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# ON HOMOGENIZATION OF ELLIPTIC EQUATIONS WITH RANDOM COEFFICIENTS

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**Abstract** In this paper, we investigate the rate of convergence of the solution $u_\varepsilon$ of the random elliptic partial difference equation $(\nabla^{\varepsilon *} a(x/\varepsilon, \omega)\nabla^\varepsilon + 1)u_\varepsilon(x,\omega) = f(x)$ to the corresponding homogenized solution. Here $x \in \varepsilon\mathbf{Z}^d$, and $\omega \in \Omega$ represents the randomness. Assuming that $a(x)$'s are independent and uniformly elliptic, we shall obtain an upper bound $\varepsilon^\alpha$ for the rate of convergence, where $\alpha$ is a constant which depends on the dimension $d \geq 2$ and the deviation of $a(x,\omega)$ from the identity matrix. We will also show that the (statistical) average of $u_\varepsilon(x,\omega)$ and its derivatives decay exponentially for large $x$.

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## 1. Introduction.

In this paper we shall be concerned with the problem of homogenization of elliptic equations in divergence form. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathbf{a} : \Omega \to \mathbf{R}^{d(d+1)/2}$ be a bounded measurable function from $\Omega$ to the space of symmetric $d \times d$ matrices. We assume that there are positive constants $\lambda, \Lambda$ such that

$$\lambda I_d \le \mathbf{a}(\omega) \le \Lambda I_d, \quad \omega \in \Omega, \tag{1.1}$$

in the sense of quadratic forms, where $I_d$ is the identity matrix in $d$ dimensions. We assume that $\mathbf{Z}^d$ acts on $\Omega$ by translation operators $\tau_x : \Omega \to \Omega,\ x \in \mathbf{Z}^d$, which are measure preserving and satisfy the properties $\tau_x \tau_y = \tau_{x+y},\ \tau_0 = \text{identity},\ x, y \in \mathbf{Z}^d$. Using these operators we can define a measurable matrix valued function on $\mathbf{Z}^d \times \Omega$ by $\mathbf{a}(x, \omega) = \mathbf{a}(\tau_x \omega), \quad x \in \mathbf{Z}^d,\ \omega \in \Omega$.

Let $\mathbf{Z}^d_\varepsilon = \varepsilon \mathbf{Z}^d$ be the $\varepsilon$ scaled integer lattice where $\varepsilon > 0$. For functions $g : \mathbf{Z}^d_\varepsilon \to \mathbf{R}$ we define the discrete derivative $\nabla^\varepsilon_i g$ of $g$ in the $i$th direction to be

$$\nabla^\varepsilon_i\ g(x) \stackrel{\text{def.}}{=} [g(x + \varepsilon \mathbf{e}_i) - g(x)]/\varepsilon, \quad x \in \mathbf{Z}^d_\varepsilon,$$

where $\mathbf{e}_i \in \mathbf{Z}^d$ is the element with entry 1 in the $i$th position and 0 in other positions. The formal adjoint of $\nabla^\varepsilon_i$ is given by $\nabla^{\varepsilon *}_i$, where

$$\nabla^{\varepsilon *}_i\ g(x) \stackrel{\text{def.}}{=} [g(x - \varepsilon \mathbf{e}_i) - g(x)]/\varepsilon, \quad x \in \mathbf{Z}^d_\varepsilon\ .$$

We shall be interested in solutions of the elliptic equation,

$$\sum_{i,j=1}^d \nabla^{\varepsilon *}_i \left[ a_{ij}\left(\frac{x}{\varepsilon},\ \omega\right) \nabla^\varepsilon_j u_\varepsilon(x, \omega) \right] + u_\varepsilon(x, \omega) = f(x), x \in \mathbf{Z}^d_\varepsilon,\ \omega \in \Omega. \tag{1.2}$$

Here $f : \mathbf{R}^d \to \mathbf{R}$ is assumed to be a smooth function with compact support and $a_{ij}(y, \omega)$ are the entries of the matrix $\mathbf{a}(y, \omega),\ y \in \mathbf{Z}^d$.

It is well known [7, 4] that, under the assumptions of ergodicity of the translation operators $\tau_x,\ x \in \mathbf{Z}^d$, the solution of (1.2) converges as $\varepsilon \to 0$ to the solution of the homogenized equation,

$$-\sum_{i,j=1}^d q_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + u(x) = f(x), \quad x \in \mathbf{R}^d. \tag{1.3}$$

Here $q = [q_{ij}]$ is a symmetric positive definite matrix determined from $\mathbf{a}(\omega),\ \omega \in \Omega$. If we denote expectation value on $\Omega$ by $\langle\ \ \rangle$ and

$$\int_{\mathbf{Z}^d_\varepsilon} dx \stackrel{\text{def.}}{=} \varepsilon^d \sum_{x \in \mathbf{Z}^d_\varepsilon},$$

then one has

$$\lim_{\varepsilon \to 0} \left\langle \int_{\mathbf{Z}^d_\varepsilon} \left| u_\varepsilon(x, \cdot) - u(x) \right|^2 dx \right\rangle = 0. \tag{1.4}$$

See [2] for extensions to unbounded, non-symmetric $a$'s.

Our goal in this paper is to estimate the rate of convergence in the limit (1.4). To do this we shall need to make rather restrictive assumptions on the matrix $\mathbf{a}(\cdot)$ beyond the uniform boundedness assumptions (1.1). We can however prove a result, just assuming (1.1), which is helpful for us in studying the rate of convergence in (1.4). To motivate it observe that since

(1.3) is a constant coefficient equation it is easy to see that the solution is $C^\infty$ and that for any $n$ tuple $\alpha = (\alpha_1, ..., \alpha_n)$ with $1 \le \alpha_i \le d,\ i = 1, ..., n$, one has

$$\sup_{x \in \mathbf{R}^d} \left| \mathrm{e}^{\delta|x|} \Big( \prod_{i=1}^{n} \frac{\partial}{\partial x_{\alpha_i}} \Big) u(x) \right| \le A_\alpha \ , \tag{1.5}$$

where $\delta > 0$ can be chosen to be fixed and $A_\alpha$ is a constant depending only on $\alpha$ and $f$. Consider now the problem of proving an inequality analogous to (1.5) for the expectation $\big\langle u_\varepsilon(x, \cdot) \big\rangle$ of the solution $u_\varepsilon(x, \omega)$ of (1.2). In view of the uniform boundedness (1.1) there exists $\delta > 0$, independent of $\varepsilon$ such that

$$\sup_{x \in \mathbf{R}^d} \left| \mathrm{e}^{\delta|x|} u_\varepsilon(x, \omega) \right| \le C, \quad \omega \in \Omega,$$

where $C$ is a constant. To prove this one needs to use the deep theory of Nash [3]. One evidently can immediately conclude that

$$\sup_{x \in \mathbf{R}^d} \left| e^{\delta|x|} \big\langle u_\varepsilon(x, \cdot) \big\rangle \right| \le C.$$

Nash's methods will not however yield similar inequalities on derivatives of $\langle u_\varepsilon(x, \cdot) \rangle$. In §3 we prove the following analogue of (1.5):

**Theorem 1.1.** *Suppose (1.1) holds, $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support and $u_\varepsilon(x, \omega)$ is the solution of (1.2). Then there exists a constant $\delta > 0$ depending only on $\lambda, \Lambda$, such that for any $n$ tuple $\alpha = (\alpha_1, ..., \alpha_n)$ with $1 \le \alpha_i \le d$, $i = 1, ..., n$, one has*

$$\sup_{x \in \mathbf{R}^d} \left| e^{\delta|x|} \left( \prod_{i=1}^{n} \nabla^\varepsilon_{\alpha_i} \right) \big\langle u_\varepsilon(x, \cdot) \big\rangle \right| \le A_\alpha \ ,$$

*where $A_\alpha$ depends only on $\lambda, \Lambda, \alpha$ and $f$.*

We can obtain a rate of convergence in (1.4) if we assume that the random matrix $\mathbf{a}(\cdot)$ satisfies (1.1) and that the matrices $\mathbf{a}(\tau_x \cdot), x \in \mathbf{Z}^d$, are independent. Our first theorem is as follows:

**Theorem 1.2.** *Suppose $\mathbf{a}(\cdot)$ satisfies (1.1), the matrices $\mathbf{a}(\tau_x \cdot), x \in \mathbf{Z}^d$, are independent, and $\gamma = 1 - \lambda/\Lambda$, $|\gamma| < 1$. Let $u_\varepsilon(x, \omega)$ be the solution of (1.2) where $f : \mathbf{R}^d \to \mathbf{R}$ is assumed to be a $C^\infty$ function of compact support. Then for $d \ge 2$ there is the inequality*

$$\left\langle \int_{\mathbf{Z}^d_\varepsilon} \left| u_\varepsilon(x, \cdot) - \big\langle u_\varepsilon(x, \cdot) \big\rangle \right|^2 dx \right\rangle \le C\varepsilon^{2\alpha},$$

*where $\alpha > 0$ is a constant depending only on $\gamma$, and $C$ only on $\gamma$ and $f$. If $d \ge 3$ then one can take $\alpha = 1$ for sufficiently small $\gamma > 0$. For $d = 2$, $\alpha$ can be taken arbitrarily close to $1$ if $\gamma > 0$ is taken sufficiently small.*

**Remark 1.** *One can see by explicit computation that when $d = 1$, then $\alpha = 1/2$.*

Theorem 1.2 is independent of the dimension $d$ when $d \ge 3$. We also have a theorem which is dimension dependent for all $d$.

**Theorem 1.3.** *Suppose $\mathbf{a}(\cdot)$ and $f$ satisfy the same conditions as in Theorem 1.2. Let $g : \mathbf{R}^d \to \mathbf{R}$ be a $C^\infty$ function of compact support. Then for $d \ge 2$ there is the inequality,*

$$\left\langle \left\{ \int_{\mathbf{Z}^d_\varepsilon} g(x) \Big[ u_\varepsilon(x, \cdot) - \big\langle u_\varepsilon(x, \cdot) \big\rangle \Big] dx \right\}^2 \right\rangle \le C\varepsilon^\beta,$$

*where $\beta > 0$ is a constant depending only on $\gamma$, and $C$ only on $\gamma, g$ and $f$. The number $\beta$ can be taken arbitrarily close to $d$ if $\gamma > 0$ is taken sufficiently small.*

**Remark 2.** *Theorem 1.3 only gives more information than Theorem 1.2 in the case when* $d \geq 3$.

We first prove Theorems 2 and 3 under the assumption that the $\mathbf{a}(x, \omega),\ x \in \mathbf{Z}^d,\ \omega \in \Omega$, are given by independent Bernoulli variables. This is accomplished in §4. We put

$$\mathbf{a}(x, \cdot) = (1 + \gamma Y_x) I_d, \quad x \in \mathbf{Z}^d, \tag{1.6}$$

where the random variables $Y_x$ , $x \in \mathbf{Z}^d$, are assumed to be iid Bernoulli, $Y_x = \pm 1$ with equal probability. We must take $\gamma$ in (1.6) to satisfy $|\gamma| < 1$ to ensure $\mathbf{a}(\cdot)$ remains positive definite. In §5 the method is extended to the general case. The methods can also be extended to deal with variables $\mathbf{a}(x, \cdot),\ x \in \mathbf{Z}^d$, which are weakly correlated. To carry this out one must have, however, rather detailed knowledge on the decay of correlation functions.

To prove theorem 1.2 we use an idea from the proof of (1.4) in [7]. Thus we make an approximation $u_\varepsilon(x, \omega) \simeq u(x) +$ random term, where $u(x)$ is the solution of (1.3). The random term is obtained from the solution $\Psi(\omega)$ of a variational problem on $\Omega$ (see lemma 2.2). The difference between $u_\varepsilon(x, \omega)$ and the above approximation is estimated in proposition 2.1. One can obtain a new proof of the homogenization result (1.4) from proposition 2.1 by using the fact that $\Psi$ is square integrable on $\Omega$ and applying the von Neumann ergodic theorem. For the proof of theorem 1.2 one needs to show that $\Psi$ is $p$ integrable for some $p < 2$. This $p$ integrability is not in the conventional sense $< |\Psi|^p > < \infty$. If $\mathbf{a}(x, \cdot)$ is given by (1.6), one expands $\Psi$ in the orthonormal basis of Walsh functions for $L^2(\Omega)$ generated by the Bernoulli variables $Y_x,\ x \in \mathbf{Z}^d$. We say that $\Psi$ is $p$ integrable if the coefficients of $\Psi$ in this basis are $p$ summable. Evidently if $p = 2$ then $p$ integrability of $\Psi$ and $< |\Psi|^p > < \infty$ are equivalent, but not if $p$ is different from 2. We use the Calderon-Zygmund theorem [3] to show that $\Psi$ is $p$ integrable for some $p < 2$.

Observe that when $\mathbf{a}(x, \cdot)$ is given by (1.6) then the entries of the matrix $\mathbf{a}(\cdot)$ generate a 2 dimensional subspace of $L^2(\Omega)$. One can readily generalise the proof described in the previous paragraph to all random matrices $\mathbf{a}(\cdot)$ whose entries generate a finite dimensional subspace of $L^2(\Omega)$. The main task of §5 is to extend the argument to the situation where the entries of $\mathbf{a}(\cdot)$ generate an infinite dimensional subspace of $L^2(\Omega)$. To carry this out we introduce the Legendre polynomials $P_l(z)$ to give us an approximately orthogonal basis for the space generated by the entries of $\mathbf{a}(\cdot)$. We use in a crucial way the fact that the ratio of the $L^\infty$ norm of $P_l$ to the $L^2$ norm, on the interval $[-1, 1]$, is polynomially bounded in $l$ (in fact $\sqrt{2l+1}$ ).

We cannot use proposition 2.1 to prove theorem 1.3. Instead, we have to make use of the Fourier representation for $u_\varepsilon(x, \omega)$ given in §3. This representation appears to have considerable power. To illustrate this we use it to prove theorem 1.1. The remainder of the proof of theorem 1.3 then follows along the same lines as the proof of theorem 1.2

The research in this paper was motivated by previous work of Naddaf and Spencer [6]. Let $\mathbf{A} : \mathbf{R} \to \mathbf{R}^{d(d+1)/2}$ be a bounded measurable function from $\mathbf{R}$ to the space of real symmetric $d \times d$ matrices. Assume that there are positive constants $\lambda, \Lambda$, such that

$$\lambda I_d \leq \mathbf{A}(\phi) \leq \Lambda I_d, \quad \phi \in \mathbf{R}.$$

Let $\phi(x, \omega),\ x \in \mathbf{Z}^d,\ \omega \in \Omega$, be an Euclidean field theory satisfying the Brascamp-Lieb inequality [5]. The matrices $\mathbf{a}(x, \omega),\ x \in \mathbf{Z}^d,\ \omega \in \Omega$, are obtained by setting

$$\mathbf{a}(x, \omega) = \mathbf{A}(\phi(x, \omega)),\ x \in \mathbf{Z}^d,\ \omega \in \Omega \quad .$$

Naddaf and Spencer prove that the results of Theorem 1.3 hold under the assumption that $\phi$ is a massive field theory and $\mathbf{A}$ has bounded derivative. They further prove that if $\gamma$ is sufficiently small then one can take $\beta = d$. They also have corresponding results when $\phi$ is assumed to be massless.

The Russian literature [8, 11] contains some previous work on the rate of convergence to homogenisation. This appears not to be rigorous.

## 2. Variational Formulation.

In this section we set out the variational formulation for the solution of (1.2) and for the effective diffusion matrix $q = [q_{ij}]$ in (1.3). Let $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ be the space of all measurable functions $u_\varepsilon : \mathbf{Z}_\varepsilon^d \times \Omega \to \mathbf{R}$ which satisfy

$$\|u_\varepsilon\|_{\mathcal{H}^1}^2 \stackrel{\text{def.}}{=} \left\langle \sum_{i=1}^d \int_{\mathbf{Z}_\varepsilon^d} \left[\nabla_i^\varepsilon u_\varepsilon(x,\cdot)\right]^2 dx + \int_{\mathbf{Z}_\varepsilon^d} u_\varepsilon(x,\cdot)^2 dx \right\rangle < \infty. \tag{2.1}$$

Evidently $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ is a Hilbert space with norm defined by (2.1). We define a functional $\mathcal{G}$ on $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ corresponding to the equation (1.2) by

$$\mathcal{G}(u_\varepsilon) \stackrel{\text{def.}}{=} \Bigg\langle \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\ \cdot\right) \nabla_i^\varepsilon u_\varepsilon(x,\cdot) \nabla_j^\varepsilon u_\varepsilon(x,\cdot) + \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} dx\ u_\varepsilon(x,\cdot)^2 - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x) u_\varepsilon(x,\cdot) \Bigg\rangle. \tag{2.2}$$

The following lemma is then a consequence of the Banach-Alaoglu theorem [9].

**Lemma 2.1.** *The functional $\mathcal{G} : \mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega) \to \mathbf{R}$ has a unique minimizer $u_\varepsilon \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ which satisfies the Euler-Lagrange equation,*

$$\Bigg\langle \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\cdot\right) \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) \nabla_j^\varepsilon u_\varepsilon(x,\cdot) + \int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon(x,\cdot) u_\varepsilon(x,\cdot) - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x) \psi_\varepsilon(x,\cdot) \Bigg\rangle = 0, \tag{2.3}$$

*for all $\psi_\varepsilon \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$.*

Next we turn to the variational formulation of the diffusion matrix $q$ in (1.3). To do this we use the translation operators $\tau_x$ on $\Omega$ to define discrete differentiation of a function on $\Omega$. Let $\varphi : \Omega \to \mathbf{R}$ be a measurable function. For $i = 1, ..., d$ we define $\partial_i \varphi$ by

$$\partial_i \varphi(\omega) \stackrel{\text{def.}}{=} \varphi(\tau_{\mathbf{e}_i}\omega) - \varphi(\omega), \qquad \omega \in \Omega.$$

The formal adjoint $\partial_i^*$ of $\partial_i$ is given by

$$\partial_i^* \varphi(\omega) \stackrel{\text{def.}}{=} \varphi(\tau_{-\mathbf{e}_i}\omega) - \varphi(\omega), \qquad \omega \in \Omega.$$

The discrete gradient of $\varphi$, $\nabla\varphi$ is then a function $\nabla\varphi : \Omega \to \mathbf{R}^d$ given by $\nabla\varphi(\omega) = (\partial_1\varphi(\omega), ..., \partial_d\varphi(\omega))$, $\omega \in \Omega$. For a function $\Psi : \Omega \to \mathbf{R}^d$ let $\|\Psi\|_2$ denote the $L^2$ norm,

$$\|\Psi\|_2^2 \stackrel{\text{def.}}{=} \left\langle \sum_{i=1}^d \Psi_i^2 \right\rangle,$$

where $\Psi = (\Psi_1, ..., \Psi_d)$. We consider the linear space

$$\mathcal{E} = \{\nabla\varphi : \Omega \to \mathbf{R}^d | \varphi : \Omega \to \mathbf{R} \text{ is measurable and } \|\varphi\|_2 < \infty\}.$$

Evidently if $\Psi \in \mathcal{E}$, $\Psi = (\Psi_1, ..., \Psi_d)$, then $\langle\Psi\rangle = 0$ and $\partial_i\Psi_j = \partial_j\Psi_i, \quad 1 \le i, j \le d$. Hence if $\mathcal{H}(\Omega)$ is the completion of $\mathcal{E}$ under the norm $\| \quad \|_2$, one also has

$$\big\langle\Psi\big\rangle = 0, \quad \partial_i\Psi_j = \partial_j\Psi_i, \quad 1 \le i, j \le d, \quad \Psi \in \mathcal{H}(\Omega). \tag{2.4}$$

For each $k$, $1 \le k \le d$, we define a functional $\mathcal{G}_k$ on $\mathcal{H}(\Omega)$ by

$$\mathcal{G}_k(\Psi) \stackrel{\text{def.}}{=} \left\langle \ \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\cdot)\Psi_i(\cdot)\Psi_j(\cdot) + \sum_{j=1}^{d} a_{kj}(\cdot)\Psi_j(\cdot) \ \right\rangle.$$

We have then again from Banach-Alaoglu:

**Lemma 2.2.** *The functional $\mathcal{G}_k : \mathcal{H}(\Omega) \to \mathbf{R}$ has a unique minimizer $\Psi^k \in \mathcal{H}(\Omega)$ which satisfies the Euler-Lagrange equation,*

$$\left\langle \sum_{i,j=1}^{d} a_{ij}(\cdot)\Psi_i(\cdot)\Psi_j^k(\cdot) + \sum_{j=1}^{d} a_{kj}(\cdot)\Psi_j(\cdot) \right\rangle = 0,$$

*for all $\Psi \in \mathcal{H}(\Omega)$.*

The matrix $q = [q_{kk'}]$ is given by

$$q_{kk'} \stackrel{\text{def.}}{=} \Big\langle [\mathbf{e}_k + \Psi^k(\cdot)]\mathbf{a}(\cdot)[\mathbf{e}_{k'} + \Psi^{k'}(\cdot)] \Big\rangle. \tag{2.5}$$

In view of the fact that $\langle\Psi^k\rangle = 0$, $k = 1, \dots, d$, it follows that $q$ is strictly positive definite. From the Euler-Lagrange equation one has the alternative expression,

$$q_{kk'} = \Big\langle a_{kk'}(\cdot) + \sum_{j=1}^{d} a_{kj}(\cdot)\Psi_j^{k'}(\cdot) \Big\rangle. \tag{2.6}$$

Next, let $\Delta$ be the discrete Laplacian on $\mathbf{Z}^d$. Thus if $u : \mathbf{Z}^d \to \mathbf{R}$, then $\Delta u$ is defined by

$$\Delta u(x) \stackrel{\text{def.}}{=} \sum_{i=1}^{d} [u(x + \mathbf{e}_i) + u(x - \mathbf{e}_i) - 2u(x)], \quad x \in \mathbf{Z}^d.$$

We denote by $G_\eta$ the Green's function for the operator $-\Delta + \eta$, where $\eta > 0$ is a parameter. Thus

$$-\Delta G_\eta(x) + \eta G_\eta(x) = \delta(x), \quad x \in \mathbf{Z}^d, \tag{2.7}$$

and $\delta$ is the Kronecker $\delta$ function; $\delta(x) = 0$ if $x \ne 0$, $\delta(0) = 1$. For any $\Psi \in \mathcal{H}(\Omega)$ we can use $G_\eta$ to define a function $\chi : \mathbf{Z}^d \times \Omega \to \mathbf{R}$ by the formula

$$\chi(x, \omega) \stackrel{\text{def.}}{=} \sum_{y\in\mathbf{Z}^d}\sum_{j=1}^{d} G_\eta(x - y)\partial_j^*\Psi_j(\tau_y\omega), \ x \in \mathbf{Z}^d, \ \omega \in \Omega. \tag{2.8}$$

**Lemma 2.3.** *For each $x \in \mathbf{Z}^d$ the function $\chi(x, \cdot)$ on $\Omega$ is in $L^2(\Omega)$. Furthermore, for $1 \le i \le d$,*

$$\nabla_i\chi(x, \omega) = \chi(x + \mathbf{e}_i, \omega) - \chi(x, \omega) = \Psi_i(\tau_x\omega) - \eta\sum_{y\in\mathbf{Z}^d} G_\eta(x - y)\Psi_i(\tau_y\omega), \tag{2.9}$$

$$x \in \mathbf{Z}^d, \omega \in \Omega.$$

*Proof.* The fact that $\chi(x,\cdot)$ is in $L^2(\Omega)$ follows easily from the exponential decay of $G_\eta$. To prove (2.9) we use the relations (2.4). Thus

$$\begin{aligned}\nabla_i\chi(x,\omega) &= \sum_{y\in\mathbf{Z}^d}\sum_{j=1}^d G_\eta(x-y)\left[\partial_j^*\psi_j(\tau_{y+e_i}\omega)-\partial_j^*\psi_j(\tau_y\omega)\right]\\ &= \sum_{y\in\mathbf{Z}^d}\sum_{j=1}^d G_\eta(x-y)\partial_j^*\partial_i\Psi_j(\tau_y\omega)\\ &= \sum_{y\in\mathbf{Z}^d}\sum_{j=1}^d G_\eta(x-y)\partial_j^*\partial_j\Psi_i(\tau_y\omega)\end{aligned}$$

from (2.4). Now a summation by parts yields

$$\nabla_i\chi(x,\omega)=\sum_{y\in\mathbf{Z}^d}-\Delta G_\eta(x-y)\Psi_i(\tau_y\omega),$$

which gives (2.9) on using (2.7). □

The proof of homogenization [7, 4] proceeds by writing the minimizer $u_\varepsilon$ in Lemma 2.1 approximately as

$$u_\varepsilon(x,\omega)\simeq u(x)+\varepsilon\sum_{k=1}^d\chi_k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^\varepsilon u(x),\ \ x\in\mathbf{Z}_\varepsilon^d,\quad\omega\in\Omega, \tag{2.10}$$

where $\chi_k$ is the function (2.8) corresponding to the minimizer $\Psi^k$ of $\mathcal{G}_k$ in Lemma 2.2. Clearly the parameter $\eta$ must be chosen appropriately, depending on $\varepsilon$. Now let $Z_\varepsilon(x,\omega)$ be the RHS of (2.10) minus the LHS,

$$Z_\varepsilon(x,\omega)\stackrel{\text{def.}}{=}u(x)+\varepsilon\sum_{k=1}^d\chi_k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^\varepsilon u(x)-u_\varepsilon(x,\omega), \tag{2.11}$$

and $\psi_\varepsilon$ be an arbitrary function in $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d\times\Omega)$. Then from Lemma 2.1 we have that

$$\begin{aligned}&\left\langle\sum_{i,j=1}^d\int_{\mathbf{Z}_\varepsilon^d}dx\,a_{ij}\left(\frac{x}{\varepsilon},\cdot\right)\nabla_i^\varepsilon\psi_\varepsilon(x,\cdot)\nabla_j^\varepsilon Z_\varepsilon(x,\cdot)+\int_{\mathbf{Z}_\varepsilon^d}dx\,\psi_\varepsilon(x,\cdot)Z_\varepsilon(x,\cdot)\right\rangle=\\ &\quad\left\langle\sum_{i,j=1}^d\int_{\mathbf{Z}_\varepsilon^d}dx\,a_{ij}\left(\frac{x}{\varepsilon},\cdot\right)\nabla_i^\varepsilon\psi_\varepsilon(x,\cdot)\Bigg[\nabla_j^\varepsilon u(x)+\sum_{k=1}^d\nabla_j\chi_k\left(\frac{x}{\varepsilon},\cdot\right)\nabla_k^\varepsilon u(x+\varepsilon\mathbf{e}_j)\right.\\ &\quad+\sum_{k=1}^d\varepsilon\chi_k\left(\frac{x}{\varepsilon},\cdot\right)\nabla_j^\varepsilon\nabla_k^\varepsilon u(x)\Bigg]+\int_{\mathbf{Z}_\varepsilon^d}dx\,\psi_\varepsilon(x,\cdot)\Bigg[u(x)+\varepsilon\sum_{k=1}^d\chi_k\left(\frac{x}{\varepsilon},\cdot\right)\nabla_k^\varepsilon u(x)\Bigg]\\ &\quad\left.-\int_{\mathbf{Z}_\varepsilon^d}dx\,f(x)\psi_\varepsilon(x,\cdot)\right\rangle.\end{aligned} \tag{2.12}$$

The first two terms on the RHS of the last equation can be rewritten as,

$$
(2.13)\quad \left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\cdot\right) \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot)\Big[\nabla_j^\varepsilon u(x) + \sum_{k=1}^{d} \Psi_j^k(\tau_{x/\varepsilon}\ \cdot)\nabla_k^\varepsilon u(x)\Big]\right\rangle
$$

$$
+\left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\cdot\right) \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) \sum_{k=1}^{d} \Psi_j^k(\tau_{x/\varepsilon}\ \cdot) \times [\nabla_k^\varepsilon u(x+\varepsilon\mathbf{e}_j) - \nabla_k^\varepsilon u(x)]\right\rangle
$$

$$
-\left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\cdot\right) \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) \sum_{k=1}^{d} \nabla_k^\varepsilon u(x+\varepsilon\mathbf{e}_j)\eta \times \sum_{y\in\mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon}-y\right)\Psi_j^k(\tau_y\cdot)\right\rangle.
$$

The first term in the last expression can be rewritten as

$$
\left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot)\nabla_j^\varepsilon u(x)\Big[a_{ij}(\tau_{x/\varepsilon}\ \cdot) + \sum_{k=1}^{d} a_{ik}(\tau_{x/\varepsilon}\ \cdot)\Psi_k^j(\tau_{x/\varepsilon}\ \cdot)\Big]\right\rangle.
$$

Observe next that

$$
\nabla_i^\varepsilon \psi_\varepsilon(x,\cdot)\nabla_j^\varepsilon u(x) = \psi_\varepsilon(x,\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(x) + \nabla_i^\varepsilon\left[\psi_\varepsilon(x,\cdot)\nabla_j^\varepsilon u(x-\varepsilon\mathbf{e}_i)\right].
$$

We have now that

$$
(2.14)\quad \left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, \nabla_i^\varepsilon[\psi_\varepsilon(x,\cdot)\nabla_j^\varepsilon u(x-\varepsilon\mathbf{e}_i)]\Big[a_{ij}(\tau_{x/\varepsilon}\ \cdot) + \sum_{k=1}^{d} a_{ik}(\tau_{x/\varepsilon}\ \cdot)\Psi_k^j(\tau_{x/\varepsilon}\ \cdot)\Big]\right\rangle
$$

$$
= \sum_{j=1}^{d}\left\langle \sum_{i=1}^{d} \partial_i\ \Phi^{i,j}(\cdot)\Big[a_{ij}(\cdot) + \sum_{k=1}^{d} a_{ik}(\cdot)\Psi_k^j(\cdot)\Big]\right\rangle,
$$

where

$$
\Phi^{i,j}(\omega) \overset{\text{def.}}{=} \varepsilon^{-1}\int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon(x,\ \tau_{-x/\varepsilon}\ \omega)\nabla_j^\varepsilon u(x-\varepsilon\mathbf{e}_i), \quad \omega\in\Omega.
$$

Since $\psi_\varepsilon \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d\times\Omega)$ and $u(x)$ can be assumed to be converging exponentially to zero as $|x|\to\infty$, it follows that $\Phi^{i,j}\in L^2(\Omega)$. Defining $\Phi^j\in L^2(\Omega)$ by

$$
\Phi^j(\omega) \overset{\text{def.}}{=} \varepsilon^{-1}\int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon\left(x,\tau_{-x/\varepsilon}\omega\right)\nabla_j^\varepsilon u(x), \quad \omega\in\Omega,
$$

it is clear that

$$
\Phi^{i,j}(\omega) = \Phi^j(\omega) + \int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon\left(x,\tau_{-x/\varepsilon}\omega\right)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(x), \quad \omega\in\Omega.
$$

We conclude now from Lemma 2.2 that (2.14) is the same as

$$
(2.15)\quad \sum_{j=1}^{d}\left\langle \sum_{i=1}^{d}\Big[\int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon\left(x,\ \tau_{-x/\varepsilon}\ \cdot\right)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(x)\Big]\partial_i^*\Big[a_{ij}(\cdot) + \sum_{k=1}^{d} a_{ik}(\cdot)\Psi_k^j(\cdot)\Big]\right\rangle.
$$

Hence the first term in (2.13) is the sum of (2.15) and

$$
(2.16)\quad \left\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon(x,\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(x)\Big[a_{ij}(\tau_{x/\varepsilon}\ \cdot) + \sum_{k=1}^{d} a_{ik}(\tau_{x/\varepsilon}\ \cdot)\Psi_k^j(\tau_{x/\varepsilon}\cdot)\Big]\right\rangle.
$$

Now, let us define $Q_{ij} \in L^2(\Omega)$ by

$$Q_{ij}(\omega) \stackrel{\text{def.}}{=} \partial_i^* \Big[ a_{ij}(\omega) + \sum_{k=1}^d a_{ik}(\omega) \Psi_k^j(\omega) \Big] + a_{ij}(\omega) + \sum_{k=1}^d a_{ik}(\omega) \Psi_k^j(\omega) - q_{ij}, \ \omega \in \Omega,$$

where $q_{ij}$ is given by (2.5). It follows from (2.6) that $\langle Q_{ij} \rangle = 0$. Furthermore, from (2.15), (2.16), we see that the first term in (2.13) is the same as

$$\sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x, \cdot) \right\rangle q_{ij} \nabla_i^{\varepsilon *} \nabla_j^\varepsilon u(x) + \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x, \cdot) Q_{ij}(\tau_{x/\varepsilon} \ \cdot) \right\rangle \nabla_i^{\varepsilon *} \nabla_j^\varepsilon u(x). \tag{2.17}$$

We can do a similar integration by parts for the second expression in (2.13). Thus,

$$\begin{aligned} \nabla_i^\varepsilon \psi_\varepsilon(x, \cdot) \Big[ \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j) - \nabla_k^\varepsilon u(x) \Big] = \\ \psi_\varepsilon(x, \cdot) \Big[ \nabla_i^{\varepsilon *} \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j) - \nabla_i^{\varepsilon *} \nabla_k^\varepsilon u(x) \Big] \\ + \nabla_i^\varepsilon \Big\{ \psi_\varepsilon(x, \cdot) \Big[ \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j - \varepsilon \mathbf{e}_i) - \nabla_k^\varepsilon u(x - \varepsilon \mathbf{e}_i) \Big] \Big\}. \end{aligned} \tag{2.18}$$

We have that

$$\begin{aligned} \Big\langle \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx \, a_{ij} \left( \frac{x}{\varepsilon}, \ \cdot \right) \sum_{k=1}^d \nabla_i^\varepsilon \Big\{ \psi_\varepsilon(x, \cdot) \left[ \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j - \varepsilon \mathbf{e}_i) - \ \nabla_k^\varepsilon u(x - \varepsilon \mathbf{e}_i) \right] \Big\} \Psi_j^k(\tau_{x/\varepsilon} \ \cdot) \Big\rangle \\ = \sum_{i,j,k=1}^d \ \Big\langle \partial_i \Phi^{i,j,k}(\cdot) \left[ a_{ij}(\cdot) \Psi_j^k(\cdot) \right] \Big\rangle, \end{aligned}$$

where

$$\Phi^{i,j,k}(\omega) \stackrel{\text{def.}}{=} \varepsilon^{-1} \int_{\mathbf{Z}_\varepsilon^d} dx \, \psi_\varepsilon(x, \tau_{-x/\varepsilon} \omega) \left[ \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j - \varepsilon \mathbf{e}_i) - \nabla_k^\varepsilon u(x - \varepsilon \mathbf{e}_i) \right], \quad \omega \in \Omega.$$

Once again it is clear that $\Phi^{i,j,k} \in L^2(\Omega)$. Integrating by parts we conclude that

$$\begin{aligned} \sum_{i,j,k=1}^d \Big\langle \partial_i \Phi^{i,j,k}(\cdot) \left[ a_{ij}(\cdot) \Psi_j^k(\cdot) \right] \Big\rangle &= \sum_{i,j,k=1}^d \Big\langle \Phi^{i,j,k}(\cdot) \partial_i^* \left[ a_{ij}(\cdot) \Psi_j^k(\cdot) \right] \Big\rangle \\ &= \sum_{i,j,k=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x, \cdot) Q_{ijk}(\tau_{x/\varepsilon} \cdot) \right\rangle \nabla_j^\varepsilon \nabla_k^\varepsilon u(x - \varepsilon \mathbf{e}_i), \end{aligned}$$

where

$$Q_{ijk}(\omega) \stackrel{\text{def.}}{=} \partial_i^* \left[ a_{ij} \Psi_j^k \right](\omega), \quad \omega \in \Omega.$$

Evidently we have $\langle Q_{ijk}\rangle = 0$. Next we take account of the second term in (2.18). Thus we have

$$\Big\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}\left(\frac{x}{\varepsilon},\ \cdot\right) \sum_{k=1}^{d} \psi_\varepsilon(x,\cdot)\left[\nabla_i^{\varepsilon *}\nabla_k^\varepsilon u(x+\varepsilon \mathbf{e}_j) - \ \nabla_i^{\varepsilon *}\nabla_k^\varepsilon u(x)\right] \Psi_j^k(\tau_{x/\varepsilon}\ .)\Big\rangle$$

$$= \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot)\right\rangle \varepsilon r_{ijk} \nabla_i^{\varepsilon *}\nabla_j^\varepsilon \nabla_k^\varepsilon u(x)$$

$$+ \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) R_{ijk}(\tau_{x/\varepsilon}\cdot)\right\rangle \varepsilon \nabla_i^{\varepsilon *}\nabla_j^\varepsilon \nabla_k^\varepsilon u(x),$$

where

$$R_{ijk}(\omega) \stackrel{\text{def.}}{=} a_{ij}(\omega)\Psi_j^k(\omega) - \left\langle a_{ij}(\cdot)\Psi_j^k(\cdot)\right\rangle, r_{ijk} = \left\langle a_{ij}(\cdot)\Psi_j^k(\cdot)\right\rangle.$$

Evidently $\langle R_{ijk}\rangle = 0$. Hence the second term in (2.13) is the same as

$$(2.19)\quad \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) Q_{ijk}(\tau_{x/\varepsilon}\cdot)\right\rangle\ \nabla_j^\varepsilon \nabla_k^\varepsilon u(x-\varepsilon \mathbf{e}_i)$$

$$+ \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) R_{ijk}(\tau_{x/\varepsilon}\cdot)\right\rangle \varepsilon \nabla_i^{\varepsilon *}\nabla_j^\varepsilon \nabla_k^\varepsilon u(x)$$

$$+ \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot)\right\rangle \varepsilon r_{ijk} \nabla_i^{\varepsilon *}\nabla_j^\varepsilon \nabla_k^\varepsilon u(x).$$

Now let us assume that $u(x)$ satisfies the partial difference equation,

$$(2.20)\qquad \varepsilon \sum_{i,j,k=1}^{d} r_{ijk} \nabla_i^{\varepsilon *}\nabla_j^\varepsilon \nabla_k^\varepsilon u(x) + \sum_{i,j=1}^{d} q_{ij} \nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(x) + u(x) = f(x), \qquad x \in \mathbf{Z}_\varepsilon^d.$$

This equation converges as $\varepsilon \to 0$ to the homogenized equation (1.3). Note that it is a singular perturbation of (1.3) and therefore needs to be carefully analyzed. In particular, we shall have to show that $u(x)$ and its derivatives converge rapidly to zero as $|x| \to \infty$. It follows now from

(2.12), (2.13), (2.17), (2.19), that

$$
\begin{aligned}
(2.21)\quad & \Big\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\ \ a_{ij}\left(\frac{x}{\varepsilon},\cdot\right) \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) \nabla_j^\varepsilon Z_\varepsilon(x,\cdot) + \int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon(x,\cdot) Z_\varepsilon(x,\cdot) \Big\rangle \\
& = \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) Q_{ij}(\tau_{x/\varepsilon}\ \cdot) \right\rangle \nabla_i^{\varepsilon *} \nabla_j^\varepsilon u(x) \\
& + \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) Q_{ijk}(\tau_{x/\varepsilon}\ \cdot) \right\rangle \nabla_j^\varepsilon \nabla_k^\varepsilon u(x - \varepsilon \mathbf{e}_i) \\
& + \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) R_{ijk}(\tau_{x/\varepsilon}\ \cdot) \right\rangle \varepsilon \nabla_i^{\varepsilon *} \nabla_j^\varepsilon \nabla_k^\varepsilon u(x) \\
& - \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j) a_{ij}(\frac{x}{\varepsilon},\cdot) \eta \sum_{y\in\mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon} - y\right) \Psi_j^k(\tau_y\ \cdot) \right\rangle \\
& + \sum_{i,j,k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \nabla_i^\varepsilon \psi_\varepsilon(x,\cdot) a_{ij}(\frac{x}{\varepsilon},\cdot) \nabla_j^\varepsilon \nabla_k^\varepsilon u(x)\ \varepsilon \chi_k(\frac{x}{\varepsilon},\cdot) \right\rangle \\
& \qquad\qquad + \sum_{k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \psi_\varepsilon(x,\cdot) \nabla_k^\varepsilon u(x) \varepsilon \chi_k\left(\frac{x}{\varepsilon},\cdot\right) \right\rangle.
\end{aligned}
$$

**Proposition 2.1.** *Let $Z_\varepsilon$ be defined by (2.11). Then*

$$
\left\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, Z_\varepsilon(x,\cdot)^2 \right\rangle \le C\,[A_1 + A_2 + A_3 + A_4 + A_5 + A_6]
$$

*where $C$ is a constant and*

$$
\begin{aligned}
A_1 &= \sum_{i,j,\ell=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[ \varepsilon \sum_{y\in\mathbf{Z}^d} \nabla_\ell^* G_0\left(\frac{x}{\varepsilon} - y\right) Q_{ij}(\tau_y\ \cdot) \nabla_i^{\varepsilon *} \nabla_j^\varepsilon u(\varepsilon y) \right]^2 \right\rangle, \\
A_2 &= \sum_{i,j,k,\ell=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[ \varepsilon \sum_{y\in\mathbf{Z}^d} \nabla_\ell^* G_0\left(\frac{x}{\varepsilon} - y\right) Q_{ijk}(\tau_y \cdot)\ \nabla_j^\varepsilon \nabla_k^\varepsilon u(\varepsilon y - \varepsilon \mathbf{e}_i) \right]^2 \right\rangle, \\
A_3 &= \sum_{i,j,k,\ell=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[ \varepsilon \sum_{y\in\mathbf{Z}^d} \nabla_\ell^* G_0\left(\frac{x}{\varepsilon} - y\right) R_{ijk}(\tau_y\ \cdot)\ \varepsilon \nabla_i^{\varepsilon *} \nabla_j^\varepsilon \nabla_k^\varepsilon u(\varepsilon y) \right]^2 \right\rangle, \\
A_4 &= \sum_{i,j,k=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[ \nabla_k^\varepsilon u(x + \varepsilon \mathbf{e}_j) \eta \sum_{y\in\mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon} - y\right) \Psi_j^k(\tau_y\ \cdot) \right]^2 \right\rangle, \\
A_5 &= \sum_{i,j,k=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx\ \left[ \nabla_j^\varepsilon \nabla_k^\varepsilon u(x) \varepsilon\ \chi_k\left(\frac{x}{\varepsilon},\cdot\right) \right]^2 \right\rangle\ , \\
A_6 &= \sum_{k=1}^{d} \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx\ \left[ \nabla_k^\varepsilon u(x) \varepsilon\ \chi_k\left(\frac{x}{\varepsilon},\cdot\right) \right]^2 \right\rangle.
\end{aligned}
$$

*Proof.* Let $g = g(x,\omega)$ be a function in $L^2(\mathbf{Z}_\varepsilon^d \times \Omega)$. Denote by $\mathcal{G}_g$ the functional (2.2) on $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ obtained by replacing the function $f(x)$ with $g(x,\omega)$ in (2.2). Then, according to Lemma 2.1 there is a minimizer of $\mathcal{G}_g$ in $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ which we denote by $\psi_\varepsilon$. It is clear that $\|\psi_\varepsilon\|_{\mathcal{H}^1} \le C\|g\|_2$ for some constant $C$, where $\|g\|_2$ is the $L^2$ norm of $g$. Furthermore, if we assume the solutions of (2.20) are rapidly decreasing then it follows that $Z_\varepsilon$ is in $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$. The Euler-Lagrange equations for $\psi_\varepsilon$ as given by Lemma 2.1 then tells us that the LHS of (2.21) is the same as

$$\left\langle \int_{\mathbf{Z}_\varepsilon^d} g(x,\cdot) Z_\varepsilon(x,\cdot) \right\rangle.$$

Next we consider the RHS of (2.21). If we use the Schwarz inequality on the sixth expression it is clear it is bounded by $C\|\psi_\varepsilon\|_{\mathcal{H}^1}\ A_6^{1/2}$ for some constant $C$, and hence by $C\|g\|_2\ A_6^{1/2}$ for a different constant $C$. Similarly the fifth and fourth terms are bounded by $C\|g\|_2\ A_5^{1/2}$ and $C\|g\|_2\ A_4^{1/2}$ respectively. To obtain bounds on the first three terms we need to represent $\psi_\varepsilon$ as an integral of its gradient. We can do this by using the Green's function $G_0$ which is the solution of (2.7) when $\eta = 0$. To see this observe that

$$\psi_\varepsilon(y',\omega) = \sum_{x' \in \mathbf{Z}^d} \delta\Big(\frac{y'}{\varepsilon} - x'\Big)\psi_\varepsilon(\varepsilon x',\omega), \quad y' \in \mathbf{Z}_\varepsilon^d,\ \omega \in \Omega. \tag{2.22}$$

Using (2.7) this is the same as

$$\begin{aligned}
\psi_\varepsilon(y',\omega) &= \sum_{x' \in \mathbf{Z}^d} -\Delta G_0\Big(\frac{y'}{\varepsilon} - x'\Big)\psi_\varepsilon(\varepsilon x',\omega) \\
&= \sum_{x' \in \mathbf{Z}^d} \sum_{\ell=1}^d \nabla_\ell^* G_0\Big(\frac{y'}{\varepsilon} - x'\Big)\varepsilon\ \nabla_\ell^\varepsilon \psi_\varepsilon(\varepsilon x',\omega) \\
&= \int_{\mathbf{Z}_\varepsilon^d} dx \sum_{\ell=1}^d \nabla_\ell^\varepsilon \psi_\varepsilon(x,\omega) \nabla_\ell^* G_0\left(\frac{x}{\varepsilon} - \frac{y'}{\varepsilon}\right)\varepsilon^{1-d}.
\end{aligned}$$

The first term on the RHS of (2.21) is therefore the same as

$$\sum_{i,j,\ell=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx \left\langle \nabla_\ell^\varepsilon \psi_\varepsilon(x,\cdot) \sum_{y \in \mathbf{Z}^d} \varepsilon\ \nabla_\ell^* G_0\left(\frac{x}{\varepsilon} - y\right) Q_{ij}(\tau_y\ \cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y) \right\rangle. \tag{2.23}$$

It follows again from the Schwarz inequality that this last expression is bounded by $C\|g\|_2 A_1^{1/2}$. A similar argument shows that the second and third terms on the RHS of (2.21) are bounded by $C\|g\|_2 A_2^{1/2}$, $C\|g\|_2 A_3^{1/2}$ respectively. The result follows now by taking $g = Z_\varepsilon$. $\square$

Proposition 2.1 will help us obtain an estimate on the variance of the minimizer $u_\varepsilon$ of Lemma 2.1. In fact there is the inequality

$$\begin{aligned}
\left\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, [u_\varepsilon(x,\cdot) - \langle u_\varepsilon(x,\cdot)\rangle]^2 \right\rangle &\le 2\left\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, Z_\varepsilon(x,\cdot)^2 \right\rangle \\
&\quad + 2\varepsilon^2 \left\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[\sum_{k=1}^d \chi_k\left(\frac{x}{\varepsilon},\cdot\right)\nabla_k^\varepsilon u(x)\right]^2 \right\rangle.
\end{aligned} \tag{2.24}$$

We shall also want to estimate the variance of the random variable

$$\int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) u_\varepsilon(x,\cdot),$$

where $g: \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support. To help us do this we reformulate the variational problem of Lemma 2.1. Suppose $u_\varepsilon$ is in $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d \times \Omega)$. We define a new function $v_\varepsilon$ by $v_\varepsilon(x,\omega) = u_\varepsilon(x, \tau_{-x/\varepsilon}\omega),\ x \in \mathbf{Z}_\varepsilon^d,\ \omega \in \Omega$. We then have

$$\begin{aligned}
\nabla_i^\varepsilon u_\varepsilon(x,\omega) &= [u_\varepsilon(x+\varepsilon \mathbf{e}_i, \omega) - u_\varepsilon(x,\omega)]/\varepsilon \\
&= [v_\varepsilon(x+\varepsilon\mathbf{e}_i\ ,\ \tau_{x/\varepsilon}\tau_{\mathbf{e}_i}\omega) - v_\varepsilon(x, \tau_{x/\varepsilon}\omega)]/\varepsilon \\
&= [v_\varepsilon(x+\varepsilon\mathbf{e}_i\ ,\ \tau_{x/\varepsilon}\tau_{\mathbf{e}_i}\omega) - v_\varepsilon(x+\varepsilon\mathbf{e}_i, \tau_{x/\varepsilon}\omega)]/\varepsilon \\
&\qquad +[v_\varepsilon(x+\varepsilon\mathbf{e}_i\ ,\ \tau_{x/\varepsilon}\omega) - v_\varepsilon(x, \tau_{x/\varepsilon}\omega)]/\varepsilon \\
&= \varepsilon^{-1}\partial_i v_\varepsilon(x+\varepsilon\mathbf{e}_i\ ,\ \tau_{x/\varepsilon}\omega) + \nabla_i^\varepsilon v_\varepsilon(x,\tau_{x/\varepsilon}\omega).
\end{aligned}$$

Hence from (2.2) we have

$$\begin{aligned}
\mathcal{G}(u_\varepsilon) = \Big\langle \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}(\cdot)[\nabla_i^\varepsilon v_\varepsilon(x,\cdot) + \varepsilon^{-1}\partial_i v_\varepsilon(x+\varepsilon\mathbf{e}_i,\cdot)]\times \\
[\nabla_j^\varepsilon v_\varepsilon(x,\cdot) + \varepsilon^{-1}\partial_j v_\varepsilon(x+\varepsilon\mathbf{e}_j,\cdot)] + \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} dx\ \ v_\varepsilon(x,\cdot)^2 - \\
\int_{\mathbf{Z}_\varepsilon^d} dx\, f(x) v_\varepsilon(x,\cdot)\Big\rangle\ .
\end{aligned}$$

Next we write $v_\varepsilon(x,\omega) = u(x) + \varepsilon\psi_\varepsilon(x,\omega)$ where $\langle \psi_\varepsilon(x,\cdot)\rangle = 0$. Then we have

$$\begin{aligned}
(2.25)\quad \mathcal{G}(u_\varepsilon) = \Big\langle\ \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\ \ a_{ij}(\cdot)[\nabla_i^\varepsilon u(x) + \partial_i\psi_\varepsilon(x+\varepsilon\mathbf{e}_i,\cdot) + \varepsilon\nabla_i^\varepsilon\psi_\varepsilon(x,\cdot)]\times \\
[\nabla_j^\varepsilon u(x) + \partial_j\psi_\varepsilon(x+\varepsilon\mathbf{e}_j,\cdot) + \varepsilon\nabla_j^\varepsilon\psi_\varepsilon(x,\cdot)] + \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} dx\, u(x)^2 + \\
\frac{1}{2}\,\varepsilon^2\int_{\mathbf{Z}_\varepsilon^d} dx\,\psi_\varepsilon(x,\cdot)^2 - \int_{\mathbf{Z}_\varepsilon^d} dx\ \ f(x)u(x)\Big\rangle \overset{\text{def.}}{=} \mathcal{F}_\varepsilon(u,\psi_\varepsilon).
\end{aligned}$$

Let $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$ be the space of functions $u: Z_\varepsilon^d \to \mathbf{R}$ which satisfy

$$\|u\|_{\mathcal{H}^1}^2 \overset{\text{def.}}{=} \int_{\mathbf{Z}_\varepsilon^d} dx\, u(x)^2 + \sum_{i=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\,[\nabla_i^\varepsilon u(x)]^2 < \infty.$$

Let $\mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ be the subspace of $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d\times\Omega)$ consisting of functions $\psi_\varepsilon(x,\omega)$ which satisfy $\langle\psi_\varepsilon(x,\cdot)\rangle = 0$, $x \in \mathbf{Z}_\varepsilon^d$.

**Lemma 2.4.** *If $\varepsilon > 0$ the functional $\mathcal{F}_\varepsilon : \mathcal{H}^1(\mathbf{Z}_\varepsilon^d) \times \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d\times\Omega) \to \mathbf{R}$ defined by (2.25) has a unique minimizer $(u,\psi_\varepsilon)$ with $u \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d),\ \psi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d\times\Omega)$. The minimizer satisfies the*

*Euler-Lagrange equation,*

$$\Big\langle \sum_{i,j=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}(\cdot)[\nabla_i^\varepsilon v(x) + \partial_i \varphi_\varepsilon(x+\varepsilon \mathbf{e}_i, \cdot) + \varepsilon \nabla_i^\varepsilon \varphi_\varepsilon(x,\cdot)]\times$$

$$[\nabla_j^\varepsilon u(x) + \partial_j \psi_\varepsilon(x+\varepsilon \mathbf{e}_j,\cdot) + \varepsilon \nabla_j^\varepsilon \psi_\varepsilon(x,\cdot)] + \int_{\mathbf{Z}_\varepsilon^d} dx\, v(x)u(x)$$

$$+ \varepsilon^2 \int_{\mathbf{Z}_\varepsilon^d} dx\, \varphi_\varepsilon(x,\cdot)\psi_\varepsilon(x,\cdot) - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x)v(x) \Big\rangle = 0,$$

*for all* $v \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$, $\varphi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$. *Further, if we put* $u_\varepsilon(x,\omega) = u(x) + \varepsilon\psi_\varepsilon(x, \tau_{x/\varepsilon}\ \omega)$, $x \in \mathbf{Z}_\varepsilon^d$, $\omega \in \Omega$, *then* $u_\varepsilon$ *is the unique minimizer of Lemma 2.1.*

*Proof.* To apply the Banach-Alaoglu theorem we need to show there are constants $C_1, C_2 > 0$ such that

$$\|v\|_{\mathcal{H}^1}^2 + \|\varphi_\varepsilon\|_{\mathcal{H}^1}^2 \le C_1 + C_2 \mathcal{F}_\varepsilon(v, \phi_\varepsilon), \quad v \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d),\ \varphi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega).$$

Observe now that

$$\mathcal{F}_\varepsilon(v,\varphi_\varepsilon) \ge \frac{\lambda}{2}\Big\langle\ \sum_{i=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, [\nabla_i^\varepsilon v(x) + \partial_i\varphi_\varepsilon(x+\varepsilon\mathbf{e}_i,\cdot) + \varepsilon\nabla_i^\varepsilon \varphi_\varepsilon(x,\cdot)]^2\ \Big\rangle - \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} f(x)^2 dx\,,$$

where $\lambda > 0$ is as in (1.1).

Now, using the fact that $\langle \varphi_\varepsilon(x,\cdot)\rangle = 0$ we conclude

$$\mathcal{F}_\varepsilon(v,\varphi_\varepsilon) \ge \frac{\lambda}{2}\ \sum_{i=1}^{d} \Big\langle\ \int_{\mathbf{Z}_\varepsilon^d} dx\, [\partial_i\varphi_\varepsilon(x+\varepsilon\mathbf{e}_i,\cdot) + \varepsilon\nabla_i^\varepsilon\varphi_\varepsilon(x,\cdot)]^2\ \Big\rangle$$

$$+ \frac{\lambda}{2}\ \sum_{i=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx\, [\nabla_i^\varepsilon v(x)]^2 - \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} f(x)^2 dx.$$

We have then from the Schwarz inequality

$$\|v\|_{\mathcal{H}^1}^2 \le \frac{2}{\lambda}\left[\frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} f(x)^2 dx + \mathcal{F}_\varepsilon(v,\varphi_\varepsilon)\right] + 4\ \left[\int_{\mathbf{Z}_\varepsilon^d} f(x)^2 dx + \mathcal{F}_\varepsilon(v,\varphi_\varepsilon)\right].$$

Using the fact that

$$\varepsilon^2 \sum_{i=1}^{d} \Big\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, [\nabla_i^\varepsilon \varphi_\varepsilon(x,\cdot)]^2\ \Big\rangle \le 8d \Big\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, \varphi_\varepsilon(x,\cdot)^2\ \Big\rangle$$

$$+ 2\sum_{i=1}^{d} \Big\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, [\partial_i\varphi_\varepsilon(x+\varepsilon\mathbf{e}_i,\cdot) + \varepsilon\nabla_i^\varepsilon\varphi_\varepsilon(x,\cdot)]^2\ \Big\rangle,$$

we have

$$\|\varphi_\varepsilon\|_{\mathcal{H}^1}^2 \le \left[\frac{4}{\lambda\varepsilon^2} + \frac{16d}{\varepsilon^4} + \frac{2}{\varepsilon^2}\right]\left[\frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} f(x)^2 dx\, + \mathcal{F}_\varepsilon(v,\varphi_\varepsilon)\right].$$

The Euler-Lagrange equation follows in the usual way. To verify that $u_\varepsilon(x,\cdot) = u(x) + \varepsilon\psi_\varepsilon(x,\tau_{x/\varepsilon}\ \cdot)$ is the minimizer of Lemma 2.1 we need only observe that, with this substitution, the Euler-Lagrange equation here is the same as that of Lemma 2.1. $\square$

If we let $\varepsilon \to 0$ in the expression (2.25) for $\mathcal{F}_\varepsilon(u,\psi_\varepsilon)$ we formally obtain a functional $\mathcal{F}_0(u,\psi)$ given by

$$\mathcal{F}_0(u,\psi) \stackrel{\text{def.}}{=} \Big\langle \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{R}^d} dx\; a_{ij}(\cdot)\left[\frac{\partial u(x)}{\partial x_i} + \partial_i\psi(x,\cdot)\right] \times \\ \left[\frac{\partial u(x)}{\partial x_j} + \partial_j\psi(x,\cdot)\right]\Big\rangle + \frac{1}{2}\int_{\mathbf{R}^d} dx\, u(x)^2 - \int_{\mathbf{R}^d} dx\, f(x)u(x).$$

Consider now the problem of minimizing $\mathcal{F}_0(u,\psi)$. If we fix $x \in \mathbf{R}^d$ and minimize over all $\psi(x,\cdot) \in L^2(\Omega)$ then by Lemma 2.2 the minimizer is given by

$$\partial_i\psi(x,\cdot) = \sum_{k=1}^d \frac{\partial u(x)}{\partial x_k}\; \Psi_i^k(\cdot), \qquad i = 1,\dots,d \tag{2.26}$$

where the $\Psi^k \in \mathcal{H}(\Omega)$ are the minimizers of Lemma 2.2. If we further minimize with respect to the function $u(x)$ then it is clear that $u(x)$ is the solution of the pde (1.3), where the matrix $q$ is given by (2.6). Hence in the minimization of $\mathcal{F}_0$ we can separate the minimization problems in $\omega$ and $x$ variables. The function $\psi$ defined by (2.26) has, however, no longer the property that $\psi(x,\cdot) \in L^2(\Omega)$.

We wish now to defined a new functional, closely related to $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$, which has the property that the minimization problems in $\omega$ and $x$ variables can be separated and also that the minimizer $\psi(x,\omega)$ has $\psi(x,\cdot) \in L^2(\Omega)$. For $u \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$ and $\psi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ we define $\mathcal{F}_{S,\varepsilon}(u,\psi_\varepsilon)$ by

$$\mathcal{F}_{S,\varepsilon}(u,\psi_\varepsilon) \stackrel{\text{def.}}{=} \Big\langle \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}(\cdot)[\nabla_i^\varepsilon u(x) + \partial_i\psi_\varepsilon(x,\cdot)] \times \\ [\nabla_j^\varepsilon u(x) + \partial_j\psi_\varepsilon(x,\cdot)] + \frac{1}{2}\int_{\mathbf{Z}_\varepsilon^d} dx\, u(x)^2 + \frac{1}{2}\varepsilon^2\int_{\mathbf{Z}_\varepsilon^d} dx\, \psi_\varepsilon(x,\cdot)^2 - \\ \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x)u(x)\Big\rangle. \tag{2.27}$$

It is clear that the formal limit of $\mathcal{F}_{S,\varepsilon}$ as $\varepsilon \to 0$ is, like the formal limit of $\mathcal{F}_\varepsilon$, given by $\mathcal{F}_0$. The advantage of $\mathcal{F}_{S,\varepsilon}$ over $\mathcal{F}_\varepsilon$ is that the minimization problem separates. We shall prove this in the following lemmas. First we have the analogue of Lemma 2.4:

**Lemma 2.5.** *If $\varepsilon > 0$ the functional $\mathcal{F}_{S,\varepsilon} : \mathcal{H}^1(\mathbf{Z}_\varepsilon^d) \times \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega) \to \mathbf{R}$ defined by (2.27) has a unique minimizer $(u,\psi_\varepsilon)$ with $u \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$, $\psi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$. The minimizer satisfies the Euler-Lagrange equation,*

$$\Big\langle \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}(\cdot)[\nabla_i^\varepsilon v(x) + \partial_i\varphi_\varepsilon(x,\cdot)][\nabla_j^\varepsilon u(x) + \partial_j\psi_\varepsilon(x,\cdot)] \\ + \int_{\mathbf{Z}_\varepsilon^d} dx\, v(x)u(x) + \varepsilon^2\int_{\mathbf{Z}_\varepsilon^d} dx\, \varphi_\varepsilon(x,\cdot)\psi_\varepsilon(x,\cdot) \\ - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x)v(x)\Big\rangle = 0, \tag{2.28}$$

*for all $v \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$, $\varphi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$.*

*Proof.* Same as Lemma 2.4. □

Next we need an analogue of Lemma 2.2. For $\varepsilon > 0$ define a functional $\mathcal{G}_{k,\varepsilon} : L^2(\Omega) \to \mathbf{R}$, where $1 \le k \le d$, by

$$\mathcal{G}_{k,\varepsilon}(\psi) \overset{\text{def.}}{=} \Big\langle \ \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\cdot)\partial_i\psi(\cdot)\partial_j\psi(\cdot) + \frac{1}{2}\,\varepsilon^2\psi(\cdot)^2 + \sum_{j=1}^{d} a_{kj}(\cdot)\partial_j\psi(\cdot) \ \Big\rangle.$$

**Lemma 2.6.** *The functional $\mathcal{G}_{k,\varepsilon} : L^2(\Omega) \to \mathbf{R}$ has a unique minimizer $\psi_{k,\varepsilon} \in L^2(\Omega)$ which satisfies $\langle \psi_{k,\varepsilon}(\cdot)\rangle = 0$ and the Euler-Lagrange equation,*

$$\left\langle \ \sum_{i,j=1}^{d} a_{ij}(\cdot)\partial_i\varphi(\cdot)\partial_j\psi_{k,\varepsilon}(\cdot) + \varepsilon^2\varphi(\cdot)\psi_{k,\varepsilon}(\cdot) + \sum_{j=1}^{d} a_{kj}(\cdot)\partial_j\varphi(\cdot) \ \right\rangle = 0,$$

*for all $\varphi \in L^2(\Omega)$.*

*Proof.* Same as Lemma 2.2. Observe that since $\mathcal{G}_{k,\varepsilon}(\psi - \langle\psi\rangle) \le \mathcal{G}_{k,\varepsilon}(\psi)$ for all $\psi \in L^2(\Omega)$ we have $\langle \psi_{k,\varepsilon}\rangle = 0$. □

Next, in analogy to (2.5) we define a matrix $q^\varepsilon$ by

$$q^\varepsilon_{kk'} = \Big\langle \ [\mathbf{e}_k + \nabla\psi_{k,\varepsilon}(\cdot)]\mathbf{a}(\cdot)[\mathbf{e}_{k'} + \nabla\psi_{k',\varepsilon}(\cdot)] \ \Big\rangle + \varepsilon^2\Big\langle \psi_{k,\varepsilon}(\cdot)\psi_{k',\varepsilon}(\cdot)\Big\rangle,$$

where the $\psi_{k,\varepsilon},\ k = 1, ..., d$ are the minimizers of Lemma 2.6. Evidently $q^\varepsilon$ is a symmetric positive definite matrix. Using the Euler-Lagrange equation we have a representation for $q^\varepsilon_{kk'}$ analogous to (2.6), namely

$$q^\varepsilon_{kk'} = \left\langle \ a_{kk'}(\cdot) + \sum_{j=1}^{d} a_{kj}(\cdot)\partial_j\psi_{k',\varepsilon}(\cdot) \right\rangle. \tag{2.29}$$

Let $\mathcal{G}_\varepsilon : \mathcal{H}^1(\mathbf{Z}^d_\varepsilon) \to \mathbf{R}$ be the functional

$$\mathcal{G}_\varepsilon(u) \overset{\text{def.}}{=} \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbf{Z}^d_\varepsilon} dx\; q^\varepsilon_{ij}\nabla^\varepsilon_i u(x)\nabla^\varepsilon_j u(x) + \frac{1}{2}\int_{\mathbf{Z}^d_\varepsilon} dx\, u(x)^2 - \int_{\mathbf{Z}^d_\varepsilon} dx\, f(x)u(x).$$

**Lemma 2.7.** *The functional $\mathcal{G}_\varepsilon : \mathcal{H}^1(\mathbf{Z}^d_\varepsilon) \to \mathbf{R}$ has a unique minimizer $u \in \mathcal{H}^1(\mathbf{Z}^d_\varepsilon)$ which satisfies the equation*

$$\sum_{i,j=1}^{d} q^\varepsilon_{ij}\nabla^{\varepsilon *}_i\nabla^\varepsilon_j u(x) + u(x) = f(x), \quad x \in \mathbf{Z}^d_\varepsilon. \tag{2.30}$$

*Proof.* Standard. □

**Proposition 2.2.** *Let $\psi_{k,\varepsilon}$ , $k = 1, ..., d$ be the minimizer of Lemma 2.6 and $u$ the minimizer of Lemma 2.7. Then $(u, \psi_\varepsilon)$, where*

$$\psi_\varepsilon(x,\omega) = \sum_{k=1}^{d} \nabla^\varepsilon_k u(x)\psi_{k,\varepsilon}(\omega), \quad x \in \mathbf{Z}^d_\varepsilon\ , \quad \omega \in \Omega$$

*is the minimizer of Lemma 2.5.*

*Proof.* Since $u(x)$ satisfies (2.30) it follows that $u(x)$ and its derivatives decrease exponentially as $|x| \to \infty$. Hence $u \in \mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$. Since also $\psi_{k,\varepsilon} \in L^2(\Omega)$ and $\langle \psi_{k,\varepsilon} \rangle = 0$ it follows that $\psi_\varepsilon \in \mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$. To show that $(u, \psi_\varepsilon)$ is the minimizer for the functional of Lemma 2.5 it will be sufficient to show $(u, \psi_\varepsilon)$ satisfies the Euler- Lagrange equation (2.28). Using the Euler-Lagrange equation of Lemma 2.6 we see that the LHS of (2.28) is the same as

$$\Big\langle \sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, a_{ij}(\cdot)[\nabla_i^\varepsilon v(x) \nabla_j^\varepsilon u(x) + \nabla_i^\varepsilon v(x) \partial_j \psi_\varepsilon(x, \cdot)] \Big\rangle$$
$$+ \int_{\mathbf{Z}_\varepsilon^d} dx\, v(x) u(x) - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x) v(x).$$

If we use now (2.29) we see this last expression is the same as

$$\sum_{i,j=1}^d \int_{\mathbf{Z}_\varepsilon^d} dx\, q_{ij}^\varepsilon \nabla_i^\varepsilon v(x) \nabla_j^\varepsilon u(x) + \int_{\mathbf{Z}_\varepsilon^d} dx\, v(x) u(x) - \int_{\mathbf{Z}_\varepsilon^d} dx\, f(x) v(x).$$

In view of (2.30) this last expression is zero. □

## 3. Analysis in Fourier Space

In this section we apply Fourier space methods to the problems of section 2. First we shall be interested in finding a solution to the partial difference equation (2.20), where $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support. If we let $\varepsilon \to 0$ in (2.20) then we get formally the singular perturbation problem,

$$-\varepsilon \sum_{i,j,k=1}^d r_{ijk} \frac{\partial^3 u(x)}{\partial x_i \partial x_j \partial x_k} - \sum_{i,j=1}^d q_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + u(x) = f(x), \ x \in \mathbf{R}^d. \tag{3.1}$$

Evidently this is a singular perturbation of the homogenized equation (1.3). It is easy to see that this equation has a solution $u(x)$, all of whose derivatives decrease exponentially fast to zero as $|x| \to \infty$. Furthermore this decrease is uniform in $\varepsilon$ as $\varepsilon \to 0$. Now let us write equation (2.20) as

$$L_\varepsilon u(x) = f(x), \ x \in \mathbf{Z}_\varepsilon^d, \tag{3.2}$$

where the operator $L_\varepsilon$ is given by the LHS of (2.20). In contrast to the case of (3.1) we cannot assert that (3.2) has a solution in general. We can however assert that it has an approximate solution.

**Proposition 3.1.** *Let $f : \mathbf{R}^d \to \mathbf{R}$ be a $C^\infty$ function with compact support. Then there exists a function $u : \mathbf{Z}_\varepsilon^d \to \mathbf{R}$ with the following properties:*

(a) *There exists a constant $\delta > 0$ such that for any $n$ tuple $\alpha = (\alpha_1, ..., \alpha_n)$ with $1 \le \alpha_i \le d, i = 1, ..., n,$ one has*

$$\sup_{x \in \mathbf{Z}_\varepsilon^d} \left| e^{\delta |x|} \left( \prod_{i=1}^n \nabla_{\alpha_i}^\varepsilon \right) u(x) \right| \le A_\alpha,$$

*and $A_\alpha$ is independent of $\varepsilon$.*

(b) *The function* $u(x)$ *satisfies the equation* $L_\varepsilon u(x) = f(x) + g_\varepsilon(x)$ *and* $g_\varepsilon(x)$ *has the property that*

$$|g_\varepsilon(x)| \le C\varepsilon^\gamma \exp[-\delta|x|], \ \ x \in \mathbf{Z}_\varepsilon^d,$$

*where* $\gamma > 0$ *can be chosen arbitrarily large and the constant* $C$ *is independent of* $\varepsilon$*.*

*Proof.* We go into Fourier variables. For $u : \mathbf{Z}_\varepsilon^d \to \mathbf{C}$, the Fourier transform of $u$ is given by

$$\hat{u}(\xi) = \int_{\mathbf{Z}_\varepsilon^d} dx\, u(x) e^{ix\cdot\xi}, \ \ \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d. \tag{3.3}$$

The function $u$ can be obtained from its Fourier transform by the formula,

$$u(x) = \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} \hat{u}(\xi) e^{-ix\cdot\xi} d\xi, \quad x \in \mathbf{Z}_\varepsilon^d.$$

The equation (3.2) is given in Fourier variables by

$$\Big\{ \sum_{i,j,k=1}^d r_{ijk}\varepsilon^{-2}(e^{i\varepsilon\mathbf{e}_i\cdot\xi} - 1)(e^{-i\varepsilon\mathbf{e}_j\cdot\xi} - 1)(e^{-i\varepsilon\mathbf{e}_k\cdot\xi} - 1) +$$
$$\sum_{i,j=1}^d q_{ij}\varepsilon^{-2}(e^{i\varepsilon\mathbf{e}_i\cdot\xi} - 1)(e^{-i\varepsilon\mathbf{e}_j\cdot\xi} - 1) + 1\Big\}\hat{u}(\xi) = \hat{f}(\xi), \ \ \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d.$$

We can rewrite this as

$$\hat{u}(\xi)\Big\{1 + 4\sum_{i,j=1}^d q_{ij} \exp[i\varepsilon(\mathbf{e}_i - \mathbf{e}_j)\cdot\xi/2] \frac{\sin(\varepsilon\mathbf{e}_i\cdot\xi/2)}{\varepsilon} \frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi/2)}{\varepsilon}$$
$$- 8i \sum_{i,j,k=1}^d r_{ijk} \exp[i\varepsilon(\mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k)\cdot\xi/2] \frac{\sin(\varepsilon\mathbf{e}_i\cdot\xi/2)}{\varepsilon} \frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi/2)}{\varepsilon} \times$$
$$\sin(\varepsilon\mathbf{e}_k\cdot\xi/2)\Big\} = \hat{f}(\xi).$$

Since the matrix $q$ is positive definite it is clear that provided $\varepsilon|\xi| \ll 1$ then the coefficient of $\hat{u}(\xi)$ in the last expression is non-zero. On the other hand if $\varepsilon|\xi| = O(1)$ then this coefficient could become zero. To get around this problem we define a function $u(x)$ by

$$
\begin{aligned}
(3.4)\quad \hat{u}(\xi)\Big\{1+4\sum_{i,j=1}^{d} q_{ij}\exp[\mathrm{i}\varepsilon(\mathbf{e}_i-\mathbf{e}_j)\cdot\xi/2]\frac{\sin(\varepsilon\mathbf{e}_i\cdot\xi/2)}{\varepsilon}\frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi/2)}{\varepsilon} \\
-8\mathrm{i}\left[1+A\Big\{\sum_{i=1}^{d}(1-\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi})(1-\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi})\Big\}^N\right]^{-1}\times \\
\sum_{i,j,k=1}^{d} r_{ijk}\exp[\mathrm{i}\varepsilon(\mathbf{e}_i-\mathbf{e}_j-\mathbf{e}_k)\cdot\xi/2]\times \\
\frac{\sin(\varepsilon\mathbf{e}_i\cdot\xi/2)}{\varepsilon}\frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi/2)}{\varepsilon}\sin(\varepsilon\mathbf{e}_k\cdot\xi/2)\Big\} = \hat{f}(\xi), \\
\xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d.
\end{aligned}
$$

In this last expression $N$ is a positive integer and $A$ a positive constant chosen large enough, depending on $N$, so that the coefficient of $\hat{u}(\xi)$ is nonzero for all $\xi\in[-\pi/\varepsilon,\pi/\varepsilon]^d$. Hence (3.4) defines a function $u:\mathbf{Z}^d_\varepsilon\to\mathbf{C}$ uniquely.

Since $f$ is $C^\infty$ with compact support it follows that $\hat{f}$ is analytic. In particular, for any positive integer $m$ and $\delta>0$ there exists a constant $C_{m,\delta}$, independent of $\varepsilon$, such that

$$
(3.5)\qquad (1+|\xi|^m)|\hat{f}(\xi+i\eta)|\le C_{m,\delta},\quad \xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d,\quad \eta\in\mathbf{R}^d, |\eta|<\delta.
$$

Observe next that the function $\hat{u}(\xi)$ defined in (3.4) is periodic in the cube $[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d$. Furthermore there exists $\delta>0$, independent of $\varepsilon$, such that $\hat{u}(\xi+\mathrm{i}\eta)$ is analytic in $\xi+\mathrm{i}\eta\in\mathbf{C}^d$ provided $|\eta|<\delta$. We have therefore that $u(x)$ is given by the formula,

$$
(3.6)\qquad u(x)=\frac{1}{(2\pi)^d}\int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d}\hat{u}(\xi+\mathrm{i}\eta)\exp[-\mathrm{i}x\cdot(\xi+\mathrm{i}\eta)]\,d\xi,\ \ x\in\mathbf{Z}^d_\varepsilon, |\eta|<\delta.
$$

Now if $A$ is large and $\delta>0$ is taken sufficiently small it is easy to see that the modulus of the coefficient of $\hat{u}(\xi+\mathrm{i}\eta)$ on the LHS of (3.4) is strictly positive for $\xi\in[-\pi/\varepsilon,\pi/\varepsilon]^d$, $|\eta|<\delta$. It follows then from (3.5), (3.6) that part (a) of proposition 3.1 holds.

To prove part (b) we use (3.6) to compute $L_\varepsilon u(x) - f(x)$. We have for $x \in \mathbf{Z}_\varepsilon^d$,

$$L_\varepsilon u(x) - f(x) = \frac{-1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi \ \hat{u}(\xi + \mathrm{i}\eta) \exp[-\mathrm{i}x \cdot (\xi + \mathrm{i}\eta)]$$

$$8\mathrm{i}A \Big\{ \sum_{i=1}^d \Big( 1 - \exp[\mathrm{i}\varepsilon \mathbf{e}_i \cdot (\xi + \mathrm{i}\eta)] \Big) \Big( 1 - \exp[-\mathrm{i}\varepsilon \mathbf{e}_i \cdot (\xi + \mathrm{i}\eta)] \Big) \Big\}^N \times$$

$$\left[ 1 + A \Big\{ \sum_{i=1}^d \Big( 1 - \exp[\mathrm{i}\varepsilon \mathbf{e}_i \cdot (\xi + \mathrm{i}\eta)] \Big) \Big( 1 - \exp[-\mathrm{i}\varepsilon \mathbf{e}_i \cdot (\xi + \mathrm{i}\eta)] \Big) \Big\}^N \right]^{-1}$$

$$\times \sum_{i,j,k=1}^d r_{ijk} \exp\left[\mathrm{i}\varepsilon(\mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k) \cdot (\xi + \mathrm{i}\eta)/2\right] \times$$

$$\frac{\sin(\varepsilon \mathbf{e}_i \cdot [\xi + \mathrm{i}\eta]/2)}{\varepsilon} \ \frac{\sin(\varepsilon \mathbf{e}_j \cdot [\xi + \mathrm{i}\eta]/2)}{\varepsilon} \ \sin(\varepsilon \mathbf{e}_k . [\xi + \mathrm{i}\eta]/2).$$

In view of (3.5) it follows that there is a constant C, independent of $\varepsilon$, such that

$$|L_\varepsilon u(x) - f(x)| \le C\varepsilon^{2N+1} \exp[\eta \cdot x], \quad x \in \mathbf{Z}_\varepsilon^d, |\eta| < \delta.$$

Part (b) follows from this last inequality. □

Next we wish to rewrite the functional $\mathcal{F}_\varepsilon$ of lemma 2.4 in Fourier variables. First observe that if we define the space $\hat{\mathcal{H}}^1([\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d)$ as the set of all functions $\hat{u} : [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d \to \mathbf{C}$ such that $\hat{u}(\xi) = \overline{\hat{u}(-\xi)}$ and

$$||\hat{u}||^2_{\hat{\mathcal{H}}^1} \ \stackrel{\text{def.}}{=} \ \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi \ \Big[ |\hat{u}(\xi)|^2 + \sum_{j=1}^d \varepsilon^{-2} \ |1 - \mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_j \cdot \xi}|^2 |\hat{u}(\xi)|^2 \Big] < \infty,$$

then $\mathcal{H}^1(\mathbf{Z}_\varepsilon^d)$ and $\hat{\mathcal{H}}^1([\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d)$ are unitarily equivalent via the Fourier transform (3.3). Similarly we can define a space $\hat{\mathcal{H}}_0^1([\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d \times \Omega)$ as all functions $\hat{\psi} : [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d \times \Omega \to \mathbf{C}$ such that $\hat{\psi}(\xi, \cdot) = \overline{\hat{\psi}(-\xi, \cdot)}$,

$$\|\hat{\psi}\|^2_{\mathcal{H}^1} \stackrel{\text{def.}}{=} \Big\langle \ \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi \ \Big[ |\hat{\psi}(\xi, \cdot)|^2 + \sum_{j=1}^d \varepsilon^{-2} |1 - \mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_j \cdot \xi}|^2 |\hat{\psi}(\xi, \cdot)|^2 \Big] \ \Big\rangle < \infty,$$

and $\langle \hat{\psi}(\xi, \cdot) \rangle = 0, \ \xi \in [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d$. Again it is clear that $\mathcal{H}_0^1(\mathbf{Z}_\varepsilon^d \times \Omega)$ and $\hat{\mathcal{H}}_0^1([\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d \times \Omega)$ are unitarily equivalent via the Fourier transform,

$$\hat{\psi}(\xi, \cdot) = \int_{\mathbf{Z}_\varepsilon^d} dx \ \psi(x, \cdot) \mathrm{e}^{\mathrm{i}x \cdot \xi}, \quad \xi \in \left[ \frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon} \right]^d.$$

It is also clear that the functional $\mathcal{F}_\varepsilon$ defined by (2.25) satisfies

$$
\begin{aligned}
(3.7)\quad \mathcal{F}_\varepsilon(u,\psi_\varepsilon) &= \hat{\mathcal{F}}_\varepsilon(\hat u,\hat\psi_\varepsilon)\\
&\overset{\text{def.}}{=} \frac{1}{(2\pi)^d}\Big\langle\ \frac12\sum_{i,j=1}^d\int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi\ a_{ij}(\cdot)\ \Big[\varepsilon^{-1}(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\hat u(\xi)+\\
&\qquad \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\partial_i\hat\psi_\varepsilon(\xi,\cdot)+(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\hat\psi_\varepsilon(\xi,\cdot)\Big]\times\\
&\Big[\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\overline{\hat u(\xi)}+\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\partial_j\overline{\hat\psi_\varepsilon}(\xi,\cdot)+(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\overline{\hat\psi_\varepsilon}(\xi,\cdot)\Big]\\
&\quad+\ \frac12\int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi\ |\hat u(\xi)|^2+\ \frac12\ \varepsilon^2\int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi\ |\hat\psi(\xi,\cdot)|^2\\
&\qquad\qquad -\int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} d\xi\ \overline{\hat f(\xi)}\hat u(\xi)\Big\rangle,
\end{aligned}
$$

for $u\in\mathcal{H}^1(\mathbf{Z}^d_\varepsilon),\ \psi_\varepsilon\in\mathcal{H}^1_o\ (\mathbf{Z}^d_\varepsilon\times\Omega)$.

We now follow the development of lemma 2.4 through proposition 2.2, but in Fourier space variables. First we have:

**Lemma 3.1.** *The functional $\hat{\mathcal{F}}_\varepsilon:\hat{\mathcal{H}}^1\ ([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)\times\hat{\mathcal{H}}^1_0([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d\times\Omega)\to\mathbf{R}$ defined by (3.7) has a unique minimizer $(\hat u,\hat\psi_\varepsilon)$ with $\hat u\in\hat{\mathcal{H}}^1([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$, $\hat\psi_\varepsilon\in\hat{\mathcal{H}}^1_0\ ([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d\times\Omega)$. The minimizer satisfies the Euler-Lagrange equation,*

$$
\begin{aligned}
(3.8)\qquad \varepsilon^2\hat\psi_\varepsilon(\xi,\cdot)\ &+\ \sum_{i,j=1}^d\Big[\mathrm{e}^{\mathrm{i}\varepsilon(\mathbf{e}_j-\mathbf{e}_i)\cdot\xi}\partial_j^*[a_{ij}(\cdot)\partial_i\hat\psi_\varepsilon(\xi,\cdot)]\\
&+\ \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\partial_j^*\ [a_{ij}(\cdot)\hat\psi_\varepsilon(\xi,\cdot)]\\
&+\ \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\Big\{a_{ij}(\cdot)\partial_i\hat\psi_\varepsilon(\xi,\cdot)-\langle a_{ij}(\cdot)\partial_i\hat\psi_\varepsilon(\xi,\cdot)\rangle\Big\}\\
&+\ (\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\Big\{a_{ij}(\cdot)\hat\psi_\varepsilon(\xi,\cdot)-\langle a_{ij}(\cdot)\hat\psi_\varepsilon(\xi,\cdot)\rangle\Big\}\\
&+\ \varepsilon^{-1}(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\hat u(\xi)\Big\{\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\partial_j^*(a_{ij}(\cdot))\\
&+\ (\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\big[a_{ij}(\cdot)-\langle a_{ij}(\cdot)\rangle\big]\Big\}\Big]=0,
\end{aligned}
$$

$$
\begin{aligned}
(3.9)\quad \hat f(\xi)=\hat u(\xi)+\sum_{i,j=1}^d\ \Big[&\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\varepsilon^{-1}\ (\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\ \Big\langle\ a_{ij}(\cdot)\Big\rangle\hat u(\xi)\\
&+\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\Big\langle a_{ij}(\cdot)\partial_i\hat\psi_\varepsilon(\xi,\cdot)\Big\rangle\\
&\qquad+\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\ (\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\Big\langle a_{ij}(\cdot)\hat\psi_\varepsilon(\xi,\cdot)\Big\rangle\Big].
\end{aligned}
$$

*Conversely if there exists $\hat u\in\hat{\mathcal{H}}^1\ ([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$ and $\hat\psi_\varepsilon\in\hat{\mathcal{H}}^1_0([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d\times\Omega)$ which satisfy the Euler-Lagrange equations (3.8), (3.9) then $(\hat u,\hat\psi_\varepsilon)$ is the unique minimizer for $\hat{\mathcal{F}}_\varepsilon$.*

*Proof.* Standard. $\square$

Next for $\varepsilon > 0, 1 \le k \le d$ and $\xi \in [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d$ we define a functional $\mathcal{G}_{\xi,k,\varepsilon} : L^2(\Omega) \to \mathbf{R}$ by

$$\mathcal{G}_{\xi,k,\varepsilon}(\hat{\psi}) \stackrel{\text{def.}}{=} \Big\langle \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\cdot) \Big[ e^{-i\varepsilon \mathbf{e}_i \cdot \xi} \partial_i \hat{\psi}(\cdot) + (e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - 1) \hat{\psi}(\cdot) \Big] \\ \Big[ e^{i\varepsilon \mathbf{e}_j \cdot \xi} \partial_j \overline{\hat{\psi}(\cdot)} + (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1) \overline{\hat{\psi}(\cdot)} \Big] + \frac{1}{2} \varepsilon^2 |\hat{\psi}(\cdot)|^2 \\ + Re \sum_{j=1}^{d} a_{kj}(\cdot) \Big[ e^{i\varepsilon \mathbf{e}_j \cdot \xi} \partial_j \overline{\hat{\psi}(\cdot)} + (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1) \overline{\hat{\psi}(\cdot)} \Big] \Big\rangle.$$

Observe that the functional $\mathcal{G}_{\xi,k,\varepsilon}$ at $\xi = 0$ is identical to the functional $\mathcal{G}_{k,\varepsilon}$ of lemma 2.6. The following lemma corresponds to lemma 2.6.

**Lemma 3.2.** *Let $L_0^2(\Omega)$ be the square integrable functions $\hat{\psi} : \Omega \to \mathbf{C}$ such that $\langle \hat{\psi}(\cdot) \rangle = 0$. The functional $\mathcal{G}_{\xi,k,\varepsilon} : L_0^2(\Omega) \to \mathbf{R}$ has a unique minimizer $\hat{\psi}_{k,\varepsilon}(\xi, \cdot) \in L_0^2(\Omega)$ which satisfies the Euler-Lagrange equation,*

(3.10)

$$\begin{aligned} \varepsilon^2 \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \quad &+ \quad \sum_{i,j=1}^{d} \Big[ e^{i\varepsilon(\mathbf{e}_j - \mathbf{e}_i)\cdot\xi} \partial_j^* (a_{ij}(\cdot) \partial_i \hat{\psi}_{k,\varepsilon}(\xi,\cdot)) \\ &+ \quad e^{i\varepsilon \mathbf{e}_j \cdot \xi} (e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - 1) \partial_j^* (a_{ij}(\cdot) \hat{\psi}_{k,\varepsilon}(\xi,\cdot)) \\ &+ \quad e^{-i\varepsilon \mathbf{e}_i \cdot \xi} (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1) \Big\{ a_{ij}(\cdot) \partial_i \hat{\psi}_{k,\varepsilon}(\xi,\cdot) - \Big\langle a_{ij}(\cdot) \partial_i \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \Big\rangle \Big\} \\ &+ \quad (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1)(e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - 1) \Big\{ a_{ij}(\cdot) \hat{\psi}_{k,\varepsilon}(\xi,\cdot) - \Big\langle a_{ij}(\cdot) \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \Big\rangle \Big\} \Big] \\ &+ \quad \sum_{j=1}^{d} \Big[ e^{i\varepsilon \mathbf{e}_j \cdot \xi} \partial_j^* (a_{kj}(\cdot)) + (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1) [a_{kj}(\cdot) - \Big\langle a_{kj}(\cdot) \Big\rangle] \Big] = 0. \end{aligned}$$

*Proof.* Standard. $\square$

Observe now that if $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ satisfy (3.10), $k = 1, \dots, d$, then $\hat{\psi}_\varepsilon(\xi,\cdot)$ defined by

$$\hat{\psi}_\varepsilon(\xi,\cdot) = \sum_{k=1}^{d} \varepsilon^{-1} (e^{-i\varepsilon \mathbf{e}_k \cdot \xi} - 1) \hat{u}(\xi) \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \tag{3.11}$$

satisfies (3.8). It follows from uniqueness of the minimizer in Lemma 3.2 that $\psi_\varepsilon(\xi,\cdot) = \overline{\psi_\varepsilon(-\xi,\cdot)}$. Making the substitution (3.11) for $\hat{\psi}_\varepsilon$ in (3.9) we see that (3.9) is the same as

$$\hat{f}(\xi) = \hat{u}(\xi) + \hat{u}(\xi) \sum_{i,j=1}^{d} \varepsilon^{-1} (e^{i\varepsilon \mathbf{e}_j \cdot \xi} - 1) \varepsilon^{-1} (e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - 1) \; q_{ij}^{\varepsilon}(\xi), \tag{3.12}$$

where

$$(3.13) \quad q^\varepsilon_{k,k'}(\xi) \stackrel{\text{def.}}{=} \Big\langle a_{k,k'}(\cdot) + \sum_{j=1}^d \mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_j\cdot\xi} a_{kj}(\cdot)\partial_j \hat{\psi}_{k',\varepsilon}(\xi,\cdot) + \sum_{j=1}^d (\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi} - 1) a_{kj}(\cdot)\hat{\psi}_{k',\varepsilon}(\xi,\cdot)\Big\rangle.$$

We can obtain an alternative expression for $q^\varepsilon_{kk'}(\xi)$ by using the Euler-Lagrange equation, (3.10). We have

$$(3.14) \quad q^\varepsilon_{kk'}(\xi) = \Big\langle \sum_{i,j=1}^d a_{ij}(\cdot)\Big(\delta_{ki} + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\partial_i\overline{\hat{\psi}_{k,\varepsilon}}(\xi,\cdot) + (\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1)\overline{\hat{\psi}_{k,\varepsilon}}(\xi,\cdot)\Big)$$
$$\Big(\delta_{k'j} + \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\partial_j\hat{\psi}_{k',\varepsilon}(\xi,\cdot) + (\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\hat{\psi}_{k',\varepsilon}(\xi,\cdot)\Big)\Big\rangle$$
$$+\, \varepsilon^2\Big\langle \overline{\hat{\psi}_{k,\varepsilon}}(\xi,\cdot)\psi_{k',\varepsilon}(\xi,\cdot)\Big\rangle.$$

It is clear from this last expression that the matrix $q^\varepsilon(\xi) = [q^\varepsilon_{kk'}(\xi)]$ is Hermitian, non-negative definite. In view of the fact that $\Big\langle\hat{\psi}_{k,\varepsilon}(\xi,\cdot)\Big\rangle = 0$ it follows that it is bounded below by the matrix $\lambda I_d$, where $\lambda$ is given by (1.1). Hence the equation (3.12) can be solved uniquely for $\hat{u}(\xi)$ in terms of $\hat{f}(\xi)$.

Suppose now that we know the minimizer $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ of Lemma 3.2 is continuous as a function from $[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d \to L^2_0(\Omega), k = 1,\dots,d$. Hence if we define $\hat{u}(\xi)$ by (3.12) then it is easy to see that $\hat{u}(\xi)$ is continuous and

$$|\hat{u}(\xi)| \le \frac{|\hat{f}(\xi)|}{1+\lambda\sum_{i=1}^d \varepsilon^{-2}|\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}-1|^2}\ , \quad \xi \in \left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d.$$

Since we are assuming $f$ is $C^\infty$ of compact support it follows that $\hat{f}$ is in $L^2([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$, whence $\hat{u}$ is in $\hat{\mathcal{H}}^1([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$. Defining $\hat{\psi}_\varepsilon(\xi,\cdot)$ by (3.11) it is easy to see that $\hat{\psi}_\varepsilon(\xi,\cdot)$ is in $\hat{\mathcal{H}}^1_0([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d\times\Omega)$. We conclude that $\hat{u}(\xi), \hat{\psi}_\varepsilon(\xi,\cdot)$ defined by (3.12) and (3.11) are the unique solution to the variational problem of lemma 3.1.

We still need to establish the continuity of the $\hat{\psi}_{k,\varepsilon}(\xi,\cdot), k = 1,\cdots,d$. We can actually assert more than this.

**Proposition 3.2.** *For $\xi\in\mathbf{R}^d$ let $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ be the minimizer of lemma 3.2. Then*
*(a) $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ regarded as a function from $\mathbf{R}^d$ to $L^2_0(\Omega)$ is continuous and periodic,*

$$\hat{\psi}_{k,\varepsilon}\left(\xi + \frac{2\pi n}{\varepsilon},\cdot\right) = \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \quad \textit{if} \;\; \xi\in\mathbf{R}^d, n\in\mathbf{Z}^d.$$

*(b) There exists $\alpha > 0$, independent of $\varepsilon$, such that $\hat{\psi}_{k,\varepsilon} : \mathbf{R}^d \to L^2_0(\Omega)$ has an analytic continuation into the region $\{\xi + \mathrm{i}\eta \in \mathbf{C}^d : \xi,\eta\in\mathbf{R}^d, |\eta| < \alpha\}$.*
*(c) For any $\delta > 0$, the number $\alpha > 0$ can be chosen, independent of $\varepsilon$, such that the matrix $q^\varepsilon(\xi)$ defined by (3.13) satisfies the inequality,*

$$|q^\varepsilon_{kk'}(\xi+\mathrm{i}\eta) - q^\varepsilon_{kk'}(\xi)| < \delta,\ 1 \le k,k' \le d,\ \xi,\ \eta\in\mathbf{R}^d,\ |\eta| < \alpha.$$

To prove proposition 3.2 we define operators from $L^2(\Omega)$ to $L^2_0(\Omega)$ as follows: If $\varphi \in L^2(\Omega)$ let $\psi(\cdot)$ be the solution of the equation,

$$(3.15) \quad \sum_{i=1}^d \Big(\partial_i^* + 1 - \mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi}\Big)\Big(\partial_i + 1 - \mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi}\Big)\psi(\cdot) + \frac{\varepsilon^2}{\Lambda}\psi(\cdot) = \Big(\mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_k\cdot\xi}\partial_k^* + \mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_k\cdot\xi} - 1\Big)\Big(\varphi(\cdot) - \langle\varphi\rangle\Big),$$

where $\Lambda$ is given in (1.1).

The operator $T_{k,\varepsilon,\xi}$ is defined by putting $\psi = T_{k,\varepsilon,\xi}\varphi$. Let $\|\ \|$ denote the usual $L^2$ norm and for $\xi \in \mathbf{R}^d$ define $\|\ \|_{\varepsilon,\xi}$ by

$$\| \psi \|^2_{\varepsilon,\xi} \overset{\text{def.}}{=} \sum_{i=1}^d \| [\partial_i + 1 - \mathrm{e}^{\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi}]\psi \|^2 + \frac{\varepsilon^2}{\Lambda} \| \psi \|^2 .$$

It is easy to see that $T_{k,\varepsilon,\xi}$ is a bounded operator from $L^2(\Omega)$, equipped with the standard norm, to $L^2_0(\Omega)$, equipped with the norm $\|\ \|_{\varepsilon,\xi}$, and that the operator norm of $T_{k,\varepsilon,\xi}$ satisfies $\| T_{k,\varepsilon,\xi} \| \le 1$.

We can rewrite the Euler-Lagrange equation (3.10) using the operators $T_{k,\varepsilon,\xi}$. First let $\mathbf{b}(\cdot)$ be the random matrix defined by $\mathbf{b}(\cdot) = [\Lambda I_d - \mathbf{a}(\cdot)]/\Lambda$. Evidently $\mathbf{b}(\cdot)$ is symmetric positive definite and $\mathbf{b}(\cdot) \le (1-\lambda/\Lambda)I_d$. Substituting for $\mathbf{a}(\cdot)$ in terms of $\mathbf{b}(\cdot)$ into (3.10) yields then the equation,

$$(3.16) \quad \hat{\psi}_{k,\varepsilon}(\xi,\cdot) - \sum_{i,j=1}^d T_{j,\varepsilon,\xi}\Big\{b_{ij}(\cdot)\Big[\mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi}\partial_i + \mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi} - 1\Big]\hat{\psi}_{k,\varepsilon}(\xi,\cdot)\Big\} + \frac{1}{\Lambda}\sum_{j=1}^d T_{j,\varepsilon,\xi}(a_{kj}(\cdot)) = 0.$$

We define an operator $T_{\mathbf{b},\varepsilon,\xi}$ on $L^2_0(\Omega)$ by

$$(3.17) \qquad T_{\mathbf{b},\varepsilon,\xi}\varphi(\cdot) = \sum_{i,j=1}^d T_{j,\varepsilon,\xi}\Big\{b_{ij}(\cdot)[\mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi}\partial_i + \mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_i\cdot\xi} - 1]\varphi(\cdot)\Big\}, \ \varphi \in L^2_0(\Omega),$$

where $\mathbf{b}(\cdot)$ is an arbitrary random real symmetric matrix. We define $\| \mathbf{b} \|$ to be

$$\| \mathbf{b} \| = \sup\Big\{ |\sum_{i,j=1}^d b_{ij}(\omega)\lambda_i\lambda_j| : \sum_{i=1}^d \lambda_i^2 = 1, \omega \in \Omega\Big\}.$$

Thus $\| \mathbf{b} \|$ is the maximum over $\omega \in \Omega$ of the spectral radii of $\mathbf{b}(\omega)$. It is easy to see now that $T_{\mathbf{b},\varepsilon,\xi}$ is a bounded operator on $L^2_0(\Omega)$ equipped with the norm $\|\ \|_{\varepsilon,\xi}$ and that the corresponding operator norm satisfies $\| T_{\mathbf{b},\varepsilon,\xi} \| \le \| \mathbf{b} \|$.

Our goal now is to show that the operators $T_{k,\varepsilon,\xi}$ and $T_{\mathbf{b},\varepsilon,\xi}$ can be analytically continued from $\xi \in \mathbf{R}^d$ to a strip $\{\xi + \mathrm{i}\eta : \xi, \eta \in \mathbf{R}^d, |\eta| < \alpha\}$ in $\mathbf{C}^d$. Furthermore, the norm bounds we have obtained continue to approximately hold.

**Lemma 3.3.** *(a) Assume $L^2(\Omega)$ is equipped with the standard norm and let $\mathcal{B}(L^2(\Omega))$ be the corresponding Banach space of bounded operators on $L^2(\Omega)$. Then there exists $\alpha > 0$, independent of $\varepsilon$, such that the mapping $\xi \to T_{k,\varepsilon,\xi}$ from $\mathbf{R}^d$ to $\mathcal{B}(L^2(\Omega))$ can be analytically continued into the region $\{\xi + \mathrm{i}\eta \in \mathbf{C}^d : \xi, \eta \in \mathbf{R}^d, |\eta| < \alpha\}$.*
*(b) For $\xi, \eta \in \mathbf{R}^d, |\eta| < \alpha$ consider $T_{k,\varepsilon,\xi+i\eta}$ as a bounded operator from $L^2(\Omega)$, equipped with the standard norm, to $L^2_0(\Omega)$, equipped with the norm $\| \ \|_{\varepsilon,\xi}$. We denote the corresponding operator norm also by $\| \ \|_{\varepsilon,\xi}$. Then, for $\alpha$ sufficiently small, independent of $\varepsilon$, there exists a constant $C_\alpha$, depending only on $\alpha$, such that*

$$\| T_{k,\varepsilon,\xi+\mathrm{i}\eta} - T_{k,\varepsilon,\xi} \|_{\varepsilon,\xi} \le C_\alpha |\eta|, \ \ \xi, \eta \in \mathbf{R}^d, |\eta| < \alpha.$$

*Proof.* We can write down the solution of (3.15) by using a Green's function. Thus, in analogy to (2.7), let $G_{\varepsilon,\xi}$ be the solution of the equation,

$$\sum_{i=1}^d \Big[\nabla_i + 1 - \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\Big]\Big[\nabla_i^* + 1 - \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\Big] G_{\varepsilon,\xi}(x) + \frac{\varepsilon^2}{\Lambda} G_{\varepsilon,\xi}(x) = \delta(x), \quad x \in \mathbf{Z}^d. \tag{3.18}$$

Then $T_{k,\varepsilon,\xi}$ is given by

$$T_{k,\varepsilon,\xi}\phi(\cdot) = \sum_{y\in\mathbf{Z}^d} \Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi}\nabla_k + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1\Big] G_{\varepsilon,\xi}(y)\Big[\phi(\tau_y\cdot) - \langle\phi\rangle\Big], \quad \phi \in L^2(\Omega). \tag{3.19}$$

The function $G_{\varepsilon,\xi}(y)$ decays exponentially as $|y| \to \infty$. Hence the RHS of (3.19) is in $L^2_0(\Omega)$. It is a simple matter to verify that $\psi = T_{k,\varepsilon,\xi}\phi$, defined by (3.19), satisfies (3.15).

To prove part (a) we need to analyze the solution of (3.18). To do this we go into Fourier variables. Thus if

$$\hat{G}_{\varepsilon,\xi}(\zeta) = \sum_{x\in\mathbf{Z}^d} G_{\varepsilon,\xi}(x) e^{ix\cdot\zeta}, \quad \zeta \in [-\pi,\pi]^d,$$

then from (3.18) we have that

$$\hat{G}_{\varepsilon,\xi}(\zeta) = \left\{ \sum_{i=1}^d \Big[\mathrm{e}^{-\mathrm{i}\mathbf{e}_i\cdot\zeta} - \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\Big]\Big[\mathrm{e}^{\mathrm{i}\mathbf{e}_i\cdot\zeta} - \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}\Big] + \varepsilon^2/\Lambda \right\}^{-1}, \ \zeta \in [-\pi,\pi]^d. \tag{3.20}$$

Taking the inverse Fourier transform we reconstruct $G_{\varepsilon,\xi}$ from $\hat{G}_{\varepsilon,\xi}$ by the formula,

$$G_{\varepsilon,\xi}(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \hat{G}_{\varepsilon,\xi}(\zeta) e^{-ix\cdot\zeta} d\zeta, \ x \in \mathbf{Z}^d. \tag{3.21}$$

Observe now that there exists $\alpha > 0$, independent of $\varepsilon$, such that $\hat{G}_{\varepsilon,\xi}(\zeta)$, regarded as a function of $\xi$ and $\zeta$, is analytic in the region $\{(\xi,\zeta) \in \mathbf{C}^{2d} : |Im\ \xi| < \alpha, |Im\ \zeta| < \varepsilon\alpha\}$. From (3.20), (3.21) we have that

$$\Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi}\nabla_k + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1\Big] G_{\varepsilon,\xi}(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} d\zeta \mathrm{e}^{-\mathrm{i}x\cdot\zeta} \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} \times$$
$$\Big[\mathrm{e}^{-\mathrm{i}\mathbf{e}_k\cdot\zeta} - \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi}\Big] \Big\{ \sum_{i=1}^d [\mathrm{e}^{-\mathrm{i}\mathbf{e}_i\cdot\zeta} - \mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}][\mathrm{e}^{\mathrm{i}\mathbf{e}_i\cdot\zeta} - \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_i\cdot\xi}] + \varepsilon^2\ /\Lambda \Big\}^{-1}.$$

By deforming the contour of integration in $\zeta$ in (3.21) into the complex space $\mathbf{C}^d$ we see that for every $x \in \mathbf{Z}^d$ the function $[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi}\nabla_k + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1]\times\ G_{\varepsilon,\xi}(x)$ is analytic in $\xi$ for $\xi \in \mathbf{C}^d$ with $|\ Im\ \xi\ | < \alpha$. Furthermore there is a universal constant $C$ such that

$$(3.22) \qquad |\ [\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi}\nabla_k + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1]G_{\varepsilon,\xi}(x)\ | \le C\exp[-\varepsilon\alpha|x|], \quad x \in \mathbf{Z}^d, \xi \in \mathbf{C}^d, |Im\ \xi| < \alpha.$$

Note that a similar inequality for $G_{\varepsilon,\xi}(x)$ holds if $d \ge 3$ but not if $d = 2$. Part (a) follows now since it is clear that the RHS of (3.19) is analytic if we have a finite summation instead of the sum over all of $\mathbf{Z}^d$. The inequality (3.22) gives uniform convergence in the norm of $\mathcal{B}(L^2(\Omega))$, whence the result of part (a).

We turn to the proof of part (b). We have from (3.19) that

$$\begin{aligned}
&\|T_{k,\varepsilon,\xi+\mathrm{i}\eta}\phi - T_{k,\varepsilon,\xi}\phi\|^2_{\varepsilon,\xi} \\
&\quad = \sum_{n\in\mathbf{Z}^d}\Gamma(n)\sum_{x\in\mathbf{Z}^d}\Big(\sum_{j=1}^d [h_j(\eta,x) - h_j(0,x)]\Big[\overline{h}_j(\eta,x-n) - \overline{h}_j(0,x-n)\Big] \\
&\qquad\qquad + \Big[h(\eta,x) - h(0,x)\Big]\Big[\overline{h}(\eta,x-n) - \overline{h}(0,x-n)\Big]\Big),
\end{aligned}$$

where $\Gamma$ is the correlation function,

$$\Gamma(n) = \Big\langle \Big[\phi(\tau_n\cdot) - \langle\phi\rangle\Big]\Big[\overline{\phi}(\tau_0\cdot) - \langle\overline{\phi}\rangle\Big]\Big\rangle, n \in \mathbf{Z}^d,$$

and the $h, h_j$ are given by

$$(3.23) \qquad h(\eta,x) = \frac{\varepsilon}{\sqrt{\Lambda}}\Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta)}\nabla_k + \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta)} - 1\Big]G_{\varepsilon,\xi+\mathrm{i}\eta}(x), \quad x \in \mathbf{Z}^d,$$

$$h_j(\eta,x) = \Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta)}\nabla_k + \mathrm{e}^{i\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta)} - 1\Big]\Big[\nabla_j^* + 1 - \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\Big]G_{\varepsilon,\xi+\mathrm{i}\eta}(x), \quad x \in \mathbf{Z}^d, 1 \le j \le d.$$

Now $\Gamma(n)$ is a positive definite function. Hence by Bochner's theorem [9] there is a finite positive measure $d\mu_\phi$ on $[-\pi,\pi]^d$ such that

$$\Gamma(n) = \int_{[-\pi,\pi]^d}\mathrm{e}^{\mathrm{i}n\cdot\zeta}d\mu_\phi(\zeta),\ n \in \mathbf{Z}^d,$$

and,

$$\int_{[-\pi,\pi]^d}d\mu_\phi(\zeta) = \|\phi - \langle\phi\rangle\|^2 \le \|\phi\|^2.$$

It follows that

$$\begin{aligned}
&\|\ T_{k,\varepsilon,\xi+\mathrm{i}\eta}\phi - T_{k,\varepsilon,\xi}\phi\ \|^2_{\varepsilon,\xi} = \\
&\quad \int_{[-\pi,\pi]^d}\Big[\sum_{j=1}^d|\hat{h}_j(\eta,\zeta) - \hat{h}_j(0,\zeta)|^2 + \ |\hat{h}(\eta,\zeta) - \hat{h}(0,\zeta)|^2\Big]d\mu_\phi(\zeta) \le \\
&\qquad \|\ \phi\ \|^2\sup_{\zeta\in[-\pi,\pi]^d}\Big\{\sum_{j=1}^d|\hat{h}_j(\eta,\zeta) - \hat{h}_j(0,\zeta)|^2 + |\hat{h}(\eta,\zeta) - \hat{h}(0,\zeta)|^2\Big\}.
\end{aligned}$$

From (3.20) , (3.23) we have that

$$\hat{h}(\eta,\zeta) = \frac{\varepsilon}{\sqrt{\Lambda}}\Big\{ \exp\Big[i\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta) - \mathbf{i}\mathbf{e}_k\cdot\zeta\Big] - 1\Big\}\hat{G}_{\varepsilon,\xi+\mathrm{i}\eta}(\zeta),$$
$$\hat{h}_j(\eta,\zeta) = \Big\{ \exp\Big[\mathrm{i}\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta) - \mathbf{i}\mathbf{e}_k\cdot\zeta\Big] - 1\Big\}\ \Big\{\mathrm{e}^{\mathbf{i}\mathbf{e}_j\cdot\zeta} - \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\Big\}\hat{G}_{\varepsilon,\xi+\mathrm{i}\eta}(\zeta).$$

Hence,

$$\sum_{j=1}^{d} |\hat{h}_j(\eta,\zeta) - \hat{h}_j(0,\zeta)|^2 + |\hat{h}(\eta,\zeta) - \hat{h}(0,\zeta)|^2 \ =$$
$$\hat{G}_{\varepsilon,\xi}(\zeta)^{-1}|\Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot(\xi+\mathrm{i}\eta)} - \mathrm{e}^{\mathbf{i}\mathbf{e}_k\cdot\zeta}\Big]\hat{G}_{\varepsilon,\xi+\mathrm{i}\eta}(\zeta) - \Big[\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - \mathrm{e}^{\mathbf{i}\mathbf{e}_k\cdot\zeta}\Big]\hat{G}_{\varepsilon,\xi}(\zeta)|^2.$$

It is easy to see that we can choose $\alpha$ sufficiently small, independent of $\varepsilon$, such that this last expression is less that $C|\eta|^2$ for all $\zeta\in[-\pi,\pi]^d, |\eta|<\alpha$, where the constant $C$ is independent of $\varepsilon$. The result follows. □

**Corollary 3.1.** *Let $T_{\mathbf{b},\varepsilon,\xi+\mathrm{i}\eta}$ be the analytic continuation of the operator $T_{\mathbf{b},\varepsilon,\xi}$ of (3.17). Then, for $\alpha$ sufficiently small, independent of $\varepsilon$, there exists a constant $C_\alpha$, depending only on $\alpha$, such that $\parallel T_{\mathbf{b},\varepsilon,\xi+\mathrm{i}\eta} - T_{\mathbf{b},\varepsilon,\xi} \parallel_{\varepsilon,\xi} \le C_\alpha|\eta| \ \ \parallel \mathbf{b} \parallel$ . Here the operator norm $\parallel \ \parallel_{\varepsilon,\xi}$ is that induced on bounded operators on $L^2_0(\Omega)$, equipped with the norm $\parallel \ \parallel_{\varepsilon,\xi}$.*

*Proof.* Follows from lemma 3.3 and Taylor expansion. □

*Proof of Proposition 3.2.* ¿From (3.16), (3.17) the function $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ can be obtained as the solution of the equation,

$$\hat{\psi}_{k,\varepsilon}(\xi,\cdot) - T_{\mathbf{b},\varepsilon,\xi}\hat{\psi}_{k,\varepsilon}(\xi,\cdot) + \frac{1}{\Lambda}\sum_{j=1}^{d} T_{j,\varepsilon,\xi}(a_{kj}(\cdot)) = 0,$$

where the matrix $\mathbf{b}$ has $\parallel \mathbf{b} \parallel \le 1-\lambda/\Lambda$. In view of lemma 3.3 and corollary 3.1 there exists $\alpha>0$, independent of $\varepsilon$, such that the equation

$$\hat{\psi}_{k,\varepsilon}(\xi+\mathrm{i}\eta,\cdot) - T_{\mathbf{b},\varepsilon,\xi+\mathrm{i}\eta}\hat{\psi}_{k,\varepsilon}(\xi+\mathrm{i}\eta,\cdot) + \frac{1}{\Lambda}\sum_{j=1}^{d} T_{j,\varepsilon,\xi+\mathrm{i}\eta}(a_{kj}(\cdot)) = 0, \tag{3.24}$$

has a unique solution $\hat{\psi}_{k,\varepsilon}(\xi+\mathrm{i}\eta,\cdot)\in L^2_0(\Omega)$, provided $\xi,\eta\in\mathbf{R}^d, |\eta|<\alpha$. Further, $\alpha$ can be chosen sufficiently small, independent of $\varepsilon$, such that

$$\parallel \hat{\psi}_{k,\varepsilon}(\xi+\mathrm{i}\eta,\cdot) - \hat{\psi}_{k,\varepsilon}(\xi,\cdot) \parallel_{\varepsilon,\xi} \le C_\alpha|\eta|, \tag{3.25}$$

where the constant $C_\alpha$ is independent of $\varepsilon$. It is easy to see that the function $\hat{\psi}_{k,\varepsilon}(\xi+\mathrm{i}\eta,\cdot)$ is the analytic continuation of $\hat{\psi}_{k,\varepsilon}(\xi,\cdot),\ \xi\in\mathbf{R}^d$. In fact we just write the solution of (3.24) as a perturbation series. A finite truncation of the series is clearly analytic in $\xi+\mathrm{i}\eta\in\mathbf{C}^d$. Now we use the fact that lemma 3.3 and corollary 3.1 gives us uniform convergence in the standard norm on $L^2_0(\Omega)$ to assert the analyticity of the entire series. This proves parts (a) and (b).

To prove part (c) we use the representation (3.13) for $q^\varepsilon(\xi)$. Thus

$$q^\varepsilon_{kk'}(\xi+\mathrm{i}\eta)-q^\varepsilon_{kk'}(\xi)=\Big\langle\sum_{j=1}^d a_{kj}(\cdot)$$

$$\Big[\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}\partial_j+(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi}-1)\Big]\Big[\hat\psi_{k',\varepsilon}(\xi+\mathrm{i}\eta,\cdot)-\hat\psi_{k',\varepsilon}(\xi,\cdot)\Big]\Big\rangle$$

$$+\Big\langle\sum_{j=1}^d a_{kj}(\cdot)(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot(\xi+\mathrm{i}\eta)}-\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_j\cdot\xi})(\partial_j+1)\hat\psi_{k',\varepsilon}(\xi+\mathrm{i}\eta,\cdot)\Big\rangle.$$

Hence from the Schwarz inequality,

$$|q^\varepsilon_{kk'}(\xi+\mathrm{i}\eta)-q^\varepsilon_{kk'}(\xi)|\le C\parallel\hat\psi_{k',\varepsilon}(\xi+\mathrm{i}\eta,\cdot)$$

$$-\hat\psi_{k',\varepsilon}(\xi,\cdot)\parallel_{\varepsilon,\xi}+C|\eta|\parallel\hat\psi_{k',\varepsilon}(\xi+\mathrm{i}\eta,\cdot)\parallel_{\varepsilon,\xi},$$

where the constant $C$ depends only on $\alpha$ and the uniform bound $\Lambda$ on the matrix $\mathbf{a}(\cdot)$. The result follows now from (3.25).

*Proof of Theorem 1.1.* From proposition 3.2 there exists $\alpha>0$ such that the matrix $q^\varepsilon(\xi)$ has an analytic continuation into the region $\Big\{\xi+\mathrm{i}\eta:|\eta|<\alpha\Big\}$. From (3.12) we have then that $\hat u(\xi)$ can also be analytically continued into this region. The result follows now by using the deformation of contour argument of proposition 3.1, the fact that $q^\varepsilon(\xi)$ is bounded below as a quadratic form by $\lambda I_d$ and part (c) of proposition 3.2.

## 4. A Bernoulli Environment.

In this section we consider a situation in which the random matrix $\mathbf{a}:\Omega\to\mathbf{R}^{d(d+1)/2}$ is generated by independent Bernoulli variables. For each $n\in\mathbf{Z}^d$ let $Y_n$ be independent Bernoulli variables, whence $Y_n=\pm1$ with equal probability. The probability space $(\Omega,\mathcal{F},\mu)$ is then the space generated by all the variables $Y_n,\ n\in\mathbf{Z}^d$. A point $\omega\in\Omega$ is a set of configurations $\{(Y_n,\ n):n\in\mathbf{Z}^d\}$. For $y\in\mathbf{Z}^d$ the translation operator $\tau_y$ acts on $\Omega$ by taking the point $\omega=\{(Y_n,\ n):n\in\mathbf{Z}^d\}$ to $\tau_y\omega=\{(Y_{n+y},\ n):n\in\mathbf{Z}^d\}$. The random matrix $\mathbf{a}$ is then defined by

$$\mathbf{a}(\omega)\overset{\text{def.}}{=}(1+\gamma Y_0)I_d,\quad \omega=\{(Y_n,n):n\in\mathbf{Z}^d\},$$

where $I_d$ is the identity $d\times d$ matrix and $\gamma$ is a number satisfying $0<\gamma<1$. Evidently, we have $\mathbf{a}(x,\omega)=(1+\gamma Y_x)I_d,\ x\in\mathbf{Z}^d$.

For $N=1,2,...$, let $\mathbf{Z}^{d,N}$ be the collection of all sets of $N$ distinct elements $\{n_1,...,n_N\}$ with $n_j\in\mathbf{Z}^d,\ 1\le j\le N$. For $1\le p<\infty$ a function $\psi_N:\mathbf{Z}^{d,N}\to\mathbf{R}$ is in $L^p(\mathbf{Z}^{d,N})$ if

$$\|\psi_N\|_p^p\overset{\text{def.}}{=}\sum_{m\in\mathbf{Z}^{d,N}}|\psi_N(m)|^p<\infty.$$

For each $y\in\mathbf{Z}^d$ we may define a translation operator $\tau_y$ on $\mathbf{Z}^{d,N}$ by

$$\tau_y\{n_1,...,n_N\}\overset{\text{def.}}{=}\{n_1-y,...,n_N-y\}.$$

We can then define the convolution of two functions $\psi_N,\varphi_N:\mathbf{Z}^{d,N}\to\mathbf{R}$. This is a function $\psi_N*\varphi_N:\mathbf{Z}^d\to\mathbf{R}$ given by

$$\psi_N*\varphi_N(y)=\sum_{m\in\mathbf{Z}^{d,N}}\psi_N(m)\varphi_N(\tau_y m),\quad y\in\mathbf{Z}^d. \tag{4.1}$$

If $N = 1$ this is just the standard discrete convolution. We have the following generalization of Young's inequality:

**Proposition 4.1.** *Suppose $p, q$ satisfy $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \le r \le \infty$. Then if $\psi_N \in L^p(\mathbf{Z}^{d,N})$ and $\varphi_N \in L^q(\mathbf{Z}^{d,N})$ it follows that $\psi_N * \varphi_N \in L^r(\mathbf{Z}^d)$ and*

$$\|\psi_N * \varphi_N\|_r \le \|\psi_N\|_p \|\varphi_N\|_q. \tag{4.2}$$

*Proof.* We follow the standard procedure. If $r = \infty$ the result follows from Hölder's inequality applied to (4.1). For $p = 1$ we have from Hölder's inequality that

$$\begin{aligned} |\psi_N * \varphi_N(y)|^q &\le \Big[ \sum_{m \in \mathbf{Z}^{d,N}} |\psi_N(m)| \Big]^{q-1} \sum_{m \in \mathbf{Z}^{d,N}} |\psi_N(m)| |\varphi_N(\tau_y m)|^q \\ &\le \|\psi_N\|_1^{q-1} \sum_{m \in \mathbf{Z}^{d,N}} |\psi_N(m)| |\varphi_N(\tau_y m)|^q. \end{aligned}$$

Now if we sum this last inequality over $y \in \mathbf{Z}^d$ we get the inequality (4.2) with $r = q$. For the general case we have

$$|\psi_N * \varphi_N(y)|^r \le \Big[ \sum_{m \in \mathbf{Z}^{d,N}} |\psi_N(m)|^p \Big]^{(1-\alpha)r/p} \times \\ \Big[ \sum_{m \in \mathbf{Z}^{d,N}} |\varphi_N(m)|^q \Big]^{(1-\beta)r/q} \sum_{m \in \mathbf{Z}^{d,N}} |\psi_N(m)|^p |\varphi_N(\tau_y m)|^q,$$

where $\alpha, \beta$ are given by $r\alpha = p,\ r\beta = q$. The result follows from this last inequality by summing over $y \in \mathbf{Z}^d$. $\square$

Next we define the Fock space corresponding to the $N$ body spaces $L^p(\mathbf{Z}^{d,N})$, $N = 1, 2, \ldots$. We denote by $\psi$ a set $\{\psi_N : N = 0, 1, 2, ...\}$ of functions $\psi_N : \mathbf{Z}^{d,N} \to \mathbf{R}$, $N = 1, 2, ...$ and $\psi_0 \in \mathbf{R}$. The $L^p$ Fock space which we denote by $\mathcal{F}^p(\mathbf{Z}^d)$ is all such $\psi$ which satisfy

$$\|\psi\|_p^p = |\psi_0|^p + \sum_{N=1}^{\infty} \|\psi_N\|_p^p < \infty.$$

For two functions $\psi = \{\psi_N : N = 0, 1, 2, ...\}$ and $\varphi = \{\varphi_N : N = 0, 1, 2, ...\}$ we define the convolution $\psi * \varphi : \mathbf{Z}^d \to \mathbf{R}$ by

$$\psi * \varphi(y) \stackrel{\text{def.}}{=} \sum_{N=1}^{\infty} \psi_N * \varphi_N(y), \qquad y \in \mathbf{Z}^d. \tag{4.3}$$

Arguing as in Proposition 4.1 we have a version of Young's inequality for this situation.

**Corollary 4.1.** *Suppose $p, q, r$ are as in Proposition 4.1, and $\psi \in \mathcal{F}^p(\mathbf{Z}^d)$, $\varphi \in \mathcal{F}^q(\mathbf{Z}^d)$. Then $\psi * \varphi \in L^r(\mathbf{Z}^d)$ and $\|\psi * \varphi\|_r \le \|\psi\|_p \|\varphi\|_q$.*

The point in defining the Fock spaces $\mathcal{F}^p(\mathbf{Z}^d)$ here is the fact that $\mathcal{F}^2(\mathbf{Z}^d)$ is unitarily equivalent to $L^2(\Omega)$. In fact if $\psi = \{\psi_N : N = 0, 1, 2, ...\}$ with $\psi_0 \in \mathbf{R}$ and $\psi_N : \mathbf{Z}^{d,N} \to \mathbf{R}$ we can define a function $U\psi$ on $\Omega$ by

$$U\psi = \psi_0 + \sum_{N=1}^{\infty} \sum_{\{m = \{n_1, \ldots, n_N\} \in \mathbf{Z}^{d,N}\}} \psi_N(m) Y_{n_1} Y_{n_2} \ldots Y_{n_N}.$$

Evidently $U : \mathcal{F}^2(\mathbf{Z}^d) \to L^2(\Omega)$ is unitary. We now define $L^p(\Omega)$ for $1 \le p < \infty$ as the image of $\mathcal{F}^p(\mathbf{Z}^d)$ under $U$ equipped with the induced norm. Evidently we can define the convolution for functions in $L^p(\Omega)$ and Young's inequality holds as in Corollary 4.1. Observe now that $L^p(\Omega)$ is contained in $L^2(\Omega)$ if $1 \le p \le 2$. We have seen in Lemma 2.2 that the minimizer $\Psi^k = (\Psi_1^k, ..., \Psi_d^k)$ has $\Psi_i^k \in L^2(\Omega)$, $i = 1, ..., d$. We can strengthen this result here as follows:

**Proposition 4.2.** *Suppose the random matrix $\mathbf{a}(\omega)$ is given by $\mathbf{a}(\omega) = (1+\gamma Y_0) I_d$ with $0 < \gamma < 1$, and $\Psi^k = (\Psi_1^k, ..., \Psi_d^k)$ is the minimizer for the corresponding variational problem as given in Lemma 2.2. Then there exists $p$, $1 < p < 2$, depending on $\gamma$ such that $\Psi_i^k \in L^p(\Omega)$, $i = 1, ..., d$. The number $p$ can be chosen arbitrarily close to 1 provided $\gamma > 0$ is taken sufficiently small.*

*Proof.* Writing $\Psi_i = \partial_i \Phi$, $i = 1, ..., d$ and assuming $\Phi$ is an arbitrary function in $L^2(\Omega)$ it is clear that the Euler-Lagrange equation in Lemma 2.2 is the same as

$$\sum_{i,j=1}^{d} \partial_i^* \left[ a_{ij}(\omega) \Psi_j^k(\omega) \right] + \sum_{j=1}^{d} \partial_j^* a_{kj}(\omega) = 0. \tag{4.4}$$

Thus if we can find $\Psi^k \in \mathcal{H}(\Omega)$ satisfying (4.4) then $\Psi^k$ is the unique solution to the variational problem of Lemma 2.2.

For any $k = 1, \ldots, d$ we define an operator $T_k : L^2(\Omega) \to \mathcal{H}(\Omega)$ as follows: Suppose $\Phi \in L^2(\Omega)$. Then, in analogy to our derivation of (4.4) we see that there is a unique $\Psi \in \mathcal{H}(\Omega)$ such that

$$\sum_{i=1}^{d} \partial_i^* \Psi_i = \partial_k^* \Phi. \tag{4.5}$$

We put $\Psi = T_k \Phi$. It is easy to see that $T_k$ is a bounded operator with $\|T_k\| \le 1$. Next, for $k = 1, ..., d$ and $\eta > 0$ we define an operator $T_{k,\eta} : L^2(\Omega) \to \mathcal{H}(\Omega)$ as follows: Suppose $\Phi \in L^2(\Omega)$. Then by using the variational argument of Lemma 2.2 one sees that there is a unique $\Phi_\eta \in L^2(\Omega)$ such that

$$\sum_{i=1}^{d} \partial_i^* \; \partial_i \; \Phi_\eta + \eta \Phi_\eta = \partial_k^* \Phi. \tag{4.6}$$

We put $\nabla \Phi_\eta = T_{k,\eta} \Phi$. It is again clear that $T_{k,\eta}$ is a bounded operator and $\|T_{k,\eta}\| \le 1$.

We can obtain a representation for the solution $\Phi_\eta$ of (4.6) with the help of the Green's function $G_\eta$ of (2.7). It is easy to see that

$$\Phi_\eta(\omega) = \sum_{y \in \mathbf{Z}^d} \nabla_k G_\eta(y) \Phi(\tau_y \omega), \qquad \omega \in \Omega, \tag{4.7}$$

is in $L^2(\Omega)$ and satisfies (4.6). From (4.5), (4.6), we see that for $\Phi \in L^2(\Omega)$, $T_{k,\eta}\Phi$ converges weakly to $T_k \Phi$ in $\mathcal{H}(\Omega)$ as $\eta \to 0$ provided the corresponding function $\Phi_\eta$ defined by (4.7) satisfies $\eta \Phi_\eta \to 0$ weakly in $L^2(\Omega)$. This last fact follows from the ergodicity of the translation operators $\tau_y$. One sees this by going to the spectral representation of the $\tau_y$ [9].

We consider now the case of the $\mathbf{a}(\omega)$ in the statement of Proposition 4.2. In this situation (4.4) can be rewritten as

$$\Psi^k + \gamma \sum_{j=1}^{d} T_j(Y_0 \Psi_j^k) + \sum_{j=1}^{d} T_j(a_{kj}) = 0.$$

Define $T : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ by the formula,

$$T\Psi = \sum_{j=1}^{d} T_j(Y_0\Psi_j), \quad \Psi \in \mathcal{H}(\Omega).$$

Since the operation of multiplication by $Y_0$ is unitary on $L^2(\Omega)$ it follows that $T$ is bounded and in fact $\|T\| \le 1$. Hence (4.4) is equivalent to solving the equation

$$(I + \gamma T)\Psi^k + \sum_{j=1}^{d} T_j(a_{kj}) = 0. \tag{4.8}$$

Since $\|T\| \le 1$ it is evident this equation has a unique solution provided $|\gamma| < 1$. We can for $\eta > 0$ define an operator $T_\eta : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ in analogy to the definition of $T$ by

$$T_\eta\Psi = \sum_{j=1}^{d} T_{j,\eta}(Y_0\Psi_j), \quad \Psi \in \mathcal{H}(\Omega).$$

It is clear that $T_\eta$ is bounded with $\|T_\eta\| \le 1$. Hence if $|\gamma| < 1$ there is a unique solution to the equation

$$(I + \gamma T_\eta)\Psi^{k,\eta} + \sum_{j=1}^{d} T_{j,\eta}(a_{kj}) = 0. \tag{4.9}$$

Furthermore the solution $\Psi^{k,\eta}$ of (4.9) converges weakly to the solution $\Psi^k$ of (4.4) as $\eta \to 0$ in $\mathcal{H}(\Omega)$.

For $1 \le p < \infty$ we define the spaces $\mathcal{H}^p(\Omega)$ as follows: Let $\mathcal{E}_p = \{\nabla\varphi : \varphi \in L^p(\Omega)\}$. For $\Psi \in \mathcal{E}_p$ we define the norm of $\Psi$, $\|\Psi\|_p$ to be given by $\|\Psi\|_p^p = \sum_{i=1}^{d} \|\Psi_i\|_p^p$, where $\Psi = (\Psi_1, ..., \Psi_d)$. The Banach space $\mathcal{H}^p(\Omega)$ is the closure of $\mathcal{E}_p$ in this norm. Evidently $\mathcal{H}^2(\Omega)$ is the same as $\mathcal{H}(\Omega)$. We can show that there exists $p < 2$, depending on $\gamma < 1$ such that (4.9) has a solution in $\mathcal{H}^p(\Omega)$. To see this observe that

$$(T_\eta\Psi)_i(\omega) = \sum_{j=1}^{d} \sum_{y \in \mathbf{Z}^d} \nabla_i^*\nabla_j G_\eta(y)(Y_0\Psi_j)(\tau_y\omega), \quad \omega \in \Omega.$$

Now the RHS of this last equation is just a singular integral. In fact if $\varphi = \{\varphi_N : N = 0, 1, 2, ...\}$ is in $L^p(\Omega)$ then the function $\psi$ defined by

$$\psi(\omega) = \sum_{y \in \mathbf{Z}^d} \nabla_i^*\nabla_j G_\eta(y)\varphi(\tau_y\omega), \quad \omega \in \Omega\ ,$$

is given by $\psi = \{\psi_N : N = 0, 1, 2, ...\}$ where $\psi_0 = 0$ and for $N \ge 1$, one has

$$\psi_N(m) = \sum_{y \in \mathbf{Z}^d} \nabla_i^*\nabla_j G_\eta(y)\varphi_N(\tau_y m), \quad m \in \mathbf{Z}^{d,N}.$$

Thus $\psi_N$ is the convolution of a second derivative of a Green's function with $\varphi_N$. We can therefore invoke the Calderon-Zygmund theorem [10] to conclude the following: Let $1 < p < \infty$. Then $T_\eta$ is a bounded operator on $\mathcal{H}^p(\Omega)$ for every $\eta > 0$. Further, there exists a bounded operator $T$ on $\mathcal{H}^p(\Omega)$ such that $\lim_{\eta\to 0}\ \|T\Psi - T_\eta\Psi\|_p = 0$ for every $\Psi \in \mathcal{H}^p(\Omega)$. There exists $p < 2$ such that $\lim_{\eta\to 0}\ \gamma\|T_\eta\|_p < 1$.

In the last statement we are using the fact that $\|T\|_2 \le 1$ and the continuity of the operator norms in $p$. Hence there exists $p < 2$ such that for all sufficiently small $\eta$ the equation (4.9) has a unique solution $\Psi^{k,\eta}$ in $\mathcal{H}^p(\Omega)$. It also follows from the above that as $\eta \to 0$, $\Psi^{k,\eta}$ converges in $\mathcal{H}^p(\Omega)$ to a function $\Psi^k \in \mathcal{H}^p(\Omega)$. Since $\mathcal{H}^p(\Omega) \subset \mathcal{H}^2(\Omega) = \mathcal{H}(\Omega)$ it follows that this $\Psi^k$ is also the solution of the variational problem. This proves the first part of Proposition 4.2. The fact that $p$ can be taken arbitrarily close to 1 for sufficiently small $\gamma$ is clearly also a direct consequence of the Calderon-Zygmund theorem. □

**Proposition 4.3.** *Suppose the random matrix* $\mathbf{a}(\omega)$ *is given by* $\mathbf{a}(\omega) = (1+\gamma Y_0)I_d$ *with* $0 < \gamma < 1$ *and* $Z_\varepsilon(x, \omega)$ *is as in Proposition 2.1. Then if* $d > 2$ *one has*

$$\left\langle \int_{\mathbf{Z}^d_\varepsilon} dx Z_\varepsilon(x, \cdot)^2 \right\rangle \le C\varepsilon^{2\alpha} \ , \tag{4.10}$$

*where* $\alpha > 0$ *is a constant depending only on* $\gamma$ *and* $C$ *is independent of* $\varepsilon$*. If* $\gamma$ *is sufficiently small one can take* $\alpha = 1$*.*

*Proof.* From Proposition 2.1 it is sufficient to obtain estimates on $A_1, A_2, A_3, A_4, A_5, A_6$. We first consider $A_1$. In view of the boundedness of the matrix $\mathbf{a}(\omega)$ and Proposition 4.2 it follows that $Q_{ij}(\omega)$ is in $L^p(\Omega)$ for some $p$, $1 < p < 2$, $i, j = 1, ..., d$. Taking the expectation value and using translation invariance we see that $A_1$ is given by the expression,

$$A_1 = \sum_{i,j,\ell=1}^{d} \int_{\mathbf{Z}^d_\varepsilon} dx\varepsilon^2 \sum_{y\in\mathbf{Z}^d} \sum_{y'\in\mathbf{Z}^d} \nabla^*_\ell G_0\left(\frac{x}{\varepsilon} - y\right) \times$$
$$\nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon y) \nabla^*_\ell G_0\left(\frac{x}{\varepsilon} - y'\right) \nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon y') \ Q_{ij} * Q_{ij}(y - y'),$$

where convolution is defined by (4.3). Making the change of variable $n = y - y'$, this becomes

$$A_1 = \sum_{i,j,\ell=1}^{d} \varepsilon^2 \sum_{n\in\mathbf{Z}^d} h_{ij\ell}(n) Q_{ij} * Q_{ij}(n),$$

where

$$h_{ij\ell}(n) = \int_{\mathbf{Z}^d_\varepsilon} dx \sum_{y\in\mathbf{Z}^d} \nabla^*_\ell G_0\left(\frac{x}{\varepsilon} - y\right) \nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon y) \times \tag{4.11}$$
$$\nabla^*_\ell G_0\left(\frac{x}{\varepsilon} - y + n\right) \nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon[y - n]).$$

We can estimate the summation with respect to $x$ from our knowledge of the properties of $G_0$. We conclude that there is a constant $C$ independent of $\varepsilon$ such that

$$\begin{aligned} |h_{ij\ell}(n)| &\le \frac{C\varepsilon^d}{1 + |n|^{d-2}} \sum_{y\in\mathbf{Z}^d} |\nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon y)| \\ |\nabla^{\varepsilon*}_i \nabla^\varepsilon_j u(\varepsilon[y - n])| &\le \frac{C}{1 + |n|^{d-2}} \ \exp[-\delta\varepsilon|n|], \end{aligned}$$

where $C, \delta > 0$ are independent of $\varepsilon$. In this last inequality we have used Proposition 3.1. From Corollary 4.1 it follows that $Q_{ij} * Q_{ij} \in L^r(\mathbf{Z}^d)$ where $1/r = 2/p - 1$. Since $1 < p < 2$ one has $1 < r < \infty$. Hence from Hölder's inequality we conclude that

$$A_1 \le C\varepsilon^2 \left[ \sum_{n\in\mathbf{Z}^d} \frac{1}{(1 + |n|^{d-2})^{r'}} \ \exp[-\delta r'\varepsilon|n|] \right]^{1/r'} \quad ,$$

where $1/r + 1/r' = 1$. Evidently this yields

$$A_1 \le C\varepsilon^2 \max\left[1,\ (1/\varepsilon)^{2\,-\,d/r}\right].$$

Since $r < \infty$ it follows that $A_1 \le C\varepsilon^{2\alpha}$ for some $\alpha > 0$. If $\gamma$ is sufficiently small we can choose $p$ close to 1 and hence $r$ close to 1. Since $d > 2$ it follows that $A_1 \le C\varepsilon^2$ in this case.

It is clear that $A_2$ and $A_3$ can be dealt with in exactly the same way as $A_1$. To deal with $A_4$ we choose $\eta = \varepsilon^2$. Arguing as before we have

$$A_4 = \sum_{i,j,k=1}^{d} \varepsilon^4 \sum_{n\in\mathbf{Z}^d} h_{ijk}(n)\Psi_j^k * \Psi_j^k(n),$$

where

$$h_{ijk}(n) = \int_{\mathbf{Z}_\varepsilon^d} dx[\nabla_k^\varepsilon u(x+\varepsilon\mathbf{e}_j)]^2 \sum_{y\in\mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon} - y\right) G_\eta\left(\frac{x}{\varepsilon} - y + n\right).$$

Again we see from known properties of Green's functions that

$$\begin{aligned}
|h_{ijk}(n)| &\le \frac{C}{1+|n|^{d-4}} \exp[-\delta\varepsilon|n|], && d > 4,\\
|h_{ijk}(n)| &\le C\log(1/\varepsilon)\exp[-\delta\varepsilon|n|], && d = 4,\\
|h_{ijk}(n)| &\le \frac{C}{\varepsilon}\exp[-\delta\varepsilon|n|], && d = 3.
\end{aligned}$$

Using these inequalities and arguing as before we see that $A_4 \le C\varepsilon^{2\alpha}$ with $\alpha > 0$ and $\alpha = 1$ if $\gamma$ is sufficiently small.

To deal with $A_5$ and $A_6$ we use the representation,

$$\chi_k\left(\frac{x}{\varepsilon},\ \cdot\right) = \sum_{y\in\mathbf{Z}^d}\sum_{j=1}^{d} \nabla_j^* G_\eta\ \left(\frac{x}{\varepsilon} - y\right)\Psi_j^k(\tau_y\ \cdot),$$

and argue as previously. □

For $d = 2$ we can get almost the same result as in Proposition 4.3. We have

**Proposition 4.4.** *Suppose $d = 2$ in the statement of Proposition 4.3. Then (4.10) holds for some $\alpha > 0$ depending on $\gamma$. The number $\alpha$ can be taken arbitrarily close to 1 provided $\gamma$ is chosen sufficiently small.*

*Proof.* We consider $A_1$ again. The problem is that when $d = 2$ the summation with respect to $x$ in (4.11) gives infinity. To get around this we replace $G_0$ in the representation (2.23) of the first term on the RHS of (2.21) by $G_\eta$ with $\eta = \varepsilon^2$. Hence from (2.7), (2.22), we have

$$\psi_\varepsilon(y',\omega) = \sum_{x'\in\mathbf{Z}^d} -\Delta G_\eta\left(\frac{y'}{\varepsilon} - x'\right)\psi_\varepsilon(\varepsilon x',\omega)$$

$$+ \sum_{x'\in\mathbf{Z}^d} \eta\, G_\eta\left(\frac{y'}{\varepsilon} - x'\right)\psi_\varepsilon(\varepsilon x',\omega), \quad y' \in \mathbf{Z}_\varepsilon^d\ ,\ \omega\in\Omega.$$

We can rewrite this as

$$\begin{aligned}
\psi_\varepsilon(y',\omega) &= \sum_{x'\in\mathbf{Z}^d}\sum_{\ell=1}^d \nabla_\ell^* G_\eta\left(\frac{y'}{\varepsilon}-x'\right)\varepsilon\ \nabla_\ell^\varepsilon\psi_\varepsilon(\varepsilon x',\omega) \\
&\quad+\sum_{x'\in\mathbf{Z}^d}\eta\, G_\eta\left(\frac{y'}{\varepsilon}-x'\right)\psi_\varepsilon(\varepsilon x',\omega) \\
&= \int_{\mathbf{Z}_\varepsilon^d} dx\sum_{\ell=1}^d \nabla_\ell^\varepsilon\psi_\varepsilon(x,\omega)\nabla_\ell^* G_\eta\left(\frac{x}{\varepsilon}-\frac{y'}{\varepsilon}\right)\varepsilon^{1-d} \\
&\quad+\int_{\mathbf{Z}_\varepsilon^d} dx\,\psi_\varepsilon(x,\omega)G_\eta\left(\frac{x}{\varepsilon}-\frac{y'}{\varepsilon}\right)\varepsilon^{2-d}.
\end{aligned}$$

The first term on the RHS of (2.21) is therefore the same as

$$\begin{aligned}
(4.12)\quad &\sum_{i,j,\ell=1}^d\int_{\mathbf{Z}_\varepsilon^d} dx\left\langle\ \nabla_\ell^\varepsilon\psi_\varepsilon(x,\cdot)\sum_{y\in\mathbf{Z}^d}\varepsilon\nabla_\ell^* G_\eta\left(\frac{x}{\varepsilon}-y\right)\ Q_{ij}(\tau_y\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y)\right\rangle \\
&\quad+\sum_{i,j=1}^d\int_{\mathbf{Z}_\varepsilon^d} dx\left\langle\ \psi_\varepsilon(x,\cdot)\sum_{y\in\mathbf{Z}^d}\varepsilon^2 G_\eta\left(\frac{x}{\varepsilon}-y\right)\ Q_{ij}(\tau_y\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y)\right\rangle.
\end{aligned}$$

If we use the Schwarz inequality in (4.12) we see the first term is bounded by $C\|\psi_\varepsilon\|_{\mathcal{H}^1}\ A_{1,0}^{1/2}$ and the second term by $C\|\psi_\varepsilon\|_{\mathcal{H}^1}\ A_{1,1}^{1/2}$, where

$$\begin{aligned}
A_{1,0} &= \sum_{i,j,\ell=1}^d\Big\langle\ \int_{\mathbf{Z}_\varepsilon^d} dx\left[\varepsilon\sum_{y\in\mathbf{Z}^d}\nabla_\ell^* G_\eta\left(\frac{x}{\varepsilon}-y\right)\ Q_{ij}(\tau_y\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y)\right]^2\ \Big\rangle\ , \\
A_{1,1} &= \sum_{i,j=1}^d\Big\langle\ \int_{\mathbf{Z}_\varepsilon^d} dx\left[\varepsilon^2\sum_{y\in\mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon}-y\right)\ Q_{ij}(\tau_y\cdot)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y)\right]^2\ \Big\rangle.
\end{aligned}$$

We proceed now as in Proposition 4.3. Thus $A_{1,0}$ can be written as

$$A_{1,0}=\sum_{i,j,\ell=1}^d\varepsilon^2\sum_{n\in\mathbf{Z}^d} h_{ij\ell}(n)Q_{ij}*Q_{ij}(n),$$

where

$$h_{ij\ell}(n)\ =\ \int_{\mathbf{Z}_\varepsilon^d} dx\sum_{y\in\mathbf{Z}^d}\nabla_\ell^* G_\eta\left(\frac{x}{\varepsilon}-y\right)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon y)\ \ \nabla_\ell^* G_\eta\left(\frac{x}{\varepsilon}-y+n\right)\nabla_i^{\varepsilon *}\nabla_j^\varepsilon u(\varepsilon[y\ -\ n]).$$

It is clear now that there are constants $C,\ \delta>0$ such that

$$|h_{i,j,\ell}(n)|\le C\ \log\left(\frac{1}{\varepsilon}\right)\exp[-\delta\varepsilon|n|].$$

Arguing as before we see that $A_{1,0} \le C\varepsilon^{2\alpha}$ for some $\alpha > 0$ and $\alpha$ can be chosen arbitrarily close to 1 if $\gamma$ is small. We have a similar representation for $A_{1,1}$ given by

$$A_{1,1} = \sum_{i,j=1}^{d} \varepsilon^4 \sum_{n \in \mathbf{Z}^d} h_{ij}(n) Q_{ij} * Q_{ij}(n),$$

where in this case $h_{ij}$ is given by

$$h_{ij}(n) \quad = \quad \int_{\mathbf{Z}_\varepsilon^d} dx \sum_{y \in \mathbf{Z}^d} G_\eta\left(\frac{x}{\varepsilon} - y\right) \nabla_i^{\varepsilon *} \nabla_j^{\varepsilon} u(\varepsilon y) \quad G_\eta\left(\frac{x}{\varepsilon} - y + n\right) \nabla_i^{\varepsilon *} \nabla_j^{\varepsilon} u(\varepsilon[y \quad - \quad n]).$$

The estimate now on $h_{ij}$ is

$$|h_{ij}(n)| \le C\, \varepsilon^{-2} [\log(1/\varepsilon)]^2 \exp[-\delta\varepsilon|n|].$$

We conclude again that $A_{1,1} \le C\varepsilon^{2\alpha}$ for some $\alpha > 0$ and $\alpha$ can be chosen arbitrarily close to 1 if $\gamma$ is small.

The second and third terms on the RHS of (2.21) can be dealt with exactly the same way we dealt with the first term. The fourth, fifth and sixth terms are handled in the same way we handled $A_4, A_5, A_6$ in Proposition 4.3. □

*Proof of Theorem 1.2.* In view of Proposition 4.3, 4.4 and the inequality (2.24), it is sufficient for us to estimate

$$\varepsilon^2 \Big\langle \int_{\mathbf{Z}_\varepsilon^d} dx \left[ \sum_{k=1}^{d} \chi_k(\frac{x}{\varepsilon}, \ \cdot) \nabla_k^\varepsilon u(x) \right]^2 \Big\rangle.$$

Since this quantity is bounded by $dA_6$, we just have to use the estimates from Propositions 4.3, 4.4.

Next we turn to the proof of Theorem 1.3. It is clear from Lemma 2.4 that if $u_\varepsilon$ is the minimizer for Lemma 2.1 and $(u, \psi_\varepsilon)$ the minimizer for Lemma 2.4 then

$$\Big\langle \Big[ \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) u_\varepsilon(x, \cdot) - \Big\langle \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) u_\varepsilon(x, \cdot) \Big\rangle \Big]^2 \Big\rangle = \varepsilon^2 \Big\langle \left[ \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) \psi_\varepsilon(x, \tau_{x/\varepsilon}\ \cdot) \right]^2 \Big\rangle.$$

We shall first prove a result when $\psi_\varepsilon$ is the minimizer for the separable problem given in Lemma 2.5.

**Proposition 4.5.** *Let $(u, \psi_\varepsilon)$ be the minimizer of the functional $\mathcal{F}_{S,\varepsilon}$ given in Lemma 2.5, and $g : \mathbf{R}^d \to \mathbf{R}$ a $C^\infty$ function of compact support. Then*

$$\varepsilon^2 \Big\langle \Big[ \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) \psi_\varepsilon(x, \tau_{x/\varepsilon}\ \cdot) \Big]^2 \Big\rangle \le C\, \varepsilon^\alpha \tag{4.13}$$

*for some $\alpha > 0$, provided $|\gamma| < 1$. The number $\alpha$ can be taken arbitrarily close to $d$ provided $\gamma$ is taken sufficiently small.*

*Proof.* From Proposition 2.2 it is sufficient for us to bound

$$\varepsilon^2 \Big\langle \Big[ \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) \nabla_k^\varepsilon u(x) \psi_{k,\varepsilon}(\tau_{x/\varepsilon}\ \cdot) \Big]^2 \Big\rangle,$$

where $\psi_{k,\varepsilon}(\cdot)$ is the minimizer of Lemma 2.6. This last expression is the same as

$$\varepsilon^{d+2} \sum_{n \in \mathbf{Z}^d} h(n) \psi_{k,\varepsilon} * \psi_{k,\varepsilon}(n),$$

where

$$h(n) = \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) \nabla_k^\varepsilon u(x) g(x - \varepsilon n) \nabla_k^\varepsilon u(x - \varepsilon n), \quad n \in \mathbf{Z}^d. \tag{4.14}$$

We shall see in the next lemma that $\varepsilon\psi_{k,\varepsilon} \in L^p(\Omega)$ for some $p$, $1 < p < 2$, and that $\|\varepsilon\psi_{k,\varepsilon}\|_p \le C$ where $C$ is independent of $\varepsilon$ as $\varepsilon \to 0$. Further, $p$ can be taken arbitrarily close to 1 for sufficiently small $\gamma > 0$. The result follows easily from this and Young's inequality, Corollary 4.1. Since $\varepsilon\psi_{k,\varepsilon}$ is in $L^p(\Omega)$ it follows that $\varepsilon^2 \psi_{k,\varepsilon} * \psi_{k,\varepsilon}$ is in $L^r(\mathbf{Z}^d)$ with $1/r = 2/p - 1$, and $\|\varepsilon^2 \psi_{k,\varepsilon} * \psi_{k,\varepsilon}\|_r \le C$ where $C$ is independent of $\varepsilon$. Hence from Hölder's inequality the LHS of (4.13) is bounded by

$$C\, \varepsilon^d \|h\|_{r'}, \quad \text{where} \;\; 1/r + 1/r' = 1.$$

It is easy now to see from (4.14) that $\|h\|_{r'} \le C\varepsilon^{-d/r'}$. Note that $r' > 1$ since $p < 2$, hence $r < \infty$.

$\square$

**Lemma 4.1.** *Suppose the random matrix $\mathbf{a}(\omega)$ is given by $\mathbf{a}(\omega) = (1 + \gamma Y_0) I_d$ with $0 < \gamma < 1$, and $\psi_{k,\varepsilon}(\cdot)$ is the minimizer for the corresponding variational problem as given in Lemma 2.6. Then there exists p, $1 < p < 2$, depending on $\gamma$ such that $\varepsilon\psi_{k,\varepsilon} \in L^p(\Omega)$ with $\|\varepsilon\psi_{k,\varepsilon}\|_p$ bounded independent of $\varepsilon$ as $\varepsilon \to 0$. The number p can be chosen arbitrarily close to 1 provided $\gamma > 0$ is taken sufficiently small.*

*Proof.* Observe that the result is trivial for $p = 2$. In fact we have

$$\begin{aligned} \|\varepsilon\psi_{k,\varepsilon}\|_2^2 &\le \Big\langle\, a_{kk}(\cdot) \Big\rangle + 2\mathcal{G}_{k,\varepsilon}(\psi_{k,\varepsilon}) \\ &\le \Big\langle\, a_{kk}(\cdot)\, \Big\rangle, \quad (\text{ since } \; \mathcal{G}_{k,\varepsilon}(0) = 0). \end{aligned}$$

To prove the $L^p$ result with $p < 2$ we proceed similarly to Proposition 4.2. First note that the Euler-Lagrange equation of Lemma 2.6 is the same as

$$\sum_{i,j=1}^d \partial_i^* [a_{ij}(\omega) \partial_j \psi_{k,\varepsilon}(\omega)] + \varepsilon^2 \psi_{k,\varepsilon}(\varepsilon) + \sum_{j=1}^d \partial_j^* a_{kj}(\omega) = 0, \quad \omega \in \Omega. \tag{4.15}$$

If we can find $\psi_{k,\varepsilon} \in L^2(\Omega)$ satisfying (4.15) then $\psi_{k,\varepsilon}$ is the unique solution to the variational problem of Lemma 2.6.

For any $k = 1, ..., d$ define an operator $T_{k,\varepsilon} : L^2(\Omega) \to L^2(\Omega)$ as follows: Suppose $\varphi \in L^2(\Omega)$. Then, in analogy to our derivation of (4.15) we see that there is a unique $\psi \in L^2(\Omega)$ satisfying

$$\sum_{i=1}^d \partial_i^* \partial_i \psi + \varepsilon^2 \psi = \partial_k^* \varphi.$$

We put $\psi = T_{k,\varepsilon}\varphi$. It is easy to see that $T_{k,\varepsilon}$ is a bounded operator with $\|T_{k,\varepsilon}\| \le 2\varepsilon^2$. We can obtain a representation of $T_{k,\varepsilon}\varphi$ using the Green's function $G_\eta$ of (2.7). We have

$$T_{k,\varepsilon}\varphi(\omega) = \sum_{y \in \mathbf{Z}^d} \nabla_k G_{\varepsilon^2}(y) \varphi(\tau_y \omega), \quad \omega \in \Omega.$$

Putting $\mathbf{a}(\omega) = (1+\gamma Y_0)I_d$ we see that (4.15) is the same as

$$\psi_{k,\varepsilon} + \gamma \sum_{j=1}^{d} T_{j,\varepsilon}(Y_0 \partial_j \psi_{k,\varepsilon}) + \sum_{j=1}^{d} T_{j,\varepsilon}(a_{k,j}) = 0. \tag{4.16}$$

Observe that (4.16) cannot necessarily be solved in $L^2(\Omega)$ since the norms of $T_{j,\varepsilon}$, as bounded operators on $L^2(\Omega)$, can become arbitrarily large as $\varepsilon \to 0$. To get around this we define a norm on $L^2(\Omega)$ which depends on $\varepsilon$. For $\psi \in L^2(\Omega)$ let $\|\psi\|_\varepsilon$ be defined by

$$\|\psi\|_\varepsilon^2 \stackrel{\text{def.}}{=} \sum_{j=1}^{d} \|\partial_j \psi\|^2 + \varepsilon^2 \|\psi\|^2.$$

Let $T_\varepsilon$ be the operator on $L^2(\Omega)$ given by

$$T_\varepsilon \psi = \sum_{j=1}^{d} T_{j,\varepsilon}(Y_0 \partial_j \psi). \tag{4.17}$$

We show that the operator norm of $T_\varepsilon$ with respect to $\| \ \|_\varepsilon$ on $L^2(\Omega)$ has $\|T_\varepsilon\|_\varepsilon \le 1$. In fact (4.17) implies that

$$\left[\sum_{i=1}^{d} \partial_i^* \partial_i + \varepsilon^2\right] T_\varepsilon \psi = \sum_{j=1}^{d} \partial_j^* (Y_0 \partial_j \psi),$$

whence

$$\|T_\varepsilon \psi\|_\varepsilon^2 \le \|T_\varepsilon \psi\|_\varepsilon \|\psi\|_\varepsilon,$$

which implies the result. We conclude that (4.16) is uniquely solvable for $\psi_{k,\varepsilon}$ in $L^2(\Omega)$ provided $|\gamma| < 1$. Following Proposition 4.2, we define a $p$ norm on $L^p(\Omega)$, $\ 1 < p < \infty$, which depends on $\varepsilon$, by

$$\|\psi\|_{\varepsilon,p}^p \stackrel{\text{def.}}{=} \sum_{j=1}^{d} \|\partial_j \psi\|_p^p + \|\varepsilon \psi\|_p^p, \ \psi \in L^p(\Omega). \tag{4.18}$$

The result follows if we can show that $T_\varepsilon$ is a bounded operator on $L^p(\Omega)$ with the norm $\|T_\varepsilon\|_{\varepsilon,p}$ induced by the vector norm (4.18), bounded independent of $\varepsilon$ as $\varepsilon \to 0$. This again is just a consequence of the Calderon-Zygmund theorem. $\square$

*Proof of Theorem 1.3.* We need to go into the Fourier representation studied in §3. Thus from (3.11) we have that

$$\begin{aligned}
\varepsilon \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x) \psi_\varepsilon(x, \tau_{x/\varepsilon}\cdot) &= \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x)\, \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} \hat{\psi}_\varepsilon(\xi, \tau_{x/\varepsilon}\cdot) e^{-ix\cdot\xi}\, d\xi \\
&= \sum_{k=1}^{d} \int_{\mathbf{Z}_\varepsilon^d} dx g(x)\, \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} (e^{-i\varepsilon \mathbf{e}_k \cdot \xi} - 1) \times \\
&\qquad \hat{u}(\xi) \hat{\psi}_{k,\varepsilon}(\xi, \tau_{x/\varepsilon}\cdot) e^{-ix\cdot\xi} d\xi
\end{aligned}$$

Hence

$$\left\langle \left[ \varepsilon \int_{\mathbf{Z}_\varepsilon^d} dx\ g(x)\ \psi_\varepsilon(x, \tau_{x/\varepsilon}\cdot) \right]^2 \right\rangle \le$$

$$d \sum_{k=1}^d \left\langle \left| \int_{\mathbf{Z}_\varepsilon^d} dx g(x) \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} (\mathrm{e}^{-\mathrm{i}\varepsilon \mathbf{e}_k\cdot\xi} - 1)\hat{u}(\xi)\hat{\psi}_{k,\varepsilon}(\xi, \tau_{x/\varepsilon}\cdot)\mathrm{e}^{-\mathrm{i}x\cdot\xi} d\xi \right|^2 \right\rangle$$

$$= d \sum_{k=1}^d \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} d\xi d\zeta (2\pi)^{-d}\varepsilon^{-1}(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1)\hat{u}(\xi)(2\pi)^{-d}\varepsilon^{-1}\times$$

$$(\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1)\overline{\hat{u}(\xi)} \sum_{n\in\mathbf{Z}^d} \varepsilon^{d+2} h_k(n, \xi, \zeta)\hat{\psi}_{k,\varepsilon}(\xi) * \overline{\hat{\psi}}_{k,\varepsilon}(\zeta)(n),$$

where $\hat{\psi}_{k,\varepsilon}(\xi), \hat{\psi}_{k,\varepsilon}(\zeta)$ denote the functions $\hat{\psi}_{k,\varepsilon}(\xi,\cdot),\ \hat{\psi}_{k,\varepsilon}(\zeta,\cdot)$ respectively in $L^2(\Omega)$, and $h_k$ is given by the formula

$$h_k(n, \xi, \zeta) = \int_{\mathbf{Z}_\varepsilon^d} dx\, g(x)g(x - \varepsilon n) \exp[\mathrm{i}x\cdot(\zeta - \xi) - \mathrm{i}\varepsilon n\cdot\zeta].$$

Since $g : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support it is easy to see that for every $r'$, $1 \le r' \le \infty$, $h_k(\cdot, \xi, \zeta)$ is in $L^{r'}(\mathbf{Z}^d)$ and $\|h_k(\cdot, \xi, \zeta)\|_{r'} \le C\ \varepsilon^{-d/r'}$, where $C$ depends only on $g$. We know now from § 3 that

$$\left| \varepsilon^{-1}(\mathrm{e}^{-\mathrm{i}\varepsilon\mathbf{e}_k\cdot\xi} - 1)\hat{u}(\xi) \right| \le \frac{C_m}{1 + |\xi|^m}, \quad \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d,$$

for any positive integer $m$, where $C_m$ depends only on $m$. Hence to finish the proof of Theorem 1.3 we need to prove an analogue of Lemma 4.1 for the function $\hat{\psi}_{k,\varepsilon}(\xi,\cdot)$ and our bounds must be uniform for $\xi \in [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d$. This is accomplished by replacing the operator $T_\varepsilon$ of (4.17) by the operator $T_{\mathbf{b},\varepsilon,\xi}$ of (3.17) and following the argument of Lemma 4.1.

## 5. More General Environments.

In this section we shall show how to generalize the methods of §4 to prove Theorem 1.2 and Theorem 1.3. Just as in § 3 we define a random matrix $\mathbf{b}(\cdot)$ by $\mathbf{b}(\cdot) = [\Lambda I_d - \mathbf{a}(\cdot)]/\Lambda$. Thus $\mathbf{b}(\cdot)$ is a symmetric positive definite matrix and $\mathbf{b}(\cdot) \le (1 - \lambda/\Lambda)I_d = \gamma I_d$ in the sense of quadratic forms. Next let $S$ be the set $S = \{\cup_{k=1}^m \{i_k, j_k\} : 1 \le i_k \le j_k \le d,\ k = 1, ..., m,\ 1 \le m < \infty\}$. For $s \in S$ we define a random variable, $b_s$ by

$$b_s(\cdot) = \prod_{k=1}^m b_{i_k,j_k}(\cdot), \quad s = \bigcup_{k=1}^m \{i_k, j_k\},$$

and a random variable $Y_{0,s} = b_s - \langle b_s\rangle$. For $s \in S, n \in \mathbf{Z}^d$ we define $Y_{n,s}$ as the translate of $Y_{0,s}$. Thus $Y_{n,s}(\cdot) = Y_{0,s}(\tau_n\ \cdot)$. It follows from our assumptions that the variables $Y_{n_1,s_1},\ Y_{n_2,s_2}$ are independent if $n_1 \ne n_2$. They are not necessarily independent if $n_1 = n_2$. We can think of the extra index $s$ on the variable $Y_{n,s}$ as denoting a spin. We are therefore led to define a Fock space $\mathcal{F}_S^p(\mathbf{Z}^d)$ of many particle functions where the particles move in $\mathbf{Z}^d$ and have spin in $S$. Thus $\psi \in \mathcal{F}_S^p(\mathbf{Z}^d)$ is a collection of $N$ particle functions $\psi = \{\psi_N : N = 0, 1, 2, ...\}$ where $\psi_0 \in \mathbf{R}$ and

$\psi_N : \mathbf{Z}^{d,N} \times S^N \to \mathbf{R}$, $N = 1, 2, ....$ Each $\psi_N$ is in $L^p(\mathbf{Z}^{d,N} \times S^N)$ with $L^p$ norm given by

$$\|\psi_N\|_p^p \stackrel{\text{def.}}{=} \sum_{m \in \mathbf{Z}^{d,N} \times S^N} |\psi_N(m)|^p.$$

The norm on $\mathcal{F}_S^p(\mathbf{Z}^d)$ is given as before by

$$\|\psi\|_p^p \stackrel{\text{def.}}{=} |\psi_0|^p + \sum_{N=1}^{\infty} \|\psi_N\|_p^p \quad .$$

For $s = \cup_{k=1}^m \{i_k, j_k\} \in S$, let $|s| = m$. Given a parameter $\alpha > 0$ we define a mapping $U_\alpha$ from $\mathcal{F}_S^2(\mathbf{Z}^d)$ to functions on $\Omega$ by

$$U_\alpha \psi \stackrel{\text{def.}}{=} \psi_0 + \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,...,n_N\} \in \mathbf{Z}^{d,N} \\ s_i \in S,\ 1 \le i \le N}} \psi_N(n_1, s_1, n_2, s_2, ..., n_N, s_N) \times \alpha^{|s_1|+...+|s_N|}\ Y_{n_1,s_1} Y_{n_2,s_2} \cdots Y_{n_N,s_N}. \tag{5.1}$$

**Lemma 5.1.** *For any $\alpha > 0$ the number $\gamma$ can be chosen sufficiently small so that $U_\alpha$ is a bounded operator from $\mathcal{F}_S^2(\mathbf{Z}^d)$ to $L^2(\Omega)$ and $\|U_\alpha\| \le 1$.*

*Proof.* Since $\langle Y_{0,s} \rangle = 0,\ s \in S$, we have

$$\begin{aligned}\|U_\alpha\ \psi\|_2^2 = \psi_0^2 + \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,...,n_N\} \in \mathbf{Z}^{d,N}, \\ s_i, s_i' \in S,\ 1 \le i \le N}} &\psi_N(n_1, s_1, n_2, s_2, ..., n_N, s_N) \times \\ &\psi_N(n_i, s_1', ..., n_N, s_N') \alpha^{|s_1|+...+|s_N|+|s_1'|+...+|s_N'|} \left\langle Y_{n_1,s_1}\ Y_{n_1,s_1'} \right\rangle \times \\ &\left\langle Y_{n_2,s_2}\ Y_{n_2,s_2'} \right\rangle \cdots \left\langle Y_{n_N,s_N}\ Y_{n_N,s_N'} \right\rangle \\ \le \psi_0^2 + \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,...,n_N\} \in \mathbf{Z}^{d,N}, \\ s_i, s_i' \in S, 1 \le i \le N}} &|\psi_N(n_1, s_1, ..., n_N, s_N)| \times \\ &|\psi_N(n_1, s_1', ..., n_N, s_N')| \delta^{|s_1|+...+|s_N|+|s_1'|+...+|s_N'|} ,\end{aligned}$$

where $\delta > 0$ can be chosen arbitrarily small provided $\gamma > 0$ is taken sufficiently small. It is evident now that for fixed $(n_1, ..., n_N) \in \mathbf{Z}^{d,N}$ one has

$$\sum_{s_i, s_i' \in S, 1 \le i \le N} |\psi_N(n_1, s_1, ..., n_N, s_N)|\ |\psi_N(n_1, s_1', ..., n_N, s_N')| \times \delta^{|s_1|+...+|s_N|+|s_1'|+...+|s_N'|} \le \left[\sum_{s \in S} \delta^{2|s|}\right]^N \sum_{s_i \in S, 1 \le i \le N} |\psi_N(n_1, s_1, ..., n_N, s_N)|^2.$$

The result follows now by taking $\delta$ small enough so that $\sum_{s \in S} \delta^{2|s|} \le 1$. $\square$

Our first goal here will be to prove an analogue of Proposition 4.2. Let $T_{\mathbf{b}} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ be the operator defined by

$$T_{\mathbf{b}}\Psi = \sum_{i,j=1}^{d} T_i\Big(b_{ij}(\cdot)\Psi_j\Big), \ \Psi \in \mathcal{H}(\Omega), \tag{5.2}$$

where the operators $T_i$ are given by (4.5). It is clear that $\|T_{\mathbf{b}}\| \le \gamma$ and that the minimizer $\Psi^k$ of Lemma 2.2 is the unique solution to the equation

$$\Psi^k = T_{\mathbf{b}}\Psi^k - \frac{1}{\Lambda}\sum_{j=1}^{d} T_j(a_{kj}). \tag{5.3}$$

For each $y \in \mathbf{Z}^d$ we can define a translation operator $\tau_y$ on $\mathcal{F}^2_S(\mathbf{Z}^d)$ as follows:

$$\tau_y\psi_0 = \psi_0,$$
$$\tau_y\psi_N(n_1, s_1, ..., n_N, s_N) = \psi_N(n_1 - y, s_1, ..., n_N - y, s_N), N \ge 1.$$

It is clear that $\tau_y U_\alpha = U_\alpha \tau_y, \ y \in \mathbf{Z}^d, \ \alpha > 0$. Just as in §2 we can use the translation operators to define derivative operators $\partial_i$ , $\partial_i^*$, $1 \le i \le d$, on $\mathcal{F}^2_S(\mathbf{Z}^d)$. We define the space of gradients of functions in $\mathcal{F}^2_S(\mathbf{Z}^d)$, which we denote by $\mathcal{H}^2_S(\mathbf{Z}^d)$, in analogy to the definition of $\mathcal{H}(\Omega)$. Thus for $\varphi \in \mathcal{F}^2_S(\mathbf{Z}^d)$ let $\nabla\varphi$ be the gradient $(\partial_1\varphi, ..., \partial_d\varphi)$. Then $\mathcal{H}^2_S(\mathbf{Z}^d)$ is the completion of the space of gradients $\{\nabla\varphi : \varphi \in \mathcal{F}^2_S(\mathbf{Z}^d)\}$ under the norm $\| \ \|_2$,

$$\|\Psi\|_2^2 = \sum_{i=1}^{d} \|\Psi_i\|_2^2, \ \Psi = (\Psi_1, ..., \Psi_d), \ \Psi_i \in \mathcal{F}^2_S(\mathbf{Z}^d), \ \ 1 \le i \le d.$$

We wish to define an operator $T_{\mathbf{b},\alpha,\mathcal{F}}$ on $\mathcal{H}^2_S(\mathbf{Z}^d)$ which has the property that if $T_{\mathbf{b}}$ is the operator of (5.2) then $U_\alpha \ T_{\mathbf{b},\alpha,\mathcal{F}} = T_{\mathbf{b}}U_\alpha$. First we define operators $T_{k,\mathcal{F}} : \mathcal{F}^2_S(\mathbf{Z}^d) \longrightarrow \mathcal{H}^2_S(\mathbf{Z}^d)$, $1 \le k \le d$. These are defined exactly as in (4.5). Thus, note that for $\Phi \in \mathcal{F}^2_S(\mathbf{Z}^d)$ there is a unique $\Psi \in \mathcal{H}^2_S(\mathbf{Z}^d)$ such that

$$\sum_{i=1}^{d} \partial_i^*\Psi_i = \partial_k^*\Phi.$$

We can see this by a variational argument as previously. We put then $\Psi = T_{k,\mathcal{F}}\Phi$ and it is easy to see that $T_{k,\mathcal{F}}$ is bounded with $\|T_{k,\mathcal{F}}\| \le 1$. It is also clear that $U_\alpha T_{k,\mathcal{F}} = T_k U_\alpha$, where $T_k$ is defined by (4.5).

Next we need to define analogues of the multiplication operators $b_{ij}(\cdot)$, $1 \le i, j \le d$. For any pair $(i, j)$ with $1 \le i, j \le d$ and $\alpha > 0$ define an operator $B_{i,j,\alpha} : \mathcal{F}^2_S(\mathbf{Z}^d) \to \mathcal{F}^2_S(\mathbf{Z}^d)$ as follows:

Suppose $\varphi \in \mathcal{F}_S^2(\mathbf{Z}^d)$, $\varphi = \{\varphi_N : N = 0, 1, 2, ...\}$. Then $B_{i,j,\alpha}\, \varphi = \psi = \{\psi_N : N = 0, 1, 2, ....\}$. Here $\psi$ is given in terms of $\varphi$ by

$$\psi_N(n_1, s_1, ..., n_N, s_N) = \langle b_{i,j}(\cdot)\rangle \varphi_N(n_1, s_1, ..., n_N, s_N) + \sum_{s\in S} [\langle b_{i,j} b_s\rangle - \langle b_{i,j}\rangle\langle b_s\rangle] \alpha^{|s|} \varphi_{N+1}(0, s, n_1, s_1, ..., n_N, s_N),$$
$$\text{if } n_k \neq 0, \quad 1 \le k \le N, \quad N \ge 0;$$
$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = 0,$$
$$\text{if } n_k \neq 0, \quad 2 \le k \le N, \quad \text{and } \{i, j\} \text{ not contained in } s_1;$$
$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = \alpha^{-1} \varphi_N(0, s_1 \backslash \{i, j\}, n_2, s_2, ..., n_N, s_N),$$
$$\text{if } n_k \neq 0,\ 2 \le k \le N, \quad \text{and } \{i, j\} \text{ strictly contained in } s_1;$$
$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = \alpha^{-1} \varphi_{N-1}(n_2, s_2, ..., n_N, s_N) - \sum_{s\in S} \langle b_s\rangle \alpha^{|s|-1}\ \varphi_N(0, s, n_2, s_2, ..., n_N, s_N),$$
$$\text{if } n_k \neq 0,\ 2 \le k \le N \quad \text{and } \{i, j\} = s_1.$$

Observe that $U_\alpha B_{i,j,\alpha} = b_{i,j}(\cdot) U_\alpha$. It is clear that if we define $T_{\mathbf{b},\alpha,\mathcal{F}}$ by

$$T_{\mathbf{b},\alpha,\mathcal{F}}\, \Psi = \sum_{i,j,=1}^{d} T_{i,\mathcal{F}}\left(B_{i,j,\alpha} \Psi_j\right), \qquad \Psi \in \mathcal{H}_S^2(\mathbf{Z}^d),$$

then $U_\alpha\ T_{\mathbf{b},\alpha,\mathcal{F}} = T_{\mathbf{b}} U_\alpha$.

We wish to obtain an equation in $\mathcal{H}_S^2(\mathbf{Z}^d)$ which corresponds to the equation (5.3) in $\mathcal{H}(\Omega)$. For $1 \le k, j \le d$ define $\Phi_{k,j} \in \mathcal{F}_S^2(\mathbf{Z}^d)$ by $\Phi_{k,j} = \{\Phi_{k,j}^N : N = 0, 1, 2, ...\}$ with $\Phi_{k,j}^N = 0$ if $N \neq 1$ and $\Phi_{k,j}^1(n, s) = 0$ if $n \neq 0$ or $s \neq \{k, j\}$, $\Phi_{k,j}^1(0, \{k, j\}) = 1$. The equation corresponding to (5.3) is given by

$$\Psi^k = T_{\mathbf{b},\alpha,\mathcal{F}} \Psi^k + \alpha^{-1} \sum_{j=1}^{d}\ T_{j,\mathcal{F}}(\Phi_{k,j}). \tag{5.4}$$

It is easy to see that we have the following:

**Lemma 5.2.** *(a) The number $\gamma$ can be chosen sufficiently small so that for some $\alpha > 1$ the operator $T_{\mathbf{b},\alpha,\mathcal{F}}$ is a bounded operator on $\mathcal{H}_S^2(\mathbf{Z}^d)$ with norm strictly less than 1.*
*(b) Suppose $\gamma, \alpha$ have been chosen so that part (a) holds and also Lemma 5.1. Then if $\Psi^k$ is the unique solution to (5.4) the function $U_\alpha\, \Psi^k$ is in $\mathcal{H}(\Omega)$ and satisfies (5.3).*

We can use the same method of proof as in Proposition 4.2 to prove the corresponding analogue for the solution of (5.4).

**Lemma 5.3.** *Suppose $\Psi^k = \left(\Psi_1^k, ..., \Psi_d^k\right)$ is the solution of (5.4) given by Lemma 5.2. Then for $\gamma, \alpha^{-1}$ sufficiently small there exists $p$, $1 < p < 2$, depending only on $\gamma, \alpha^{-1}$ such that $\Psi_i^k \in \mathcal{F}_S^p(\mathbf{Z}^d)$, $i = 1, ..., d$. The number $p$ can be chosen arbitrarily close to 1 provided $\gamma, \alpha^{-1}$ are taken small enough.*

**Lemma 5.4.** *Let $p$ satisfy $1 < p < 2$ and $\psi \in \mathcal{F}_S^p(\mathbf{Z}^d) \subset \mathcal{F}_S^2(\mathbf{Z}^d)$ where $\psi = \{\psi_N : N = 0, 1, 2, ...\}$ with $\psi_0 = 0$. Assume $\gamma > 0$ is chosen small enough so that $U_\alpha \psi \in L^2(\Omega)$. Let $g : \mathbf{Z}^d \to \mathbf{R}$ be the function $g(n) = \langle U_\alpha \psi(\cdot) U_\alpha \psi(\tau_n\, \cdot)\rangle$, $n \in \mathbf{Z}^d$. Then $\gamma > 0$ can be chosen small enough so that $g \in L^r(\mathbf{Z}^d)$ where $1/r = 2/p - 1$.*

*Proof.* Similarly to Lemma 5.1 we have that

$$|g(n)| \le \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,...,n_N\}\in \mathbf{Z}^{d,N}, \\ s_i,s_i'\in S,\ 1\le i\le N}} |\psi_N(n_1,s_1,...,n_N,s_N)| \times$$
$$|\tau_n\psi_N(n_1,s_1',...,n_N,s_N')|\delta^{|s_1|+...+|s_N|+|s_1'|+...+|s_N'|},$$

where $\delta > 0$ can be taken arbitrarily small. We now use the method of Proposition 4.1 to finish the proof. □

The previous three lemmas can be used to prove part of Theorem 1.2, the case when $\gamma$ may be taken arbitrarily small. This is done by simply following through the corresponding proofs for the Bernoulli case given in §4. Next we wish to consider the case of Theorem 1.2 when $\gamma$ is assumed only to be strictly less than 1. First we shall deal with the case where the variables $b_s(\cdot),\ s\in S$, are finitely generated. This means there exist variables $Y_k(\cdot),\ k=0,1,...,M$ such that $\langle Y_kY_{k'}\rangle = \delta_{k,k'},\ 1\le k,k'\le M$ and $Y_0\equiv 1$. Furthermore, for any $i,j,\ 1\le i,j\le d$, and $k$, $0\le k\le M$, the variable $b_{i,j}(\cdot)Y_k$ is in the linear span of the variables $Y_{k'},\ 0\le k'\le M$. This was the situation in § 4 where we could take $M=1$.

We proceed now as before, taking our spin space $S$ to be the set of integers $\{1,2,...,M\}$. Letting $Y_{0,s}=Y_s,\ s=1,...,M$, we may define as before the spaces $\mathcal{F}_S^p(\mathbf{Z}^d)$ and the transformation $U_1$. It is clear that the following holds:

**Lemma 5.5.** *$U_1$ is a bounded operator from $\mathcal{F}_S^2(\mathbf{Z}^d)$ to $L^2(\Omega)$ and $\|U_1\|\le 1$.*

Next, let $\mathcal{F}_{S,d}^p(\mathbf{Z}^d)$ be the space of vectors $\Psi=(\Psi_1,...,\Psi_d)$ with $\Psi_i\in\mathcal{F}_S^p(\mathbf{Z}^d)$, $1\le i\le d$. The norm of $\Psi$ is given by

$$\|\Psi\|_p^p = \sum_{i=1}^{d} \|\Psi_i\|_p^p\ .$$

We can define an operator $B$ on $\mathcal{F}_{S,d}^p(\mathbf{Z}^d)$ with the property that

$$(U_1B\Psi)_i = \sum_{j=1}^{d} b_{i,j}(\cdot)(U_1\Psi)_j, \qquad 1\le i\le d.$$

To do this let us write

$$b_{ij}(\cdot)Y_k = \sum_{k'=0}^{M} B_{i,j,k,k'}\ Y_{k'}, \qquad 1\le i,j\le d, \quad 0\le k\le M.$$

For a function $\varphi \in \mathcal{F}_S^p(\mathbf{Z}^d)$, $\varphi = \{\varphi_N : N = 0,1,2,...\}$, we put $B_{i,j}\varphi = \psi = \{\psi_N : N = 0,1,2,...\}$, where $\psi$ is given in terms of $\varphi$ as follows:

$$
\begin{aligned}
(5.5)\quad \psi_0 &= B_{i,j,0,0}\ \varphi_0 + \sum_{k=1}^{M} B_{i,j,k,0}\varphi_1(0,k);\\
\psi_N(n_1,k_1,...,n_N,k_N) &= B_{i,j,0,0}\varphi_N(n_1,k_1,...,n_N,k_N)\\
&\quad + \sum_{k=1}^{M} B_{i,j,k,0}\varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N),\\
&\qquad\qquad \text{if } n_k \neq 0, 1 \le k \le N, \quad N \ge 1;\\
\psi_N(n_1,k_1,...,n_N,k_N) &= B_{i,j,0,k_1}\varphi_{N-1}(n_2,k_2,...,n_N,k_N)\\
&\quad + \sum_{k=1}^{M} B_{i,j,k,k_1}\varphi_N(n_1,k,n_2,k_2,...,n_N,k_N),\\
&\qquad\qquad \text{if } n_1 = 0.
\end{aligned}
$$

It is clear that for any $\varphi \in \mathcal{F}_p^2(\mathbf{Z}^d)$ one has $U_1(B_{i,j}\varphi) = b_{i,j}(\cdot)U_1\ \varphi$. It follows that the operator $B$ is given by

$$
(B\Psi)_i = \sum_{j=1}^{d} B_{i,j}\Psi_j, \qquad \Psi = (\Psi_1,...,\Psi_d) \in \mathcal{F}_{S,d}^2(\mathbf{Z}^d).
$$

**Lemma 5.6.** *There exists $p_0 < 2$ depending only on $\gamma, M$ such that if $p_0 \le p \le 2$ then $B$ is a bounded operator on $\mathcal{F}_{S,d}^p(\mathbf{Z}^d)$ with norm, $\|B\| \le 2\gamma/(1+\gamma)$.*

*Proof.* For $1 \le i \le d$, $0 \le k \le M$, let $\lambda_{i,k}$ be real parameters. Then it follows from the orthogonality of the variables $Y_k$ and the bound on the quadratic form $\mathbf{b}(\cdot)$ that

$$
\sum_{i=1}^{d}\sum_{k=0}^{M}\left[\sum_{j=1}^{d}\sum_{k'=0}^{M}\ B_{i,j,k',k}\lambda_{j,k'}\right]^2 \le \gamma^2 \sum_{i=1}^{d}\sum_{k=0}^{M}\lambda_{i,k}^2.
$$

Suppose now $\varphi = (\varphi^{(1)},...,\varphi^{(d)}) \in \mathcal{F}_{S,d}^2(\mathbf{Z}^d)$. From (5.5) and the previous inequality we have

$$
\begin{aligned}
&\sum_{i=1}^{d}\Bigg\{\Bigg[\sum_{j=1}^{d} B_{i,j}\varphi_N^{(j)}(n_1,k_1,...,n_N,k_N)\Bigg]^2\\
&\qquad + \sum_{k=1}^{M}\Bigg[\sum_{j=1}^{d} B_{i,j}\varphi_{N+1}^{(j)}(0,k,n_1,k_1,...,n_N,k_N)\Bigg]^2\Bigg\}\\
&\qquad \le \gamma^2 \sum_{i=1}^{d}\Bigg\{\varphi_N^{(i)}(n_1,k_1,...,n_N,k_N)^2 + \sum_{k=1}^{M}\varphi_N^{(i)}(0,k,n_1,k_1,...,n_N,k_N)^2\Bigg\},
\end{aligned}
$$

if $n_r \neq 0, \ \ 1 \le r \le N$. Applying Hölder's inequality this yields for $1 < p < 2$ the inequality

$$\sum_{i=1}^{d} \left\{ \left[ \sum_{j=1}^{d} B_{i,j} \varphi_N^{(j)}(n_1, k_1, ..., n_N, k_N) \right]^p + \sum_{k=1}^{M} \left[ \sum_{j=1}^{d} B_{i,j} \varphi_{N+1}^{(j)}(0, k, n_1, k_1, ..., n_N, k_N) \right]^p \right\}$$

$$\le [d(M+1)]^{1-p/2} \gamma^2 \sum_{i=1}^{d} \left\{ \varphi_N^{(i)}(n_1, k_1, ..., n_N, k_N)^p \right.$$

$$\left. + \sum_{k=1}^{M} \varphi_N^{(i)}(0, k, n_1, k_1, ..., n_N, k_N)^p \right\}.$$

The result follows now by summing the last inequality over $n_j, k_j, 1 \le j \le N, \ N = 0, 1, 2, ...$ .

□

Next we define an operator $T_{\mathcal{F}} : \mathcal{F}^2_{S,d}(\mathbf{Z}^d) \to \mathcal{H}^2_S(\mathbf{Z}^d)$ by

$$T_{\mathcal{F}} \Psi \stackrel{\text{def.}}{=} \sum_{i=1}^{d} T_{i,\mathcal{F}}(\Psi_i), \ \ \Psi \in \mathcal{F}^2_{S,d}(\mathbf{Z}^d),$$

where the operators $T_{i,\mathcal{F}}$ are defined as before. It is easy to see that $T_{\mathcal{F}}$ is bounded with $\|T_{\mathcal{F}}\| \le 1$. Let $\Phi_{k,j} \in \mathcal{F}^2_S(\mathbf{Z}^d)$ have the property that $U_1 \Phi_{k,j} = b_{k,j}(\cdot)$. Then if $\Psi^k \in \mathcal{H}^2_S(\mathbf{Z}^d)$ satisfies the equation

$$\Psi^k = T_{\mathcal{F}}(B\Psi^k) + \sum_{j=1}^{d} T_{j,\mathcal{F}}(\Phi_{k,j}), \tag{5.6}$$

the function $U_1 \Psi^k \in \mathcal{H}(\Omega)$ is the unique solution to (5.3). It is clear from Lemma 5.6 that the following holds:

**Lemma 5.7.** *Suppose $\Psi^k = (\Psi^k_1, ..., \Psi^k_d)$ is the solution of (5.6). Then there exists $p, \ 1 < p < 2$, depending only on $\gamma < 1$ such that $\Psi^k_i \in \mathcal{F}^p_S(\mathbf{Z}^d), \ 1 \le i \le d$.*

Theorem 1.2, in the case when $\gamma$ is close to 1, follows from lemma 5.7 just as before. To complete the proof of Theorem 1.2, in the case when $\gamma$ is close to 1, we need to deal with the situation where the variables $b_{i,j}(\cdot)$ are not finitely generated. To do this let $V_0$ be the subspace of $L^2(\Omega)$ generated by the constant function. For $k \ge 1$ we define the linear space $V_k$ inductively as the span of the spaces $V_{k-1}$ and $b_{i,j}(\cdot)V_{k-1}, \ 1 \le i, j \le d$. By our assumption $V_{k-1}$ is strictly contained in $V_k, \ k \ge 1$. We suppose further for the moment that $b_{i,j}(\cdot) = b(\cdot)\delta_{i,j}, \ 1 \le i, j \le d$, whence $|b(\cdot)| \le \gamma$. It follows that the dimension of the space $V_k$ is $k+1, \ k \ge 0$. Let $Y_0, Y_1, ...$ be an orthonormal set of variables in $L^2(\Omega)$ with the property that $Y_0 \equiv 1$ and $V_k$ is spanned by the variables $Y_{k'}, \ 0 \le k' \le k, \ k = 0, 1, 2, ...$ . For $k = 0, 1, 2, ...$ we write

$$b(\cdot) Y_k = \sum_{k'=0}^{k+1} B_{k,k'} Y_{k'} \ . \tag{5.7}$$

Let $S$ be the set of integers $\{1, 2, ...\}$ and $\mathcal{F}^p_S(\mathbf{Z}^d)$ be the corresponding Fock space defined as before. For $s \in S$ let the modulus of $s$, $|s| = s$. Then for any $\alpha > 0$ we can define the mapping $U_\alpha$ from $\mathcal{F}^2_S(\mathbf{Z}^d)$ to functions on $\Omega$ by (5.1). Let $B_\alpha$ be the operator on $\mathcal{F}^2_S(\mathbf{Z}^d)$ with the property that

$$U_\alpha \, B_\alpha \, \varphi \ = \ b(\cdot) U_\alpha \varphi, \ \ \ \varphi \in \mathcal{F}^2_S(\mathbf{Z}^d).$$

Then $B_\alpha$ is given as follows: For $\varphi = \{\varphi_N : N = 0,1,2,...\}$ we put $B_\alpha\varphi = \psi = \{\psi_N : N = 0,1,2,...\}$, where $\psi$ is defined in terms of $\varphi$ by,

$$
\begin{aligned}
(5.8)\quad & \psi_0 = B_{0,0}\varphi_0 + \sum_{k=1}^{\infty} \alpha^k B_{k,0}\varphi_1(0,k) \ ; \\
& \psi_N(n_1,k_1,...,n_N,k_N) = B_{0,0}\varphi_N(n_1,k_1,...,n_N,k_N) \\
& \qquad\qquad + \sum_{k=1}^{\infty} \alpha^k B_{k,0}\varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N), \\
& \qquad\qquad\qquad\qquad \text{if } n_j \neq 0,\ 1 \le j \le N,\ N \ge 1; \\
& \psi_N(0,1,n_2,k_2,...,n_N,k_N) = \alpha^{-1}B_{0,1}\varphi_{N-1}(n_2,k_2,...,n_N,k_N) \\
& \qquad\qquad + \sum_{k=1}^{\infty} \alpha^{k-1} B_{k,1}\varphi_N(0,k,n_2,k_2,...,n_N,k_N), \\
& \qquad\qquad\qquad\qquad \text{if } n_j \neq 0,\ 2 \le j \le N,\ N \ge 1; \\
& \psi_N(0,k_1,n_2,k_2,...,n_N,k_N) = \sum_{k=k_1-1}^{\infty} \alpha^{k-k_1} B_{k,k_1}\varphi_N(0,k,n_2,k_2,...,n_N,k_N), \\
& \qquad\qquad\qquad\qquad \text{if } n_j \neq 0,\ 2 \le j \le N,\ k_1 > 1,\ N \ge 1.
\end{aligned}
$$

**Lemma 5.8.** *There exists $\alpha, p_0$, $0 < \alpha < 1$, $1 < p_0 < 2$, depending only on $\gamma$ such that if $p_0 \le p \le 2$, then $B_\alpha$ is a bounded operator on $\mathcal{F}^p_S(\mathbf{Z}^d)$ with norm, $\|B_\alpha\| \le 2\gamma/(1+\gamma)$.*

*Proof.* For $k = 0,1,2...$ let $\lambda_k$ be real parameters. Then it follows from (5.7) that

$$
(5.9)\qquad \left[\sum_{k'=0}^{\infty} B_{k',0}\lambda_{k'}\right]^2 + \sum_{k=1}^{\infty}\left[\sum_{k'=k-1}^{\infty} B_{k',k}\lambda_{k'}\right]^2 \le \gamma^2 \sum_{k=0}^{\infty}\lambda_k^2.
$$

Observe now from (5.8) that if $n_j \neq 0$, $1 \le j \le N$, then

$$
\begin{aligned}
(5.10)\quad & |B_\alpha\varphi_N(n_1,k_1,...,n_N,k_N)|^p + \sum_{k=1}^{\infty} |B_\alpha\ \varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N)|^p \\
& \qquad \le \left|\sum_{k'=0}^{\infty} \alpha^{k'} B_{k',0}\lambda_{k'}\right|^p + \sum_{k=1}^{\infty}\left|\sum_{k'=k-1}^{\infty} \alpha^{k'-k} B_{k',k}\lambda_{k'}\right|^p,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_0 &= \varphi_N(n_1,k_1,...,n_N,k_N), \\
\lambda_k &= \varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N),\ k \ge 1.
\end{aligned}
$$

We can use (5.9) to obtain a bound on the RHS of (5.10) when $p = 2$. To do this let $g_\alpha(k)$ be defined for $k = 0,1,2,...$ by

$$
\begin{aligned}
g_\alpha(0) &= \left[\sum_{k'=0}^{\infty} \alpha^{k'} B_{k',0}\lambda_{k'}\right]^2, \\
g_\alpha(k) &= \left[\sum_{k'=k-1}^{\infty} \alpha^{k'} B_{k',k}\lambda_{k'}\right]^2, k \ge 1.
\end{aligned}
$$

Then, when $p=2$, the RHS of (5.10) is

$$\sum_{k=0}^{\infty}\alpha^{-2k}g_\alpha(k)=\sum_{k=0}^{\infty}g_\alpha(k)+\sum_{k=1}^{\infty}\left[\alpha^{-2k}-\alpha^{-2(k-1)}\right]\sum_{m=k}^{\infty}g_\alpha(m).$$

It follows from (5.9) that

$$\begin{aligned}\sum_{k=0}^{\infty}g_\alpha(k) &\le \gamma^2\sum_{k=0}^{\infty}\alpha^{2k}\lambda_k^2,\\ \sum_{m=k}^{\infty}g_\alpha(m) &\le \gamma^2\sum_{m=k-1}^{\infty}\alpha^{2m}\lambda_m^2\ ,\ k\ge 1.\end{aligned}$$

We conclude that

$$\sum_{k=0}^{\infty}\alpha^{-2k}g_\alpha(k)\le\alpha^{-2}\gamma^2\sum_{k=0}^{\infty}\lambda_k^2. \tag{5.11}$$

Next let $M\ge 1$ be an integer. We have that

$$\sum_{k=0}^{\infty}\left|\sum_{k'=k+M}^{\infty}\alpha^{k'-k}B_{k',k}\lambda_{k'}\right|^p\le\sum_{k=0}^{\infty}\left[\sum_{k'=k+M}^{\infty}\alpha^{k'-k}|B_{k',k}|^{p/(p-1)}\right]^{p-1}\sum_{k'=k+M}^{\infty}\alpha^{k'-k}|\lambda_{k'}|^p,$$

by Hölder's inequality. From (5.9) it follows that

$$\sum_{k'=0}^{\infty}B_{k,k'}^2\le 1,\quad k=0,1,2,...,$$

whence it follows that

$$\sum_{k'=k+M}^{\infty}\alpha^{k'-k}|B_{k',k}|^{p/(p-1)}\le\alpha^M/(1-\alpha).$$

We conclude from these last inequalities that

$$\sum_{k=0}^{\infty}\left|\sum_{k'=k+M}^{\infty}\alpha^{k'-k}B_{k',k}\lambda_{k'}\right|^p\le\frac{\alpha^{pM}}{(1-\alpha)^p}\sum_{k=0}^{\infty}|\lambda_k|^p\ . \tag{5.12}$$

This last inequality, with $p=2$, and (5.11) yields the inequality,

$$\begin{aligned}\left[\sum_{k'=0}^{M-1}\alpha^{k'}B_{k',0}\lambda_{k'}\right]^2 &+ \sum_{k=1}^{\infty}\left[\sum_{k'=k-1}^{k+M-1}\alpha^{k'-k}B_{k',k}\lambda_{k'}\right]^2\\ &\le \left[\alpha^{-2}\gamma^2(1+\delta)+\frac{\alpha^{2M}}{(1-\alpha)^2}(1+\delta^{-1})\right]\sum_{k=0}^{\infty}\lambda_k^2\ ,\end{aligned}$$

for any $\delta>0$.

Next, let $N$ be an integer, $N\gg M$, and $n\ge 0$ be an integer. Then we have from the previous inequality that

$$\left[\sum_{k'=0}^{M-1}\alpha^{k'}B_{k',0}\lambda_{k'}\right]^2+\sum_{k=1}^{N+n-1}\left[\sum_{k'=k-1}^{k+M-1}\alpha^{k'-k}B_{k',k}\lambda_{k'}\right]^2\le C_{\alpha,\gamma,\delta,M}\sum_{k=0}^{N+n+M-2}\lambda_k^2,$$

and

$$\sum_{k=N+n+rN}^{N+n+(r+1)N-1} \Big[\sum_{k'=k-1}^{k+M-1} \alpha^{k'-k} B_{k',k}\lambda_{k'}\Big]^2 \le C_{\alpha,\gamma,\delta,M} \sum_{k=N+n+rN-1}^{N+n+(r+1)N+M-2} \lambda_k^2, \qquad r=0,1,2,...,$$

where

$$C_{\alpha,\gamma,\delta,M} = \alpha^{-2}\gamma^2(1+\delta) + \frac{\alpha^{2M}}{(1-\alpha)^2}(1+\delta^{-1}).$$

If we use Hölder's inequality this last inequality implies that for any $p$, $1<p<2$, one has

$$\Big|\sum_{k'=0}^{M-1} \alpha^{k'} B_{k',0}\lambda_{k'}\Big|^p + \sum_{k=1}^{N+n-1} \Big|\sum_{k'=k-1}^{k+M-1} \alpha^{k'-k} B_{k',k}\lambda_{k'}\Big|^p \le (N+n)^{1-p/2}\ C_{\alpha,\gamma,\delta,M}^{p/2} \sum_{k=0}^{N+n+M-2} |\lambda_k|^p,$$

and

$$\sum_{k=N+n+rN}^{N+n+(r+1)N-1} \Big|\sum_{k'=k-1}^{k+M-1} \alpha^{k'-k} B_{k',k}\lambda_{k'}\Big|^p$$
$$\le N^{1-p/2}\ C_{\alpha,\gamma,\delta,M}^{p/2} \sum_{k=N+n+rN-1}^{N+n+(r+1)N+M-2} |\lambda_k|^p\ , \qquad r=0,1,2,....$$

If we sum this last inequality with respect to $r$ and average over $n$, $0\le n\le N$, we obtain the inequality,

$$\Big|\sum_{k'=0}^{M-1} \alpha^{k'} B_{k',0}\ \lambda_{k'}\Big|^p + \sum_{k=1}^{\infty} \Big|\sum_{k'=k-1}^{k+M-1} \alpha^{k'-k} B_{k',k}\ \lambda_{k'}\Big|^p$$
$$\le (2N)^{1-p/2} C_{\alpha,\gamma,\delta,M}^{p/2}\ \frac{1}{N+1}\ \times$$
$$\sum_{n=0}^{N}\Big[\sum_{k=0}^{N+n+M-2} |\lambda_k|^p + \sum_{r=0}^{\infty} \sum_{k=N+n+rN-1}^{N+n+(r+1)N+M-2} |\lambda_k|^p\Big].$$

This last inequality together with (5.12) imply that we can choose $\alpha, p_0$, $0<\alpha<1$, such that the RHS of (5.10) is bounded by

$$\left(\frac{2\gamma}{1+\gamma}\right)^p \sum_{k=0}^{\infty} |\lambda_k|^p.$$

To see this we need to see how to choose the constants $\alpha,\delta,M,N,p_0$. Evidently we can take $\alpha=(1+\sqrt{\gamma})/2$. Next we pick $\delta$ small enough so that $\alpha^{-1}(1+\delta)^{1/2} < 2/(1+\gamma)$. Then we choose $M$ to be large enough so that $C_{\alpha,\gamma,\delta,M} < 4\gamma^2/(1+\gamma)^2$. Finally we choose $N,p_0$ so that for $p_0\le p\le 2$ one has

$$(2N)^{1-p/2}(1+10M/N)C_{\alpha,\gamma,\delta,M}^{p/2} < [2\gamma/(1+\gamma)]^p.$$

We conclude from (5.10) that for this choice of $\alpha, p_0$, if $p_0 \le p \le 2$, then

$$|B_\alpha \varphi_N(n_1,k_1,...,n_N,k_N)|^p + \sum_{k=1}^\infty |B_\alpha \varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N)|^p \le$$
$$\left(\frac{2\gamma}{1+\gamma}\right)^p \left[ |\varphi_N(n_1,k_1,...,n_N,k_N)|^p + \sum_{k=1}^\infty |\varphi_{N+1}(0,k,n_1,k_1,...,n_N,k_N)|^p \right],$$

if $n_j \ne 0,\ 1 \le j \le N$. The result follows now by summing the last inequality with respect to the $n_j,\ 1 \le j \le N$, and $N$. □

Next we extend the previous method to the case where the random matrix $\mathbf{b}(\cdot)$ is assumed to be a diagonal matrix. We cannot now compute the dimension of the linear spaces $V_k$ defined after Lemma 5.7. We can however estimate their dimension. It is easy to see that the dimension of $V_k$ is bounded above by $(k+1)^d$. Let $Y_{k,j},\ 0 \le j \le J_k,\ k = 0,1,2,...$, be an orthonormal set of variables in $L^2(\Omega)$ with the property that $Y_{0,0} \equiv 1$ and $V_k$ is spanned by the variables $Y_{k',j}$ $, 0 \le j \le J_{k'},\ 0 \le k' \le k,\ k = 0,1,2,...$ . For a variable $Y_{k,j}$ let us denote by $s = (k,j)$ the spin of that variable with modulus $|s| = k$. It follows from the definition of the spaces $V_k$ that then,

$$b_{ii}(\cdot) Y_s = \sum_{|s'| \le |s|+1} B_{i,s,s'} Y_{s'}, \quad 1 \le i \le d, \tag{5.13}$$

for appropriate constants $B_{i,s,s'}$. Evidently the variables $Y_s,\ |s| \ge 0$, span the space generated by the $b_{ii}(\cdot),\ 1 \le i \le d$.

Let $S$ be the set of spins $s$ defined in the previous paragraph such that $|s| > 0$. For any integer $M \ge 1$, let $S_M$ be the set,

$$S_M = \Big\{ s \in S : |s| \le M \} \cup \{ (r_1,...,r_d,s') : s' \in S, |s'| = M,$$
$$r_i \ \text{ non-negative integers },\ 1 \le i \le d, r_1 + \cdots + r_d > 0 \Big\}.$$

We associate a modulus $|s|$ with each $s \in S_M$. If $s \in S \cap S_M$ then the modulus of $s$ is as in the previous paragraph. Otherwise if $s = (r_1,...,r_d,s')$ then $|s| = r_1+...+r_d+|s'| = r_1+..+r_d+M > M$. We can also associate a variable to each $s \in S_M$. For $s \in S \cap S_M$ we put $Y_{s,M} = Y_s$. If $s = (r_1,..,r_d,s')$ we put

$$Y_{s,M} = \left[ \prod_{i=1}^d b_{ii}(\cdot)^{r_i} Y_{s'} - \left\langle \prod_{i=1}^d b_{ii}(\cdot)^{r_i} Y_{s'} \right\rangle \right] \gamma^{-r_1 - r_2 - \cdots - r_d}.$$

It is clear that $\langle Y_{s,M} \rangle = 0,\ \langle Y_{s,M}^2 \rangle \ \le 1,\ s \in S_M$.

In analogy to before we define for $n \in \mathbf{Z}^d,\ s \in S_M$, variables $Y_{n,s,M}$ by $Y_{n,s,M}(\cdot) = Y_{s,M}(\tau_n \cdot)$. We may also define the Fock space $\mathcal{F}^2_{S_M}(\mathbf{Z}^d)$ and a mapping $U_{\alpha,M}$ corresponding to (5.1). Thus for $\psi \in \mathcal{F}^2_{S_M}(\mathbf{Z}^d)$ one has,

$$U_{\alpha,M}\ \psi = \psi_0 + \sum_{N=1}^\infty \sum_{\substack{\{n_1,..,n_N\} \in \mathbf{Z}^{d,N}, \\ s_i \in S_M,\ 1 \le i \le N}} \psi_N(n_1,s_1,n_2,s_2,...,n_N,s_N) \times$$
$$\alpha^{|s_1|+\cdots+|s_N|} Y_{n_1,s_1,M}\ Y_{n_2,s_2,M} \cdots Y_{n_N,s_N,M}.$$

**Lemma 5.9.** *Suppose $0 < \alpha < 1$. Then $M$ can be chosen sufficiently large, depending only on $\alpha$, such that $U_{\alpha,M}$ is a bounded operator from $\mathcal{F}^2_{S_M}(\mathbf{Z}^d)$ to $L^2(\Omega)$ and $\|U_{\alpha,M}\| \le 1$.*

*Proof.* Just as in Lemma 5.1 we have that

$$
\begin{aligned}
(5.14)\quad \|U_{\alpha,M}\ \psi\|_2^2 = \psi_0^2 + &\\
&\sum_{N=1}^{\infty} \sum_{\substack{\{n_1,\dots,n_N\}\in\mathbf{Z}^{d,N},\\ s_i,s_i'\in S_M,\ 1\le i\le N}} \psi_N(n_1,s_1,...,n_N,s_N)\psi_N(n_1,s_1',\dots,n_N,s_N')\times\\
&\qquad \alpha^{|s_1|+\cdots+|s_N|+|s_1'|+\cdots+|s_N'|}\delta_M(s_1,s_1')\cdots\delta_M(s_N,s_N')\times\\
&\qquad\qquad \left\langle Y_{n_1,s_1,M}\ Y_{n_1,s_1',M}\right\rangle\cdots\left\langle Y_{n_N,s_N.M}\ Y_{n_N,s_M',M}\right\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_M(s,s') &= 0, \qquad |s|,|s'|\le M, \quad s\ne s',\\
\delta_M(s,s') &= 1, \qquad \text{otherwise.}
\end{aligned}
$$

If we use the Schwarz inequality on the RHS of (5.14) we have

$$
\begin{aligned}
\|U_{\alpha,M}\ \psi\|_2^2 \le \psi_0^2 + \sum_{N=1}^{\infty} &\sum_{\substack{\{n_1,\dots,n_N\}\in\mathbf{Z}^{d,N},\\ s_i,s_i'\in S_M,\ 1\le i\le N}} \psi_N^2(n_1,s_1,\dots,n_N,s_N)\times\\
&\alpha^{|s_1|+\cdots+|s_N|+|s_1'|+\cdots+|s_N'|}\delta_M(s_1,s_1')\cdots\delta_M(s_N,s_N')\times\\
&\qquad \left\langle Y^2_{n_1,s_1,M}\right\rangle\cdots\left\langle Y^2_{n_N,s_N,M}\right\rangle.
\end{aligned}
$$

We consider now the sum,

$$
(5.15)\qquad \sum_{s'\in S_M} \alpha^{|s|+|s'|}\delta_M(s,s'), \quad s\in S_M.
$$

If $|s| > M$ this sum is bounded by

$$
\begin{aligned}
\alpha^{|s|}\sum_{s'\in S_M}\alpha^{|s'|} \le \alpha^{|s|}\sum_{k=1}^{M}(k+1)^d\alpha^k&\\
+\ \alpha^{|s|}\sum_{r_1+..+r_d\ge 1}&\alpha^{r_1+..+r_d}(M+1)^d\alpha^M \le 2\ \alpha^{|s|}(M+1)^d/(1-\alpha)^d
\end{aligned}
$$

If $|s| \le M$ then (5.15) is bounded by

$$
\alpha^{2|s|} + \alpha^{|s|}\sum_{|s'|>M}\alpha^{|s'|} \le \alpha^{2|s|} + \alpha^{|s|+M}(M+1)^d/(1-\alpha)^d.
$$

It is clear from these last two inequalities that we may choose $M$, depending only on $\alpha$, such that (5.15) is bounded above by 1 for all $s \in S_M$. The result follows now since $\langle Y^2_{n_j,s_j,M}\rangle \le 1$, $1 \le j \le N,\ N = 1, 2, \dots$ . ☐

Next, in analogy to the development following Lemma 5.7, we define for any $i$, $1 \le i \le d$, an operator $B_{i,\alpha,M}$ on $\mathcal{F}^2_{S_M}(\mathbf{Z}^d)$ which has the property,

$$
U_{\alpha,M}B_{i,\alpha,M} = b_{ii}(\cdot)U_{\alpha,M}.
$$

For a function $\varphi \in \mathcal{F}^2_{S_M}(\mathbf{Z}^d)$, $\varphi = \{\varphi_N : N = 0, 1, 2, ...\}$ we put $B_{i,\alpha,M}\ \varphi = \psi = \{\psi_N : N = 0, 1, 2, ...\}$, where $\psi$ is given in terms of $\varphi$ as follows:

Let $B_{i,s,s'}$ be the parameters defined by (5.13). Then,

- If $n_j \neq 0,\ 1 \leq j \leq N,\ N \geq 0$,

$$\begin{aligned}\psi_N(n_1, s_1, ..., n_N, s_N) &= B_{i,0,0}\varphi(n_1, s_1, ..., n_N, s_N) \\ &+ \sum_{s \in S_M, |s|<M} \alpha^{|s|} B_{i,s,0}\varphi_{N+1}(0, s, n_1, s_1, ..., n_N, s_N) \\ &+ \sum_{s \in S_M, |s|\geq M} \alpha^{|s|} \langle b_{ii}(\cdot) Y_{s,M}\rangle\, \varphi_{N+1}(0, s, n_1, s_1, ..., n_N, s_N).\end{aligned}$$

- Suppose $|s_1| > M,\ s_1 = (r_1, .., r_d, s_1'), n_j \neq 0,\ 2 \leq j \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = 0 \quad \text{if} \quad r_i = 0,$$
$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = \alpha^{-1}\gamma\varphi_N(0, \bar{s}_1, n_2, s_2, ..., n_N, s_N),$$

  where $\bar{s}_1 = (r_1, ..., r_{i-1}, r_i - 1, r_{i+1}, ..., r_d, s_1')$ if $r_i \geq 1$.
- Suppose $1 < |s_1| \leq M$, $n_j \neq 0,\ 2 \leq j \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = \sum_{|s_1|-1 \leq |s| < M} \alpha^{|s|-|s_1|} B_{i,s,s_1}\ \varphi_N(0, s, n_2, s_2, ..., n_N, s_N).$$

- Suppose $|s_1| = 1,\ n_j \neq 0,\ 2 \leq j \leq N$. Then

$$\begin{aligned}\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) &= B_{i,0,s_1}\alpha^{-1}\varphi_{N-1}(n_2, s_2, ..., n_N, s_N) \\ &+ \sum_{1 \leq |s| < M} \alpha^{|s|-1} B_{i,s,s_1}\varphi_N(0, s, n_2, s_2, ..., n_N, s_N) - \alpha^{M-1} B_{i,0,s_1} \times \\ &\sum_{\substack{|s|=M, \\ r_1+..+r_d>0}} (\alpha\gamma^{-1})^{r_1+..+r_d} \left\langle \prod_{j=1}^{d} b_{jj}(\cdot)^{r_j} Y_s \right\rangle \times \\ &\varphi_N\big(0, (r_1, ..., r_d, s), n_2, s_2, ..., n_N, s_N\big).\end{aligned}$$

**Lemma 5.10.** *There exists $\alpha$, $M_0, 0 < \alpha < 1$, $M_0$ a positive integer, depending only on $\gamma$, such that if $M \geq M_0$ then $B_{i,\alpha,M}$ is a bounded operator on $\mathcal{F}^2_{S_M}(\mathbf{Z}^d)$ with norm $\|B_{i,\alpha,M}\| \leq 2\gamma/(1+\gamma)$, $1 \leq i \leq d$.*

*Proof.* We have just as in Lemma 5.8 that if $\lambda_s,\ 0 \leq |s| < M$, are parameters then

$$\sum_{0 \leq |s| \leq M} \left[\sum_{0 \leq |s'| < M} B_{i,s',s}\lambda_{s'}\right]^2 \leq \gamma^2 \sum_{0 \leq |s| < M} \lambda_s^2. \tag{5.16}$$

Arguing as in Lemma 5.8 we conclude from the above inequality that

$$\sum_{0 \leq |s| \leq M} \left[\sum_{0 \leq |s'| < M} \alpha^{|s'|-|s|} B_{i,s',s}\lambda_{s'}\right]^2 \leq \alpha^{-2}\gamma^2 \sum_{0 \leq |s| < M} \lambda_s^2. \tag{5.17}$$

Suppose $\varphi \in \mathcal{F}^2_{S_M}(\mathbf{Z}^d)$. We fix points $\{n_1,...,n_N\} \in Z^{d,N}$ with $n_j \neq 0,\ 1 \le j \le N$, and $s_j \in S_M,\ 1 \le j \le N$. We then define parameters $\lambda_s,\ s \in S_M$ , and $\lambda_0$ by

$$
\begin{aligned}
\lambda_0 &= \varphi(n_1, s_1, \ldots, n_N, s_N), \\
\lambda_s &= \varphi_{N+1}(0, s, n_1, s_1, \ldots, n_N, s_N), \quad s \in S_M.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|B_{i,\alpha,M}\varphi_N(n_1, s_1, \ldots, n_N, s_N)|^2 + \sum_{s \in S_M} |B_{i,\alpha,M}\varphi_{N+1}(0, s, n_1, s_1, \ldots, n_N, s_N)|^2 \\
&= \left[\sum_{0 \le |s| < M} \alpha^{|s|} B_{i,s,0}\lambda_s + \sum_{|s| \ge M} \alpha^{|s|} < b_{ii}(\cdot) Y_{s,M} > \lambda_s\right]^2 \\
&\qquad + \sum_{|s|=1} \Bigg[\sum_{0 \le |s'| < M} \alpha^{|s'|-1} B_{i,s',s}\lambda_{s'} - \alpha^{M-1} B_{i,0,s} \times \\
&\qquad \sum_{\substack{|s'|=M, \\ r_1+\cdots+r_d>0}} (\alpha\gamma^{-1})^{r_1+..+r_d} \left\langle \prod_{j=1}^d b_{jj}(\cdot)^{r_j} Y_{s'} \right\rangle \lambda_{(r_1,\ldots,r_d,s')}\Bigg]^2 \\
&\qquad + \sum_{1 < |s| \le M} \left[\sum_{|s|-1 \le |s'| < M} \alpha^{|s'|-|s|} B_{i,s',s}\lambda_{s'}\right]^2 + \alpha^{-2}\gamma^2 \sum_{|s| \ge M} \lambda_s^2.
\end{aligned}
$$

From the Schwarz inequality the RHS of the last equation is bounded above by

$$
\begin{aligned}
(5.18) \quad &(1+\delta) \sum_{0 \le |s| \le 1} \left[\sum_{0 \le |s'| < M} \alpha^{|s'|-|s|} B_{i,s',s}\lambda_{s'}\right]^2 \\
&+ \sum_{1 < |s| \le M} \left[\sum_{0 \le |s'| < M} \alpha^{|s'|-|s|} B_{i,s',s}\lambda_{s'}\right]^2 + \alpha^{-2}\gamma^2 \sum_{|s| \ge M} \lambda_s^2 \\
&+ (1+\delta^{-1}) \left[\sum_{|s| \ge M} \alpha^{|s|} \langle b_{ii}(\cdot) Y_{s,M} \rangle \lambda_s\right]^2 + (1+\delta^{-1}) \times \\
&\sum_{|s|=1} \left[\alpha^{M-1} B_{i,0,s} \sum_{\substack{|s'|=M, \\ r_1+\cdots+r_d>0}} (\alpha\gamma^{-1})^{r_1+\cdots+r_d} \left\langle \prod_{j=1}^d b_{jj}(\cdot)^{r_j} Y_{s'} \right\rangle \lambda_{(r_1\ldots,r_d,s')}\right]^2,
\end{aligned}
$$

for any $\delta > 0$. In view of (5.17) the sum of the first three terms in the last expression is bounded above by

$$
(1+\delta)\alpha^{-2}\gamma^2 \sum_{|s| \ge 0} \lambda_s^2.
$$

The fourth term is bounded above by

$$
(1+\delta^{-1})\gamma^2 \left[\sum_{|s| \ge M} \alpha^{2|s|}\right] \sum_{|s| \ge M} \lambda_s^2 \le (1+\delta^{-1})\gamma^2 \alpha^{2M} \frac{(M+1)^d}{(1-\alpha^2)^d} \sum_{|s| \ge M} \lambda_s^2.
$$

The final term in the expression is bounded above by

$$(1+\delta^{-1})\alpha^{2M}\frac{(M+1)^d}{(1-\alpha)^{2d}}\left[\sum_{|s|=1} B_{i,0,s}^2\right]\left[\sum_{|s|>M}\lambda_s^2\right].$$

It is also clear from (5.16) that

$$\sum_{|s|=1} B_{i,0,s}^2 \le \gamma^2.$$

The result follows from the last set of inequalities by first picking $\alpha, \delta,\ 0<\alpha<1,\ \delta>0$, such that $(1+\delta)^{1/2}\alpha^{-1}\gamma < 2\gamma/(1+\gamma)$. Then $M_0$ can be chosen large enough, depending on $\delta, \alpha$ so that the sum (5.18) is bounded above by

$$\left(\frac{2\gamma}{1+\gamma}\right)^2 \sum_{|s|\ge 0} \lambda_s^2.$$

□

We may easily deduce from the proof of Lemma 5.10 the analogue of Lemma 5.8.

**Lemma 5.11.** *There exists $\alpha,\ M_0,\ p_0,\ 0<\alpha<1,\ M_0$ a positive integer, $1<p_0<2$, depending only on $\gamma$, such that if $p_0\le p\le 2$, then $B_{i,\alpha,M_0}$ is a bounded operator on $\mathcal{F}^p_{S_{M_0}}(\mathbf{Z}^d)$ with norm*

$$\|B_{i,\alpha,M_0}\| \le 2\gamma/(1+\gamma) \quad .$$

If we follow the development after Lemma 5.5 we can deduce from Lemma 5.11 the analogue of Lemma 5.6. Thus we may define a space of vector valued functions $\mathcal{F}^p_{S_M,d}(\mathbf{Z}^d)$ and an operator $B_{\alpha,M}$ on it corresponding to the operators $B_{i,\alpha,M}, i=1,\dots,d$. We then have

**Corollary 5.1.** *There exists $\alpha,\ M_0,\ p_0,\ 0<\alpha<1,\ M_0$ a positive integer, $1<p_0<2$, depending only on $\gamma$, such that if $p_0\le p\le 2$, then $B_{\alpha,M_0}$ is a bounded operator on $\mathcal{F}^p_{S_{M_0},d}(\mathbf{Z}^d)$ with norm*

$$\|B_{\alpha,M_0}\| \le 2\gamma/(1+\gamma).$$

Theorem 1.2, with $\gamma$ close to 1, follows from Corollary 5.1 provided we assume $\mathbf{b}(\cdot)$ is diagonal. Next we deal with the case of nondiagonal $\mathbf{b}(\cdot)$. We restrict ourselves first to the case $d=2$. An arbitrary real symmetric $2\times 2$ matrix can be written as

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \lambda\cos^2\theta+\mu\ \sin^2\theta & (\mu-\lambda)\sin\theta\cos\theta \\ (\mu-\lambda)\sin\theta\cos\theta & \mu\cos^2\theta+\lambda\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda+\mu}{2}+\frac{\lambda-\mu}{2}\cos 2\theta & \frac{\mu-\lambda}{2}\sin 2\theta \\ \frac{\mu-\lambda}{2}\sin 2\theta & \frac{\lambda+\mu}{2}+\frac{\mu-\lambda}{2}\cos 2\theta \end{bmatrix}.$$

The random matrix

$$\mathbf{b}(\cdot) = \begin{bmatrix} b_{11}(\cdot) & b_{12}(\cdot) \\ b_{12}(\cdot) & b_{22}(\cdot) \end{bmatrix}$$

induces random values of the variables $\lambda, \mu, \theta$ as follows:

$$\begin{aligned} \lambda(\cdot)+\mu(\cdot) &= b_{11}(\cdot)+b_{22}(\cdot), \\ \lambda(\cdot)-\mu(\cdot) &= \left\{[b_{11}(\cdot)-b_{22}(\cdot)]^2+4b_{12}(\cdot)^2\right\}^{1/2}. \end{aligned}$$

If $\lambda(\cdot) > \mu(\cdot)$ then $\theta(\cdot)$ is the unique angle, $0 \le \theta < \pi$, such that

$$[\lambda(\cdot)-\mu(\cdot)]\cos 2\theta = b_{11}(\cdot)-b_{22}(\cdot),$$
$$[\lambda(\cdot)-\mu(\cdot)]\sin 2\theta = -2b_{12}(\cdot).$$

If $\lambda(\cdot)=\mu(\cdot)$ then we take $\theta(\cdot)=0$.

Since $\lambda(\cdot), \mu(\cdot)$ are the eigenvalues of $\mathbf{b}(\cdot)$ it follows that $|\lambda(\cdot)|, |\mu(\cdot)| \le \gamma < 1$. For $\ell = 0,1,2,\dots$, let $P_\ell(z), z \in \mathbf{R}$, be the Legendre polynomials.

Let $\ell, m, r$ be integers with the property $\ell, m \ge 0$. We associate with the 3-tuple $(\ell, m, r)$ a variable $X_{\ell,m,r}$ as follows:
If $r > 0$ then,

$$X_{\ell,m,r} = \sqrt{2\ell+1}P_\ell\left(\frac{\lambda(\cdot)}{\gamma}\right)\sqrt{2m+1}P_m\left(\frac{\mu(\cdot)}{\gamma}\right)\sqrt{2}\sin r\theta(\cdot).$$

If $r = 0$ then

$$X_{\ell,m,r} = \sqrt{2\ell+1}P_\ell\left(\frac{\lambda(\cdot)}{\gamma}\right)\sqrt{2m+1}P_m\left(\frac{\mu(\cdot)}{\gamma}\right).$$

If $r < 0$ then

$$X_{\ell,m,r} = \sqrt{2\ell+1}P_\ell\left(\frac{\lambda(\cdot)}{\gamma}\right)\sqrt{2m+1}P_m\left(\frac{\mu(\cdot)}{\gamma}\right)\sqrt{2}\cos r\theta(\cdot).$$

Observe that if $\lambda(\cdot), \mu(\cdot), \theta(\cdot)$ are independent variables, with $\lambda(\cdot), \mu(\cdot)$ uniformly distributed on $[-\gamma, \gamma]$ and $\theta(\cdot)$ uniformly distributed on $[-\pi, \pi]$, then the variables $X_{\ell,m,r}$ form an orthonormal set. This set includes the constant function $X_{0,0,0}$.

We can use the recurrence relation for Legendre polynomials,

$$(2\ell+1)zP_\ell(z) = \ell P_{\ell-1}(z) + (\ell+1)P_{\ell+1}(z),$$

to be found in [1], and the addition theorems for trigonometric functions to obtain the result of multiplying the variables $X_{\ell,m,r}$ by the components of the matrix $\mathbf{b}(\cdot)$. In particular we have

$$b_{ij}(\cdot)X_{\ell,m,r} = \sum_{\substack{|\ell'-\ell|=1,\\ |m'-m|=1,\\ ||r'|-|r||=2}} \Gamma_{i,j,\ell,m,r,\ell',m',r'}X_{\ell',m',r'},$$

where the parameters $\Gamma_{i,j,\ell,m,r,\ell',m',r'}$ are explicitly computable. In view of the fact that the variables $X_{\ell,m,r}$ are orthonormal when $\lambda(\cdot), \mu(\cdot), \theta(\cdot)$ are uniformly distributed, it follows that

$$\sum_{i=1}^{2}\sum_{\ell,m=0}^{\infty}\sum_{r=-\infty}^{\infty}\left[\sum_{j=1}^{2}\sum_{\ell',m'=0}^{\infty}\sum_{r'=-\infty}^{\infty}\Gamma_{i,j,\ell',m',r',\ell,m,r}\lambda_{j,\ell',m',r'}\right]^2 \le \gamma^2\sum_{i=1}^{2}\sum_{\ell,m=0}^{\infty}\sum_{r=-\infty}^{\infty}\lambda_{i,\ell,m,r}^2, \tag{5.19}$$

where the $\lambda_{i,\ell,m,r}$ are arbitrary parameters.

We consider again the linear spaces $V_k$ defined after Lemma 5.7. The dimension of $V_k$ is bounded by $(k+1)^{d(d+1)/2}$ which is $(k+1)^3$ when $d=2$. Just as before we let $Y_{k,j}$, $0 \le j \le J_k$, $k =$

$0, 1, 2, ...$ be an orthonormal set of variables in $L^2(\Omega)$ with the property that $Y_{0,0} \equiv 1$ and $V_k$ is spanned by the variables $Y_{k',j},\ 0 \le j \le J_{k'},\ 0 \le k' \le k,\ k = 0, 1, 2, ....$ . For a variable $Y_{k,j}$ we denote by $s = |s| = k$. It follows from the definition of the spaces $V_k$ that then,

$$b_{ij}(\cdot)Y_s = \sum_{|s'| \le |s|+1} B_{i,j,s,s'}\ Y_{s'}, \quad 1 \le i, j \le d, \tag{5.20}$$

for appropriate constants $B_{i,j,s,s'}$. We conclude in the same way we obtained (5.19) that

$$\sum_{i=1}^{d} \sum_{s \in S} \left[ \sum_{j=1}^{d} \sum_{s' \in S} B_{i,j,s',s} \lambda_{j,s'} \right]^2 \le \gamma^2 \sum_{i=1}^{d} \sum_{s \in S} \lambda_{i,s}^2 \quad , \tag{5.21}$$

where $S$ is the set of spins $s = (k, j)$ and $\lambda_{i,s}$ are arbitrary parameters.

For integers $M, K \ge 1$ let $S_{M,K}$ be the set,

$$S_{M,K} = \Big\{ s \in S : 0 < |s| < M \} \cup \{(\ell, m, r, k, s') : s' \in S, |s'| = M, \\ \ell, m, r, k \text{ integers with } \ell, m \ge 0,\ 0 \le k < K \Big\}.$$

We associate a modulus $|s|$ with each $s \in S_{M,K}$. If $s \in S \cap S_{M,K}$ then the modulus of $s$ is as in the previous paragraph. Otherwise if $s = (\ell, m, r, k, s')$ then $|s| = \ell + m + |r| + |s'| = \ell + m + |r| + M$. We associate a variable to each $s \in S_{M,K}$. For $s \in S \cap S_{M,K}$ we put $Y_{s,M,K} = Y_s$. If $s = (\ell, m, r, k, s')$ we put

$$Y_{s,M,K} = X_{\ell,m,r}\ Y_{s'} - \langle X_{\ell,m,r}\ Y_{s'} \rangle .$$

It is clear that $\langle Y_{s,M,K} \rangle = 0$. If we use the fact that the $L_\infty$ norm of the Legendre polynomials is 1 (see, for example,[1]) then we see also that $\langle Y_{s,M,K}^2 \rangle \le 8|s|^2 + 1,\ s \in S_{M,K}$.

In analogy to before we define for $n \in \mathbf{Z}^2,\ s \in S_{M,K}$, variables $Y_{n,s,M,K}$ by $Y_{n,s,M,K}(\cdot) = Y_{s,M,K}(\tau_n\ \cdot)$. We may also define the Fock space $\mathcal{F}^2_{S_{M,K}}(\mathbf{Z}^d)$ and a mapping $U_{\alpha,M,K}$ corresponding to (5.1). Thus for $\psi \in \mathcal{F}_{S_{M,K}}(\mathbf{Z}^d)$ one has

$$U_{\alpha,M,K}\ \psi = \psi_0 + \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,..,n_N\} \in \mathbf{Z}^{d,N}, \\ s_i \in S_{M,K}, 1 \le i \le N}} \psi_N(n_1, s_1, n_2, s_2, ..., n_N, s_N) \times \\ \alpha^{|s_1| + \cdots + |s_N|} Y_{n_1,s_1,M,K}\ Y_{n_2,s_2,M,K} \cdots Y_{n_N,s_N,M,K}.$$

**Lemma 5.12.** *Suppose $0 < \alpha < 1$ and $K \ge 1$. Then $M$ can be chosen sufficiently large, depending only on $\alpha$ and $K$, such that $U_{\alpha,M,K}$ is a bounded operator from $\mathcal{F}^2_{S_{M,K}}(\mathbf{Z}^d)$ to $L^2(\Omega)$ and $\|U_{\alpha,M,K}\| \le 1$.*

*Proof.* We can use the same argument as in Lemma 5.9 since we know that $\left\langle Y_{s,M,K}^2 \right\rangle \le 8|s|^2 + 1$. □

Next we define for any $i, j,\ 1 \le i, j \le 2$, an operator $B_{i,j,\alpha,M,K}$ on $\mathcal{F}^2_{S_{M,K}}(\mathbf{Z}^2)$ which has the property,

$$U_{\alpha,M,K}\ B_{i,j,\alpha,M,K} = b_{ij}(\cdot)U_{\alpha,M,K}.$$

For a function $\varphi \in \mathcal{F}^2_{S_{M,K}}(\mathbf{Z}^2),\ \varphi = \{\varphi_N : N = 0, 1, 2, ...\}$ we put

$$B_{i,j,\alpha,M,K}\varphi = \psi = \{\psi_N : N = 0, 1, 2, ...\},$$

where $\psi$ is given in terms of $\varphi$ as follows: Let $B_{i,j,s,s'}$ be defined as in (5.20). Then

- If $n_t \neq 0,\ 1 \leq t \leq N,\ N \geq 0.$

$$\psi_N(n_1, s_1, \ldots, n_N, s_N) = B_{i,j,0,0}\varphi_N(n_1, s_1, ..., n_N, s_N) + \sum_{\substack{s \in S_{M,K}, \\ |s|<M}} \alpha^{|s|} B_{i,j,s,0}\varphi_{N+1}(0, s, n_1, s_1, \ldots, n_N, s_N) + \sum_{\substack{s \in S_{M,K}, \\ |s|\geq M}} \alpha^{|s|} \langle b_{ij}(\cdot) Y_{s,M,K}\rangle \varphi_{N+1}(0, s, n_1, s_1, \ldots, n_N, s_N) \quad ,$$

- Suppose $|s_1| > M,\ s_1 = (\ell, m, r, k, s_1'),\ n_t \neq 0,\ 2 \leq t \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, ..., n_N, s_N) = \sum_{\ell', m', r'} \Gamma_{i,j,\ell',m',r',\ell,m,r} \alpha^{\ell'+m'+|r'|-\ell-m-|r|}\ \varphi_N\Big(0, (\ell', m', r', k, s_1'), n_2, s_2, \ldots, n_N, s_N\Big).$$

- Suppose $1 < |s_1| < M,\ n_t \neq 0,\ 2 \leq t \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, \ldots, n_N, s_N) = \sum_{|s_1|-1 \leq |s| < M} \alpha^{|s|-|s_1|} B_{i,j,s,s_1} \times \varphi_N(0, s, n_2, s_2, \ldots, n_N, s_N).$$

- Suppose $|s_1| = M,\ s_1 = (0, 0, 0, k, s_1'),\ n_t \neq 0,\ 2 \leq t \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, \ldots, n_N, s_N) = \sum_{\ell', m', r'} \Gamma_{i,j,\ell',m',r',0,0,0} \times \alpha^{\ell'+m'+|r'|}\varphi_N\Big(0, (\ell', m', r', k, s_1'), n_2, s_2, ..., n_N, s_N\Big) + \frac{1}{K} \sum_{|s|=|s_1|-1} \alpha^{-1} B_{i,j,s,s_1}\varphi_N(0, s, n_2, s_2, ..., n_N, s_N).$$

- Suppose $|s_1| = 1,\ n_t \neq 0,\ 2 \leq t \leq N$. Then

$$\psi_N(0, s_1, n_2, s_2, \ldots, n_N, s_N) = B_{i,j,0,s_1}\ \alpha^{-1}\varphi_{N-1}(n_2, s_2, ..., n_N, s_N) + \sum_{1 \leq |s| < M} \alpha^{|s|-1} B_{i,j,s,s_1}\varphi_N(0, s, n_2, s_2, ..., n_N, s_N) - \alpha^{M-1} B_{i,j,0,s_1} \sum_{|s|=M} \sum_{k=0}^{K-1} \sum_{\ell,m=0}^{\infty} \sum_{r=-\infty}^{\infty} \alpha^{\ell+m+|r|} \langle X_{\ell,m,r} Y_s\rangle\ \varphi_N\Big(0, (\ell, m, r, k, s), n_2, s_2, \ldots, n_N, s_N\Big).$$

Next we may define as previously the space $\mathcal{F}^p_{S_{M,K},2}(\mathbf{Z}^2)$ of vector valued functions $\Psi = (\Psi_1, \Psi_2)$ on Fock space. We define an operator $B_{\alpha,M,K}$ by

$$(B_{\alpha,M,K}\ \Psi)_i = \sum_{j=1}^{2} B_{i,j,\alpha,M,K}\Psi_j, \quad i = 1, 2.$$

**Lemma 5.13.** *There exists* $\alpha, K, M_0,\quad 0<\alpha<1,\ K, M_0$ *positive integers, depending only on* $\gamma$*, such that if* $M \geq M_0$ *then* $B_{\alpha,M,K}$ *is a bounded operator on* $\mathcal{F}^2_{S_{M,K},2}(\mathbf{Z}^2)$ *with norm* $\|B_{\alpha,M,K}\| \leq 2\gamma/(1+\gamma)$.

*Proof.* Suppose $\Psi \in \mathcal{F}^2_{S_{M,K}}(\mathbf{Z}^2)$. We fix points $\{n_1,...,n_N\} \in \mathbf{Z}^{2,N}$ with $n_j \neq 0,\ 1 \leq j \leq N$ and $s_j \in S_{M,K}$ , $1 \leq j \leq N$. We define parameters $\lambda_{i,s},\ s \in S_{M,K}$ and $\lambda_{i,0}, i=1,2$ by

$$\begin{aligned}
\lambda_{i,0} &= \Psi_{i,N}(n_1,s_1,...,n_N,s_N), \quad i=1,2, \\
\lambda_{i,s} &= \Psi_{i,N+1}(0,s,n_1,s_1,...,n_N,s_N), \quad \in S_{M,K}, \quad i=1,2.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\sum_{j=1}^{2} B_{i,j,\alpha,M,K}\Psi_{j,N}(n_1,s_1,...,n_N,s_N)\right]^2 \\
&\quad + \sum_{s\in S_{M,K}}\sum_{i=1}^{2}\left[\sum_{j=1}^{2} B_{i,j,\alpha,M,K}\Psi_{j,N+1}(0,s,n_1,s_1,...,n_N,s_N)\right]^2 \\
&= \sum_{i=1}^{2}\left[\sum_{j=1}^{2}\left\{\sum_{0\leq|s|<M}\alpha^{|s|}B_{i,j,s,0}\lambda_{j,s} + \sum_{|s|\geq M}\alpha^{|s|}\left\langle b_{ij}(\cdot)Y_{s,M,K}\right\rangle\lambda_{j,s}\right\}\right]^2 \\
&\quad + \sum_{|s|=1}\sum_{i=1}^{2}\Bigg[\sum_{j=1}^{2}\Bigg\{\sum_{0\leq|s'|<M}\alpha^{|s|-1}B_{i,j,s',s}\lambda_{j,s'} \\
&\quad -\alpha^{M-1}B_{i,j,0,s}\sum_{|s'|=M}\sum_{k=0}^{K-1}\sum_{\ell,m=0}^{\infty}\sum_{r=-\infty}^{\infty}\alpha^{\ell+m+|r|}\left\langle X_{\ell,m,r},Y_{s'}\right\rangle\lambda_{j,(\ell,m,r,k,s')}\Bigg\}\Bigg]^2 \\
&\quad + \sum_{1<|s|<M}\sum_{i=1}^{2}\left[\sum_{j=1}^{2}\sum_{|s|-1\leq|s'|<M}\alpha^{|s'|-|s|}B_{i,j,s',s}\lambda_{j,s'}\right]^2 \\
&\quad + \sum_{|s|=M}\sum_{k=0}^{K-1}\sum_{i=1}^{2}\Bigg[\sum_{j=1}^{2}\Bigg\{\sum_{\ell',m',r'}\Gamma_{i,j,\ell',m'r',0,0,0}\alpha^{\ell'+m'+|r'|}\lambda_{j,(\ell',m',r',k,s)} \\
&\qquad\qquad + \frac{1}{K}\sum_{|s'|=|s|-1}\alpha^{-1}\ B_{i,j,s',s}\ \lambda_{j,s'}\Bigg\}\Bigg]^2 \\
&\quad + \sum_{|s|=M}\sum_{k=0}^{K-1}\sum_{\ell+m+|r|>0}\sum_{i=1}^{2}\Bigg[\sum_{j=1}^{2}\Bigg\{\sum_{\ell',m',r'}\Gamma_{i,j,\ell',m'r',\ell,m,r}\times \\
&\qquad\qquad \alpha^{\ell'+m'+|r'|-\ell-m-|r|}\lambda_{j,(\ell',m',r',k,s)}\Bigg\}\Bigg]^2.
\end{aligned}$$

From the Schwarz inequality the RHS of the last equation is bounded above by

$$(1+\delta) \sum_{0\le|s|\le 1} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{0\le|s'|<M} \alpha^{|s'|-|s|} B_{i,j,s',s} \lambda_{j,s'} \right]^2$$

$$+ \sum_{1<|s|<M} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{|s|-1\le|s'|<M} \alpha^{|s'|-|s|} B_{i,j,s',s} \lambda_{j,s'} \right]^2$$

$$+(1+\delta) \sum_{|s|=M} \sum_{k=0}^{K-1} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{\ell',m',r'} \Gamma_{i,j,\ell',m'r',0,0,0} \alpha^{\ell'+m'+|r'|} \lambda_{j,(\ell',m',r',k,s)} \right]^2$$

$$+ \sum_{|s|=M} \sum_{k=0}^{K-1} \sum_{\ell+m+|r|>0} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{\ell',m',r'} \Gamma_{i,j,\ell',m'r',\ell,m,r} \times \right.$$

$$\left. \alpha^{\ell'+m'+|r'|-\ell-m-|r|} \lambda_{j,(\ell',m',r',k,s)} \right]^2$$

$$+(1+\delta^{-1}) \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{|s|\ge M} \alpha^{|s|} \langle b_{i,j}(\cdot) Y_{s,M,K} \rangle \lambda_{j,s} \right]^2$$

$$+(1+\delta^{-1}) \sum_{|s|=1} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \left\{ \alpha^{M-1} B_{i,j,0,s} \right. \right.$$

$$\left. \left. \sum_{|s'|=M} \sum_{k=0}^{K-1} \sum_{\ell,m=0}^{\infty} \sum_{r=-\infty}^{\infty} \alpha^{\ell+m+|r|} \langle X_{\ell,m,r} Y_{s'} \rangle \lambda_{j,(\ell,m,r,k,s')} \right\} \right]^2$$

$$+ \frac{(1+\delta^{-1})}{K} \sum_{s\in S,|s|=M} \sum_{i=1}^{2} \left[ \sum_{j=1}^{2} \sum_{|s'|=|s|-1} \alpha^{-1} B_{i,j,s',s} \lambda_{j,s'} \right]^2 .$$

for any $\delta > 0$. In view of (5.21) it follows that the sum of the first two terms in the last expression is bounded above by

$$(1+\delta)\alpha^{-2}\gamma^2 \sum_{i=1}^{2} \sum_{0\le|s|<M} \lambda_{i,s}^2 .$$

In view of (5.19) the sum of the next two terms is bounded above by

$$(1+\delta)\alpha^{-6}\gamma^2 \sum_{i=1}^{2} \sum_{|s|\ge M} \lambda_{i,s}^2 .$$

The fifth term is bounded above by

$$(1+\delta^{-1})\gamma^2 \sum_{i=1}^{2} \left\langle \Big[ \sum_{|s|\ge M} \alpha^{|s|} \lambda_{i,s} Y_{s,M,K} \Big]^2 \right\rangle \le (1+\delta^{-1})\gamma^2 \Big[ \sum_{|s|\ge M} (8|s|^2+1)\alpha^{2|s|} \Big] \sum_{i=1}^{2} \sum_{|s|\ge M} \lambda_{i,s}^2 .$$

The sixth term is bounded above by

$$(1+\delta^{-1})\gamma^2 \sum_{i=1}^{2} \Big[ \sum_{|s|\ge M} \alpha^{|s|-1} \ \sqrt{8|s|^2+1} \ \ |\lambda_{i,s}| \Big]^2$$

$$\le (1+\delta^{-1})\gamma^2 \Big[ \sum_{|s|\ge M} (8|s|^2+1)\alpha^{2|s|-2} \Big] \sum_{i=1}^{2} \sum_{|s|\ge M} \lambda_{i,s}^2 .$$

The final term is bounded by

$$\frac{(1+\delta^{-1})}{K}\gamma^2\alpha^{-2} \sum_{i=1}^{2} \sum_{|s|=M-1} \lambda_{i,s}^2 \quad .$$

The result follows now exactly as in Lemma 5.10 by choosing $\alpha,\ 0<\alpha<1$, such that $\alpha^{-3}\gamma < 2\gamma/(1+\gamma)$, then choosing $\delta$ small and $K$ large so that $(1+\delta^{-1})/K$ is small. □

We can easily extend the argument in Lemma 5.13 to obtain:

**Corollary 5.2.** *There exists $\alpha,\ M_0, K_0, p_0,\ 0<\alpha<1,\ M_0, K_0$ positive integers, $1<p_0<2$, depending only on $\gamma$ such that if $p_0 \le p \le 2$ then $B_{\alpha,M_0,K_0}$ is a bounded operator on $\mathcal{F}^p_{S_{M_0,K_0},2}(\mathbf{Z}^2)$ with norm $\|B_{\alpha,M_0,K_0}\| \le 2\gamma/(1+\gamma)$.*

Theorem 1.2, with $\gamma$ close to 1, follows from Corollary 5.2 just as before. Since it is clear that one can extend the previous argument to $d>2$, the proof of Theorem 1.2 is complete. The proof of Theorem 1.3 follows in a similar manner.

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