{-(1-1\wedge 1/\rho)}$, $C_2 = c_2 2^{(\rho-3)_+}((1/2)c^{-(1-2(1\wedge 1/\rho))_+} \vee 1)$, $\mu = 0$ and $\nu = (\rho-2)_+$. So this case follows immediately from Theorem 2.2(iii).

5.2. Case (V)$_1$

In this subsection we suppose condition (V)$_1$ on $V(x)$.

Let us adopt an idea in [D-I-Tam]. Take again a nonnegative $h \in C_0^\infty$ with support in $\{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} h(x)dx = 1$. For $0 < \varepsilon \leq 1/4$, set

$$V_\varepsilon(x) := \Big(\frac{1}{\varepsilon\langle x\rangle^\eta}\Big)^d \int_{\mathbb{R}^d} h\Big(\frac{x-y}{\varepsilon\langle x\rangle^\eta}\Big) V(y)dy,$$

where $\eta := ((\rho-1)\vee 0)\wedge 1$. Then V_ε is a smooth function and it satisfies the following:

Lemma 5.1. (i) $V_\varepsilon(x) \geq c'\langle x\rangle^\rho$ *where* $c' = c/4^\rho$.

(ii) $|V_\varepsilon(x) - V(x)| \leq C'\varepsilon\langle x\rangle^{(\rho-1)_+ + \eta}$ *where* $C' = c_1(5/4)^{(\rho-1)_+}$.

(iii) $|\nabla V_\varepsilon(x)| \leq c_1'\langle x\rangle^{(\rho-1)_+}$ *where* $c_1' = c_1(5/4)^{\rho\vee 1}$.

(iv) $|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \leq (1/\varepsilon)c_2'\{\langle x\rangle^{(\rho-2\lambda)_+} + |x-y|^{(\rho-2\lambda)_+}\}|x-y|$ *where* $\lambda := (1+\eta)/2$ *and* $c_2' = c_1(5/4)^{(\rho-1)_+}2^{(\rho-3)_+}(5d/16+2)$.

The proof is not difficult, so is omitted (cf. [Tak]).

As a consequence of Lemma 5.1, it is easily seen that V_ε satisfies condition (A)$_2$, i.e.

$$\begin{aligned}(\mathrm{A})_{2,\varepsilon} \qquad |\nabla V_\varepsilon(x)| &\leq C_1' V_\varepsilon(x)^{1-1\wedge\lambda/\rho},\\ |\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| &\leq \tfrac{1}{\varepsilon}C_2'\Big\{V_\varepsilon(x)^{(1-2(1\wedge\lambda/\rho))_+} + |x-y|^{(\rho-2)_+}\Big\}|x-y|\end{aligned}$$

where $C_1' = c_1' c'^{-(1-1\wedge\lambda/\rho)}$ and $C_2' = c_2'(c'^{-(1-2(1\wedge\lambda/\rho))_+} \vee 1)$. Indeed, by the definition of λ, we have $\rho - \rho\wedge\lambda \geq (\rho-1)_+$, $(\rho - 2(\rho\wedge\lambda))_+ = (\rho-2\lambda)_+ = (\rho-2)_+$. Hence (A)$_{2,\varepsilon}$ follows, because, by (i) with $\langle x\rangle \geq 1$,

$$V_\varepsilon(x)^{1-1\wedge\lambda/\rho} \geq (c')^{1-1\wedge\lambda/\rho}\langle x\rangle^{\rho-\rho\wedge\lambda} \geq (c')^{1-1\wedge\lambda/\rho}\langle x\rangle^{(\rho-1)_+},$$

$$V_\varepsilon(x)^{(1-2(1\wedge\lambda/\rho))_+} \geq (c')^{(1-2(1\wedge\lambda/\rho))_+}\langle x\rangle^{(\rho-2(\rho\wedge\lambda))_+} = (c')^{(1-2(1\wedge\lambda/\rho))_+}\langle x\rangle^{(\rho-2\lambda)_+}.$$

In what follows *we write* c, C, c_1, c_2, C_1 *and* C_2 *simply for* c', C', c_1', c_2', C_1' *and* C_2'.

Now let $K_\varepsilon(t) := e^{-tV_\varepsilon/2}e^{-tH_0}e^{-tV_\varepsilon/2}$, $G_\varepsilon(t) := e^{-tV_\varepsilon}e^{-tH_0}$ and $R_\varepsilon(t) := e^{-tH_0/2}e^{-tV_\varepsilon}e^{-tH_0/2}$.

Claim 5.1. *Let $t \geq 0$ and $n \in \mathbb{N}$. Then with $H^\varepsilon = H_0 + V_\varepsilon$*

$$
\begin{aligned}
&\|K_\varepsilon(\tfrac{t}{n})^n - e^{-tH^\varepsilon}\|_{p\to p} \\
&\leq \text{ const}(\rho, d)\Big[{C_1}^2((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2(1\wedge\lambda/\rho)})n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}\Big\{(C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^j] \\
&\quad + (C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}] + (C_2\tfrac{1}{\varepsilon}(\tfrac{1}{n})^{1\wedge 2\lambda/\rho}\, t^{1\wedge 2\lambda/\rho})^j n\mathbb{E}[\sigma_{t/n}^j]\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\|G_\varepsilon(\tfrac{t}{n})^n - e^{-tH^\varepsilon}\|_{p\to p},\ \|R_\varepsilon(\tfrac{t}{n})^n - e^{-tH^\varepsilon}\|_{p\to p} \\
&\leq \text{ const}(\rho, d)\Big[\tfrac{1}{n}C_1(t + t^{1\wedge\lambda/\rho})\mathbb{E}[\sigma_t^{1/2}] + C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t^{1\wedge 2\lambda/\rho}\mathbb{E}[\sigma_t] \\
&\quad + C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2\vee\rho)/2}]) + C_1(\tfrac{t}{n} + (\tfrac{t}{n})^{1\wedge\lambda/\rho})\mathbb{E}[\sigma_{t/n}^{1/2}] \\
&\quad + {C_1}^2((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2(1\wedge\lambda/\rho)})n\mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^{2}\Big\{(C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^j] + (C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}] \\
&\quad + (C_2\tfrac{1}{\varepsilon}(\tfrac{1}{n})^{1\wedge 2\lambda/\rho}t^{1\wedge 2\lambda/\rho})^j n\mathbb{E}[\sigma_{t/n}^j]\Big\}\Big].
\end{aligned}
$$

This is obvious from $(\mathrm{A})_{2,\varepsilon}$ and Theorem 2.2(iii).

Claim 5.2. *Let $t \geq 0$ and $n \in \mathbb{N}$. Then*

$$
\begin{aligned}
&\|e^{-tH} - e^{-tH^\varepsilon}\|_{p\to p}, \\
&\|K(\tfrac{t}{n})^n - K_\varepsilon(\tfrac{t}{n})^n\|_{p\to p},\ \|G(\tfrac{t}{n})^n - G_\varepsilon(\tfrac{t}{n})^n\|_{p\to p},\ \|R(\tfrac{t}{n})^n - R_\varepsilon(\tfrac{t}{n})^n\|_{p\to p} \\
&\quad \leq \text{ const}(C, c, \rho)\,\varepsilon\, t^{2/((\rho\wedge 2)\vee 1)-1}.
\end{aligned}
$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^d)$. By (3.4), (3.5) and (3.6) with (A.6),

$$
\begin{aligned}
&|(e^{-tH} - e^{-tH^\varepsilon})f(x)| \\
&\leq \mathbb{E}\Big[\Big|\exp\Big(-\int_0^t V(x + X_s)ds\Big) - \exp\Big(-\int_0^t V_\varepsilon(x + X_s)ds\Big)\Big||f(x + X_t)|\Big],
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&|(K(\tfrac{t}{n})^n - K_\varepsilon(\tfrac{t}{n})^n)f(x)| \\
&\leq \mathbb{E}\Big[\Big|\exp\Big(-\tfrac{t}{2n}\sum_{k=1}^{n}(V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\Big) \\
&\qquad - \exp\Big(-\tfrac{t}{2n}\sum_{k=1}^{n}(V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n}))\Big)\Big||f(x + X_t)|\Big],
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&|(G(\tfrac{t}{n})^n - G_\varepsilon(\tfrac{t}{n})^n)f(x)| \\
&\leq \mathbb{E}\Big[\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x + X_{(k-1)t/n})\Big) \\
&\qquad - \exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V_\varepsilon(x + X_{(k-1)t/n})\Big)\Big||f(x + X_t)|\Big],
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&|(R(\tfrac{t}{n})^n - R_\varepsilon(\tfrac{t}{n})^n)f(x)| \\
&\le \mathbb{E}\Big[\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^n V(x+X_{(2k-1)t/2n})\Big) \\
&\qquad - \exp\Big(-\tfrac{t}{n}\sum_{k=1}^n V_\varepsilon(x+X_{(2k-1)t/2n})\Big)\Big||f(x+X_t)|\Big].
\end{aligned}
\tag{5.4}
$$

By a formula

$$
e^{-a} - e^{-b} = \int_0^1 (b-a)e^{-\theta a}e^{-(1-\theta)b}d\theta, \quad a, b \in \mathbb{R}
$$

and Lemma 5.1, we have

$$
\begin{aligned}
&\Big|\exp\Big(-\int_0^t V(x+X_s)ds\Big) - \exp\Big(-\int_0^t V_\varepsilon(x+X_s)ds\Big)\Big| \\
&\le \int_0^1 d\theta \int_0^t |V_\varepsilon(x+X_s) - V(x+X_s)|ds \\
&\qquad \times \exp\Big(-\theta\int_0^t V(x+X_s)ds\Big)\exp\Big(-(1-\theta)\int_0^t V_\varepsilon(x+X_s)ds\Big) \\
&\le C\varepsilon \int_0^t \langle x+X_s\rangle^{(\rho-1)_+ + \eta}ds\ \exp\Big(-c\int_0^t \langle x+X_s\rangle^\rho ds\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\exp\Big(-\tfrac{t}{2n}\sum_{k=1}^n (V(x+X_{(k-1)t/n}) + V(x+X_{kt/n}))\Big) \\
&- \exp\Big(-\tfrac{t}{2n}\sum_{k=1}^n (V_\varepsilon(x+X_{(k-1)t/n}) + V_\varepsilon(x+X_{kt/n}))\Big)\Big| \\
&\le \int_0^1 d\theta\, \tfrac{t}{2n}\sum_{k=1}^n \Big(|V_\varepsilon(x+X_{(k-1)t/n}) - V(x+X_{(k-1)t/n})| \\
&\qquad\qquad + |V_\varepsilon(x+X_{kt/n}) - V(x+X_{kt/n})|\Big) \\
&\qquad \times \exp\Big(-\theta\tfrac{t}{2n}\sum_{k=1}^n (V(x+X_{(k-1)t/n}) + V(x+X_{kt/n}))\Big) \\
&\qquad \times \exp\Big(-(1-\theta)\tfrac{t}{2n}\sum_{k=1}^n (V_\varepsilon(x+X_{(k-1)t/n}) + V_\varepsilon(x+X_{kt/n}))\Big) \\
&\le C\varepsilon\Big(\tfrac{t}{2n}\sum_{k=1}^n \langle x+X_{(k-1)t/n}\rangle^{(\rho-1)_+ + \eta} + \tfrac{t}{2n}\sum_{k=1}^n \langle x+X_{kt/n}\rangle^{(\rho-1)_+ + \eta}\Big) \\
&\quad \times \exp\Big(-c\tfrac{t}{2n}\sum_{k=1}^n \langle x+X_{(k-1)t/n}\rangle^\rho\Big)\exp\Big(-c\tfrac{t}{2n}\sum_{k=1}^n \langle x+X_{kt/n}\rangle^\rho\Big).
\end{aligned}
$$

Similarly

$$\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(k-1)t/n})\Big)-\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V_\varepsilon(x+X_{(k-1)t/n})\Big)\Big|$$
$$\le\ C\varepsilon\tfrac{t}{n}\sum_{k=1}^{n}\langle x+X_{(k-1)t/n}\rangle^{(\rho-1)_++\eta}\exp\Big(-c\tfrac{t}{n}\sum_{k=1}^{n}\langle x+X_{(k-1)t/n}\rangle^{\rho}\Big),$$

$$\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(2k-1)t/2n})\Big)-\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V_\varepsilon(x+X_{(2k-1)t/2n})\Big)\Big|$$
$$\le\ C\varepsilon\tfrac{t}{n}\sum_{k=1}^{n}\langle x+X_{(2k-1)t/2n}\rangle^{(\rho-1)_++\eta}\exp\Big(-c\tfrac{t}{n}\sum_{k=1}^{n}\langle x+X_{(2k-1)t/2n}\rangle^{\rho}\Big).$$

By Jensen's inequality and (3.25),

$$\Big|\exp\Big(-\int_0^t V(x+X_s)ds\Big)-\exp\Big(-\int_0^t V_\varepsilon(x+X_s)ds\Big)\Big|,$$
$$\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(k-1)t/n})\Big)-\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V_\varepsilon(x+X_{(k-1)t/n})\Big)\Big|,$$
$$\Big|\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V(x+X_{(2k-1)t/2n})\Big)-\exp\Big(-\tfrac{t}{n}\sum_{k=1}^{n}V_\varepsilon(x+X_{(2k-1)t/2n})\Big)\Big|$$
$$\le\ C\varepsilon\, t^{1-((\rho-1)_++\eta)/\rho}\Big(\tfrac{(\rho-1)_++\eta}{\rho}\tfrac{1}{ec}\Big)^{((\rho-1)_++\eta)/\rho},$$

$$\Big|\exp\Big(-\tfrac{t}{2n}\sum_{k=1}^{n}(V(x+X_{(k-1)t/n})+V(x+X_{kt/n}))\Big)$$
$$-\exp\Big(-\tfrac{t}{2n}\sum_{k=1}^{n}(V_\varepsilon(x+X_{(k-1)t/n})+V_\varepsilon(x+X_{kt/n}))\Big)\Big|$$
$$\le\ C\varepsilon(\tfrac{t}{2})^{1-((\rho-1)_++\eta)/\rho}2\Big(\tfrac{(\rho-1)_++\eta}{\rho}\tfrac{1}{ec}\Big)^{((\rho-1)_++\eta)/\rho},$$

where for $\rho=0$ we understand $((\rho-1)_++\eta)/\rho=0$. Substituting these into (5.1), (5.2), (5.3) and (5.4), respectively, we have

$$|(e^{-tH}-e^{-tH^\varepsilon})f(x)|,$$
$$|(K(\tfrac{t}{n})^n-K_\varepsilon(\tfrac{t}{n})^n)f(x)|,\ |(G(\tfrac{t}{n})^n-G_\varepsilon(\tfrac{t}{n})^n)f(x)|,\ |(R(\tfrac{t}{n})^n-R_\varepsilon(\tfrac{t}{n})^n)f(x)|$$
$$\le\ \mathrm{const}(C,c,\rho)\,\varepsilon\, t^{2/((\rho\wedge 2)\vee 1)\,-1}\mathbb{E}[|f(x+X_t)|],$$

which imply the estimates in Claim 5.2 and the proof is complete. □

Proof of Theorem 2.3(i). By Claims 5.1 and 5.2

$$\|K(\tfrac{t}{n})^n-e^{-tH}\|_{p\to p}$$

$$\le \|K(\tfrac{t}{n})^n - K_\varepsilon(\tfrac{t}{n})^n\|_{p\to p} + \|K_\varepsilon(\tfrac{t}{n})^n - e^{-tH^\varepsilon}\|_{p\to p} + \|e^{-tH^\varepsilon} - e^{-tH}\|_{p\to p}$$

$$\le \mathrm{const}(\rho, C, c, d)\Big[\varepsilon\, t^{2/((\rho\wedge 2)\vee 1)\,-1} + {C_1}^2((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2(1\wedge\,\lambda/\rho)})n\mathbb{E}[\sigma_{t/n}]$$

$$+ \sum_{j=1}^{2}\Big\{(C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^j] + (C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]$$

$$+ (C_2\tfrac{1}{\varepsilon}(\tfrac{1}{n})^{1\wedge\,2\lambda/\rho}\,t^{1\wedge\,2\lambda/\rho})^j n\mathbb{E}[\sigma_{t/n}^j]\Big\}\Big],$$

$$\|G(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p},\ \|R(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p}$$

$$\le \mathrm{const}(\rho, C, c, d)\Big[\varepsilon\, t^{2/((\rho\wedge 2)\vee 1)\,-1} + \tfrac{1}{n}C_1(t + t^{1\wedge\,\lambda/\rho})\mathbb{E}[\sigma_t^{1/2}]$$

$$+ C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t^{1\wedge\,2\lambda/\rho}\mathbb{E}[\sigma_t] + C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2\vee\rho)/2}])$$

$$+ C_1(\tfrac{t}{n} + (\tfrac{t}{n})^{1\wedge\,\lambda/\rho})\mathbb{E}[\sigma_{t/n}^{1/2}] + {C_1}^2((\tfrac{t}{n})^2 + (\tfrac{t}{n})^{2(1\wedge\,\lambda/\rho)})n\mathbb{E}[\sigma_{t/n}]$$

$$+ \sum_{j=1}^{2}\Big\{(C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^j] + (C_2\tfrac{1}{\varepsilon}\tfrac{1}{n}t)^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]$$

$$+ (C_2\tfrac{1}{\varepsilon}(\tfrac{1}{n})^{1\wedge\,2\lambda/\rho}\,t^{1\wedge\,2\lambda/\rho})^j n\mathbb{E}[\sigma_{t/n}^j]\Big\}\Big].$$

Now let $n \ge 2^{2(2\vee\rho)}$ and $\varepsilon := n^{-(1/2)\wedge(\lambda/\rho)} = n^{-1/2\vee\rho}$. Then $\varepsilon \le 1/4$, $\varepsilon^{-1}n^{-1\wedge\,2\lambda/\rho} = n^{-1/2\vee\rho}$ and $\varepsilon^{-1}n^{-1} \le n^{-1/2\vee\rho}$. Therefore we have

$$\|K(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p}$$

$$\le \mathrm{const}(\rho, C, c, C_1, C_2, d)\,(\tfrac{1}{n})^{1/2\vee\rho}\Big[t^{2/((\rho\wedge 2)\vee 1)\,-1} + (t^2 + t^{2(1\wedge((\rho\wedge 2)\vee 1)/2\rho)})n\mathbb{E}[\sigma_{t/n}]$$

$$+ \sum_{j=1}^{2}\Big\{(t^j + t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big\}\Big],$$

$$\|G(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p},\ \|R(\tfrac{t}{n})^n - e^{-tH}\|_{p\to p}$$

$$\le \mathrm{const}(\rho, C, c, C_1, C_2, d)\,(\tfrac{1}{n})^{1/2\vee\rho}\Big[t^{2/((\rho\wedge 2)\vee 1)\,-1} + (t + t^{1\wedge((\rho\wedge 2)\vee 1)/2\rho})\mathbb{E}[\sigma_t^{1/2}]$$

$$+ t^{2/2\vee\rho}\mathbb{E}[\sigma_t] + t(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2\vee\rho)/2}]) + (t^2 + t^{2(1\wedge((\rho\wedge 2)\vee 1)/2\rho)})n\mathbb{E}[\sigma_{t/n}]$$

$$+ \sum_{j=1}^{2}\Big\{(t^j + t^{j2/2\vee\rho})n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{j(2\vee\rho)/2}]\Big\}\Big],$$

and the proof is complete. □

6. Proof of Theorem 2.4

For $a > 0$, the proof will be given, divided into the three cases $a = 1$, $a > 1$ and $0 < a < 1$. First we note that for every $a > 0$

$$\mathbb{E}[\sigma_t^a] < \infty. \tag{6.1}$$

In fact, it is enough to show when $a = \nu$ is a positive integer. To do so, let φ_t be the characteristic function of σ_t, i.e., $\varphi_t(\xi) = \mathbb{E}[e^{\sqrt{-1}\xi\sigma_t}]$. We have $\varphi_t(\xi) = e^{-tf(\xi)}$, where

$$f(\xi) \;=\; \int_{(0,\infty)} (1 - e^{\sqrt{-1}\,\xi l}) e^{-l/2} n(dl).$$

Since smoothness of $\varphi_t(\xi)$ near $\xi = 0$ implies existence of moments of σ_t (cf. Exercise 2.6(viii) in [It]), we have only to show that φ_t or f is in C^∞ near $\xi = 0$. But this is obvious, because, by a property of the Lévy measure n, the integral $\int_{(0,\infty)} l^\nu e^{-l/2} n(dl)$ is convergent, so that by the Lebesgue convergence theorem

$$(\tfrac{d}{d\xi})^\nu f(\xi) \;=\; -\int_{(0,\infty)} (\sqrt{-1}\, l)^\nu e^{\sqrt{-1}\,\xi l} e^{-l/2} n(dl).$$

By Itô's formula (e.g. [Ik-Wa]),

$$\begin{aligned} \sigma_t^a \;&=\; \int_0^{t+}\int_{(0,\infty)} \Big\{(\sigma_{s-} + l)^a - \sigma_{s-}^a\Big\} N(dsdl) \\ &=\; \int_0^{t+}\int_{(0,\infty)} a \int_0^1 (\sigma_{s-} + \theta l)^{a-1} d\theta\, l\, N(dsdl), \end{aligned}$$

and hence, by taking expectation $\mathbb{E}$

$$\mathbb{E}[\sigma_t^a] \;=\; \int_0^t ds \int_{(0,\infty)} l e^{-l/2} n(dl)\, a \int_0^1 \mathbb{E}\Big[(\sigma_s + \theta l)^{a-1}\Big] d\theta. \tag{6.2}$$

This is further, by the change of variable $r = \frac{s}{t}$, rewritten as

$$\tfrac{1}{t}\,\mathbb{E}[\sigma_t^a] \;=\; \int_0^1 dr \int_{(0,\infty)} l e^{-l/2} n(dl)\, a \int_0^1 \mathbb{E}\Big[(\sigma_{tr} + \theta l)^{a-1}\Big] d\theta. \tag{6.3}$$

6.1. The case $a = 1$

By (6.3), it is clear that

$$\tfrac{1}{t}\,\mathbb{E}[\sigma_t] \;=\; \int_{(0,\infty)} l e^{-l/2} n(dl) \;\in\; (0,\infty). \tag{6.4}$$

6.2. The case $a > 1$

By (6.1) and (6.3), $\mathbb{E}[(\sigma_r + \theta l)^{a-1}]$ is of course integrable on $(0,\infty) \times [0,1] \times [0,1]$ w.r.t. $le^{-l/2} n(dl) dr d\theta$. Since σ_t is increasing in t with $\sigma_{0+} = \sigma_0 = 0$ and $a - 1 > 0$, we have $(\sigma_{tr} + \theta l)^{a-1} \downarrow \theta^{a-1} l^{a-1}$ as $t \downarrow 0$. It follows by the Lebesgue convergence theorem that

$$\tfrac{1}{t}\mathbb{E}[\sigma_t^a] \downarrow \int_{(0,\infty)} l^a e^{-l/2} n(dl) \;\in\; (0,\infty). \tag{6.5}$$

6.3. The case $0 < a < 1$

By the same reason as above (but in this case, $a-1<0$), we have $(\sigma_{tr}+\theta l)^{a-1} \uparrow \theta^{a-1}l^{a-1}$ as $t \downarrow 0$, and hence, by the monotone convergence theorem

$$\tfrac{1}{t}\mathbb{E}[\sigma_t^a] \uparrow \int_{(0,\infty)} l^a e^{-l/2} n(dl) \in (0,\infty]. \tag{6.6}$$

This time the integral on the RHS is not always convergent. To find the exact asymptotics we suppose assumption (L).

We start with a remark on (L) and $\psi(\lambda)$ defined by (1.3):

Fact. (i) *If* $0 \le \alpha < 1$, *then*

$$\psi(\lambda) \sim \Gamma(1-\alpha)\lambda^\alpha L(\lambda) \qquad \text{as } \lambda \uparrow \infty.$$

(ii) *If* $\alpha = 1$, *then* $\int_0^\cdot n((s,\infty))ds$ *is slowly varying at zero,* $L(1/t) = o(\int_0^t n((s,\infty))ds)$ *as* $t \downarrow 0$ *and*

$$\psi(\lambda) \sim \lambda \int_0^{1/\lambda} n((s,\infty))ds \qquad \text{as } \lambda \uparrow \infty.$$

Proof. First of all note that

$$\infty > \int_{(0,\infty)} l \wedge 1\, n(dl) = \int_0^1 n((t,\infty))dt, \tag{6.7}$$

$$\psi(\lambda) = \lambda \int_0^\infty e^{-\lambda t} d\Big(\int_0^t n((s,\infty))ds\Big). \tag{6.8}$$

By (1.6), $n((1/y,\infty)) \sim y^\alpha L(y)$ as $y \uparrow \infty$, and by (6.7),

$$\int_x^\infty \tfrac{1}{y^2} n((\tfrac{1}{y},\infty))dy = \int_0^{1/x} n((s,\infty))ds < \infty \qquad \text{for any } x > 0.$$

Let us apply Lemma and Theorem 1 of §VIII.9 in [Fe]. These say that $\int_\cdot^\infty 1/y^2 n((1/y,\infty))dy$ is regularly varying with exponent $-1+\alpha$ and

$$\frac{(1/x)n((1/x,\infty))}{\int_x^\infty 1/y^2 n((1/y,\infty))dy} \longrightarrow 1-\alpha \qquad \text{as } x \uparrow \infty.$$

Combining these with (1.6), we see that when $0 \le \alpha < 1$

$$\int_0^t n((s,\infty))ds \sim \tfrac{1}{1-\alpha}\, t\, n((t,\infty)) \sim \tfrac{1}{1-\alpha} t^{1-\alpha} L(\tfrac{1}{t}) \qquad \text{as } t \downarrow 0,$$

and that when $\alpha = 1$, $\int_0^\cdot n((s,\infty))ds$ is slowly varying at zero and $L(1/t) = o(\int_0^t n((s,\infty))ds)$ as $t \downarrow 0$.

By virtue of (6.8), if we apply the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), the asymptotics of ψ follow from those of $\int_0^\cdot n((s,\infty))ds$. □

Remark. Conversely, when $0 \le \alpha < 1$, we have (1.6) by Fact (i) by the Tauberian theorem.

Recall functions ϕ, L_1 and L_2 around assumption (L) in Section 1. By Fact, L_1 is slowly varying at infinity and

$$\phi(\lambda) \sim \lambda^\alpha L_1(\lambda) \qquad \text{as } \lambda \uparrow \infty. \tag{6.9}$$

As ψ is strictly increasing with $\psi(0) = 0$ and $\psi(\infty) = \infty$, so is ϕ, so that the inverse ϕ^{-1} exists. By (6.9), if $0 < \alpha \le 1$,

$$\phi^{-1}(x) \sim x^{1/\alpha} L_2(x) \qquad \text{as } x \uparrow \infty. \tag{6.10}$$

Since, by (6.9) again, ϕ is regularly varying with exponent α, so is ϕ^{-1} with exponent $1/\alpha$, and hence L_2 and $\int_0^\cdot (\phi^{-1}(\theta))^{-\alpha} d\theta$ $(0 < \alpha < 1)$ are also slowly varying at infinity.

Now we are in a position to show the asymptotics of $\mathbb{E}[\sigma_t^a]$ for $0 < a < 1$.

Claim 6.1. (i) *If $0 < a < \alpha$,*

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1-a)} t^{a/\alpha} L_2(\tfrac{1}{t})^{-a} \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1-a)} \phi^{-1}(\tfrac{1}{t})^{-a} \qquad \textit{as } t \downarrow 0.$$

(ii) *If $a = \alpha$,*

$$\mathbb{E}[\sigma_t^\alpha] \sim \frac{1}{\Gamma(1-\alpha)} t \int_0^{1/t} \Big(\phi^{-1}(\theta)\Big)^{-\alpha} d\theta \qquad \textit{as } t \downarrow 0.$$

(iii) *If $\alpha < a < 1$, then $\int_0^\infty \lambda^{-1-a}\phi(\lambda)d\lambda \in (0,\infty)$ and*

$$\mathbb{E}[\sigma_t^a] \sim t \frac{a}{\Gamma(1-a)} \int_0^\infty \lambda^{-1-a}\phi(\lambda)d\lambda \qquad \textit{as } t \downarrow 0.$$

Proof. To rewrite (6.2), we see first with (2.2)

$$\begin{aligned}
\mathbb{E}\Big[(\sigma_s + \theta l)^{a-1}\Big] &= \frac{1}{\Gamma(1-a)} \int_0^\infty \lambda^{-a} e^{-\lambda\theta l} \mathbb{E}\Big[e^{-\lambda\sigma_s}\Big] d\lambda \\
&= \frac{1}{\Gamma(1-a)} \int_0^\infty \lambda^{-a} e^{-\lambda\theta l} e^{-s\phi(\lambda)} d\lambda,
\end{aligned}$$

and then we have

$$\begin{aligned}
\mathbb{E}[\sigma_t^a] &= \frac{a}{\Gamma(1-a)} \int_0^t ds \int_0^\infty \lambda^{-1-a} e^{-s\phi(\lambda)} d\lambda \int_{(0,\infty)} (1 - e^{-\lambda l}) e^{-l/2} n(dl) \\
&= \frac{a}{\Gamma(1-a)} \int_0^t ds \int_0^\infty \lambda^{-1-a} \phi(\lambda) e^{-s\phi(\lambda)} d\lambda.
\end{aligned}$$

The λ-integral in the last line is further computed by the change of variable $\lambda = \phi^{-1}(x)$ as follows:

$$\begin{aligned}
&\int_0^\infty \lambda^{-1-a}\phi(\lambda)e^{-s\phi(\lambda)}d\lambda \\
&= \int_0^\infty (\phi^{-1}(x))^{-1-a} x e^{-sx}(\phi^{-1})'(x)dx \\
&= \Big[-\tfrac{1}{a}(\phi^{-1}(x))^{-a} x e^{-sx}\Big]_0^\infty + \tfrac{1}{a}\int_0^\infty (\phi^{-1}(x))^{-a}(e^{-sx} - sxe^{-sx})dx \\
&= \tfrac{1}{a}\Big\{\int_0^\infty (\phi^{-1}(x))^{-a} e^{-sx}dx - s\int_0^\infty (\phi^{-1}(x))^{-a} x e^{-sx}dx\Big\} \\
&= \tfrac{1}{a}\Big\{\int_0^\infty e^{-sx}d\Big(\textstyle\int_0^x (\phi^{-1}(\theta))^{-a}d\theta\Big) - s\displaystyle\int_0^\infty e^{-sx}d\Big(\textstyle\int_0^x (\phi^{-1}(\theta))^{-a}\theta d\theta\Big)\Big\} \\
&= \tfrac{1}{a}\Big\{L\Big(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}d\theta\Big) - sL\Big(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}\theta d\theta\Big)\Big\}.
\end{aligned}$$

Here $L(\cdot, G)$ denotes the Laplace transform of a right-continuous increasing function $G : [0,\infty) \to [0,\infty)$: $L(s, G) := \int_0^\infty e^{-sx}dG(x)$. The last fourth and third equalities are respectively because $0 \le (\phi^{-1}(x))^{-a}xe^{-sx} \le (\psi'(1/2))^a x^{1-a}e^{-sx} \to 0$ as $x \downarrow 0$, and because for $b > a-1$, $\int_0^\infty (\phi^{-1}(x))^{-a}x^b e^{-sx}dx \le (\psi'(1/2))^a \int_0^\infty x^{b-a}e^{-sx}dx = (\psi'(1/2))^a s^{a-b-1}\Gamma(b-a+1) < \infty$. Hence (6.2) is rewritten as follows:

$$\mathbb{E}[\sigma_t^a] = \frac{1}{\Gamma(1-a)}\int_0^t \Big(L(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}d\theta) - sL(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a}\theta d\theta)\Big)ds. \tag{6.11}$$

1° The case $0 < a < \alpha$. Then $0 < \alpha \le 1$. By (6.10), $(\phi^{-1}(\cdot))^{-a}$ is regularly varying with exponent $-a/\alpha \in (-1, 0)$. By Theorem 1 of §VIII.9 in [Fe],

$$\frac{x^2(\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a}\theta d\theta} \longrightarrow 2 - \tfrac{a}{\alpha} > 0, \quad \frac{x(\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a}d\theta} \longrightarrow \tfrac{\alpha-a}{\alpha} > 0$$

as $x \uparrow \infty$. Hence, by combining this with (6.10),

$$\begin{aligned}
\int_0^x (\phi^{-1}(\theta))^{-a}\theta d\theta &\sim \frac{1}{2-a/\alpha}x^2(\phi^{-1}(x))^{-a} \sim \frac{1}{2-a/\alpha}x^{2-a/\alpha}L_2(x)^{-a}, \\
\int_0^x (\phi^{-1}(\theta))^{-a}d\theta &\sim \frac{\alpha}{\alpha-a}x(\phi^{-1}(x))^{-a} \sim \frac{\alpha}{\alpha-a}x^{1-a/\alpha}L_2(x)^{-a}
\end{aligned}$$

as $x \uparrow \infty$. By applying the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), this implies that

$$\begin{aligned}
L(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}\theta d\theta) &\sim \dfrac{\Gamma(2-a/\alpha+1)}{2-a/\alpha}s^{-2+a/\alpha}L_2(\tfrac{1}{s})^{-a} = \Gamma(2-\tfrac{a}{\alpha})s^{-2+a/\alpha}L_2(\tfrac{1}{s})^{-a}, \\
L(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}d\theta) &\sim \tfrac{\alpha}{\alpha-a}\Gamma(2-\tfrac{a}{\alpha})s^{-1+a/\alpha}L_2(\tfrac{1}{s})^{-a}
\end{aligned}$$

as $s \downarrow 0$, and hence

$$\begin{aligned}
L(s, \textstyle\int_0^\cdot (\phi^{-1}(\theta))^{-a}d\theta) - sL(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a}\theta d\theta) &\sim \Big(\tfrac{\alpha}{\alpha-a} - 1\Big)\Gamma(2-\tfrac{a}{\alpha})s^{-1+a/\alpha}L_2(\tfrac{1}{s})^{-a} \\
&= \tfrac{a}{\alpha}\Gamma(\tfrac{\alpha-a}{\alpha})s^{-1+a/\alpha}L_2(\tfrac{1}{s})^{-a} \quad \text{as } s \downarrow 0.
\end{aligned}$$

Now if, for simplicity, we set

$$Z(x) \;:=\; L(\tfrac{1}{x}, \int_0^{\cdot}(\phi^{-1}(\theta))^{-a}d\theta) - \tfrac{1}{x}L(\tfrac{1}{x}, \int_0^{\cdot}(\phi^{-1}(\theta))^{-a}\theta d\theta),$$

then, by (6.11)

$$\mathbb{E}[\sigma_t^a] \;=\; \frac{1}{\Gamma(1-a)}\int_0^t Z(\tfrac{1}{s})ds \;=\; \frac{1}{\Gamma(1-a)}\int_{1/t}^{\infty} x^{-2}Z(x)dx$$

and also,

$$Z(x) \;\sim\; \tfrac{a}{\alpha}\Gamma(\tfrac{\alpha-a}{\alpha})\, x^{1-\,a/\alpha}L_2(x)^{-a} \qquad \text{as } x\uparrow\infty.$$

Therefore, applying Theorem 1 of §VIII.9 in [Fe] again, we have

$$\frac{(1/t)^{-2+1}Z(1/t)}{\mathbb{E}[\sigma_t^a]} \longrightarrow \Gamma(1-a)\tfrac{a}{\alpha} \qquad \text{as } t\downarrow 0,$$

and consequently

$$\mathbb{E}[\sigma_t^a] \;\sim\; \frac{\Gamma((\alpha-a)/\alpha)}{\Gamma(1-a)}\, t^{a/\alpha}L_2(\tfrac{1}{t})^{-a},$$

which is just the assertion (i).

2° The case $a=\alpha$. Then $0<\alpha<1$ and hence, by (6.10), $(\phi^{-1}(\cdot))^{-\alpha}$ is regularly varying with exponent -1. Once again, by Theorem 1 of §VIII.9 in [Fe],

$$\frac{x^2(\phi^{-1}(x))^{-\alpha}}{\int_0^x(\phi^{-1}(\theta))^{-\alpha}\theta d\theta} \;\longrightarrow\; 1, \quad \frac{x(\phi^{-1}(x))^{-\alpha}}{\int_0^x(\phi^{-1}(\theta))^{-\alpha}d\theta} \;\longrightarrow\; 0$$

as $x\uparrow\infty$, and $\int_0^{\cdot}(\phi^{-1}(\theta))^{-\alpha}d\theta$ is slowly varying at infinity. By combining this with (6.10)

$$\int_0^x (\phi^{-1}(\theta))^{-\alpha}\theta d\theta \;\sim\; x^2(\phi^{-1}(x))^{-\alpha} \;\sim\; xL_2(x)^{-\alpha},$$

$$L_2(x)^{-\alpha} \;\sim\; x(\phi^{-1}(x))^{-\alpha} \;=\; o(\int_0^x(\phi^{-1}(\theta))^{-\alpha}d\theta)$$

as $x\uparrow\infty$, and hence, by the Abelian theorem

$$L\Big(s, \int_0^{\cdot}(\phi^{-1}(\theta))^{-\alpha}\theta d\theta\Big) \;\sim\; s^{-1}L_2(\tfrac{1}{s})^{-\alpha},$$

$$L\Big(s, \int_0^{\cdot}(\phi^{-1}(\theta))^{-\alpha}d\theta\Big) \;\sim\; \int_0^{1/s}(\phi^{-1}(\theta))^{-\alpha}d\theta$$

as $s\downarrow 0$. Therefore

$$Z(\tfrac{1}{s}) \;=\; L\Big(s, \int_0^{\cdot}(\phi^{-1}(\theta))^{-\alpha}d\theta\Big) - sL\Big(s, \int_0^{\cdot}(\phi^{-1}(\theta))^{-\alpha}\theta d\theta\Big)$$

$$\sim\; \int_0^{1/s}(\phi^{-1}(\theta))^{-\alpha}d\theta \qquad \text{as } s\downarrow 0.$$

In exactly the same way as in 1° we eventually have

$$\frac{(1/t)^{-2+1}Z(1/t)}{\mathbb{E}[\sigma_t^\alpha]} \longrightarrow \Gamma(1-\alpha) \qquad \text{as } t \downarrow 0,$$

from which the assertion (ii) is easily seen.

3° The case $\alpha < a < 1$. Then $0 \le \alpha < 1$. By (6.6), it is enough to show that

$$\int_{(0,\infty)} l^a e^{-l/2} n(dl) \;=\; \frac{a}{\Gamma(1-a)} \int_0^\infty \lambda^{-1-a}\phi(\lambda)d\lambda \;<\; \infty.$$

First this identity is seen from the following computation:

$$\begin{aligned}\int_0^\infty \lambda^{-1-a}\phi(\lambda)d\lambda &= \int_0^\infty \lambda^{-1-a}\Big(\psi(\lambda+\tfrac{1}{2}) - \psi(\tfrac{1}{2})\Big)d\lambda \\ &= \int_{(0,\infty)} e^{-l/2}n(dl)\int_0^\infty \lambda^{-1-a}(1-e^{-\lambda l})d\lambda \\ &= \frac{\Gamma(1-a)}{a}\int_{(0,\infty)} l^a e^{-l/2}n(dl).\end{aligned}$$

Next this integral is convergent. Indeed, since $\phi(\lambda) \le \psi'(1/2)\lambda$ $(\lambda \ge 0)$,

$$\int_0^R \lambda^{-1-a}\phi(\lambda)d\lambda \;\le\; \psi'(\tfrac{1}{2})\int_0^R \lambda^{-a}d\lambda \;=\; \psi'(\tfrac{1}{2})\,\frac{R^{1-a}}{1-a} \;<\; \infty$$

for any $R > 0$. On the other hand, since $\phi(\lambda) \sim \lambda^\alpha L_1(\lambda)$ as $\lambda \uparrow \infty$, and $L_1(\cdot)$ is slowly varying at infinity, there exists an $R_\varepsilon > 0$ for $0 < \varepsilon < a - \alpha$ (cf. Lemma 2 of §VIII.8 in [Fe]) such that $\phi(\lambda) \le 2\lambda^\alpha L_1(\lambda)$ and $L_1(\lambda) < \lambda^\varepsilon$ for any $\lambda \ge R_\varepsilon$. Hence

$$\int_{R_\varepsilon}^\infty \lambda^{-1-a}\phi(\lambda)d\lambda \;\le\; \int_{R_\varepsilon}^\infty \lambda^{-1-a}2\lambda^{\alpha+\varepsilon}d\lambda \;=\; \frac{2}{a-\alpha-\varepsilon}\Big(\frac{1}{R_\varepsilon}\Big)^{a-\alpha-\varepsilon} \;<\; \infty.$$

□

Appendix: Semigroups $e^{-t(H_0^\psi+V)}$ and their generators in $L_p(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$

In this appendix we suppose only that $V : \mathbb{R}^d \to [0,\infty)$ is a continuous function. The main result is Theorem A.1, which follows from Lemma A.2 (Kato's inequality).

Let $M(dsdx)$ be a Poisson random measure on $[0,\infty)\times(\mathbb{R}^d\setminus\{0\})$ with intensity measure $dsJ(dx)$, where

$$J(dx) \;:=\; \int_{(0,\infty)} e^{-l/2}p(l,x)n(dl)dx, \tag{A.1}$$

$$p(l,x) \;:=\; \Big(\frac{1}{2\pi l}\Big)^{d/2}\exp\Big(-\frac{|x|^2}{2l}\Big). \tag{A.2}$$

This $M(\cdot)$ may be defined on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ as in Section 2. Note that for $p \in [1,\infty)$ the $2p$-th order absolute moment of J is finite, i.e.,

$$\int_{\mathbb{R}^d\setminus\{0\}} |x|^{2p}J(dx) \;<\; \infty. \tag{A.3}$$

Following the notation in [Ik-Wa], we set

$$\widehat{M}(dsdx) \ := \ dsJ(dx), \quad \widetilde{M}(dsdx) \ := \ M(dsdx) - \widehat{M}(dsdx)$$

and define an $\mathbb{R}^d$-valued right-continuous process $(X_t)_{t\geq 0}$ by

$$X_t \ := \int_0^{t+}\int_{\mathbb{R}^d\setminus\{0\}} x1_{|x|\geq 1}M(dsdx) + \int_0^{t+}\int_{\mathbb{R}^d\setminus\{0\}} x1_{|x|<1}\widetilde{M}(dsdx), \tag{A.4}$$

where the second term on the RHS is a stochastic integral w.r.t. $\widetilde{M}$. This is a d-dimensional time-homogeneous Lévy process starting at the origin such that

$$\mathbb{E}[e^{\sqrt{-1}\langle p,X_t\rangle}] \ = \ e^{-t\left(\psi((|p|^2+1)/2)-\psi(1/2)\right)},$$

which is easily seen by Itô's formula (cf. [Ik-Wa]), so that

$$(X_t)_{t\geq 0} \overset{\mathcal{L}}{\sim} (B(\sigma_t))_{t\geq 0}. \tag{A.5}$$

We now define a system of operators $P_t^{\psi,V}, t\geq 0$, by the Feynman-Kac formula:

$$P_t^{\psi,V}f(x) \ := \ \mathbb{E}\Big[\exp\Big(-\int_0^t V(x+X_s)ds\Big)f(x+X_t)\Big]. \tag{A.6}$$

From this definition the following is easily seen:

(i) If f is a nonnegative Borel measurable function, so is $P_t^{\psi,V}f$, and it satisfies

$$P_t^{\psi,V}(P_s^{\psi,V}f) \ = \ P_{t+s}^{\psi,V}f, \tag{A.7}$$

$$\int_{\mathbb{R}^d}|P_t^{\psi,V}f(x)|^p dx \ \leq \ \int_{\mathbb{R}^d}|f(x)|^p dx, \qquad 1\leq p<\infty. \tag{A.8}$$

(ii) If $f\in C_\infty(\mathbb{R}^d)$, then $P_t^{\psi,V}f\in C_\infty(\mathbb{R}^d)$ and

$$\|P_t^{\psi,V}f\|_\infty \ \leq \ \|f\|_\infty, \tag{A.9}$$

$$\|P_t^{\psi,V}f-f\|_\infty \ \to \ 0 \quad \text{as } t\downarrow 0. \tag{A.10}$$

(iii) For two nonnegative Borel measurable functions f,g

$$\int_{\mathbb{R}^d}P_t^{\psi,V}f(x)g(x)dx \ = \ \int_{\mathbb{R}^d}f(x)P_t^{\psi,V}g(x)dx. \tag{A.11}$$

By (i) and (ii), $\{P_t^{\psi,V}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^d)$. By the Riesz-Banach theorem there exists a finite measure $P^{\psi,V}(t,x,dy)$ on $\mathbb{R}^d$ such that

$$P_t^{\psi,V}f(x) \ = \ \int_{\mathbb{R}^d}f(y)P^{\psi,V}(t,x,dy), \quad f\in C_\infty(\mathbb{R}^d). \tag{A.12}$$

Indeed, by noting (A.5), $P^{\psi,V}(t,x,dy)$ is absolutely continuous w.r.t. the Lebesgue measure dy on $\mathbb{R}^d$ and expressed as

$$P^{\psi,V}(t,x,dy) \;=\; \mathbb{E}\Big[\exp\Big(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\Big)p(\sigma_t, x-y)\Big]dy, \tag{A.13}$$

where $B_{0,x}^{\tau,y}(\theta)$ is defined in (3.13).

By (i) and (ii) again $P_t^{\psi,V}$ is uniquely extended to a bounded operator on $L_p(\mathbb{R}^d)$, which is denoted by the same $P_t^{\psi,V}$, and thus $\{P_t^{\psi,V}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $L_p(\mathbb{R}^d)$. Clearly, for $f \in L_p(\mathbb{R}^d)$

$$P_t^{\psi,V}f(x) \;=\; \mathbb{E}\Big[\exp\Big(-\int_0^t V(x+X_s)ds\Big)f(x+X_t)\Big] \quad \text{a.e. } x$$

and, when $p=2$, $P_t^{\psi,V}$ is symmetric.

Let $\mathfrak{G}_p^{\psi,V}$ be the infinitesimal generator of $\{P_t^{\psi,V}\}_{t\geq 0}$ on $L_p(\mathbb{R}^d)$ for $1 \leq p < \infty$, and on $C_\infty(\mathbb{R}^d)$ for $p=\infty$. Their domains are denoted by $\mathfrak{D}(\mathfrak{G}_p^{\psi,V})$.

Put

$$H_0^\psi f(x) \;:=\; -\int_{\mathbb{R}^d\setminus\{0\}} \{f(x+y) - f(x) - \langle y, \nabla f(x)\rangle 1_{|y|<1}\} J(dy), \tag{A.14}$$

$$H^\psi f(x) \;:=\; H_0^\psi f(x) + V(x)f(x). \tag{A.15}$$

Claim A.1. (i) *For $f \in \mathcal{S}(\mathbb{R}^d)$, $H_0^\psi f$ is in $\mathcal{S}(\mathbb{R}^d)$, and hence, for $f \in C_0^\infty(\mathbb{R}^d)$, $H^\psi f \in C_\infty(\mathbb{R}^d) \cap \bigcap_{1\leq p<\infty} L_p(\mathbb{R}^d)$.*

(ii) *For $f \in C^\infty(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ (where $1 \leq p < \infty$), $H_0^\psi f$ is well-defined, i.e., the integral in* (A.14) *is convergent for a.e. x, and $H_0^\psi f \in L_p^{loc}(\mathbb{R}^d)$. Also, for $f \in C^\infty(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, the integral in* (A.14) *is convergent for every x and $H_0^\psi f \in C(\mathbb{R}^d)$.*

For the proof, cf. [I1].

Claim A.2. $C_0^\infty(\mathbb{R}^d) \subset \bigcap_{1\leq p\leq\infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,V})$, *and for $f \in C_0^\infty(\mathbb{R}^d)$, $\mathfrak{G}_p^{\psi,V} f = -H^\psi f$.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^d)$.

We start with the proof that

$$\tfrac{1}{t}(P_t^{\psi,V}f - f) \underset{t\downarrow 0}{\longrightarrow} -H^\psi f \qquad \text{in } C_\infty(\mathbb{R}^d). \tag{A.16}$$

Since $H^\psi f \in C_\infty(\mathbb{R}^d)$ by Claim A.1, it is enough to check pointwise convergence (cf. Lemma 31.7 in [Sa]). To do so we apply Itô's formula for (A.4) to obtain

$$\exp\Big(-\int_0^t V(x+X_s)ds\Big)f(x+X_t)$$

$$
\begin{aligned}
&= f(x) - \int_0^t \exp\Big(-\int_0^s V(x+X_r)dr\Big)V(x+X_s)f(x+X_s)ds \\
&+ \int_0^{t+}\int_{\mathbb{R}^d\setminus\{0\}} \exp\Big(-\int_0^s V(x+X_r)dr\Big)\Big(f(x+X_{s-}+y)-f(x+X_{s-})\Big)1_{|y|\geq 1}M(dsdy) \\
&+ \int_0^{t+}\int_{\mathbb{R}^d\setminus\{0\}} \exp\Big(-\int_0^s V(x+X_r)dr\Big)\Big(f(x+X_{s-}+y)-f(x+X_{s-})\Big)1_{|y|<1}\widetilde{M}(dsdy) \\
&+ \int_0^{t}\int_{\mathbb{R}^d\setminus\{0\}} \exp\Big(-\int_0^s V(x+X_r)dr\Big)\Big(f(x+X_s+y)-f(x+X_s) \\
&\qquad\qquad - \langle y, \nabla f(x+X_s)\rangle\Big)1_{|y|<1}\widehat{M}(dsdy).
\end{aligned}
$$

Note that the third term on the RHS is a martingale, so that the expectation is zero. Taking expectation and changing the variable $s = t\sigma$ we have

$$
\begin{aligned}
&\tfrac{1}{t}(P_t^{\psi,V}f(x)-f(x)) + \int_0^1 \mathbb{E}\Big[\exp\Big(-\int_0^{t\sigma} V(x+X_r)dr\Big)(Vf)(x+X_{t\sigma})\Big]d\sigma \\
&= \int_0^1 d\sigma \int_{|y|\geq 1} \mathbb{E}\Big[\exp\Big(-\int_0^{t\sigma} V(x+X_r)dr\Big)\Big(f(x+X_{t\sigma}+y)-f(x+X_{t\sigma})\Big)\Big]J(dy) \\
&+ \int_0^1 d\sigma \int_{0<|y|<1} \mathbb{E}\Big[\exp\Big(-\int_0^{t\sigma} V(x+X_r)dr\Big)\Big(f(x+X_{t\sigma}+y)-f(x+X_{t\sigma}) \\
&\qquad\qquad - \langle y, \nabla f(x+X_{t\sigma})\rangle\Big)\Big]J(dy) \\
&= \int_0^1 d\sigma \int_{\mathbb{R}^d\setminus\{0\}} \mathbb{E}\Big[\exp\Big(-\int_0^{t\sigma} V(x+X_r)dr\Big) \\
&\qquad\qquad \times \int_0^1 (1-\theta)\langle y, \nabla^2 f(x+X_{t\sigma}+\theta y)y\rangle d\theta\Big]J(dy),
\end{aligned}
\tag{A.17}
$$

where the second equality is due to Taylor's theorem with the aid of symmetry of $J(dy)$. On letting $t \downarrow 0$ in the first equality of (A.17) we have (A.16) pointwise.

Next we prove for $1 \leq p < \infty$ that

$$
\tfrac{1}{t}(P_t^{\psi,V}f - f) \underset{t\downarrow 0}{\longrightarrow} -H^\psi f \qquad \text{in } L_p(\mathbb{R}^d). \tag{A.18}
$$

Since $H^\psi f \in L_p(\mathbb{R}^d)$ by Claim A.1, it is enough to check weak convergence (cf. Lemma 32.3 in [Sa]).

First of all, we note by (A.17) that

$$
\sup_{t>0} \|\tfrac{1}{t}(P_t^{\psi,V}f - f)\|_p < \infty \quad \text{for } 1 \leq p < \infty \tag{A.19}
$$

and that

$$
\lim_{R\to\infty}\limsup_{t\downarrow 0} \int_{|x|>R} |\tfrac{1}{t}(P_t^{\psi,V}f(x) - f(x))|dx = 0. \tag{A.20}
$$

Indeed, by the second equality of (A.17)

$$|\tfrac{1}{t}(P_t^{\psi,V}f(x)-f(x))| \le \int_0^1 \mathbb{E}[|(Vf)(x+X_{t\sigma})|]d\sigma + \int_0^1 d\sigma \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 J(dy) \int_0^1 (1-\theta)\mathbb{E}[|\nabla^2 f(x+X_{t\sigma}+\theta y)|]d\theta. \tag{A.21}$$

Hence, by Minkowski's inequality, Jensen's inequality and Fubini's theorem

$$\Big(\int_{\mathbb{R}^d} |\tfrac{1}{t}(P_t^{\psi,V}f(x)-f(x))|^p dx\Big)^{1/p} \le \|Vf\|_p + \tfrac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} |y|^2 J(dy)\,\|\nabla^2 f\|_p,$$

which shows (A.19). To show (A.20), take $R_0>0$ such that $\operatorname{supp} f \subset \{x\in\mathbb{R}^d; |x|<R_0\}$, and let $R>R_0$. Note that $1_{|x|>R}h(x+y) = 1_{|x|>R}h(x+y)1_{|y|\ge R-R_0}$ for $h=f$, ∇f or $\nabla^2 f$. Hence, by (A.21),

$$\begin{aligned}
&\int_{|x|>R} |\tfrac{1}{t}(P_t^{\psi,V}f(x)-f(x))|dx \\
&\le \int_0^1 \mathbb{E}\Big[\int_{|x|>R} |(Vf)(x+X_{t\sigma})|dx\,;\,|X_{t\sigma}|\ge R-R_0\Big]d\sigma \\
&+ \int_0^1 d\sigma \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 J(dy)\int_0^1 (1-\theta)d\theta \\
&\qquad \times \mathbb{E}\Big[\int_{|x|>R} |\nabla^2 f(x+X_{t\sigma}+\theta y)|dx\,;\,|X_{t\sigma}+\theta y|\ge R-R_0\Big] \\
&\le \|Vf\|_1 \int_0^1 \mathbb{P}(|X_{t\sigma}|\ge R-R_0)d\sigma \\
&+ \tfrac{1}{2}\|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 J(dy)\,\mathbb{P}(|X_{t\sigma}|+|y|\ge R-R_0).
\end{aligned}$$

Since $\lim_{t\downarrow 0} X_{t\sigma}=0$ a.s., by the Lebesgue-Fatou inequality

$$\begin{aligned}
&\limsup_{t\downarrow 0}\int_{|x|>R} |\tfrac{1}{t}(P_t^{\psi,V}f(x)-f(x))|dx \\
&\le \tfrac{1}{2}\|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 J(dy)\ \limsup_{t\downarrow 0}\mathbb{P}(|X_{t\sigma}|+|y|\ge R-R_0) \\
&\le \tfrac{1}{2}\|\nabla^2 f\|_1 \int_{|y|\ge R-R_0} |y|^2 J(dy),
\end{aligned}$$

and thus (A.20) follows.

Now we show weak convergence in $L_p(\mathbb{R}^d)$ of (A.18). When $1<p<\infty$, let q be the conjugate exponent of p. For each $g\in L_q(\mathbb{R}^d)$ and $R>0$

$$|\langle \tfrac{1}{t}(P_t^{\psi,V}f-f)+H^\psi f, g\rangle| \le \|\tfrac{1}{t}(P_t^{\psi,V}f-f)+H^\psi f\|_\infty \int_{|x|\le R} |g(x)|dx$$

$$+ \big(\sup_{t>0} \|\tfrac{1}{t}(P_t^{\psi,V} f - f)\|_p + \|H^\psi f\|_p\big)\Big(\int_{|x|>R} |g(x)|^q dx\Big)^{1/q}.$$

By (A.16), the first term tends to zero as $t \downarrow 0$ for fixed $R > 0$, and the second term tends to zero as $R \uparrow \infty$. This shows weak convergence in $L_p(\mathbb{R}^d)$. Next, when $p = 1$, for each $g \in L_\infty(\mathbb{R}^d)$ and $R > 0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \Big(\tfrac{1}{t}(P_t^{\psi,V} f(x) - f(x)) + H^\psi f(x)\Big) g(x) dx\Big| \\
&\le \|\tfrac{1}{t}(P_t^{\psi,V} f - f) + H^\psi f\|_\infty \int_{|x|\le R} |g(x)| dx \\
&\quad + \Big(\int_{|x|>R} |\tfrac{1}{t}(P_t^{\psi,V} f(x) - f(x))| dx + \int_{|x|>R} |H^\psi f(x)| dx\Big) \|g\|_\infty.
\end{aligned}$$

Therefore, by (A.16) and (A.20), similarly we can show weak convergence in $L_1(\mathbb{R}^d)$. The proof of Claim A.2 is complete. □

Remark. When V is further a C^∞-function and all its derivatives have polynomial growth, it can be shown in exactly the same way as above that $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1\le p\le\infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,V})$ and $\mathfrak{G}_p^{\psi,V} = -H^\psi$ on $\mathcal{S}(\mathbb{R}^d)$.

By Claim A.2, H^ψ on $C_0^\infty(\mathbb{R}^d)$ is closable as an operator in $L_p(\mathbb{R}^d)$, $1 \le p < \infty$, or $C_\infty(\mathbb{R}^d)$. It is natural to ask whether or not its smallest closed extension agrees with $-\mathfrak{G}_p^{\psi,V}$. The following theorem is an affirmative answer.

Theorem A.1. *The smallest closed extension of $H^\psi = -\mathfrak{G}_p^{\psi,V}|_{C_0^\infty(\mathbb{R}^d)}$ in $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$ (resp. $H^\psi = -\mathfrak{G}_\infty^{\psi,V}|_{C_0^\infty(\mathbb{R}^d)}$ in $C_\infty(\mathbb{R}^d)$) agrees with $-\mathfrak{G}_p^{\psi,V}$ (resp. $-\mathfrak{G}_\infty^{\psi,V}$). In other words, $C_0^\infty(\mathbb{R}^d)$ is a core of $\mathfrak{G}_p^{\psi,V}$ $(1 \le p \le \infty)$.*

Needless to say, this theorem for $p = 2$, the L_2-case, says nothing but that $H_0^\psi + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

In the same way as in [I1] and [I-Tsu] we prove this theorem. Take $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\rho \ge 0$, $\operatorname{supp}\rho \subset \{x \in \mathbb{R}^d; |x| \le 1\}$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. For $0 < \delta \le 1$ set $\rho_\delta(x) := (1/\delta)^d \rho(x/\delta)$. For $u \in L_1^{loc}(\mathbb{R}^d)$, we denote the convolution $u * \rho_\delta$ by u^δ. Clearly $u^\delta \in C^\infty(\mathbb{R}^d)$ and $u^\delta \to u$ in $L_1^{loc}(\mathbb{R}^d)$ as $\delta \downarrow 0$.

Lemma A.1. *Let $1 \le q \le \infty$. Suppose $u \in L_q(\mathbb{R}^d)$ is such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$, i.e., for some $f \in L_1^{loc}(\mathbb{R}^d)$ it holds that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} f(x)\varphi(x) dx = \int_{\mathbb{R}^d} u(x) H_0^\psi \varphi(x) dx. \tag{A.22}$$

Then $H_0^\psi u^\delta \to H_0^\psi u$ in $L_1^{loc}(\mathbb{R}^d)$ as $\delta \downarrow 0$.

Proof. Since $u \in L_q(\mathbb{R}^d)$, $u^\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$. By Claim A.1, $H_0^\psi u^\delta \in L_q^{loc}(\mathbb{R}^d)$ or $\in C(\mathbb{R}^d)$ according as $1 \le q < \infty$ or $q = \infty$, and hence $H_0^\psi u^\delta \in L_1^{loc}(\mathbb{R}^d)$. For the proof, it is enough to check that $H_0^\psi u^\delta = (H_0^\psi u)^\delta$.

By (A.22)

$$\begin{aligned}
(H_0^\psi u)^\delta(x) &= \int_{\mathbb{R}^d} (H_0^\psi u)(y)\rho_\delta(x-y)dy \\
&= \int_{\mathbb{R}^d} u(y) H_0^\psi \rho_\delta(x-\cdot)(y)dy \\
&= \int_{\mathbb{R}^d} u(y)dy\Big(-\int_{\mathbb{R}^d\setminus\{0\}} \Big\{\rho_\delta(x-y-z) - \rho_\delta(x-y) \\
&\qquad - \langle z, \nabla\rho_\delta(x-\cdot)(y)\rangle 1_{|z|<1}\Big\} J(dz)\Big). \qquad \text{(A.23)}
\end{aligned}$$

The integral on the RHS is convergent, because with $\rho_\delta(x-\cdot) =: g \in C_0^\infty(\mathbb{R}^d)$, it is bounded by $\int_{|z|\ge 1} J(dz)\, \|u\|_q\, 2\|g\|_{q/(q-1)} + (1/2)\int_{0<|z|<1} |z|^2 J(dz)\, \|u\|_q\, \|\nabla^2 g\|_{q/(q-1)}$. Here when $q = 1$ or ∞ we understand $\|\cdot\|_{q/(q-1)} = \|\cdot\|_\infty$ or $\|\cdot\|_1$. Hence by noting that $\nabla\rho_\delta(x-\cdot)(y) = -(\nabla\rho_\delta)(x-y)$, Fubini's theorem gives us that

$$\begin{aligned}
(H_0^\psi u)^\delta(x) &= -\int_{\mathbb{R}^d\setminus\{0\}} J(dz)\Big(\int_{\mathbb{R}^d} u(y)\rho_\delta(x-z-y)dy - \int_{\mathbb{R}^d} u(y)\rho_\delta(x-y)dy \\
&\qquad - 1_{|z|<1}\Big\langle -z, \int_{\mathbb{R}^d} u(y)(\nabla\rho_\delta)(x-y)dy\Big\rangle\Big) \\
&= -\int_{\mathbb{R}^d\setminus\{0\}} (u^\delta(x+z) - u^\delta(x) - 1_{|z|<1}\langle z, \nabla u^\delta(x)\rangle) J(dz) \\
&= H_0^\psi u^\delta(x),
\end{aligned}$$

where the symmetry of $J(dz)$ has been used. The proof is complete. □

Lemma A.2. (Kato's inequality). *Let $1 \le q \le \infty$. Suppose $u \in L_q(\mathbb{R}^d)$ is such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$. Then the following distributional inequality holds:*

$$\operatorname{sgn} u H_0^\psi u \ \ge\ H_0^\psi |u|,$$

i.e. for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \ge 0$,

$$\int_{\mathbb{R}^d} (\operatorname{sgn} u)(x) H_0^\psi u(x)\varphi(x)dx \ \ge\ \int_{\mathbb{R}^d} |u(x)| H_0^\psi \varphi(x)dx.$$

Here $\operatorname{sgn} u$ is a bounded function on $\mathbb{R}^d$ defined by

$$(\operatorname{sgn} u)(x) := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if} \quad u(x) \neq 0 \\ 0 & \text{if} \quad u(x) = 0. \end{cases}$$

Proof. First let $u \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$. By Claim A.1 $H_0^\psi u \in L_q^{loc}(\mathbb{R}^d)$ or $\in C(\mathbb{R}^d)$ according as $1 \le q < \infty$ or $q = \infty$. For $\varepsilon > 0$, set $u_\varepsilon(x) := \sqrt{|u(x)|^2 + \varepsilon^2}$. Clearly $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $u_\varepsilon \ge \varepsilon$. Since $|u(x)|\,|u(x+y)| \le u_\varepsilon(x)\,u_\varepsilon(x+y) - \varepsilon^2$, we have

$$-|u(x)|\,|u(x+y)| + u(x)^2 \;\ge\; -u_\varepsilon(x)\,u_\varepsilon(x+y) + u_\varepsilon(x)^2.$$

By noting that $2u(x)\nabla u(x) = \nabla|u(x)|^2 = \nabla u_\varepsilon(x)^2 = 2u_\varepsilon(x)\nabla u_\varepsilon(x)$, this inequality gives us that

$$\begin{aligned}
&-u(x)\{u(x+y) - u(x) - \langle y, \nabla u(x)\rangle 1_{|y|<1}\}\\
&= \; -u(x)u(x+y) + u(x)^2 + \langle y, u(x)\nabla u(x)\rangle 1_{|y|<1}\\
&\ge \; -u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2 + \langle y, u_\varepsilon(x)\nabla u_\varepsilon(x)\rangle 1_{|y|<1}\\
&= \; -u_\varepsilon(x)\{u_\varepsilon(x+y) - u_\varepsilon(x) - \langle y, \nabla u_\varepsilon(x)\rangle 1_{|y|<1}\}.
\end{aligned}$$

Integrating both sides by $J(dy)$, we have $u(x)H_0^\psi u(x) \ge u_\varepsilon(x)H_0^\psi u_\varepsilon(x)$, or

$$\tfrac{u(x)}{u_\varepsilon(x)}\, H_0^\psi u(x) \;\ge\; H_0^\psi u_\varepsilon(x). \tag{A.24}$$

Second let $u \in L_q(\mathbb{R}^d)$ be such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$. Since $u^\delta = u * \rho_\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$, it holds by (A.24) that

$$\begin{aligned}
\int_{\mathbb{R}^d} \tfrac{u^\delta(x)}{(u^\delta)_\varepsilon(x)}\, H_0^\psi u^\delta(x)\varphi(x)dx \;&\ge\; \int_{\mathbb{R}^d} H_0^\psi (u^\delta)_\varepsilon(x)\,\varphi(x)dx\\
&= \int_{\mathbb{R}^d} (u^\delta)_\varepsilon(x) H_0^\psi \varphi(x)dx
\end{aligned} \tag{A.25}$$

for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$. In (A.25) let $\delta \downarrow 0$ first and $\varepsilon \downarrow 0$ second. As $\delta \downarrow 0$, $H_0^\psi u^\delta \to H_0^\psi u$ in $L_1^{loc}(\mathbb{R}^d)$ by Lemma A.1, and $u^\delta \to u$ in $L_1^{loc}(\mathbb{R}^d)$. By taking a subsequence if necessary we may suppose that $u^\delta \to u$ a.e. Since $|(u^\delta)_\varepsilon - u_\varepsilon| \le |u^\delta - u|$ and $|u^\delta/(u^\delta)_\varepsilon| \le 1$, $u^\delta/(u^\delta)_\varepsilon \to u/u_\varepsilon$ boundedly a.e. Hence, letting $\delta \downarrow 0$ in (A.25), we have

$$\int_{\mathbb{R}^d} \tfrac{u(x)}{u_\varepsilon(x)}\, H_0^\psi u(x)\varphi(x)dx \;\ge\; \int_{\mathbb{R}^d} u_\varepsilon(x) H_0^\psi \varphi(x)dx. \tag{A.26}$$

Finally, by $|u_\varepsilon - |u|| \le \varepsilon$ and $|u/u_\varepsilon| \le 1$, we obtain that $u/u_\varepsilon \to \operatorname{sgn} u$ boundedly as $\varepsilon \downarrow 0$. Consequently, letting $\varepsilon \downarrow 0$ in (A.26) yields that

$$\int_{\mathbb{R}^d} (\operatorname{sgn} u)(x) H_0^\psi u(x)\varphi(x)dx \;\ge\; \int_{\mathbb{R}^d} |u(x)| H_0^\psi \varphi(x)dx$$

and the proof is complete. □

Proof of Theorem A.1. First consider the L_p-case, $1 \le p < \infty$. It suffices to show that $\operatorname{Im}(H_0^\psi + V + 1) = (H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d))$ is dense in $L_p(\mathbb{R}^d)$. By the Hahn-Banach theorem, this is further reduced to show the following: Let q be the conjugate exponent of p. If $v \in L_q(\mathbb{R}^d)$ satisfies that $\langle v, (H_0^\psi + V + 1)\varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $v = 0$ in $L_q(\mathbb{R}^d)$.

Let $v \in L_q(\mathbb{R}^d)$ be as above. Then $H_0^\psi v = -(V+1)v$ and hence $H_0^\psi v \in L_1^{loc}(\mathbb{R}^d)$. By Lemma A.2, it is seen that for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} |v(x)| H_0^\psi \varphi(x) dx \;&\leq\; \int_{\mathbb{R}^d} (\operatorname{sgn} v)(x) H_0^\psi v(x)\varphi(x) dx \\ &=\; -\int_{\mathbb{R}^d} (V(x)+1)|v(x)|\varphi(x) dx,\end{aligned}$$

and hence

$$\int_{\mathbb{R}^d} |v(x)|(H_0^\psi + 1)\varphi(x) dx \;\leq\; 0. \tag{A.27}$$

Each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be approximated by a sequence $\{\varphi_n\}_{n=1}^\infty$ of $C_0^\infty(\mathbb{R}^d)$ in the sense that $\varphi_n \to \varphi$ and $(H_0^\psi+1)\varphi_n \to (H_0^\psi+1)\varphi$ in $L_p(\mathbb{R}^d)$. If φ is moreover nonnegative, so are $\{\varphi_n\}_{n=1}^\infty$. Therefore (A.27) is valid for nonnegative $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Now note that the resolvent $(1-\mathfrak{G}_p^{\psi,0})^{-1}$ is expressed as

$$(1-\mathfrak{G}_p^{\psi,0})^{-1} f(x) \;=\; \int_0^\infty e^{-t}\mathbb{E}[f(x+X_t)]dt.$$

Then it is not difficult to check that if $f \in \mathcal{S}(\mathbb{R}^d)$, then $(1-\mathfrak{G}_p^{\psi,0})^{-1} f \in \mathcal{S}(\mathbb{R}^d)$ and further, if f is nonnegative, so is $(1-\mathfrak{G}_p^{\psi,0})^{-1} f$. Also, by Remark following Claim A.2 (with $V(x) \equiv 0$) $f = (1-\mathfrak{G}_p^{\psi,0})(1-\mathfrak{G}_p^{\psi,0})^{-1} f = (H_0^\psi+1)(1-\mathfrak{G}_p^{\psi,0})^{-1} f$. Hence, by (A.27)

$$\int_{\mathbb{R}^d} |v(x)| f(x) dx \;\leq\; 0,$$

whence it immediately follows that $v = 0$ and the proof in the L_p-case is complete.

Next let us consider the C_∞-case. In the same reason as above we have only to show that $(H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d))$ is dense in $C_\infty(\mathbb{R}^d)$. For this let $\nu \in C_\infty(\mathbb{R}^d)^*$, the dual of $C_\infty(\mathbb{R}^d)$, be such that $\langle \nu, (H_0^\psi + V + 1)\varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. By the Riesz-Banach theorem, ν is regarded as a finite signed Borel measure on $\mathbb{R}^d$, and thus

$$\int_{\mathbb{R}^d} (H_0^\psi + V + 1)\varphi(x)\nu(dx) \;=\; 0 \qquad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{A.28}$$

Let $\nu^\delta = \nu * \rho_\delta$, i.e., $\nu^\delta(x) := \int_{\mathbb{R}^d} \rho_\delta(x-y)\nu(dy), x \in \mathbb{R}^d$. Then $\nu^\delta \in C_b^\infty(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. It follows by Claim A.1 and by (A.28) that $H_0^\psi \nu^\delta \in C(\mathbb{R}^d)$ and

$$H_0^\psi \nu^\delta \;=\; -\nu^\delta(1+V) - \int_{\mathbb{R}^d} (V(y) - V(\cdot))\rho_\delta(\cdot - y)\nu(dy).$$

By Lemma A.2, this implies that for nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{R}^d} |\nu^\delta(x)| H_0^\psi \varphi(x) dx \\ &\leq \int_{\mathbb{R}^d} (\operatorname{sgn} \nu^\delta)(x) H_0^\psi \nu^\delta(x)\varphi(x) dx \\ &\leq -\int_{\mathbb{R}^d} |\nu^\delta(x)|\varphi(x) dx + \int_{\mathbb{R}^d} \varphi(x) dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy),\end{aligned}$$

where $|\nu|$ is the total variation of ν, and hence we have that for nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi(x)dx \ \le \ \int_{\mathbb{R}^d} \varphi(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy). \qquad \text{(A.29)}$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$ be nonnegative. Then, $\varphi := (1 - \mathfrak{G}_\infty^{\psi,0})^{-1} f$ is in $\mathcal{S}(\mathbb{R}^d)$ and nonnegative, and $(H_0^\psi + V)\varphi = f$. Set $\varphi_n(x) := \varphi(x)\chi(|x|^2/n^2)$ with a $\chi \in C^\infty([0,\infty) \to \mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ $(0 \le t \le 1)$ and $\chi(t) = 0$ $(t \ge 2)$. Clearly $\varphi_n \in C_0^\infty(\mathbb{R}^d)$, $0 \le \varphi_n \le \varphi$ and $\operatorname{supp} \varphi_n \subset \{x; |x| \le \sqrt{2}n\}$. Moreover $\|\varphi_n - \varphi\|_\infty$ and $\|H_0^\psi \varphi_n - H_0^\psi \varphi\|_\infty \to 0$ as $n \to \infty$. From (A.29) and this observation it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nu^\delta(x)| f(x)dx \ &= \ \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi(x)dx \\
&= \ \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi_n(x)dx + \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)(\varphi - \varphi_n)(x)dx \\
&\le \ \int_{\mathbb{R}^d} \varphi_n(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\
&\quad + \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty |\nu|(\mathbb{R}^d) \\
&\le \ \|\varphi\|_\infty \int_{|x| \le \sqrt{2}n} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\
&\quad + \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty |\nu|(\mathbb{R}^d).
\end{aligned}$$

Here, recalling that $\rho_\delta(z)$ has support in $\{z; |z| \le \rho\}$, we see that for each $n \in \mathbb{N}$

$$\begin{aligned}
&\int_{|x| \le \sqrt{2}n} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\
&\le \ \int_{|y| \le \sqrt{2}n + \delta} |\nu|(dy) \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)dx \\
&= \ \int_{|y| \le \sqrt{2}n + \delta} |\nu|(dy) \int_{|x| \le 1} |V(y) - V(y + \delta x)|\rho(x)dx \\
&\longrightarrow 0 \quad \text{as } \delta \downarrow 0.
\end{aligned}$$

On the other hand, noting that $\nu^\delta(x)dx \to \nu(dx)$ weakly, we see that

$$\int_{\mathbb{R}^d} |\nu^\delta(x)| f(x)dx \ \ge \ \Big|\int_{\mathbb{R}^d} f(x)\nu^\delta(x)dx\Big| \longrightarrow \Big|\int_{\mathbb{R}^d} f(x)\nu(dx)\Big| \quad \text{as } \delta \downarrow 0.$$

Therefore it follows that $\int_{\mathbb{R}^d} f(x)\nu(dx) = 0$ for $f \in \mathcal{S}(\mathbb{R}^d)$, $f \ge 0$, which implies that $\nu = 0$, and the proof in the C_∞-case is complete. □

In this paper we have denoted the semigroups $P_t^{\psi,0}$ and $P_t^{\psi,V}$ by $e^{-tH_0^\psi}$ and $e^{-t(H_0^\psi + V)}$, respectively, taking Theorem A.1 into account. With the general theory ([Trot], [Ch]) we have taken for granted that the Trotter product formula holds in the strong topology of $L_p(\mathbb{R}^d)$ or $C_\infty(\mathbb{R}^d)$:

$$\begin{aligned}
s\text{-}\lim_{n\to\infty} (e^{-tH_0^\psi/n} e^{-tV/n})^n \ &= \ s\text{-}\lim_{n\to\infty} (e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n \\
&= \ s\text{-}\lim_{n\to\infty} (e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n \ = \ e^{-t(H_0^\psi + V)}.
\end{aligned}$$

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