Electronic Journal of Probability

Vol. 14 (2009), Paper no. 47, pages 1372–1416.

Journal URL
http://www.math.washington.edu/~ejpecp/

One-dimensional random field Kac's model: weak large deviations principle *

Enza Orlandi [1] and Pierre Picco [2]

Abstract

We present a quenched weak large deviations principle for the Gibbs measures of a Random Field Kac Model (RFKM) in one dimension. The external random magnetic field is given by symmetrically distributed Bernouilli random variables. The results are valid for values of the temperature and magnitude of the field in the region where the free energy of the corresponding random Curie Weiss model has only two absolute minimizers. We give an explicit representation of the large deviation rate function and characterize its minimizers. We show that they are step functions taking two values, the two absolute minimizers of the free energy of the random Curie Weiss model. The points of discontinuity are described by a stationary renewal process related to the $h-$extrema of a bilateral Brownian motion studied by Neveu and Pitman, where h depends on the temperature and magnitude of the random field. Our result is a complete characterization of the typical profiles of RFKM (the ground states) which was initiated in [4] and extended in [6]

Key words: phase transition, large deviations random walk, random environment, Kac potential

AMS 2000 Subject Classification: Primary 60K35, secondary 82B20,82B43.

Submitted to EJP on April 17, 2008, final version accepted May 26, 2009.

* Supported by: CNR-CNRS-Project 8.005, INFM-Roma; Prin07: 20078XYHYS.; GDRE 224 GREFI-MEFI, CNRS-INdAM. Dedicated to A.V. Skorohod for the fiftieth birthday of his fundamental paper [17].

[1] Dipartimento di Matematica, Universitá di Roma Tre, L.go S.Murialdo 1, 00146 Roma, Italy. orlandi@mat.uniroma3.it

[2] LATP, CMI, UMR 6632, CNRS, Université de Provence, 39 rue Frederic Joliot Curie, 13453 Marseille Cedex 13, France. picco@cmi.univ-mrs.fr

1 Introduction

We consider a one-dimensional spin system interacting via a ferromagnetic two-body Kac potential and an external random magnetic field given by symmetrically distributed Bernoulli random variables. Kac's potential is a two-body ferromagnetic interaction of strength γ and range $\frac{1}{\gamma}$, where γ is a dimensionless parameter. It is normalized so that when $\gamma \downarrow 0$, i.e. very long range, the total interaction between one spin and all the others is one.

Kac's potentials were introduced in [9], and then generalized in [12] and [14] to provide a rigorous derivation of the van der Waals theory of liquid-vapor phase transition. For motivations and discussions concerning existing results we refer to [6], see also the heuristic discussion later in this section.

The first result of this paper is an extension of the results of [6]. Let β be the inverse temperature, θ the magnitude of the random field and Ω the probability space in which the random field is defined. Take $(\beta, \theta) \in \mathcal{E}$, where $\mathcal{E}$, see (2.20), is the region where the free energy of the corresponding random Curie Weiss model has only two absolute minimizers: $\{m_\beta, Tm_\beta\}$, $m_\beta := (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta := (-m_{\beta,2}, -m_{\beta,1})$. The first minimizer m_β is associated to positive magnetization $\tilde{m}_\beta = \frac{(m_{\beta,1}+m_{\beta,2})}{2}$, the other Tm_β to negative magnetization $-\tilde{m}_\beta$. We exhibit a set of magnetization profiles typical for the infinite volume random Gibbs measure $\mu^\omega_{\beta,\theta,\gamma}$ for $\omega \in \Omega$ and $(\beta, \theta) \in \mathcal{E}$. Such a set is a suitable neighborhood of a properly defined locally bounded variation function $u^*_\gamma : I\!R \times \Omega \to I\!R^2$, which according to the realizations of the random field, takes values in $\{m_\beta, Tm_\beta\}$ and performs, with an overwhelming probability with respect to the distribution of the random fields when $\gamma \downarrow 0$, a finite number of jumps in any finite interval, *i.e.* $u^*_\gamma(\cdot, \omega) \in BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$. Next, we prove that, when $\gamma \downarrow 0$, the limiting distribution of the interdistance between the jump points of u^*_γ with respect to the distribution of the random magnetic fields is the same as that determined by Neveu-Pitman [13] when studying the stationary renewal process of $h(\beta, \theta)$-extrema of a bilateral Brownian motion. In addition we define the limiting (in Law) typical profile $u^*(\cdot, \omega)$ that belongs to $BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$. In particular, the distribution of the position of the jump of u^* nearest to the origin is the same as the limiting distribution of the point of localization of Sinai's random walk in random environment, (RWRE), see (2.41), determined independently by Kesten [11] and Golosov [8]. This is a surprising link between RFKM and RWRE. The last natural question concerns the large deviations from the typical profile u^*_γ of the random Gibbs measure $\mu^\omega_{\beta,\theta,\gamma}$. When the profile $u \in BV_{\rm loc}(I\!R, \{Tm_\beta, m_\beta\})$ is a suitable local perturbation of the typical profile u^*_γ we identify the large deviations functional $\Gamma(\cdot)$ by showing that

$$\lim_{\gamma \downarrow 0} \left[-\frac{\gamma}{\beta} \log \mu^\omega_{\beta,\theta,\gamma}[\mathcal{A}(u)] \right] = \Gamma(u), \tag{1.1}$$

where $\mathcal{A}(u)$ is a suitable neighborhood of $u \in BV_{\rm loc}(I\!R, \{Tm_\beta, m_\beta\})$, see (2.28). The functional $\Gamma(\cdot)$ is a positive random functional and the convergence (1.1) holds in Law. This differs from to the large deviation functional associated to the global empirical magnetization, see [4], which is not random. The $\Gamma(\cdot)$ is determined by two distinct contributions: the free energy cost $\mathcal{F}^*$ to undergo a phase change (the surface tension) and a bulk contribution due to the presence of the random magnetic field. It represents, in the chosen limit, the random cost for the system to deviate from the equilibrium value u^*. We have then proved a *weak* large deviation principle for $\{\mu^\omega_{\beta,\theta,\gamma}\}_\gamma$.

Heuristic To explain the result, in particular the correct scaling and the form of the large deviations functional $\Gamma(\cdot)$ in (1.1), it might be instructive to recall the results on the same spin model without the presence of the random external field, i.e $\theta = 0$, see [3]. In [3] a large deviations principle for Gibbs measures was established. The typical magnetization profiles are constant near one of the two values of the minimizer of the Curie-Weiss canonical free energy, over intervals of length of the order $e^{\frac{\beta}{\gamma}F(\beta)}$ where $F(\beta)$ represents the cost of the excess free energy functional to go from one phase to the other. i.e the surface tension.

Moreover, suitably marking the locations of the phase changes of the typical profiles and scaling the space by $e^{-\frac{\beta}{\gamma}F(\beta)}$, when $\gamma \downarrow 0$ one obtains as limiting Gibbs distribution of the marks, the Poisson Point Process. The thermal fluctuations are responsible for the stochastic behavior on this scale.

When the random fields are added to the system, *i.e* $\theta \neq 0$, the macroscopic picture of the system changes drastically. There is an interplay between the ferromagnetic two-body interaction which attracts spins alike and the presence of the random field which would like to have the spins aligned according to its sign. It is relatively easy to see that the relative cost of a phase change is $e^{-\frac{\beta}{\gamma}\mathcal{F}^*}$. This is of the same order as in the case without external random field. Here $\mathcal{F}^* = \mathcal{F}^*(\beta, \theta)$ is a deterministic quantity, the surface tension (see (2.23)). It is also relatively easy to see that the fluctuations of the random field over intervals $\frac{1}{\gamma^2}$ should play an important role.

Namely take a homogeneous configuration of spins in a volume V. The random fields contribution to the hamiltonian is $\sum_{i\in V} h_i$. The fluctuations of this term are up to a multiplicative constant equal to $\sqrt{|V|}$. To be able to see a competition between these fluctuations and the previous cost of a phase change, one needs at least that

$$\left|\sum_{i\in V} h_i\right| > \frac{\beta}{\gamma}\mathcal{F}^*, \tag{1.2}$$

that is $|V|$ of the order $1/\gamma^2$. Once the scale is determined a more subtle analysis is needed to characterize the typical profile. A simple picture of it should be a function taking value m_β over intervals of order $1/\gamma^2$ then making a phase change in intervals of order $\frac{1}{\gamma}$ to the value Tm_β and keeping such a value over intervals of order $1/\gamma^2$ and so on. Such intervals should have random length and therefore the ends of these intervals, *i.e.* the points detecting a phase change, should be random variables with respect to the σ–algebra generated by the random magnetic fields. The goal is to find their limiting distributions when $\gamma \downarrow 0$ after scaling down the space by γ^2.

To do so the above argument is clearly too rough. First of all it is merely the fluctuations of the random field contribution of configurations near m_β or Tm_β that should be relevant. A non trivial fact is that the relative Gibbs weight of configurations near Tm_β on a volume B with respect to the one which near m_β on the same volume is in fact roughly speaking $\exp(-c(\beta,\theta)\sum_{i\in B} h_i)$ for a constant $c(\beta,\theta)$. The minus sign corresponding to the fact that if $\sum_{i\in B} h_i$ is large and positive, one should expect that on B the system will prefer to be near m_β. Using this fact, the relative Gibbs weight of coarse grained spin configurations that stay near Tm_β in a volume B with respect to the one where a phase m_β is created on the very same volume B is roughly speaking

$$e^{+\frac{\beta}{\gamma}\mathcal{F}^*} e^{-c(\beta,\theta)\sum_{i\in B} h_i} e^{+\frac{\beta}{\gamma}\mathcal{F}^*}. \tag{1.3}$$

Considering the same volume B, one wants to avoid to create a phase Tm_β on a volume $C \subset B$ within a phase m_β on B. The relative Gibbs weight of these new coarse grained spin configurations with respect to the ones that stay near m_β on the volume B is

$$e^{-\frac{\beta}{\gamma}\mathcal{F}^*} e^{-c(\beta,\theta)\sum_{i\in C} h_i} e^{-\frac{\beta}{\gamma}\mathcal{F}^*}. \tag{1.4}$$

Therefore if

$$\begin{aligned} &\sum_{i\in B} h_i > \frac{2\beta\mathcal{F}^*}{\gamma c(\beta,\theta)} \quad \text{and} \\ &\sum_{i\in C} h_i > -\frac{2\beta\mathcal{F}^*}{\gamma c(\beta,\theta)} \quad \forall C \in B \end{aligned} \tag{1.5}$$

the right hand sides of (1.3) and (1.4) go to zero exponentially. In this case the typical profile should be near m_β in the volume B. Applying similar arguments, if

$$\begin{aligned} &\sum_{i\in B} h_i < -\frac{2\beta F^*}{\gamma c(\beta,\theta)} \quad \text{and} \\ &\sum_{i\in C} h_i < \frac{2\beta F^*}{\gamma c(\beta,\theta)} \quad \forall C\subset B \end{aligned} \tag{1.6}$$

the typical profile should be near Tm_β on B.

So going left to right, from a region, say $B_0(+)$ where (1.5) is satisfied to a region, say $B_1(-)$ where (1.6) is satisfied, there should be a local maximum of the random walk $\sum_i h_i$ and going from this $B_1(-)$ to a region $B_2(+)$ on its right there should be a local minimum. So a candidate for the typical profile will stay near m_β between the corresponding minimizer of such local minima and the maximizer of such local maxima; then near to Tm_β between the maximizer of such a local maxima and the minimizer of the next such local minima. The region $B_0(+)$ and the nearest region $B_1(-)$ on his right are adjacent and therefore the changes of phases could be associated to these local maxima and local minima. In [6], a way to construct the typical profiles was given. Here we give the limiting distribution of the localization of the jumps of the typical profile.

Once the typical profile u^* is determined, the next step is to identify a large deviations functional. Consider an $u\in BV_{\mathrm{loc}}(\{m_\beta, Tm_\beta\})$ which is a local modification of the typical profile u^*. Using the same arguments that lead to (1.3) and (1.4), we get that the large deviations functional should have a term of the type $\exp(-N(u,u^*)\beta\mathcal{F}^*/\gamma)$ where $N(u,u^*)$ is the difference between the number of jumps of u and those of u^*. Notice that this term is independent of the position of the jumps. The second contribution is related to the fluctuations of the random field over those intervals in which the profile u is in a different phase from u^* For the precise definition of $\Gamma(u)$ see (2.47). Obviously this is a very rough explanation.

The plan of this paper is the following: In Section 2 we give the description of the model and present the main results. In Section 3 we recall the coarse graining procedure. In Section 4 we prove the main estimates to derive upper and lower bound to deduce the large deviation estimates. In Section 5 we prove some probability estimates and the extension of the [6] results. In Section 6 we prove the main results.

Acknowledgments We are indebted to Errico Presutti who, years ago, gave us the expression of the random functional, see (2.47). We thank Jean–François Le Gall for mentioning to us the article by Neveu and Pitman, Jean Bertoin, Zhang Shi and Isaac Meilijson for illuminating discussions.

2 Model, notations and main results

2.1. The model

Let $(\Omega,\mathcal{A},I\!P)$ be a probability space on which we define $h\equiv\{h_i\}_{i\in\mathbb{Z}}$, a family of independent, identically distributed Bernoulli random variables with $I\!P[h_i=+1]=I\!P[h_i=-1]=1/2$. The spin configuration space is $\mathcal{S}\equiv\{-1,+1\}^{\mathbb{Z}}$. If $\sigma\in\mathcal{S}$ and $i\in\mathbb{Z}$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by a Kac potential of the form $J_\gamma(i-j)\equiv\gamma J(\gamma(i-j))$, $\gamma>0$. For sake of simplicity we fix $J(r)=1\!\!1_{[|r|\le 1/2]}(r)$, where we denote by $1\!\!1_A(\cdot)$ the indicator function of the set A.

For $\Lambda\subseteq\mathbb{Z}$ we set $\mathcal{S}_\Lambda=\{-1,+1\}^\Lambda$; its elements are denoted by σ_Λ; also, if $\sigma\in\mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda\subset\mathbb{Z}$ finite and a realization of the magnetic fields, the Hamiltonian in the volume

Λ, with free boundary conditions, is the random variable on $(\Omega, \mathcal{A}, I\!P)$ given by

$$H_\gamma(\sigma_\Lambda)[\omega] = -\frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J_\gamma(i-j)\sigma_i\sigma_j - \theta \sum_{i\in\Lambda} h_i[\omega]\sigma_i. \tag{2.1}$$

In the following we usually drop the ω from the notation. The corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$ and free boundary condition is then a random variable with values on the space of probability measures on $\mathcal{S}_\Lambda$. We denote it by $\mu_{\beta,\theta,\gamma,\Lambda}$ and it is defined by

$$\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \exp\{-\beta H_\gamma(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.2}$$

where $Z_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor called partition function. To take into account the interaction between the spins in Λ and those outside Λ we set

$$W_\gamma(\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i\in\Lambda} \sum_{j\in\Lambda^c} J_\gamma(i-j)\sigma_i\sigma_j. \tag{2.3}$$

If $\tilde\sigma \in \mathcal{S}$, the Gibbs measure on the finite volume Λ and boundary condition $\tilde\sigma_{\Lambda^c}$ is the random probability measure on $\mathcal{S}_\Lambda$, denoted by $\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ and defined by

$$\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}} \exp\left\{-\beta(H_\gamma(\sigma_\Lambda) + W_\gamma(\sigma_\Lambda, \tilde\sigma_{\Lambda^c}))\right\}, \tag{2.4}$$

where again the partition function $Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor.

Given a realization of h and $\gamma > 0$, there is a unique weak-limit of $\mu_{\beta,\theta,\gamma,\Lambda}$ along a family of volumes $\Lambda_L = [-L, L] \cap Z\!\!Z$, $L \in I\!N$; such limit is called the infinite volume Gibbs measure $\mu_{\beta,\theta,\gamma}$. The limit does not depend on the boundary conditions, which may be taken h-dependent, but it is a random element, i.e., different realizations of h give a priori different infinite volume Gibbs measures.

2.2. Scales

As in [3], [4] and [6], the analysis of the configurations that are typical for $\mu_{\beta,\theta,\gamma}$ in the limit $\gamma \downarrow 0$, involves a block spin transformation which transforms the microscopic system on $Z\!\!Z$ in a system on $I\!R$. We have three main different scales and according to the case it is better to work with one or the other. There will be also intermediate scales that we will discuss later.

- **2.2.1** *The microscopic and macroscopic scales.*

The basic space is the "microscopic space", i.e. the lattice $Z\!\!Z$ whose elements are denoted by i, j and so on. The microscopic scale corresponds to the length measured according to the lattice distance. The spin σ_i are indexed by $Z\!\!Z$ and the range of interaction in this scale is of order $\frac{1}{\gamma}$.
The macroscopic regions correspond to intervals of $I\!R$ of order $\frac{1}{\gamma}$ in the microscopic scale ; *i.e.* if $I \subset I\!R$, is an interval in the macroscopic scale then it will correspond to the interval $\frac{I}{\gamma}$ in the microscopic scale. In the macroscopic scale the range of the interaction becomes of order 1.

- **2.2.2** *The Brownian scale*

The Brownian scale is linked to the random magnetic fields. The Brownian regions correspond to intervals of $I\!R$ of order $\frac{1}{\gamma^2}$ in microscopic scale; i.e. if $[-Q, Q] \subset I\!R$, $Q > 0$ is an interval in Brownian scale then it corresponds to $[-\frac{Q}{\gamma^2}, \frac{Q}{\gamma^2}]$ in microscopic scale. In Brownian scale the range of interaction is of order γ.

• **2.2.3** *The partition of $I\!R$.*

Given a rational positive number δ, $\mathcal{D}_\delta$ denotes the partition of $I\!R$ into intervals $\tilde{A}_\delta(u) = [u\delta, (u+1)\delta)$ for $u \in Z\!\!Z$. If $\delta = n\delta'$ for some $n \in I\!N$, then $\mathcal{D}_\delta$ is coarser than $\mathcal{D}_{\delta'}$. For $r \in I\!R$, we denote by $D_\delta(r)$ the interval of $\mathcal{D}_\delta$ that contains r. Note that for any $r \in [u\delta, (u+1)\delta)$, we have that $D_\delta(r) = \tilde{A}_\delta(u)$. To avoid rounding problems in the following, we will consider intervals that are always $\mathcal{D}_\delta$–measurable. If $I \subseteq I\!R$ denotes a macroscopic interval we set

$$\mathcal{C}_\delta(I) = \{u \in Z\!\!Z; \tilde{A}_\delta(u) \subseteq I\}. \tag{2.5}$$

• **2.2.4** *The mesoscopic scales*

The smallest mesoscopic scale involves a parameter $0 < \delta^* < 1$, so that $\delta^*\gamma^{-1} \uparrow \infty$ when $\gamma \downarrow 0$. We require $\frac{1}{\gamma}$, $\frac{\delta^*}{\gamma}$ in $I\!N$. When considering another mesoscopic scale, say $\delta > \delta^*$, we assume $\delta = k\delta^*$ for $k \in I\!N$. The elements of $\mathcal{D}_{\delta^*}$ will be denoted by $\tilde{A}(x) \equiv [x\delta^*, (x+1)\delta^*)$, with $x \in Z\!\!Z$. The partition $\mathcal{D}_{\delta^*}$ induces a partition of $Z\!\!Z$ into blocks $A(x) = \{i \in Z\!\!Z; i\gamma \in \tilde{A}(x)\} \equiv \{a(x), \dots, a(x+1)-1\}$ with length of order $\delta^*\gamma^{-1}$ in the microscopic scale. For notational simplicity, if no confusion arises, we omit to write the explicit dependence on γ, δ^*.

2.3 Basic Notations.

• *block-spin magnetization*

Given a realization of h and for each configuration σ_Λ, we could have defined for each block $A(x)$ a pair of numbers where the first is the average magnetization over the sites with positive h and the second to those with negative h. However it appears, [4], to be more convenient to use another random partition of $A(x)$ into two sets of the same cardinality. This allows to separate on each block the expected contribution of the random field from its local fluctuations. More precisely we have the following.

Given a realization $h[\omega] \equiv (h_i[\omega])_{i\in Z\!\!Z}$, we set $A^+(x) = \big\{i \in A(x); h_i[\omega] = +1\big\}$ and $A^-(x) = \big\{i \in A(x); h_i[\omega] = -1\big\}$. Let $\lambda(x) \equiv \mathrm{sgn}(|A^+(x)| - (2\gamma)^{-1}\delta^*)$, where sgn is the sign function, with the convention that $\mathrm{sgn}(0) = 0$. For convenience we assume $\delta^*\gamma^{-1}$ to be even, in which case:

$$I\!P\left[\lambda(x) = 0\right] = 2^{-\delta^*\gamma^{-1}} \binom{\delta^*\gamma^{-1}}{\delta^*\gamma^{-1}/2}. \tag{2.6}$$

We note that $\lambda(x)$ is a symmetric random variable. When $\lambda(x) = \pm 1$ we set

$$l(x) \equiv \inf\{l \ge a(x) \,:\, \sum_{j=a(x)}^{l} 1\!\!1_{\{A^{\lambda(x)}(x)\}}(j) \ge \delta^*\gamma^{-1}/2\} \tag{2.7}$$

and consider the following decomposition of $A(x)$: $B^{\lambda(x)}(x) = \big\{i \in A^{\lambda(x)}(x); i \le l(x)\big\}$ and $B^{-\lambda(x)}(x) = A(x) \setminus B^{\lambda(x)}(x)$. When $\lambda(x) = 0$ we set $B^+(x) = A^+(x)$ and $B^-(x) = A^-(x)$. We set $D(x) \equiv A^{\lambda(x)}(x) \setminus B^{\lambda(x)}(x)$. In this way, the set $B^\pm(x)$ depends on the realizations of the random field, but the cardinality $|B^\pm(x)| = \delta^*\gamma^{-1}/2$ is the same for all realizations. We then denote

$$m^{\delta^*}(\pm, x, \sigma) = \frac{2\gamma}{\delta^*} \sum_{i \in B^\pm(x)} \sigma_i. \tag{2.8}$$

We call block spin magnetization of the block $A(x)$ the vector

$$m^{\delta^*}(x, \sigma) = (m^{\delta^*}(+, x, \sigma), m^{\delta^*}(-, x, \sigma)). \tag{2.9}$$

The total empirical magnetization of the block $A(x)$ is given by

$$\frac{\gamma}{\delta^*}\sum_{i\in A(x)}\sigma_i = \frac{1}{2}(m^{\delta^*}(+,x,\sigma)+m^{\delta^*}(-,x,\sigma)) \tag{2.10}$$

and the contribution of the magnetic field to the Hamiltonian (2.1) is

$$\frac{\gamma}{\delta^*}\sum_{i\in A(x)}h_i\sigma_i = \frac{1}{2}(m^{\delta^*}(+,x,\sigma)-m^{\delta^*}(-,x,\sigma))+\lambda(x)\frac{2\gamma}{\delta^*}\sum_{i\in D(x)}\sigma_i, \tag{2.11}$$

where $D(x)\equiv A^{\lambda(x)}(x)\setminus B^{\lambda(x)}(x)$.

- *spaces of the magnetization profiles*

Given a volume $\Lambda\subseteq \mathbb{Z}$ in microscopic scale, it corresponds to the macroscopic volume $I=\gamma\Lambda=\{\gamma i; i\in \Lambda\}$, assumed to be $\mathcal{D}_{\delta^*}$–measurable. The block spin transformation, as considered in [4] and [6], is the random map which associates to the spin configuration σ_Λ the vector $(m^{\delta^*}(x,\sigma))_{x\in\mathcal{C}_{\delta^*}(I)}$, see (2.9), with values in the set

$$\mathcal{M}_{\delta^*}(I)\equiv\prod_{x\in\mathcal{C}_{\delta^*}(I)}\left\{-1,-1+\frac{4\gamma}{\delta^*},-1+\frac{8\gamma}{\delta^*},\dots,1-\frac{4\gamma}{\delta^*},1\right\}^2. \tag{2.12}$$

We use the same notation $\mu_{\beta,\theta,\gamma,\Lambda}$ to denote both, the Gibbs measure on $\mathcal{S}_\Lambda$, and the probability measure induced on $\mathcal{M}_{\delta^*}(I)$, through the block spin transformation. Analogously, the infinite volume limit (as $\Lambda\uparrow\mathbb{Z}$) of the laws of the block spin $(m^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)}$ under the Gibbs measure will also be denoted by $\mu_{\beta,\theta,\gamma}$.

We denote a generic element in $\mathcal{M}_{\delta^*}(I)$ by

$$m_I^{\delta^*}\equiv(m^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)}\equiv(m_1^{\delta^*}(x),m_2^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)}. \tag{2.13}$$

Since I is assumed to be $\mathcal{D}_{\delta^*}$–measurable, we can identify $m_I^{\delta^*}$ with the element of

$$\mathcal{T}=\{m\equiv(m_1,m_2)\in L^\infty(\mathbb{R})\times L^\infty(\mathbb{R});\|m_1\|_\infty\vee\|m_2\|_\infty\le 1\} \tag{2.14}$$

piecewise constant, equal to $m^{\delta^*}(x)$ on each $\tilde{A}(x)=[x\delta^*,(x+1)\delta^*)$ for $x\in\mathcal{C}_{\delta^*}(I)$, and vanishing outside I. Elements of $\mathcal{T}$ are called magnetization profiles. Recalling that $I=\gamma\Lambda$, the block spin transformation can be identified with a map from the space of spin configurations $\{-1,+1\}^\Lambda$ into the subset of $\mathcal{D}_{\delta^*}$–measurable functions of $L^\infty(I)\times L^\infty(I)$. For $\delta=k\delta^*$, $m=(m_1,m_2)\in\mathcal{T}$ we define for $r\in\mathbb{R}$

$$m_i^\delta(r)=\frac{1}{\delta}\int_{D_\delta(r)}m_i(s)ds,\quad i\in\{1,2\}. \tag{2.15}$$

See 2.2.3 for $D_\delta(r)$. This defines a map from $\mathcal{T}$ into the subset of $\mathcal{D}_\delta$–measurable functions of $\mathcal{T}$. We define also a map from $\mathcal{T}$ into itself by

$$(Tm)(x)=(-m_2(x),-m_1(x))\quad\forall x\in\mathbb{R}. \tag{2.16}$$

In the following we denote the total magnetization at the site $x\in\mathbb{R}$

$$\tilde{m}(x)=\frac{m_1(x)+m_2(x)}{2}. \tag{2.17}$$

- *Excess free energy functional.*

We introduce the so called "excess free energy functional" $\mathcal{F}(m)$, $m \in \mathcal{T}$:

$$\begin{aligned}\mathcal{F}(m) &= \mathcal{F}(m_1, m_2) \\ &= \frac{1}{4}\int\int J(r-r')\,[\tilde{m}(r)-\tilde{m}(r')]^2\,drdr' + \int [f_{\beta,\theta}(m_1(r),m_2(r)) - f_{\beta,\theta}(m_{\beta,1},m_{\beta,2})]\,dr\end{aligned} \tag{2.18}$$

where $f_{\beta,\theta}(m_1,m_2)$ is the canonical free energy of the Random Field Curie-Weiss model derived in [4],

$$f_{\beta,\theta}(m_1,m_2) = -\frac{(m_1+m_2)^2}{8} - \frac{\theta}{2}(m_1-m_2) + \frac{1}{2\beta}(\mathcal{I}(m_1)+\mathcal{I}(m_2)) \tag{2.19}$$

with $\mathcal{I}(m) = \frac{(1+m)}{2}\log\left(\frac{1+m}{2}\right) + \frac{(1-m)}{2}\log\left(\frac{1-m}{2}\right)$. In Section 9 of [6], it was proved that

$$\mathcal{E} = \begin{cases} 0 < \theta < \theta_{1,c}(\beta), & \text{for } 1 < \beta < \frac{3}{2}; \\ 0 < \theta \le \theta_{1,c}(\beta) & \text{for} \beta \ge \frac{3}{2}, \end{cases} \tag{2.20}$$

where $\theta_{1,c}(\beta) = \frac{1}{\beta}\,\mathrm{arctanh}\,(1-\frac{1}{\beta})^{1/2}$, is the maximal region of the two parameters (β,θ), whose closure contains $(1,0)$ in which $f_{\beta,\theta}(\cdot,\cdot)$ has exactly three critical points $m_\beta, 0, Tm_\beta$. The two equal minima correspond to $m_\beta = (m_{\beta,1} > 0, m_{\beta,2} > 0)$ and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$ and 0 a local maximum. Moreover, for all $(\beta,\theta) \in \mathcal{E}$ there exists $\kappa(\beta,\theta) > 0$ so that for each $m \in [-1,+1]^2$

$$f_{\beta,\theta}(m) - f_{\beta,\theta}(m_\beta) \ge \kappa(\beta,\theta)\min\{\|m-m_\beta\|_1^2, \|m-Tm_\beta\|_1^2\}, \tag{2.21}$$

where $\|\cdot\|_1$ is the ℓ^1 norm in $I\!R^2$. Clearly, the absolute minimum of $\mathcal{F}$ is attained at the functions constantly equal to m_β (or constantly equal to Tm_β), the minimizers of $f_{\beta,\theta}$. We denote $\tilde{m}_\beta = \frac{m_{\beta,1}+m_{\beta,2}}{2}$.

- *The surface tension*

We denote by surface tension the free energy cost needed by the system to undergo to a phase change. It follows from [5] that under the condition $m_1(0)+m_2(0) = 0$, and for $(\beta,\theta) \in \mathcal{E}$, there exists a unique minimizer $\bar{m} = (\bar{m}_1, \bar{m}_2)$, of $\mathcal{F}(m)$ over the set

$$\mathcal{M}_\infty = \{(m_1,m_2) \in \mathcal{T}; \limsup_{r\to-\infty} m_i(r) < 0 < \liminf_{r\to+\infty} m_i(r), i = 1,2\}. \tag{2.22}$$

Without the condition $m_1(0)+m_2(0) = 0$, there is a continuum of minimizers obtained by translating $\bar{m}$. The minimizer $\bar{m}(\cdot)$ converges exponential fast, as $r \uparrow +\infty$ (resp. $-\infty$) to the limit value m_β, (resp.Tm_β). Since $\mathcal{F}$ is invariant by the T-transformation, see (2.16), interchanging $r \uparrow +\infty$ and $r \downarrow -\infty$ in (2.22), there exists one other family of minimizers obtained translating $T\bar{m}$. We denote $\mathcal{F}^*$ the surface tension:

$$\mathcal{F}^* \equiv \mathcal{F}^*(\beta,\theta) = \mathcal{F}(\bar{m}) = \mathcal{F}(T\bar{m}) > 0. \tag{2.23}$$

- *how to detect local equilibrium*

As in [4], the description of the profiles is based on the behavior of local averages of $m^{\delta^*}(x)$ over k successive blocks in the block spin representation, where $k \ge 2$ is a positive integer. Let $\delta = k\delta^*$ be such that $1/\delta \in I\!N$. Let $\ell \in Z\!\!Z$, $[\ell, \ell+1)$ be a macroscopic block of length 1, $\mathcal{C}_\delta([\ell, \ell+1))$, as in (2.5), and $\zeta > 0$. We define the block spin variable

$$\eta^{\delta,\zeta}(\ell) = \begin{cases} 1, & \text{if } \forall_{u \in \mathcal{C}_\delta([\ell,\ell+1))} \frac{\delta^*}{\delta} \sum_{x \in \mathcal{C}_{\delta^*}([u\delta,(u+1)\delta))} \|m^{\delta^*}(x,\sigma) - m_\beta\|_1 \le \zeta; \\ -1, & \text{if } \forall_{u \in \mathcal{C}_\delta([\ell,\ell+1))} \frac{\delta^*}{\delta} \sum_{x \in \mathcal{C}_{\delta^*}([u\delta,(u+1)\delta))} \|m^{\delta^*}(x,\sigma) - Tm_\beta\|_1 \le \zeta; \\ 0, & \text{otherwise.} \end{cases} \tag{2.24}$$

where for a vector $v = (v_1, v_2)$, $\|v\|_1 = |v_1| + |v_2|$. When $\eta^{\delta,\zeta}(\ell) = 1$, (resp. -1), we say that a spin configuration $\sigma \in \{-1,1\}^{\frac{1}{\gamma}[\ell,\ell+1)}$ has magnetization close to m_β, (resp. Tm_β), with accuracy (δ,ζ) in $[\ell, \ell+1)$. Note that $\eta^{\delta,\zeta}(\ell) = 1$ (resp -1) is equivalent to

$$\forall y \in [\ell, \ell+1) \qquad \frac{1}{\delta} \int_{D_\delta(y)} dx \|m^{\delta^*}(x,\sigma) - v\|_1 \le \zeta \tag{2.25}$$

for $v = m_\beta$ (resp. Tm_β). Since for any $u \in \mathcal{C}_\delta([\ell, \ell+1))$, for all $y \in [u\delta, (u+1)\delta) \subset [\ell, \ell+1)$, $D_\delta(y) = [u\delta, (u+1)\delta)$.

When $\eta^{\delta,\zeta}(\ell) = 1$, (resp. -1), we say that a spin configuration $\sigma \in \{-1,1\}^{\frac{1}{\gamma}[\ell,\ell+1)}$ has magnetization close to m_β, (resp. Tm_β), with accuracy (δ,ζ) in $[\ell, \ell+1)$. In the following the letter ℓ will always indicate an element of $Z\!\!Z$. We then say that a magnetization profile $m^{\delta^*}(\cdot)$, in a macroscopic interval $I \subseteq I\!R$, is close to the equilibrium phase τ, for $\tau \in \{-1,+1\}$, with accuracy (δ,ζ) when

$$\{\eta^{\delta,\zeta}(\ell) = \tau, \forall \ell \in I \cap Z\!\!Z\} \equiv \{\eta^{\delta,\zeta}(\ell) = \tau, \forall \ell \in I\}. \tag{2.26}$$

In view of the results on the typical configurations obtained in [6] a candidate for the limiting support of $\mu_{\beta,\theta,\gamma}$, when $\gamma \downarrow 0$, is an appropriate neighborhood of functions on $I\!R$, (considered in the Brownian scale), taking two values m_β or Tm_β that have finite variation. So we define, for any bounded interval $[a,b) \subset I\!R$ (in the Brownian scale) $BV([a,b)) \equiv BV([a,b), \{m_\beta, Tm_\beta\})$ the set of right continuous bounded variation functions on $[a,b)$ with value in $\{m_\beta, Tm_\beta\}$. We call the jump at r the quantity $Du(r) = u(r) - u(r_-)$ where $u(r_-) = \lim_{s \uparrow r} u(s)$. If r is such that $Du(r) \neq 0$ we call r a point of jump of u, and in such a case $\|Du(r)\|_1 = 4\tilde{m}_\beta$. We denote by $N_{[a,b)}(u)$ the number of jumps of u on $[a,b)$ and by $V_a^b(u)$ the variation of u on $[a,b)$, i.e.

$$V_a^b(u) \equiv \sum_{a \le r < b} \|Du(r)\|_1 = N_{[a,b)}(u) 2[m_{\beta,1} + m_{\beta,2}] = 4\tilde{m}_\beta N_{[a,b)}(u) < \infty. \tag{2.27}$$

By right continuity if $\|Du(a)\|_1 \neq 0$ then $a \in \mathcal{N}_{[a,b)}(u)$, while if $\|Du(b)\|_1 \neq 0$ then $b \notin \mathcal{N}_{[a,b)}(u)$. We denote by $BV_{\text{loc}} \equiv BV_{\text{loc}}(I\!R, \{m_\beta, Tm_\beta\})$ the set of functions from $I\!R$ with values in $\{m_\beta, Tm_\beta\}$ which restricted to any bounded interval have bounded variation.

Warning on notation Whenever we deal with functions in $\mathcal{T}$, see (2.14), we always assume that their argument varies on macroscopic scale. So $m \in \mathcal{T}$ means $m(x), x \in I$ where $I \subset I\!R$ is an interval in the macroscopic scale. Whenever we deal with bounded variation functions, if not further specified, we always assume that their argument varies on the Brownian scale. Therefore $u \in BV([a,b))$ means $u(r), r \in [a,b)$ and $[a,b)$ considered in the Brownian scale. In macroscopic scale we must write $u(\gamma x)$ for $x \in [\frac{a}{\gamma}, \frac{b}{\gamma})$.

Definition 2.1 Partition associated to BV functions *Take $u \in BV([a,b))$, $\rho > \delta$, with $8\rho + 8\delta$ smaller than the minimal distance between two points of jumps of u. Let $C_i(u)$, $i = 1, .., N_{[a,b)}(u)$, (see (2.27)), be* *the smallest $\mathcal{D}_\delta$ measurable interval that contains an interval of diameter 2ρ, centered at the $i-th$ jump of u in $[a,b)$. We have $C_i(u) \cap C_j(u) = \emptyset$ for all $i \neq j$.*

Set $C(u) = \cup_{i=1}^{N_{[a,b)}(u)} C_i(u)$, $B(u) = [a,b) \setminus C(u)$. We denote by $C_{i,\gamma}(u) = \gamma^{-1}C_i(u)$, $C_\gamma(u) = \gamma^{-1}C(u)$ and $B_\gamma(u) = \gamma^{-1}B(u)$ the elements of the induced partition on the macroscopic scale.

Since a phase change can be more precisely described in macro units, we state the following definition which corresponds to Definition 2.3 of [6].

Definition 2.2 The macro–interfaces *Given an interval $[\ell_1, \ell_2]$ (in macro-scale) and a positive integer $2R_2 \leq |\ell_2 - \ell_1|$, we say that a single phase change occurs within $[\ell_1, \ell_2]$ on a length R_2 if there exists $\ell_0 \in (\ell_1 + R_2, \ell_2 - R_2)$ so that $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_1) \in \{-1,+1\}, \forall \ell \in [\ell_1, \ell_0 - R_2]$; $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_2) = -\eta(\ell_1), \forall \ell \in [\ell_0 + R_2, \ell_2]$, and $\{\ell \in [\ell_0 - R_2, \ell_0 + R_2] : \eta^{\delta,\zeta}(\ell) = 0\}$ is a set of consecutive integers. We denote by $\mathcal{W}_1([\ell_1,\ell_2], R_2, \zeta)$ the set of configurations $\eta^{\delta,\zeta}$ with these properties.*

In words, on $\mathcal{W}_1([\ell_1,\ell_2], R_2, \zeta)$, there is an unique run of $\eta^{\delta,\zeta} = 0$, with no more than $2R_2$ elements, inside the interval $[\ell_1, \ell_2]$.

Given $[a,b)$ (in the Brownian scale), $\rho > \delta$, $\zeta > 0$ and u in $BV([a,b))$ satisfying the condition of Definition 2.1, we say that a spin configuration $\sigma \in \{-1,1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})}$ has magnetization profile close to u with accuracy (δ,ζ) and *fuzziness* ρ if $\sigma \in \mathcal{P}^\rho_{\delta,\gamma,\zeta,[a,b)}(u)$ where

$$\mathcal{P}^\rho_{\delta,\gamma,\zeta,[a,b)}(u) =$$

$$\left\{ \sigma \in \{-1,1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})} : \forall y \in B_\gamma(u),\ \frac{1}{\delta} \int_{D^\delta(y)} \| m^{\delta^*}(x,\sigma) - u^{\gamma,\delta^*}(x) \|_1 \, dx \leq \zeta \right\} \bigcap_{i=1}^{N_{[a,b)}(u)} \mathcal{W}_1([C_{i,\gamma}(u)], R_2, \zeta), \tag{2.28}$$

and

$$u^{\gamma,\delta^*}(x) = \frac{1}{\delta^*} \int_{D_{\delta^*}(x)} u(\gamma s)\, ds. \tag{2.29}$$

In (2.28) we consider the spin configurations close with accuracy (δ,ζ) to m_β or Tm_β in $B_\gamma(u)$ according to the value of $u^{\gamma,\delta^*}(\cdot)$. In $C_\gamma(u)$ we require that the spin configurations have only one jump in each interval $C_{i,\gamma}(u)$, $i = 1,..N$, and are close with accuracy (δ,ζ) to the right and to the left of this interval to the value of u in those intervals of $B_\gamma(u)$ that are adjacent to $C_{i,\gamma}(u)$.

Remark 2.3 . *The notion of fuzziness ρ deserves some explanations. As mentioned in the heuristics, the localization of the phase change is determined by minimizers and maximizers of a random walk. For a random walk there are various possible choices to define these points. One can take as maximizer the point where the random walk attains for the first time its running maximum or the point where it did for the last time. We introduced in [6] the parameter ρ that takes into account this intrinsic ambiguity on localizations of minimizers or maximizers of the random walk. In fact ρ is chosen in such a way that if one makes a choice for the maximizer and excludes an interval of length 2ρ (in the Brownian scale) centered around it then outside this interval the random walk is at least at distance ρ^{2+a} from this maximal value with an overwhelming probability see Theorem 5.1 in [6]. Note that ρ is a function of γ, see (2.56). Other possible localizations (the "last time" for example) are necessarily within such an interval of length 2ρ.*

On the other hand R_2, in the definition 2.2 and in the last term in (2.28), will be chosen in a γ–dependent way such that the Gibbs probability of runs of $\eta^{\delta,\zeta}$ larger than R_2 is exponentially small. One has (in the

Brownian scale) $\rho >> \gamma R_2$ *i.e a very rough upper bound on the space needed to make a typical macro–interface is much smaller than the imprecision linked to previous mentioned localizations of minimizers or maximizers.*

With all these definitions in hand we can slightly improve the main results of [6]. To facilitate the reading we will not write explicitly in the statement of Theorem 2.4 the choice the parameters $\delta, \delta^*, \zeta, g, R_2, Q, \rho$. All of them, but ζ depend on γ. We recall it in Subsection 2.5.

Theorem 2.4 [COPV] *Given* $(\beta,\theta) \in \mathcal{E}$*, see (2.20), there exists* $\gamma_0(\beta,\theta)$ *so that for* $0 < \gamma \le \gamma_0(\beta,\theta)$*, choosing the parameters as in Subsection 2.5, there exists* $\Omega_1 \subset \Omega$ *with*

$$I\!P[\Omega_1] \ge 1 - K(Q)\left(\frac{1}{g(\delta^*/\gamma)}\right)^{\frac{a}{8(2+a)}} \tag{2.30}$$

where a is as in (2.56),

$$K(Q) = 2 + 5(V(\beta,\theta)/(\mathcal{F}^*)^2)Q\log[Q^2 g(\delta^*/\gamma)], \tag{2.31}$$

$\mathcal{F}^* = \mathcal{F}^*(\beta,\theta)$ *is defined in (2.23) and*

$$V(\beta,\theta) = \log\frac{1 + m_{\beta,2}\tanh(2\beta\theta)}{1 - m_{\beta,1}\tanh(2\beta\theta)}. \tag{2.32}$$

For $\omega \in \Omega_1$ *we explicitly construct* $u^*_\gamma(\omega) \in BV([-Q,Q])$ *so that the minimal distance between jumps of* u^*_γ *within* $[-Q,+Q]$ *is bounded from below by* $8\rho + 8\delta$*,*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,Q]}(u^*_\gamma(\omega))\right) \ge 1 - 2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}, \tag{2.33}$$

and

$$V^Q_{-Q}(u^*_\gamma) \le 4\tilde m_\beta K(Q). \tag{2.34}$$

The proof of Theorem 2.4 is sketched at the end of Section 5. It is a direct consequence of Theorem 2.1, Theorem 2.2 and Theorem 2.4 proven in [6], together with Lemma 5.7 that gives the value (2.31). The $u^*_\gamma(\omega)$ in Theorem 2.4 is a function in $BV([-Q,Q])$ associated to the sequence of maximal elongations and their sign as determined in [6] Section 5. For the moment it is enough to know that it is possible to determine random points $\alpha^*_i = \alpha^*_i(\gamma,\omega)$ and a random sign ± 1 associated to intervals $[\epsilon\alpha^*_i, \epsilon\alpha^*_{i+1})$ in the Brownian scale, where $\epsilon = \epsilon(\gamma)$ has to be suitably chosen. These random intervals are called maximal elongations. We denote

$$u^*_\gamma(\omega)(r) \equiv \begin{cases} m_\beta, & r \in [\epsilon\alpha^*_i, \epsilon\alpha^*_{i+1}) \quad \text{if the sign of elongation } [\epsilon\alpha^*_i, \epsilon\alpha^*_{i+1}) \text{ is } = +1 \\ Tm_\beta, & r \in [\epsilon\alpha^*_i, \epsilon\alpha^*_{i+1}) \quad \text{if the sign of elongation } [\epsilon\alpha^*_i, \epsilon\alpha^*_{i+1}) \text{ is} = -1, \end{cases} \tag{2.35}$$

for $i \in \{\kappa^*(-Q)+1, \dots, -1, 0, 1, \dots, \kappa^*(Q)-1\}$, where

$$\kappa^*(Q) = \inf(i \ge 0 : \epsilon\alpha^*_i > Q), \qquad \kappa^*(-Q) = \sup(i \le 0 : \epsilon\alpha^*_i < -Q) \tag{2.36}$$

with the convention that $\inf(\emptyset) = +\infty, \sup(\emptyset) = -\infty$. We set $\epsilon\alpha^*_0 < 0$ and $\epsilon\alpha^*_1 > 0$, that is just a relabeling of the points determined in [6], Section 5. Lemma 5.7 gives that, with a $I\!P$–probability absorbed in (2.30), we have $|\kappa^*(-Q)| \vee \kappa^*(Q) \le K(Q)$, with $K(Q)$ given in (2.31).

2.4. The main results

Denote by $(W(r), r \in I\!R)$ the Bilateral Brownian motion (BBM) , *i.e.* the Gaussian process with independent increments that satisfies $I\!E(W(r)) = 0$, $I\!E(W^2(r)) = |r|$ for $r \in I\!R$ and by $\mathcal{P}$ the Wiener measure on $(C(I\!R), \mathcal{B})$ with $\mathcal{B}$ the borel σ–algebra for the topology of uniform convergence on compact. Let $h > 0$ and denote, as in Neveu - Pitman [13], by $\{S_i \equiv S_i^{(h)}; \in Z\!\!Z\}$ the points of $h-$extrema of the BBM with the labeling convention that $\dots S_{-1} < S_0 \le 0 < S_1 < S_2 \dots$. The Neveu - Pitman construction to determine them is recalled in Section 5.2. We recall their properties below Theorem 2.5.

Theorem 2.5 *Given $(\beta, \theta) \in \mathcal{E}$, see (2.20), choosing the parameters as in Subsection 2.5, setting $h = \frac{2\mathcal{F}^*}{V(\beta,\theta)}$, we have that*

$$\lim_{\gamma \to 0} \epsilon(\gamma) \alpha_i^*(\gamma) \stackrel{\text{Law}}{=} S_i^{(h)} \equiv S_i \tag{2.37}$$

for $i \in Z\!\!Z$.

The $\{S_i, i \in Z\!\!Z\}$ is a stationary renewal process on $I\!R$. The $S_{i+1} - S_i$, (and $S_{-i} - S_{-i-1}$) for $i > 1$ are independent, equidistributed, with Laplace transform

$$I\!E[e^{-\lambda(S_{i+1}-S_i)}] = [\cosh(h\sqrt{2\lambda})]^{-1} \text{ for } \lambda \ge 0 \tag{2.38}$$

(mean h^2) and distribution given by

$$\frac{d}{dx}\left(I\!P[S_2 - S_1 \le x]\right) = \frac{\pi}{2} \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)}{h^4} \exp\left[-(2k+1)^2 \frac{\pi^2}{8} \frac{x}{h^2}\right] \quad \text{for x} > 0. \tag{2.39}$$

Moreover S_1 and $-S_0$ are equidistributed, have Laplace transform

$$I\!E[e^{-\lambda S_1}] = \frac{1}{h^2 \lambda}\left(1 - \frac{1}{\cosh(h\sqrt{2\lambda})}\right) \text{ for } \lambda \ge 0 \tag{2.40}$$

and distribution given by

$$\frac{d}{dx}\left(I\!P[S_1 \le x]\right) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)h^2} \exp\left[-(2k+1)^2 \frac{\pi^2}{8} \frac{x}{h^2}\right] \quad \text{for x} > 0. \tag{2.41}$$

The formula (2.38) is given in [13].

Let $u^* \equiv u^*(W)$ be the following random function:

$$u^*(r) = \begin{cases} \begin{cases} m_\beta & \text{for} \quad r \in [S_i, S_{i+1}), \\ Tm_\beta & \text{for} \quad r \in [S_{i+1}, S_{i+2}). \end{cases} & \text{if} \quad S_i \quad \text{is a point of } h\text{–minimum for } W; \\ \begin{cases} Tm_\beta & \text{for} \quad r \in [S_i, S_{i+1}), \\ m_\beta & \text{for} \quad r \in [S_{i+1}, S_{i+2}). \end{cases} & \text{if} \quad S_i \quad \text{is a point of } h\text{–maximum for } W; \end{cases} \tag{2.42}$$

Since $\mathcal{P}$ a.s the number of $h-$extrema of the BBM is finite in any finite interval, we have that $\mathcal{P}$ a.s., $u^* \in BV_{\text{loc}}$.

Corollary 2.6 *Under the same hypothesis of Theorem 2.5, for the topology induced by the Skorohod metric that makes BV_{loc} a complete separable space, see (5.5), we have*

$$\lim_{\gamma \downarrow 0} u_\gamma^* \stackrel{\text{Law}}{=} u^*. \tag{2.43}$$

The proof of Theorem 2.5 and Corollary 2.6 are given in Section 6. Let u^* be the function defined in (2.42), $u \in BV_{\rm loc}$ and $[a,b) \subset I\!R$ a finite interval. Denote by $\mathcal{N}_{[a,b)}(u,u^*)$ the points of jump of u or u^* in $[a,b)$:

$$\mathcal{N}_{[a,b)}(u,u^*) = \{r \in [a,b) : \|Du(r)\|_1 \neq 0 \text{ or } \|Du^*(r)\|_1 \neq 0\}. \tag{2.44}$$

Since u and u^* are $BV_{\rm loc}$ functions $\mathcal{N}_{[a,b)}(u,u^*)$ is a finite set of points. We index in increasing order the points in $\mathcal{N}_{[a,b)}(u,u^*)$ and by an abuse of notation we denote $\{i \in \mathcal{N}_{[a,b)}(u,u^*)\}$ instead of $\{i : r_i \in \mathcal{N}_{[a,b)}(u,u^*)\}$. Define for $u \in BV_{\rm loc}$, the following finite volume functional

$$\begin{aligned}&\Gamma_{[a,b)}(u|u^*,W)\\ &= \frac{1}{2\tilde{m}_\beta} \sum_{i\in\mathcal{N}_{[a,b)}(u,u^*)} \left\{ \frac{\mathcal{F}^*}{2}\left[\|Du(r_i)\|_1 - \|Du^*(r_i)\|_1\right] - V(\beta,\theta)(\tilde{u}(r_i) - \tilde{u}^*(r_i))[W(r_{i+1}) - W(r_i)] \right\}.\end{aligned} \tag{2.45}$$

The functional in (2.45) is always well defined since it is $\mathcal{P}$–a.s a finite sum of terms each one being $\mathcal{P}$–a.s finite. An extension of (2.45) to $I\!R$ is given by

$$\Gamma(u|u^*,W) = \sum_{j\in\mathbb{Z}} \Gamma_{[S_j,S_{j+1})}(u|u^*,W). \tag{2.46}$$

In Theorem 2.7 stated below, we prove that the sum is positive and therefore the functional in (2.46) is well defined although it may be infinite. One can formally write the functional (2.46) as

$$\Gamma(u|u^*,W) = \frac{1}{2\tilde{m}_\beta}\left\{\frac{\mathcal{F}^*}{2}\int_{I\!R} dr\left[\|Du(r)\|_1 - \|Du^*(r)\|_1\right] - V(\beta,\theta)\int_{I\!R}(\tilde{u}(r) - \tilde{u}^*(r))dW(r)\right\}, \tag{2.47}$$

but the stochastic integral should be defined. We have the following result:

Theorem 2.7 *$\mathcal{P}$ a.s., for all $u \in BV_{\rm loc}$, $\Gamma(\cdot|u^*,W) \geq 0$ and u^* defined in (2.42) is the unique minimizer of $\Gamma(\cdot|u^*,W)$.*

Remark 2.8 . *The functional defined in (2.46) is lower semi-continuous. It is not too difficult but rather long to check that this occurs in the following sense: If u_n and u are in $BV_{\rm loc}$ and $\lim_{n\uparrow\infty} u_n = u$ in probability (i.e. such that $\forall\epsilon > 0, \lim_{n\uparrow\infty} I\!P[d(u_n,u) \geq \epsilon] = 0$ where $d(\cdot,\cdot)$ is defined in (5.5)) then $\liminf_{n\uparrow\infty}\Gamma(u_n|u^*,W) \geq \Gamma(u|u^*,W)$ in probability.*

To state our next result, that is the identification of the functional (2.46) as a large deviation rate some restrictions on the functions u in $BV_{\rm loc}$ we consider is needed. First of all to inject $BV([-Q,+Q])$ into $BV(I\!R)$, we define for $u \in BV([-Q,+Q])$

$$u^Q(r) = \begin{cases} u(r\wedge Q), & \text{if } r \geq 0;\\ u(r\vee(-Q)) & \text{if } r < 0,\end{cases} \tag{2.48}$$

so that $V_{-Q}^{Q}(u) = V_{-\infty}^{\infty}(u^Q)$. This allows to consider $u_\gamma^* \in BV([-Q,+Q])$ as an element of $BV(I\!R)$. Let $F(Q)$ be a positive increasing real function. Denote

$$\mathcal{U}_Q(u_\gamma^*) = \left\{u \in BV_{\rm loc};\ u^Q(r) = u_\gamma^{*Q}(r), \forall\,|r| \geq Q-1;\ W_Q(u) \geq 8\rho + 8\delta;\ V_{-Q}^{Q}(u) \leq V_{-Q}^{Q}(u_\gamma^*)F(Q)\right\}, \tag{2.49}$$

where $W_Q(u) \equiv \inf(r - r', -Q \le r' < r \le Q; |Du(r')| \ne 0, |Du(r)| \ne 0)$. The last requirement in (2.49) imposes that the number of jumps of u does not grow too fast with respect to the ones of u^*_γ, see Remark 2.10 for more explanations.

We denote by $u(\gamma) \equiv u(\gamma, \omega)$ a generic element of $\mathcal{U}_Q(u^*_\gamma(\omega))$.

Theorem 2.9 *Take $(\beta, \theta) \in \mathcal{E}$, u^* in (2.42), the parameters as in Subsection 2.5, then*

$$\lim_{\gamma \downarrow 0} \left[-\gamma \log \mu_{\beta,\theta,\gamma} \left(\mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u(\gamma)) \right) \right] \overset{\text{Law}}{=} \Gamma(u|u^*, W). \tag{2.50}$$

for all $u(\gamma, \omega) \in \mathcal{U}_Q(u^*_\gamma(\omega))$ such that $\lim_{\gamma \downarrow 0}(u(\gamma), u^*_\gamma) = (u, u^*)$ in Law, when

$$F(Q) = e^{(\frac{1}{8+4a} - b)(\log Q)(\log\log Q)}, \tag{2.51}$$

with $0 < b < 1/(8 + 4a)$ and a as in (2.56).

Here are some examples of elements of $\mathcal{U}_Q(u^*_\gamma(\omega))$:

$$u_1(\gamma, \omega)(r) = v(r) 1\!\!\mathrm{I}_{[-L,L]}(r) + u^*_\gamma(\omega)(r) 1\!\!\mathrm{I}_{[-Q,Q] \setminus [-L,L]}(r), \tag{2.52}$$

where $L > 0$ and $v \in BV_{\text{loc}}$ is a given non random function. When L is a fixed number independent on γ then $(u_1(\gamma), u^*_\gamma)$ converges in Law, as $\gamma \downarrow 0$, to $(u_1(W), u^*(W))$ where

$$u_1(W)(r) = v(r) 1\!\!\mathrm{I}_{[-L,L]}(r) + u^*(W)(r) 1\!\!\mathrm{I}_{I\!\!R \setminus [-L,L]}(r)$$

and the functional in the r.h.s. of (2.50) is computed on $u_1(W)$. When $L \equiv L(\gamma)$ goes to infinity, as $\gamma \downarrow 0$, then $(u_1(\gamma), u^*_\gamma)$ converges in Law to (v, u^*) and the functional in the r.h.s. of (2.50) is computed on v.

Theorem 2.9 is a consequence of accurate estimates stated in Proposition 4.1 where error terms are given.

Remark 2.10 . If a profile makes more than $F(Q)$ times the jumps of u^*_γ Theorem 2.9 cannot be applied. In this case the contribution of the error terms that one gets in Proposition 4.1 is too large. Further consider the last two conditions in (2.49). The (2.51) implies for any positive integer p, $F(Q) > Q^p$, so the last requirement in (2.49) can be satisfied for functions $u = u(\gamma)$ that are wildly oscillating. In particular, in (2.49) there are functions that makes $F(Q)$ jumps within each of all maximal elongations considered. Using (4.63), one can check that choosing ρ as in (2.56), $\rho << 1/F(Q)$. Therefore dividing each maximal elongation in $F(Q)$ equal intervals and taking the sequence of functions $\{u(\gamma)\}_\gamma$ which jump at each beginning of such interval, one gets a sequence of function with minimal distance between two jumps of order $1/F(Q)$. This sequence of functions satisfies both the requirements in (2.49). However it does not converge in Law in BV_{loc} endowed with the Skorohod topology. Therefore is not considered in Theorem 2.9, even though the estimates in Proposition 4.1 are valid.

On the other hand if one takes a sequence of functions $\{u(\gamma)\}_\gamma$, $u(\gamma) \in BV_{\text{loc}}$ that does not satisfy $W_Q(u) \ge 8\rho + 8\delta$, *i.e* such that $u(\gamma)$ has just two jumps at distance less than $8\rho + 8\delta$ for example in the middle of $[\epsilon\alpha^*_1, \epsilon\alpha^*_2)$ (see (2.35)) but coincides with the typical profile then since $8\rho + 8\delta \downarrow 0$, see (2.56) and (2.55), this sequence of functions does not converge in Law in BV_{loc} endowed with the Skorohod topology, see [2] pg 112–113 for example. This sequence does not satisfy the hypothesis of Theorem 2.9 and Proposition 4.1.

2.5 Choice of the parameters We regroup here the choice of the parameters done all along this work. This choice is similar to the one done in [6], see page 794. The g is a positive function from $(1, \infty)$ so that

$g(x) > 1, g(x)/x \le 1, \forall x > 1$ and $\lim_{x\uparrow\infty} x^{-1} g^{38}(x) = 0$. (*The exponent* 38 *has no reason to be optimal, it just works well*).

Any increasing function slowly varying at infinity can be modified to satisfy such constraints. A possible choice is $g(x) = 1 \vee \log x$ or any iterated of it. The remaining parameters can be chosen, ignoring the arithmetic constraints, as it follows. We take for δ^*, which represents the smallest coarse graining scale,

$$\delta^* = \gamma^{\frac{1}{2}+d^*} \quad \text{for some}\, 0 < d^* < 1/2. \tag{2.53}$$

For (ζ, δ), the accuracy chosen to determine how close is the local magnetization to the equilibrium values, see (2.24) and (2.28), there exists a $\zeta_0 = \zeta_0(\beta, \theta)$ such that so that for $\kappa(\beta, \theta)$ given in (2.21),

$$\frac{1}{[\kappa(\beta,\theta)]^{1/3} g^{1/6}(\frac{\delta^*}{\gamma})} < \zeta \le \zeta_0 \tag{2.54}$$

and

$$\delta = \frac{1}{5(g(\frac{\delta^*}{\gamma}))^{1/2}}. \tag{2.55}$$

The fuzziness ρ is chosen as

$$\rho = \left(\frac{5}{g(\delta^*/\gamma)}\right)^{1/(2+a)}, \tag{2.56}$$

where a is an arbitrary positive number. Furthermore ϵ that appears in (2.35) is chosen as

$$\epsilon = (5/g(\delta^*/\gamma))^4. \tag{2.57}$$

R_2 that appears in Definition 2.1 is chosen as

$$R_2 = c(\beta,\theta)(g(\delta^*/\gamma))^{7/2} \tag{2.58}$$

for some positive $c(\beta, \theta)$, and

$$Q = \exp[(\log g(\delta^*/\gamma))/\log\log g(\delta^*/\gamma)]. \tag{2.59}$$

Since g is slowly varying at infinity $\gamma R_2 \downarrow 0$ when $\gamma \downarrow 0$.

Remark 2.11 . *Note that the only constraint on ζ is (2.54). In particular one can pick up a ζ which is γ–independent. However it is also possible to choose for example*

$$\zeta = \zeta(\gamma) \equiv \frac{1}{2}\frac{1}{[\kappa(\beta,\theta)]^{1/3} g^{1/6}(\frac{\delta^*}{\gamma})} \tag{2.60}$$

that goes to zero with γ. Let us examine what these two choices mean for Theorem 2.4. Since the left hand side of (2.33) is clearly a decreasing function of ζ, the choice (2.60) gives a stronger result than the γ–independent one. Since

$$\zeta \to -\gamma \log \mu_{\beta,\theta,\gamma}\left(\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,Q]}(u(\gamma))\right)$$

is increasing, the γ–independent choice implies (2.50) with the choice (2.60). Therefore in Theorem 2.9, the γ–independent choice of ζ gives a stronger result. At last note that with the γ–independent choice of ζ, we get as a limit the rate of large deviation evaluated at u even if the neighborhood does not shrink to u.

3 The block spin representation.

In this section we state the results of the coarse graining procedure. The computations are explicitly done in Section 3 and 4 of [6].

With $C_{\delta^*}(V)$ as in (2.5), let $\Sigma_V^{\delta^*}$ denote the sigma–algebra of $\mathcal{S}$ generated by $m_V^{\delta^*}(\sigma) \equiv (m^{\delta^*}(x,\sigma),\ x \in \mathcal{C}_{\delta^*}(V))$, where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$, cf. (2.8). Take $I = [i^-, i^+) \subseteq I\!R$ with $i^\pm \in Z\!\!Z$ to be $\mathcal{D}_{\delta^*}$–measurable and set $\partial^+ I \equiv \{x \in I\!R: i^+ \le x < i^+ + 1\}$, $\partial^- I \equiv \{x \in I\!R: i^- - 1 \le x < i^-\}$, and $\partial I = \partial^+ I \cup \partial^- I$. Let F^{δ^*} be a $\Sigma_I^{\delta^*}$-measurable bounded function, $m_{\partial I}^{\delta^*} \in \mathcal{M}_{\delta^*}(\partial I)$ and $\mu_{\beta,\theta,\gamma}\left(F^{\delta^*} | \Sigma_{\partial I}^{\delta^*}\right)$ the conditional expectation of F^{δ^*} given the σ–algebra $\Sigma_{\partial I}^{\delta^*}$. We obtain, see formula 3.14 of [6],:

Lemma 3.1

$$\mu_{\beta,\theta,\gamma}\left(F^{\delta^*} \mid \Sigma_{\partial I}^{\delta^*}\right)(m_{\partial I}^{\delta^*}) = \frac{e^{\pm\frac{\beta}{\gamma}2\delta^*}}{Z_{\beta,\theta,\gamma,I}(m_{\partial I}^{\delta^*})} \sum_{m_I^{\delta^*} \in \mathcal{M}_{\delta^*}(I)} F^{\delta^*}(m^{\delta^*}) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*}) + \gamma \mathcal{G}(m_I^{\delta^*}) + \gamma V(m_I^{\delta^*})\right\}}, \quad (3.1)$$

where equality has to be interpreted as an upper bound for $\pm = +1$ and a lower bound for $\pm = -1$, and

$$Z_{\beta,\gamma,\theta,I}(m_{\partial I}^{\delta^*}) = \sum_{m_I^{\delta^*} \in \mathcal{M}_{\delta^*}(I)} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*}) + \gamma \mathcal{G}(m_I^{\delta^*}) + \gamma V(m_I^{\delta^*})\right\}}. \quad (3.2)$$

We recall the definition of the quantities in (3.1). For $(m_I^{\delta^*}, m_{\partial I}^{\delta^*})$ in $\mathcal{M}_{\delta^*}(I \cup \partial I)$, cf. (2.12),

$$\begin{aligned} \widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*}) =& E(m_I^{\delta^*}) + E(m_I^{\delta^*}, m_{\partial I}^{\delta^*}) - \frac{\theta\delta^*}{2} \sum_{x \in \mathcal{C}_{\delta^*}(I)} (m_1^{\delta^*}(x) - m_2^{\delta^*}(x)) \\ &- \delta^* \sum_{x \in \mathcal{C}_{\delta^*}(I)} \frac{\gamma}{\beta\delta^*} \log \binom{\delta^*\gamma^{-1}/2}{\frac{1+m_1^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2} \binom{\delta^*\gamma^{-1}/2}{\frac{1+m_2^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2}, \end{aligned} \quad (3.3)$$

where for $\widetilde{m}^{\delta^*}(x) = (m_1^{\delta^*}(x) + m_2^{\delta^*}(x))/2,\ J_{\delta^*}(x) = \delta^* J(\delta^* x)$,

$$E(m_I^{\delta^*}) \equiv -\frac{\delta^*}{2} \sum_{(x,y) \in \mathcal{C}_{\delta^*}(I) \times \mathcal{C}_{\delta^*}(I)} J_{\delta^*}(x-y) \tilde{m}^{\delta^*}(x) \tilde{m}^{\delta^*}(y), \quad (3.4)$$

$$E(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}) \equiv -\delta^* \sum_{x \in \mathcal{C}_{\delta^*}(I)} \sum_{y \in \mathcal{C}_{\delta^*}(\partial^\pm I)} J_{\delta^*}(x-y) \tilde{m}^{\delta^*}(x) \tilde{m}^{\delta^*}(y), \quad (3.5)$$

$$\mathcal{G}(m_I^{\delta^*}) \equiv \sum_{x \in \mathcal{C}_{\delta^*}(I)} \mathcal{G}_{x, m^{\delta^*}(x)}(\lambda(x)). \quad (3.6)$$

For each $x \in \mathcal{C}_{\delta^*}(I)$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ is the cumulant generating function:

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) \equiv -\frac{1}{\beta} \log \left\{ \frac{\sum_\sigma e^{2\beta\theta\lambda(x)\sum_{i \in D(x)} \sigma_i} 1\!\!1_{\{m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\}}}{\sum_\sigma 1\!\!1_{\{m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\}}} \right\}, \quad (3.7)$$

the sum being over $\sigma \in \{-1,+1\}^{A(x)}$. Finally denote

$$V(m_I^{\delta^*}) \equiv V_I(m_I^{\delta^*}, h) = -\frac{1}{\beta} \log I\!E_{m_I^{\delta^*}}\Big[\prod_{\substack{x \neq y \\ x,y \in \mathcal{C}_{\delta^*}(I) \times \mathcal{C}_{\delta^*}(I)}} e^{-\beta U(\sigma_{A(x)}, \sigma_{A(y)})} \Big] \quad (3.8)$$

where

$$U(\sigma_{A(x)},\sigma_{A(y)}) = -\sum_{i\in A(x),j\in A(y)} \gamma\big[J(\gamma|i-j|) - J(\delta^*|x-y|)\big]\sigma_i\sigma_j \tag{3.9}$$

and

$$I\!E_{m_I^{\delta^*}}[f] \equiv \frac{\sum_{\sigma_{\gamma^{-1}I}} \prod_{x_1\in\mathcal{C}_{\delta^*}(I)} 1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}} e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i} f(\sigma)}{\sum_{\sigma_{\gamma^{-1}I}} \prod_{x_1\in\mathcal{C}_{\delta^*}(I)} 1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}} e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}}. \tag{3.10}$$

Notice, for future reference, that for $m_I^{\delta^*} \in \mathcal{M}_{\delta^*}(I)$ one easily obtains

$$\left| H(\sigma_{\gamma^{-1}I}) + \theta \sum_{i\in\gamma^{-1}I} h_i\sigma_i - \frac{1}{\gamma}E(m_I^{\delta^*}) \right| = \frac{1}{\beta}\left| \log\Big[\prod_{x\in\mathcal{C}_{\delta^*}(I)} \prod_{y\in\mathcal{C}_{\delta^*}(I)} e^{-\beta U(\sigma_{A(x)},\sigma_{A(y)})}\Big] \right| \le |I|\delta^*\gamma^{-1}, \tag{3.11}$$

for $\sigma \in \{\sigma\in\gamma^{-1}I : m^{\delta^*}(x,\sigma) = m^{\delta^*}(x), \forall x\in\mathcal{C}_{\delta^*}(I)\}$. In the following we deal with ratios of quantities (partition functions) of the type (3.2) with boundary conditions that might be different between numerator and denominator. For this reason it is convenient to introduce the following notation, see (3.2),:

$$Z_{\beta,\theta,\gamma,I}\left(m_{\partial-I}^{\delta^*} = m_{s_1}, m_{\partial+I}^{\delta^*} = m_{s_2}\right) \equiv Z_I^{m_{s_1},m_{s_2}} \tag{3.12}$$

where $(m_{s_1},m_{s_2}) \in \{m_-,0,m_+\}^2$ and for $m_{s_1}=0$, we set in (3.3) $E(m_I^{\delta^*}, m_{\partial-I}^{\delta^*}) = 0$ while for $m_{s_2}=0$ we set $E(m_I^{\delta^*}, m_{\partial+I}^{\delta^*}) = 0$. In a similar way, recalling (3.1), if F^{δ^*} is $\Sigma_I^{\delta^*}$–measurable we set

$$Z_I^{m_{s_1},m_{s_2}}(F^{\delta^*}) \equiv \sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} F(m_I^{\delta^*}) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial-I}^{\delta^*}=m_{s_1} m_{\partial+I}^{\delta^*}=m_{s_2}) + \gamma\mathcal{G}(m_I^{\delta^*}) + \gamma V(m_I^{\delta^*})\right\}}. \tag{3.13}$$

Furthermore, we denote by $m_\beta^{\delta^*}$ one of the points in $\left\{-1,-1+\frac{4\gamma}{\delta^*},\dots,1-\frac{4\gamma}{\delta^*},1\right\}^2$ which is closest to m_β. Let $m_{\beta,I}^{\delta^*}$ be the function which coincides with $m_\beta^{\delta^*}$ on I and vanishes outside I and for $\eta\in\{-1,+1\}$,

$$\mathcal{R}^{\delta,\zeta}(\eta,I) \equiv \left\{\eta^{\delta,\zeta}(\ell) = \eta, \forall \ell\in I\right\} \tag{3.14}$$

the set of configurations which are close with accuracy (δ,ζ), see (2.26), to m_β when $\eta=1$ and to Tm_β when $\eta=-1$. If no ambiguity arises, we drop I from the notation in (3.14). By definition, $|m_\beta^{\delta^*} - m_\beta| \le 8\gamma/\delta^*$ and choosing suitably γ and ζ, $m_\beta^{\delta^*} \in \mathcal{R}^{\delta,\zeta}(+1)$ and $Tm_\beta^{\delta^*} \in \mathcal{R}^{\delta,\zeta}(-1)$. The only stochastic contribution relevant to the problem comes from ratios as

$$\frac{Z_I^{0,0}(1\!\!1_{T\mathcal{R}^{\delta,\zeta}(\eta)})}{Z_I^{0,0}(1\!\!1_{\mathcal{R}^{\delta,\zeta}(\eta)})} = \frac{Z_I^{0,0}(1\!\!1_{\mathcal{R}^{\delta,\zeta}(-\eta)})}{Z_I^{0,0}(1\!\!1_{\mathcal{R}^{\delta,\zeta}(\eta)})} \equiv e^{\beta\Delta^\eta\mathcal{G}(m_{\beta,I}^{\delta^*})} \frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))} \tag{3.15}$$

where

$$\Delta^\eta\mathcal{G}(m_{\beta,I}^{\delta^*}) \equiv \eta\left[\mathcal{G}(m_{\beta,I}^{\delta^*}) - \mathcal{G}(Tm_{\beta,I}^{\delta^*})\right] = -\eta\sum_{x\in\mathcal{C}_{\delta^*}(I)} X(x), \tag{3.16}$$

$$X(x) = \mathcal{G}_{x,m_\beta^{\delta^*}}(\lambda(x)) - \mathcal{G}_{x,Tm_\beta^{\delta^*}}(\lambda(x)). \tag{3.17}$$

and

$$\begin{aligned} &\frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))} \\ &\equiv \frac{\sum_{m_I^{\delta^*}\in\mathcal{M}^{\delta^*}(I)} 1\!\!1_{\{\mathcal{R}^{\delta,\zeta}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*},0) + \gamma\Delta_0^{-\eta}\mathcal{G}(m_I^{\delta^*}) + \gamma V(Tm_I^{\delta^*})\right\}}}{\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} 1\!\!1_{\{\mathcal{R}^{\delta,\zeta}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*},0) + \gamma\Delta_0^{\eta}\mathcal{G}(m_I^{\delta^*}) + \gamma V(m_I^{\delta^*})\right\}}}. \end{aligned} \tag{3.18}$$

The first equality in (3.15) follows from $T\mathcal{R}^{\delta,\zeta}(\eta)=\mathcal{R}^{\delta,\zeta}(-\eta)$, see (2.16). We will prove that all the other stochastic contributions appearing in the problem can be considered as error terms. For the remaining term in (3.15) it is sufficient to know that $\log \frac{Z^{0,0}_{I,0}(\mathcal{R}(-\eta))}{Z^{0,0}_{I,0}(\mathcal{R}(\eta))}(h)$ is a symmetric random variable having therefore mean zero and its stochastic contribution is small therefore considered also as part of the error terms. To help the connection with [6], notice that this term was denoted $\frac{Z_{-\eta,0,\delta,\zeta}(I)}{Z_{\eta,0,\delta,\zeta}(I)}$ there. Next, we define for $\alpha\in\mathbb{Z}$ the truncated variables:

$$\chi(\alpha)\equiv\gamma\sum_{x:\delta^* x\in\tilde{A}_{\epsilon/\gamma}(\alpha)} X(x)\mathbb{1}_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}, \tag{3.19}$$

where $\tilde{A}_{\epsilon/\gamma}(\alpha)=[\alpha\frac{\epsilon}{\gamma},(\alpha+1)\frac{\epsilon}{\gamma})$ and $p(x)\equiv p(x,\omega)=|D(x)|/|B^{\lambda(x)}(x)|=2\gamma|D(x)|/\delta^*$. The $\chi(\alpha)$ is a symmetric random variable, see Section 5 of [6],

$$\mathbb{E}[\chi(\alpha)]=0,\quad \mathbb{E}[\chi^2(\alpha)]=\epsilon c(\beta,\theta,\gamma/\delta^*), \tag{3.20}$$

where $c(\beta,\theta,\gamma/\delta^*)$ satisfies

$$(V(\beta,\theta))^2\left[1-(\gamma/\delta^*)^{\frac{1}{5}}\right]^2\le c(\beta,\theta,\gamma/\delta^*)\le(V(\beta,\theta))^2\left[1+(\gamma/\delta^*)^{\frac{1}{5}}\right]^2 \tag{3.21}$$

and $V(\beta,\theta)$ is defined in (2.32). It was proved in [6], Lemma 5.4, that there exists $d_0(\beta,\theta)>0$ such that if $\gamma/\delta^*\le d_0(\beta,\theta)$ then for all $\lambda\in\mathbb{R}$ we have

$$\mathbb{E}\left[e^{\lambda\chi(\alpha)}\right]\le e^{\frac{3\lambda^2}{4}\epsilon V^2(\beta,\theta)}. \tag{3.22}$$

Warning The truncation done in (3.19) is essential to get (3.22). Namely, depending on the values of $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ (which appears in the definition of $X(x)$, see (3.17)) has a behavior that corresponds to the classical Gaussian, Poissonian, or Binomial regimes. It turns out, see Remark 4.11 of [6], that we need accurate estimates only for those values of $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$ for which $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ is in the Gaussian regime. In this regime we obtain a more convenient representation of $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$, see Proposition 3.5 of [6], and therefore of $X(x)$, see formula (4.53) of [6]. This result holds with $\mathbb{P}\ge 1-e^{-\frac{1}{32}\left(\frac{\delta^*}{2\gamma}\right)^{\frac{1}{2}}}$ for all $x\in C_{\delta^*}(I)$, for all I so that

$$\left(\frac{2\gamma}{\delta^*}\right)^{1/2}\log\frac{|I|}{\delta^*}\le\frac{1}{32}. \tag{3.23}$$

The probability estimate can be derived taking in account that $|D(x)|=|\frac{1}{2}\sum_{i\in A(x)}h_i|$, see the end of Section 2.2, and one gets easily

$$\mathbb{P}\left[\sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x)>(2\gamma/\delta^*)^{\frac{1}{4}}\right]\le e^{-\frac{1}{32}\left(\frac{\delta^*}{2\gamma}\right)^{\frac{1}{2}}}. \tag{3.24}$$

In particular, (3.23) holds for intervals I so that $|I|=2\frac{Q}{\gamma}$ or any multiple of this, see Section 5.

4 Finite volume estimates

In this section, we give upper and lower bounds of the infinite volume random Gibbs probability $\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,Q]}(u)\big)$ in term of finite volume quantities, see Proposition 4.1. This is the fundamental ingredient in the proof of Theorem 2.9. Take $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$, see (2.49). We assume, with no loss of generality, that there exists a positive integer L, $L < Q$ such that

$$u(r) = u^*_\gamma(r), \ \ \forall |r| \geq L. \tag{4.1}$$

Denote

$$r_1 = \inf(r : r > -Q, \ \|Du(r) - Du^*_\gamma(r)\|_1 > 0); \quad r_{last} = \sup(r : r < Q, \ \|Du(r) - Du^*_\gamma(r)\|_1 > 0) \tag{4.2}$$

where Du is defined before (2.27), r_i, $i = 1, ..N_1 - 1$ the elements of $\mathcal{N}_{[r_1, r_{last})}(u, u^*_\gamma)$, see (2.44), indexed by increasing order and $r_{N_1} \equiv r_{last}$. According to (2.44), r_i, $i = 2, ..N_1 - 1$ could be a point where u and $u^*_\gamma(\omega)$ make the same jump, hence

$$N_1 \leq N_{[-L,+L]}(u) + N_{[-L,+L]}(u^*_\gamma(\omega)). \tag{4.3}$$

Proposition 4.1 *Given $(\beta, \theta) \in \mathcal{E}$, see (2.20), there exists $\gamma_0(\beta, \theta)$ so that for $0 < \gamma \leq \gamma_0(\beta, \theta)$, choosing the parameters as in Subsection 2.5, Ω_1 as in Theorem 2.4, Ω_3 defined in (4.6), Ω_4 defined in Lemma 4.10 and Ω_5 defined in (4.59) we have the following : On $\Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5)$, with $I\!P[\Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5)] \geq 1 - 4(g(\frac{\delta^*}{\gamma}))^{-\frac{1 \wedge a}{10(2+a)}}$, a as in (2.56), for all $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$ with $F(Q)$ as in (2.51), $0 < b < 1/(8 + 4a)$, we have*

$$\begin{aligned} &\frac{\gamma}{\beta} \log \left[\mu_{\beta,\theta,\gamma} \Big(\mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u) \Big) \right] = \\ &- \mathcal{F}^* \sum_{i=1}^{N_1} \left[\frac{\|Du(r_i)\|_1 - \|Du^*_\gamma(r_i)\|_1}{4\tilde{m}_\beta} \right] + \sum_{i=1}^{N_1} \frac{\tilde{u}(r_i) - \tilde{u}^*_\gamma(r_i)}{2\tilde{m}_\beta} \left[\sum_{\alpha:\, \epsilon\alpha \in [r_i, r_{i+1})} \chi(\alpha) \right] \pm g(\delta^*/\gamma)^{-b}. \end{aligned} \tag{4.4}$$

The (4.4) is an upper bound for $\pm = +1$ and a lower bound for $\pm = -1$.

We split the proof of Proposition 4.1 in three steps. The first step is a reduction to finite volume, see Lemma 4.5. The second step, see Lemma 4.6, is to replace up to some errors ratios of finite volume constrained partition functions by sum of products, see (4.42), estimated in Lemma 4.10 and Lemma 4.11. The last step is to collect all the estimates. To avoid the case that a jump of $u^*_\gamma(\omega)$ occurs at L or $-L$, we require that

$$\{-L\} \cup \{L\} \notin \cup_{i=\kappa^*(-Q)}^{\kappa^*(Q)} [\epsilon\alpha^*_i - 2\rho, \epsilon\alpha^*_i + 2\rho], \tag{4.5}$$

where $\kappa^*(\pm Q)$ are defined in (2.36) and ρ is chosen as in (2.56). Define

$$\Omega_3 \equiv \Omega_3(Q) = \bigcup_{L \in [1,Q] \cap \mathbb{Z}} \left\{ \omega : \{-L\} \cup \{L\} \in \cup_{i=k(-Q)}^{k(Q)} [\epsilon\alpha^*_i - 2\rho, \epsilon\alpha^*_i + 2\rho] \right\}. \tag{4.6}$$

Lemma 4.2 *There exist $\gamma_0(\beta, \theta) > 0$ and $a > 0$ such that for $\gamma \leq \gamma_0 = \gamma_0(\beta, \theta)$ we have*

$$I\!P[\Omega_3] \leq \frac{Q}{(g(\frac{\delta^*}{\gamma}))^{\frac{1 \wedge a}{8(2+a)}}} \leq \frac{1}{(g(\frac{\delta^*}{\gamma}))^{\frac{1 \wedge a}{10(2+a)}}}. \tag{4.7}$$

Proof: We have

$$\Omega_3 \subset \bigcup_{L \in [1,Q] \cap \mathbb{Z}} \{\exists i \in \{\kappa^*(-Q), \ldots, \kappa^*(Q)\}, \quad \epsilon\alpha^*_i \in [L - 2\rho, L + 2\rho] \cup [-L - 2\rho, -L + 2\rho]\}. \tag{4.8}$$

To estimate the probability of the event (4.8), we use Lemma 5.7 where it is proven that uniformly with respect to Q and with $I\!P$–probability larger than $1-(5/g(\delta^*/\gamma))^{\frac{a}{8(2+a)}}$, $\kappa^*(Q)$ and $\kappa^*(-Q)$ are bounded by $K(Q)$ given in (2.31). The other ingredient is the estimate of the probability that $\epsilon\alpha_0^*$ or $\epsilon\alpha_1^* \in [-2\rho, +2\rho]$. This is done in Theorem 5.1 of [6] (see formula 5.29, 5.30 and 6.66 of [6]). Then for some $c(\beta,\theta)$, $a>0$, when $\gamma \le \gamma_0(\beta,\theta)$ we have the following:

$$
\begin{aligned}
&I\!P\left[\exists i \in \{\kappa(^*-Q),\ldots,\kappa^*(Q)\} : \epsilon\alpha_i^* \in [L-2\rho, L+2\rho]\right] \\
&\le 2c(\beta,\theta)K(Q)[g(\delta^*/\gamma)]^{-1/(4(2+a))} + \left(\frac{5}{g(\frac{\delta^*}{\gamma})}\right)^{\frac{a}{8(2+a)}} \le \frac{1}{(g(\frac{\delta^*}{\gamma}))^{\frac{1\wedge a}{8(2+a)}}}.
\end{aligned}
\tag{4.9}
$$

By subadditivity one gets (4.7), recalling that $Q = \exp\left[\log(g(\frac{\delta^*}{\gamma}))/\log\log(g(\frac{\delta^*}{\gamma}))\right]$. ■

Definition 4.3 Partition associated to a couple (u,v) of $BV([a,b])$. *Let u and v be in $BV([a,b])$, ρ and δ chosen according to Definition 2.1. We associate to (u,v) the partition of $[a,b]$ obtained by taking $C(u,v) = C(u) \cup C(v)$ and $B(u,v) = [a,b] \setminus C(u,v)$. The $C(u)$ and $C(v)$ are elements of the partitions in Definition 2.1. We set $C(u,v) = \cup_{i=1}^{\bar N_{[a,b]}} C_i(u,v)$, where $\bar N_{[a,b]} \equiv \bar N(u,v,[a,b])$ is the number of disjoint intervals in $C(u,v)$, $\max\{N_{[a,b]}(u), N_{[a,b]}(v)\} \le \bar N_{[a,b]} \le N_{[a,b]}(u) + N_{[a,b]}(v)$.*

By definition, for $i \ne j$, $C_i(u) \cap C_j(u) = \emptyset$ and $C_i(v) \cap C_j(v) = \emptyset$, however when u and v have jumps at distance less than ρ, $C_i(u) \cap C_j(v) \ne \emptyset$ for some $i \ne j$ and in this case one element of $C(u,v)$ is $C_i(u) \cup C_j(v)$.

Remark 4.4 . The condition that the distance between two successive jumps of u or v is larger than $8\rho + 8\delta$, see Definition 2.1, implies that the distance between any two distinct $C_i(u,v)$ is at least $2\rho + 2\delta$. This means that in a given $C_i(u,v)$ there are at most two jumps, one of u and the other of v.

The partition in Definition 4.3 induces a partition on the rescaled (macro) interval $\frac{1}{\gamma}[a,b] = C_\gamma(u,v) \cup B_\gamma(u,v)$ where $C_\gamma(u,v) = \cup_{i=1}^{\bar N_{[a,b]}} C_{i,\gamma}(u,v)$ and $C_{i,\gamma}(u,v) = \gamma^{-1}C_i(u,v)$. We apply Definition 4.3 to the couple $(u, u^*_\gamma(\omega))$, $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$, in the interval $[a,b] = [r_1, r_{last}]$, see (4.2). To short notation, set

$$
\bar N(u, u^*_\gamma(\omega), [r_1, r_{last}]) \equiv \bar N.
$$

Of course, $N_1 \ge \bar N$, see (4.3). Since the estimates to prove Proposition 4.1 are done in intervals written in macroscale we make the following convention, see (4.1), (4.2):

$$
C_\gamma(u,u^*_\gamma) = \cup_{i=1}^{\bar N}[a_i,b_i), \quad [a_i,b_i) \cap [a_j,b_j) = \emptyset \qquad 1 \le i \ne j \le \bar N, \tag{4.10}
$$

$$
\begin{aligned}
&m(x) = u(\gamma x), \quad m^*(x) = u^*_\gamma(\gamma x) \quad \text{for} \quad x \in \frac{1}{\gamma}[-Q,Q] \\
&q_1 = -\frac{Q}{\gamma}, \quad q_2 = \frac{Q}{\gamma}; \quad v_1 = -\frac{L}{\gamma}, \quad v_2 = \frac{L}{\gamma}; \quad x_i = \frac{r_i}{\gamma}, \quad i = 1,\ldots,N_1 \\
&\mathcal{P}^\rho_{[q_1,q_2]}(m) \equiv \mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u_\gamma).
\end{aligned}
\tag{4.11}
$$

Furthermore, let us define

$$
\eta(\ell,v) = \begin{cases} 0 & \text{if} \quad \ell \in C_\gamma(v); \\ 1 & \text{when } m(x) \text{ equal to } m_\beta \text{ for } x \in B_\gamma(v); \\ -1 & \text{when } m(x) \text{ equal to } Tm_\beta \text{ for } x \in B_\gamma(v). \end{cases}
$$

The following lemma can be proven applying Proposition 4.9 and Theorem 2.4.

Lemma 4.5 (reduction to finite volume) *Under the same hypothesis of Proposition 4.1 and on the probability space $\Omega_1 \setminus \Omega_3$, for ζ_5, *, that satisfies*

$$\delta\zeta_5^3 \geq 384(1+\zeta\frac{\gamma}{\delta^*}+\theta)\frac{1}{\kappa(\beta,\theta)\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\log\frac{\delta^*}{\gamma} \tag{4.12}$$

where $\kappa(\beta,\theta)$ and $\alpha(\beta,\theta,\zeta_0)$ are constant, strictly positive when $\beta,\theta\in\mathcal{E}$ (see (2.21) and formula 6.1 in [6]); we have

$$\begin{aligned}\mu^\omega_{\beta,\theta,\gamma}\Big(\mathcal{P}^\rho_{[q_1,q_2]}(m)\Big) \geq{}& e^{-\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\Big(1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}\Big)\times\\ &\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m^*),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}\end{aligned} \tag{4.13}$$

and

$$\mu^\omega_{\beta,\theta,\gamma}\Big(\mathcal{P}^\rho_{[q_1,q_2]}(m)\Big)\leq 2e^{-\frac{\beta}{\gamma}\left\{L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3\right\}}+e^{\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\sum_{\substack{v_1-L_1-1\leq n'_0\leq v_1\\ v_2\leq n'_{\bar N+1}\leq v_2+L_1+1}}\mathcal{Z}(n'_0,n'_{\bar N+1}) \tag{4.14}$$

where

$$\mathcal{Z}(n'_0,n'_{\bar N+1})\equiv\frac{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^\rho_{[n'_0,n'_{\bar N+1}]}(m),\eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^\rho_{[n'_0,n'_{\bar N+1}]}(m^*),\eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}, \tag{4.15}$$

L_1 in (4.14) satisfies $L_1+\ell_0\leq\rho/\gamma$ and

$$\ell_0=\frac{\log(\delta^*/\gamma)}{\alpha(\beta,\theta,\zeta_0)}. \tag{4.16}$$

The configurations in $\mathcal{P}^\rho_{[v_1,v_2]}(m)$ and $\mathcal{P}^\rho_{[v_1,v_2]}(m^*)$ are long runs of $\eta^{\delta,\zeta}(\ell)\neq 0$ followed by phase changes in the intervals $[a_i,b_i)$, for $i=1,\dots\bar N$, see (4.10). To estimate the ratio of the partition functions in (4.13) and (4.14), one separates the contribution given by those intervals in which the spin configurations undergo to a phase change, *i.e* in which the block spin variables are $\eta^{\delta,\zeta}(\ell)=0$, from those intervals in which the block spin variables are $\eta^{\delta,\zeta}(\ell)\neq 0$, where we use (3.15). We start proving an upper bound for (4.14). We have the following:

Lemma 4.6 *Under the same hypothesis of Proposition 4.1, on the probability space $\Omega_1\setminus(\Omega_3\cup\Omega_4)$, and for ζ_5 as in (4.12), we have*

$$\begin{aligned}\mathcal{Z}(n'_0,n'_{\bar N+1})\leq{}& e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{-L\leq r\leq L}\left[|D\tilde u(r)|-|D\tilde u^*_\gamma(r)|\right]}e^{\frac{\beta}{\gamma}\sum_{i=1}^{\bar N}\frac{\tilde u(r_i)-\tilde u^*_\gamma(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]}\\ &\times e^{\frac{\beta}{\gamma}\bar N\left[4\zeta_5+8\delta^*+\gamma\log\frac{\rho}{\gamma}+\gamma\log L_1+\frac{20V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/4(2+a)}}+32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}}\right]}\\ &+\bar N^2e^{\bar N\log\frac{\rho}{\gamma}}e^{\frac{\beta}{\gamma}(8\delta^*+4\zeta)}e^{-\frac{\beta}{\gamma}L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}.\end{aligned} \tag{4.17}$$

* The same parameter was denoted ζ_5 in [6]. We kept the same notation to facilitate references to [6].

Proof: Recalling (2.28) and (4.10), one sees that in each interval $[a_i, b_i]$, there is a single phase change on a length R_2 for m or m^*. There are three possible cases:
Case 1 $[a_i, b_i] \in C_\gamma(u)$ and $[a_i, b_i] \in B_\gamma(u^*_\gamma)$. Therefore

$$\eta(a_i, m) = -\eta(b_i, m) \neq 0, \qquad \eta(a_i, m^*) = \eta(b_i, m^*) \neq 0. \tag{4.18}$$

Case 2 $[a_i, b_i] \in B_\gamma(u)$ and $[a_i, b_i] \in C_\gamma(u^*_\gamma)$. Therefore

$$\eta(a_i, m) = \eta(b_i, m) \neq 0, \qquad \eta(a_i, m^*) = -\eta(b_i, m^*) \neq 0. \tag{4.19}$$

Case 3 $[a_i, b_i] \in C_\gamma(u)$ and $[a_i, b_i] \in C_\gamma(u^*_\gamma)$. Therefore

$$\eta(a_i, m) = -\eta(b_i, m) \neq 0, \qquad \eta(a_i, m^*) = -\eta(b_i, m^*) \neq 0. \tag{4.20}$$

In the first two cases there exists an unique $x_i \in [a_i, b_i]$, see (4.11), so that, in the the case 1, $|D\tilde{m}(x_i))| > 0$ and in the case 2, $|D\tilde{m}^*(x_i)| > 0$. In the case 3, both m and m^* have one jump in $[a_i, b_i]$. We denote, see Definition 2.2

$$\mathcal{W}_1(\ell_i, m) \equiv \mathcal{W}_1([\ell_i - R_2, \ell_i + R_2], R_2, \zeta) \cap \{\eta^{\delta,\zeta}(\ell_i - R_2) = \eta(a_i, m), \eta^{\delta,\zeta}(\ell_i + R_2) = \eta(b_i, m)\}, \tag{4.21}$$

the set of configurations undergoing to a phase change induced by m in $[\ell_i - R_2, \ell_i + R_2]$. We denote in the cases 1 and 3,

$$\mathcal{P}^\rho_{[a_i,b_i]}(m, \ell_i, i) \equiv \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [a_i, \ell_i - R_2 - 1]) \cap \mathcal{W}_1(\ell_1, m) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, b_i]) \tag{4.22}$$

where $\mathcal{R}^{\delta,\zeta}$ is defined in (3.14), and in the case 2

$$\begin{aligned}&\mathcal{P}^\rho_{[a_i,b_i]}(m, \ell_i, i) \equiv \\ &\mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [a_i, \ell_i - R_2 - 1]) \cap \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [\ell_i - R_2, \ell_i + R_2]) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, b_i]).\end{aligned} \tag{4.23}$$

The set $\mathcal{P}^\rho_{[a_i,b_i]}(m, \ell_i, i)$ denotes the spin configurations which, in the case 1 and 3, have a jump in the interval $[a_i, b_i]$, starting after the point $\ell_i - R_2$ and ending before $\ell_i + R_2$ and close to different equilibrium values in $[a_i, b_i] \setminus [\ell_i - R_2, \ell_i + R_2]$. In the case 2, it denotes the spin configurations which are in all $[a_i, b_i]$ close to one equilibrium value, namely they do not have jumps. The ℓ_i in this last case is written for future use. We use for both m and m^* the notation (4.22) and (4.23). In the case 3 both m and m^* have a jump in $[a_i, b_i]$. Obviously we have

$$\mathcal{P}^\rho_{[a_i,b_i]}(m) \subset \bigcup_{\ell_i \in [a_i + R_2 + 1, b_i - R_2 - 1]} \mathcal{P}^\rho_{[a_i,b_i]}(m, \ell_i, i). \tag{4.24}$$

To prove (4.17), we use the subadditivity of the numerator in (4.15) to treat the $\cup$ in (4.24) obtaining a sum over $\ell_i \in [a_i + R_2 + 1, b_i - R_2 - 1]$. For the denominator we obtain an upper bound simply restricting to the subset of configurations which is suitable for us, namely

$$\mathcal{P}^\rho_{[a_i,b_i]}(m^*) \supset \mathcal{P}^\rho_{[a_i,b_i]}(m^*, \ell_i, i). \tag{4.25}$$

To short notation, let $\{\underline{\ell} \subset [\underline{a}, \underline{b}]\} \equiv \{\ell_i \in [a_i + R_2 + 1, b_i - R_2 - 1], \forall i, 1 \le i \le \bar{N}\}$ and set

$$\mathcal{A}(m, \underline{\ell}) \equiv \Big(\mathcal{P}^\rho_{[n'_0, n'_{\bar{N}+1}]}(m) \cap_{i=1}^{\bar{N}} \mathcal{P}^\rho_{[a_i,b_i]}(m, \ell_i, i),\ \eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m^*),\ \eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m^*)\Big), \tag{4.26}$$

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}) \equiv \frac{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m, \underline{\ell})\Big)}{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m^*, \underline{\ell})\Big)}. \tag{4.27}$$

Therefore, recalling (4.15), we can write

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}) \le \sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}). \tag{4.28}$$

The number of terms in the sum in (4.28) does not exceed $\prod_{i=i}^{\bar{N}} (b_i - a_i) \le \exp(\bar{N} \log(\rho/\gamma))$. For future use, when B is an event let us define

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}; B) \equiv \frac{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m, \underline{\ell}) \cap B\Big)}{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m^*, \underline{\ell})\Big)}. \tag{4.29}$$

For ℓ_0 defined in (4.16), for the very same L_1 to be chosen later, ζ_5 that satisfies (4.12), $\mathcal{O}^{\delta,\zeta_5}$ as in (4.68), $\bar{R}_2 = R_2 + \ell_0$, define

$$\begin{aligned} &\mathcal{D}(m, \underline{\ell}) \equiv \\ &\cup_{1 \le i \le \bar{N}} \left(\mathcal{R}^{\delta,\zeta}(\eta(\ell_i - R_2, m), [\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right) \cup \\ &\cup_{1 \le i \le \bar{N}} \left(\mathcal{R}^{\delta,\zeta}(\eta(\ell_i + R_2, m), [\ell_i + R_2, \ell_i + R_2 + 2\ell_0 + L_1]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])\right). \end{aligned} \tag{4.30}$$

The $\mathcal{D}(m, \underline{\ell})$ is the set of configurations which are simultaneously ζ close and ζ_5 distant, (recall $\zeta > \zeta_5$), from the equilibrium values in the interval $[\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2] \cup [\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1]$ where ℓ_i are chosen as in (4.25). Recalling (4.29) and Proposition 4.9 we have

$$\sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \mathcal{D}(m, \underline{\ell})) \le \bar{N}^2 e^{\bar{N} \log \frac{\rho}{\gamma}} e^{\frac{\beta}{\gamma}(8\delta^* + 4\zeta)} e^{-\frac{\beta}{\gamma} L_1 \frac{\kappa(\beta,\theta)}{8} \delta \zeta_5^3}. \tag{4.31}$$

To get (4.31) one uses a procedure which is standard in the study of Kac's models. We call it *the cutting at point ℓ.* We describe it next, but it is important to stress that the estimate of the error that the cutting produces is harmless only when $\eta^{\delta,\zeta}(\ell) \neq 0$, and in this case it is proportional to ζ. We specialize the description at the point $\ell_i + R_2$ in (4.30). Denote by I the union of the two adjacent blocks of length 1 containing $\ell_i + R_2 - 1$ and $\ell_i + R_2$. Applying (3.11), one can replace the hamiltonian in these two blocks by $(1/\gamma)E(m^{\delta^*})$. This produces an error term $2\delta^*/\gamma$. Next, associate the interaction term between these two blocks only to the block $\ell_i + R_2 - 1$. Since on $\mathcal{D}(m, \underline{\ell})$, at the point $\ell_i + R_2$, $\eta^{\delta,\zeta}(\ell_i + R_2) = \eta(\ell_i + R_2, m) \neq 0$, using that for all σ and all σ' such that $\eta^{\delta,\zeta}(\ell_i + R_2)(\sigma') = \eta(\ell_i + R_2, m)$ we obtain

$$|E(m^{\delta^*}_{[\ell_i + R_2 - 1)}(\sigma), m^{\delta^*}_{[\ell_i + R_2)}(\sigma')) - E(m^{\delta^*}_{[\ell_i + R_2 - 1)}(\sigma), T^{\frac{1 - \eta(\ell_i + R_2, m)}{2}} m^{\delta^*}_{\beta, [\ell_i + R_2)})| \le \zeta, \tag{4.32}$$

where we denoted by $T^1 := T$, the map defined in (2.16) and $T^0 := I$ where I is the identity map from $\mathcal{T}$ to $\mathcal{T}$. The $m^{\delta^*}_{\beta,[\ell_i + R_2)}$ is the profile constantly equals to $m^{\delta^*}_\beta$ on the block indexed by $\ell_i + R_2$, where $m^{\delta^*}_\beta$ is one of the points in $\{-1, -1 + 4\gamma/\delta^*, \ldots, 1 - 4\gamma/\delta^*, 1\}^2$ which is closest to m_β. By (4.32), the interaction between the two blocks is replaced by the one with a constant profile. This produces a boundary condition $T^{\frac{1 - \eta(\ell_i + R_2, m)}{2}} m^{\delta^*}_{\beta,[\ell_i + R_2)}$ for a constrained partition function in a volume that contains the block indexed by $\ell_i + R^2 - 1$ and creates an error term $e^{\frac{\beta}{\gamma}(2\delta^* + \zeta)}$.

Coming back to (4.31), one cuts (in the denominator and in the numerator) at the points $\ell_i + R_2$ and $\ell_i + R_2 + 2\ell_0 + L_1$ for the set $\mathcal{R}^{\delta,\zeta}(\eta(\ell_i + R_2, m), [\ell_i + R_2, \ell_i + R_2 + 2\ell_0 + L_1]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])$, and at the points $\ell_i - R_2 - L_1 - 2\ell_0$ and $\ell_i - R_2$ for the set $\mathcal{R}^{\delta,\zeta}(\eta(\ell_i - R_2, m), [\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])$. Notice that we cut at points where $\eta^{\delta,\zeta} \neq 0$ and we make the error $e^{\frac{\beta}{\gamma}(8\delta^* + 4\zeta)}$. This is the only place where making an error of order ζ does not cause a problem. Namely we can choose L_1 suitable in (4.31) so that $L_1 \frac{\kappa(\beta,\theta)}{16} \delta \zeta_5^3 > (8\delta^* + 4\zeta)$.

Furthermore denote

$$\mathcal{B}(\underline{\ell}) \equiv \cap_{1 \le i \le \bar{N}} \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right)^c \cap \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])\right)^c . \tag{4.33}$$

Since for each $\underline{\ell}$, $\mathcal{A}(m, \underline{\ell}) \cap \mathcal{D}(m, \underline{\ell})^c \subset \mathcal{A}(m, \underline{\ell}) \cap \mathcal{B}(\underline{\ell})$, we are left to estimate

$$\sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \mathcal{B}(\underline{\ell})\right) .$$

On each $\mathcal{A}(m, \underline{\ell}) \cap \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right)^c$, $1 \le i \le \bar{N}$, there exists at least one block, say $[n'_i, n'_i + 1)$ contained in $[\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2)$ with $\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i, m)$. Making the same on the right of ℓ_i and indexing n''_i the corresponding block where $\eta^{\delta,\zeta_5}(n''_i) = \eta(b_i, m)$, one gets

$$\begin{aligned} &\sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}, \mathcal{B}(\underline{\ell})\right) \le \\ &\sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \sum_{\substack{\underline{n}' \subset [\underline{\ell} - \bar{R}_2 - L_1, \underline{\ell} - \bar{R}_2] \\ \underline{n}'' \subset [\underline{\ell} + \bar{R}_2, \underline{\ell} + \bar{R}_2 + L_1]}} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \cap_{1 \le i \le \bar{N}} \{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i, m), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i, m)\}\right) . \end{aligned} \tag{4.34}$$

The number of terms in the second sum of (4.34) does not exceed $\exp(2\bar{N}(\log L_1))$. Consider now a generic term in (4.34),

$$\mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \cap_{1 \le i \le \bar{N}} \{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i, m), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i, m)\}\right) . \tag{4.35}$$

In the denominator of (4.29), we cut at the points $\underline{n}'$ and $\underline{n}''$ to get an upper bound. Each time we cut, we get the error term $e^{\frac{\beta}{\gamma}(2\delta^* + \zeta_5)}$. In the denominator, see (4.27), restrict the configurations to be in

$$\mathcal{A}(m^*, \underline{\ell}) \cap_{1 \le i \le \bar{N}} \left\{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i, m^*), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i, m^*)\right\} \tag{4.36}$$

and then cut at all the points $\underline{n}'$ and $\underline{n}''$. In this way we obtain an upper bound for (4.35). We use notation (4.22) (case 1 and 3) and (4.23) (case 2) after cutting at n'_i and n''_i. Note that $\eta(n'_i + 1) = \eta(a_i, m)$ and $\eta(n''_i - 1) = \eta(b_i, m)$ therefore we have in the case 1 and 3, see (4.22),

$$\begin{aligned} &\mathcal{P}^{\rho}_{[n'_i+1, n''_i-1]}(m, \ell_i, i) = \\ &\quad \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [n'_i + 1, \ell_i - R_2 - 1]) \cap \mathcal{W}_1(\ell_i, m) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, n''_i - 1]), \end{aligned} \tag{4.37}$$

in the case 2, see (4.23),

$$\begin{aligned} \mathcal{P}^{\rho}_{[n'_i+1, n''_i-1]}(m, \ell_i, i) &= \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [n'_i + 1, \ell_i - R_2 - 1]) \cap \\ &\mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [\ell_i - R_2, \ell_i + R_2]) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, n''_i - 1]). \end{aligned} \tag{4.38}$$

For the remaining parts corresponding to runs between two phase changes, *i.e* the intervals $[n''_i, n'_{i+1}]$, $n''_i \in [a_i, b_i]$ and $n'_{i+1} \in [a_{i+1}, b_{i+1}]$, for $i \in \{1, \ldots, \bar{N}\}$, we denote

$$\mathcal{P}^{\rho}_{[n''_i, n'_{i+1}]}(m, \zeta_5) \equiv \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [n''_i + 1, n'_{i+1} - 1]) \cap \{\eta^{\delta,\zeta_5}(n''_i) = \eta^{\delta,\zeta_5}(n'_{i+1}) = \eta(b_i, m)\}. \tag{4.39}$$

Doing similarly in the intervals $[n'_0, n'_1]$, and $[n''_{\bar N}, n'_{\bar N+1}]$, we set

$$\begin{aligned}&\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5) \equiv \mathcal{R}^{\delta,\zeta}(\eta(v_1,m),[n'_0,n'_1]) \cap \{\eta^{\delta,\zeta_5}(n'_0) = \eta^{\delta,\zeta_5}(n'_1) = \eta(v_1,m)\}\\ &= \mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5)\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}&\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5) \equiv \mathcal{R}^{\delta,\zeta}(\eta(v_2,m),[n''_{\bar N},n'_{\bar N+1}]) \cap \{\eta^{\delta,\zeta_5}(n''_{\bar N}) = \eta^{\delta,\zeta_5}(n'_{\bar N+1}) = \eta(v_2,m)\}\\ &= \mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5).\end{aligned} \tag{4.41),$$

We obtain

$$\begin{aligned}&\mathcal{Z}\left(n'_0, n'_{\bar N+1}, \underline{\ell}; \cap_{1\le i\le \bar N}\{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i,m), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i,m)\}\right) \le \\
&e^{+\bar N \frac{\beta}{\gamma}(4\zeta_5+8\delta^*)} \frac{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5)\right)}{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5)\right)} \times \\
&\prod_{i=1}^{\bar N-1}\left(\frac{Z^{m,m}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m^*,\ell_i,i)\right)} \frac{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m^*,\zeta_5)\right)}\right) \times \\
&\frac{Z^{m,m}_{[n'_{\bar N}+1,n''_{\bar N}-1]}\left(\mathcal{P}^{\rho}_{[n'_{\bar N}+1,n''_{\bar N}-1]}(m,\ell_{\bar N},\bar N)\right)}{Z^{m^*,m^*}_{[n'_{\bar N}+1,n''_{\bar N}-1]}\left(\mathcal{P}^{\rho}_{[n'_{\bar N}+1,n''_{\bar N}-1]}(m^*,\ell_{\bar N},\bar N)\right)} \frac{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5)\right)}.\end{aligned} \tag{4.42}$$

Now, the goal is to estimate separately all the ratios in the right hand side of (4.42). It follows from (4.40), and (4.41) that

$$\frac{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5)\right)}{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5)\right)} = \frac{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5)\right)} = 1.$$

The remaining ratios are estimated in Lemma 4.10 and Lemma 4.11 given below.

Collecting We insert the results of Lemma 4.10 and Lemma 4.11 in (4.42). To write in a unifying way the contributions of the jumps we note that for (4.72)

$$-\mathcal{F}^* = -\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{a_i\le s\le b_i}|D\tilde m(s)| = -\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{a_i\le s\le b_i}(|D\tilde m(s)| - |D\tilde m^*(s)|) \tag{4.43}$$

since in the case 1, see (4.18), $\sum_{a_i\le s\le b_i}|D\tilde m^*(s)| = 0$. For (4.73)

$$+\mathcal{F}^* = \frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{a_i\le s\le b_i}|D\tilde m^*(s)| = -\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{a_i\le s\le b_i}(|D\tilde m(s)| - |D\tilde m^*(s)|) \tag{4.44}$$

since in the case 2, see (4.19), $\sum_{a_i\le s\le b_i}|D\tilde m(s)| = 0$. Moreover, since neither $\tilde m$ nor $\tilde m^*$ have jump in $[b_i+1, a_{i+1}]$ for $i \in \{1,\dots,\bar N\}$, in $[v_1, a_1-1]$, and in $[b_{\bar N}+1, v_2]$, one gets simply

$$\prod_{i=1}^{\bar N} e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{a_i\le s\le b_i}[|D\tilde m(s)|-|D\tilde m^*(s)|]} = e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{-L\le r\le L}\left[|D\tilde u(r)|-|D\tilde u^*_\gamma(r)|\right]}. \tag{4.45}$$

Using (4.71), the random term gives a contribution

$$e^{\frac{\beta}{\gamma}\sum_{i=1}^{\bar N}\frac{\tilde u(r_i)-\tilde u^*_\gamma(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]}. \tag{4.46}$$

It remains to collect the error terms, see (4.31), (4.34), (4.42), (4.71), and Lemma 4.11. Denote

$$\mathcal{E}_1 \equiv \bar N\left[4\zeta_5+8\delta^*+\gamma\log\frac{\rho}{\gamma}+\gamma\log L_1+\frac{20V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/4(2+a)}}+32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}}\right], \tag{4.47}$$

$$-\mathcal{A}_2 \equiv 2\frac{\gamma}{\beta}\log\bar N+\frac{\gamma}{\beta}\bar N\log\frac{\rho}{\gamma}+8\delta^*+4\zeta-L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3, \tag{4.48}$$

and

$$\mathcal{A}\equiv\frac{\mathcal{F}^*}{2\tilde m_\beta}\sum_{-L\le r\le L}\left[|D\tilde u(r)|-|D\tilde u^*_\gamma(r)|\right]-\sum_{i=1}^{\bar N}\frac{\tilde u(r_i)-\tilde u^*_\gamma(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]. \tag{4.49}$$

We have proved

$$\mathcal{Z}(n_0',n_{\bar N+1}')\le e^{-\frac{\beta}{\gamma}\mathcal{A}}e^{\frac{\beta}{\gamma}\mathcal{E}_1}+e^{-\frac{\beta}{\gamma}\mathcal{A}_2} \tag{4.50}$$

that entails (4.17). ■

Next we estimate from below the r.h.s. of (4.13).

Lemma 4.7 *Under the same hypothesis of Proposition 4.1 and on the probability space $\Omega_1\setminus(\Omega_3\cup\Omega_4)$, for ζ_5 as in (4.12), we have*

$$\begin{aligned}&\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m^*),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}\\&\ge\Big(e^{\frac{\beta}{\gamma}(\mathcal{A}+\mathcal{E}_1)}+e^{-\frac{\beta}{\gamma}\mathcal{A}_2}\Big)^{-1}\end{aligned} \tag{4.51}$$

where $\mathcal{A}$, $\mathcal{E}_1$, and $\mathcal{A}_2$ are defined in (4.49), (4.47), and (4.48) respectively.

Proof: Obviously one can get the lower bound simply proving an upper bound for the inverse of l.h.s. of (4.51), i.e.

$$\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta([v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}. \tag{4.52}$$

Note that $\eta(v_1-1,m^*)=\eta(v_1-1,m)$ and $\eta(v_2+1,m^*)=\eta(v_2+1,m)$ and in the proof of the upper bound, see Lemma 4.6, we never used that m^* in the denominator is the one given in Theorem 2.4. Then (4.52) is equal to

$$\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta([v_1-1,m),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^\rho_{[v_1,v_2]}(m),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m)\Big)}. \tag{4.53}$$

Then by Lemma 4.6 we obtain (4.51). ■

Proof of Proposition 4.1 To prove (4.4), we use Lemma 4.5, then Lemma 4.7 to get a lower bound and Lemma 4.6 to get an upper bound. For the lower bound we get applying (4.13) and (4.51)

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big) \geq e^{-\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\left(1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3}\right)\left(e^{\frac{\beta}{\gamma}\mathcal{A}}e^{\frac{\beta}{\gamma}\mathcal{E}_1}+e^{-\frac{\beta}{\gamma}\mathcal{A}_2}\right)^{-1}. \quad (4.54)$$

For the upper bound we get

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big) \leq e^{-\frac{\beta}{\gamma}\mathcal{A}}e^{+\frac{\beta}{\gamma}\mathcal{E}_1}+2e^{-\frac{\beta}{\gamma}\mathcal{A}_2} \quad (4.55)$$

where $\mathcal{A}_2$ is defined in (4.48). To get (4.4) from (4.55), one needs $\mathcal{A}_2 > \mathcal{A}$, this will be a consequence of an upper bound on $\mathcal{A}$ and a lower bound on $\mathcal{A}_2$. We start estimating the terms of $\mathcal{A}$. We easily obtain

$$\frac{\mathcal{F}^*}{4\tilde{m}_\beta}\sum_{-L\leq r\leq L}\left[\|Du(r)\|_1-\|Du^*_\gamma(r)\|_1\right] \leq \mathcal{F}^*\left[N_{[-L,L]}(u)+N_{[-Q,Q]}(u^*_\gamma)\right]. \quad (4.56)$$

We use that $N_{[-Q,Q]}(u^*_\gamma) \leq K(Q)$, see (5.65), where $K(Q)$ is given in (2.31). If L is finite for all γ, then $N_{[-L,L]}(u)$ is bounded since $u \in BV_{loc}$. When L diverges as $\gamma \downarrow 0$ from the assumption (2.49) we have that

$$\bar{N} \leq N_{[-L,L]}(u)+N_{[-Q,Q]}(u^*_\gamma) \leq [F(Q)+1]K(Q) \quad (4.57)$$

where $F(Q)$ is given in (2.51). The second term of $\mathcal{A}$ can be estimated as

$$\begin{aligned}\left|\sum_{i=1}^{\bar{N}}\frac{\tilde{u}(r_i)-\tilde{u}^*_\gamma(r_i)}{2\tilde{m}_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]\right| &\leq \bar{N}\max_{\{-\frac{Q}{\epsilon}\leq\alpha_0\leq\frac{Q}{\epsilon}\}}\max_{\{\alpha_0\leq\bar{\alpha}\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=\alpha_0}^{\bar{\alpha}}\chi(\alpha)\right| \\ &\leq 2\bar{N}\max_{\{-\frac{Q}{\epsilon}\leq\bar{\alpha}\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=-\frac{Q}{\epsilon}}^{\bar{\alpha}}\chi(\alpha)\right|.\end{aligned} \quad (4.58)$$

Applying the Levy inequality and the exponential Markov inequality we get, for the last term in (4.58),

$$I\!P\Big[\max_{\{-\frac{Q}{\epsilon}\leq\bar{\alpha}\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=-\frac{Q}{\epsilon}}^{\bar{\alpha}}\chi(\alpha)\right| \geq \sqrt{3}V(\beta,\theta)\sqrt{[2Q+1]\log(g(\frac{\delta^*}{\gamma}))}\Big] \leq 4e^{-\log(g(\frac{\delta^*}{\gamma}))}=\frac{4}{g(\frac{\delta^*}{\gamma})}. \quad (4.59)$$

Denote Ω_5 the probability space for which (4.59) holds. Then for $\omega \in \Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5)$ and γ_0 small enough, one has

$$\mathcal{A} \leq 2[F(Q)+1]K(Q)V(\beta,\theta)\sqrt{(2Q+1)\log(g(\frac{\delta^*}{\gamma}))} \leq \bar{c}(\beta,\theta)F(Q)Q^3 \quad (4.60)$$

for some $\bar{c}(\beta,\theta)$. The last inequality in (4.60) is obtained taking $F(Q)$ as in (2.51), $K(Q)$ as in (2.31) and Q as in (2.59). We have , from (2.59), $Q^2 g(\delta^*/\gamma) \leq g^2(\delta^*/\gamma)$. Notice that L_1 enters in $\mathcal{A}_2$, see (4.48). We make the following choices

$$L_1 = \left(g(\frac{\delta^*}{\gamma})\right)^{19/2}, \quad (4.61)$$

$$\zeta_5 = \frac{1}{2^{18}c^6(\beta,\theta)}\frac{1}{g^3(\delta^*/\gamma)} \quad (4.62)$$

for some constant $c(\beta,\theta)$. The (4.61) satisfies the requirement of Proposition 4.5, *i.e.* $L_1 < \frac{\rho}{\gamma}$, see (2.56). The choice (4.62), already done in in [6], satisfies requirement (4.12) provided ζ is chosen according (2.54). Since $Q = g(\delta^*/\gamma)^{\frac{1}{\log\log g(\delta^*/\gamma)}}$, see (2.59), we have $\log g(\delta^*/\gamma) = (\log Q)(\log\log g(\delta^*/\gamma))$. Iterating this equation, for γ_0 small enough to have $\log\log\log g(\delta^*/\gamma) > 0$, one gets easily

$$\log g(\delta^*/\gamma) \geq (\log Q)(\log\log Q). \tag{4.63}$$

Recalling (2.55) and using (2.51) one can check that

$$L_1 \frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3 \geq c(\beta,\theta)F(Q)Q^3. \tag{4.64}$$

implies

$$L_1 \frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3 > 2\gamma\log\bar{N} + \gamma\bar{N}\log\frac{\rho}{\gamma} + 8\delta^* + 4\zeta. \tag{4.65}$$

Therefore, recalling (4.48), (4.64) entails $\mathcal{A}_2 > \mathcal{A}$ and finally one gets

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big) \leq e^{-\frac{\beta}{\gamma}\mathcal{A}} e^{\frac{\beta}{\gamma}\mathcal{E}_1}\left(1 + 2e^{-\frac{\beta}{\gamma}\left\{L_1\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3\right\}}\right). \tag{4.66}$$

It remains to check that $\mathcal{E}_1 \downarrow 0$. By (2.56), one has $\gamma\log(\rho/\gamma) \leq (g(\delta^*/\gamma))^{-1}$. Recalling (2.58) one has $(R_2+\ell_0)\sqrt{\gamma/\delta^*} \leq (g(\delta^*/\gamma))^{-1}$. By (4.63), (2.51) and $0 < b < 1/(8+4a)$, see Proposition 4.1, one has

$$\mathcal{E}_1 \leq K(Q)(F(Q)+1)\big[\zeta_5 + 32\theta L_1\sqrt{\frac{\gamma}{\delta^*}} + \frac{c(\beta,\theta)}{(g(\delta^*/\gamma))^{1/(8+4a)}}\big] \leq (g(\delta^*/\gamma))^{-b}. \tag{4.67}$$

So one gets the upper bound in (4.4). By (4.54), the corresponding lower bound is easily derived. ■

We state now the estimates used above proved in [6].

Definition 4.8 *For δ and ζ positive, for two integers $p_1 < p_2$ define*

$$\mathcal{O}^{\delta,\zeta}([p_1,p_2]) \equiv \big\{\eta^{\delta,\zeta}(\ell) = 0,\ \forall \ell \in [p_1,p_2]\big\}. \tag{4.68}$$

Using a simple modification of the rather involved proof of Theorem 7.4 in [6] one gets the following.

Proposition 4.9 *There exists $\gamma_0(\beta,\theta)$ and ζ_0 so that for $0 < \gamma \leq \gamma_0(\beta,\theta)$, choosing the parameters as in Subsection 2.5, for all $\omega \in \Omega_1 \setminus \Omega_3$, with Ω_1 in Theorem 2.4 and Ω_3 defined in (4.6), for all $\bar{\eta} \in \{-1,+1\}$, for all $\ell_0 \in I\!N$, for all ζ, ζ_5 with $\zeta_0 > \zeta > \zeta_5 \geq 8\gamma/\delta^*$, for all $[\bar{p}_1,\bar{p}_2] \subset [p_1,p_2] \subset [q_1,q_2]$ with $\bar{p}_1 - p_1 \geq \ell_0$, $p_2 - \bar{p}_2 \geq \ell_0$ we have, see (3.14),*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta},[p_1,p_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\bar{p}_1,\bar{p}_2])\right) \leq e^{-\frac{\beta}{\gamma}\left\{(\bar{p}_2-\bar{p}_1)\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3 - 48(1+\theta)\sqrt{\frac{\gamma}{\delta^*}}\right) - 2\zeta e^{-\alpha(\beta,\theta,\zeta_0)2\ell_0} - 4\ell_0\sqrt{\frac{\gamma}{\delta^*}}\right\}}. \tag{4.69}$$

Here $\alpha(\beta,\theta,\zeta_0)$ is a strictly positive constant for all $(\beta,\theta) \in \mathcal{E}$, $\kappa(\beta,\theta)$ is the same as in (2.21). Moreover

$$\sup_{[p_1,p_2]\subseteq[-\gamma^{-p},\gamma^{-p}]} \frac{Z^{0,0}_{[p_1,p_2]}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta},[p_1,p_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\bar{p}_1,\bar{p}_2])\right)}{Z^{0,0}_{[p_1,p_2]}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta},[p_1,p_2])\right)} \tag{4.70}$$

satisfies the same estimates as (4.69).

Next we state the lemmas used for estimating the different ratios in (4.42).

Lemma 4.10 *Under the same hypothesis of Proposition 4.1 and on the probability space* $\Omega_1 \setminus (\Omega_3 \cup \Omega_4)$ *with* $I\!P(\Omega_4) \le e^{-(\log g(\delta^*/\gamma))\left(1-\frac{1}{\log\log g(\delta^*/\gamma)}\right)}$ *for all* $1 \le i \le \bar N - 1$, *for all* n''_i, n'_i, *see (4.34), we have*

$$\frac{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m^*,\zeta_5)\right)} = \begin{cases} 1 & \text{when } \eta(b_i,m)=\eta(b_i,m^*); \\ e^{\pm\frac{\beta}{\gamma}\frac{V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/(8+4a)}}} e^{\frac{\beta}{\gamma}\frac{\tilde u(r_i)-\tilde u^*(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]} & \\ \text{when } \quad \eta(b_i,m)=-\eta(b_i,m^*). \end{cases} \tag{4.71}$$

where in the last term we have an upper bound for $\pm = +$ *and a lower bound for* $\pm = -$.

Proof: When $\eta(b_i,m)=\eta(b_i,m^*)$ the (4.71) is immediate, see definition (4.39). When $\eta(b_i,m)=-\eta(b_i,m^*)$ the estimate is a consequence of the proof of Lemma 6.3 in [6]. One gets a similar expression as in the right hand side of (4.71) with $\sum_{\alpha:\,\epsilon\alpha\in\gamma[n''_i+1,n'_{i+1}-1]}\chi(\alpha)$ instead of $\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)$. We then apply the Levy inequality to control the uniformity with respect to $1 \le i \le \bar N - 1$ and n''_i, n'_i, and the exponential Markov inequality, similarly to what we did in (4.59). ■

Lemma 4.11 *On* $\Omega_1 \setminus \Omega_3$, *choosing the parameters as in Subsection 2.5, for all* $1 \le i \le \bar N$, *for all* n''_i, n'_i, *see (4.34), in the case 1, we have*

$$\frac{Z^{m,m}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m^*,\ell_i,i)\right)} = e^{-\frac{\beta}{\gamma}(\mathcal{F}^*\pm 32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \tag{4.72}$$

In the case 2, we have

$$\frac{Z^{m,m}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m^*,\ell_i,i)\right)} = e^{+\frac{\beta}{\gamma}(\mathcal{F}^*\pm 32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \tag{4.73}$$

In the case 3, we have

$$\frac{Z^{m,m}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m^*,\ell_i,i)\right)} = e^{\pm\frac{\beta}{\gamma}(64\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \tag{4.74}$$

Proof: The proof of (4.72) and (4.73) follows from Lemma 7.3 in [6] which is highly non trivial. The (4.74) is a consequence of (4.72) and (4.73). ■

Remark 4.12 . Note that here one needs to have $L_1\sqrt{\frac{\gamma}{\delta^*}} \downarrow 0$.

5 Probability estimates and Proof of Theorem 2.4

We divide the section into several subsections. In the first by a direct application of a Donsker invariance principle in the Skorohod space, we prove that the main random contribution identified in (3.16) suitably rescaled, converges in law to a Bilateral Brownian process, see (5.6). In the second subsection we adapt to a random walk the construction done by Neveu-Pitman, [13], to determine the h-extrema for BBM. In Subsection 5.3 we state definitions and main properties of the maximal b–elongations with excess f introduced in [6]. In Subsection 5.4, we identify them with the h–extrema of Neveu–Pitman restricting suitably the probability space. We then prove Theorem 2.4. Here b, f, and h are positive constants which will be specified. In the last subsection we present rough estimates on the number of maximal $b-$elongation with excess f that are within intervals $[0, R]$.

5.1. Convergence to a Bilateral Brownian Motion

Let $\epsilon \equiv \epsilon(\gamma)$, $\lim_{\gamma\to 0}\epsilon(\gamma) = 0$, $\frac{\epsilon}{\gamma^2} > \frac{\delta^*}{\gamma}$, so that each block of length $\frac{\epsilon}{\gamma^2}$ contains at least one block $A(x)$. Define

$$\mathcal{Y}(\alpha) \equiv \begin{cases} \sum_{\tilde\alpha\in[1,\alpha]} \chi(\tilde\alpha), & \text{if} \quad \alpha \ge 1; \\ 0 & \text{if} \quad \alpha = 0; \\ -\sum_{\tilde\alpha\in(\alpha,-1)} \chi(\tilde\alpha), & \text{if} \quad \alpha \le -1 \end{cases} \qquad \alpha \in \mathbb{Z}, \tag{5.1}$$

where $\chi(\alpha)$ is defined in (3.19). Denote by $\{\hat W^\epsilon(t); t \in \mathbb{R}\}$ the following continuous time random walk:

$$\hat W^\epsilon(t) \equiv \frac{1}{\sqrt{c(\beta,\theta,\gamma/\delta^*)}} \mathcal{Y}([\frac{t}{\epsilon}]), \tag{5.2}$$

where $[x]$ is the integer part of x and $c(\beta,\theta,\gamma/\delta^*)$ is estimated in (3.21). Definition (5.2) allows to see $\hat W^\epsilon(\cdot)$ as a trajectory in the space of real functions on the line that are right continuous and have left limit, *i.e* the space $D(\mathbb{R},\mathbb{R})$. We endowed it with the Skorohod topology which makes it separable and complete. Next, we recall the Skorohod distance, see [2] chapter 3 or [7] chapter 3 where the case of $D[0,\infty)$ is considered. Denote Λ_{Lip} the set of strictly increasing Lipschitz continuous function λ mapping $\mathbb{R}$ onto $\mathbb{R}$ such that

$$\|\lambda\| = \sup_{s\neq t} \Big| \log \frac{\lambda(t)-\lambda(s)}{t-s} \Big| < \infty. \tag{5.3}$$

For $v \in D(\mathbb{R},\mathbb{R})$ and $T \ge 0$, define

$$v^T(t) = \begin{cases} v(t\wedge T), & \text{if } t \ge 0; \\ v(t \vee (-T)), & \text{if } t < 0. \end{cases} \tag{5.4}$$

For v and w in $D(\mathbb{R},\mathbb{R})$ denote

$$d(v,w) \equiv \inf_{\lambda\in\Lambda_{\text{Lip}}} \left[\|\lambda\| \vee \int_0^\infty e^{-T} \sup_{t\in\mathbb{R}} (1 \wedge (|v^T(t) - w^T(\lambda(t))|)\, \mathrm{d}T \right]. \tag{5.5}$$

Taking in account that $\chi(\alpha)$ depends on $\epsilon = \epsilon(\gamma)$, one can prove, following step by step the proof of Billingsley [2], pg 137, a Donsker Invariance Principle. As trivial consequence one obtains that for any a and b in $\mathbb{R}$

$$\lim_{\gamma\to 0} \left[\hat W^{\epsilon(\gamma)}(b) - \hat W^{\epsilon(\gamma)}(a) \right] \stackrel{\text{Law}}{=} [W(b) - W(a)], \tag{5.6}$$

where $W(\cdot)$ is the BBM, see Subsection 2.4.

5.2. The Neveu-Pitman, [13], construction of the h–extrema for the random walk $\{\hat{W}^\epsilon\}$

We shortly recall the Neveu-Pitman construction [13], used to determine the h–extrema for the bilateral Brownian Motion $(W_t, t \in I\!R)$. Realize it as the coordinates of the set Ω of real valued functions ω on $I\!R$ which vanishes at the origin. Denote by $(\theta_t, t \in I\!R)$, the flow of translation : $[\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)]$ and by ρ the time reversal $\rho\omega(t) = \omega(-t)$. For $h > 0$, the trajectory ω of the BBM admits an h–minimum at the origin if $W_t(\omega) \geq W_0(\omega) = 0$ for $t \in [-T_h(\rho\omega), T_h(\omega)]$ where $T_h(\omega) = \inf[t : t > 0, W_t(\omega) > h]$, and $-T_h(\rho\omega) = -\inf[t > 0 : W_{-t}(\omega) > h] \equiv \sup[t < 0, W_{-t}(\omega) > h]$. The trajectory ω of the BBM admits an h–minimum (resp. h maximum) at $t_0 \in I\!R$ if $W\, o\, \theta_{t_0}$ (resp. $-W\, o\, \theta_{t_0}$) admits an h minimum at 0.

To define the point process of h–extrema for the BBM, Neveu-Pitman consider first the one sided Brownian motion $(W_t, t \geq 0, W_0 = 0)$, *i.e* the part on the right of the origin of the BBM. Denote its running maximum by

$$M_t = \big(\max(W_s\,;\, 0 \leq s \leq t), t \geq 0\big) \tag{5.7}$$

and define

$$\begin{aligned} &\tau = \min(t\,;\, t \geq 0, M_t - W_t = h), \\ &\beta = M_\tau, \\ &\sigma = \max(s\,;\, 0 \leq s \leq \tau\,,\, W_s = \beta). \end{aligned} \tag{5.8}$$

The stopping time τ is the first time that the Brownian motion achieves a drawdown of size h, see [18,19]. Its Laplace transform is given by $I\!E[\exp(-\lambda\tau)] = (\cosh(h\sqrt{2\lambda}))^{-1}$, $\lambda > 0$. This is consequence of the celebrated Lévy Theorem [10] which states that $(M_t - W_t; 0 \leq t < \infty)$ and $(|W_t|; 0 \leq t < \infty)$ have the same law. Therefore τ has the same law as the first time a reflected Brownian motion reaches h. The Laplace transform of this last one is obtained applying the optional sampling theorem to the martingale $\cosh(\sqrt{2\lambda}|W_t|)\exp(-\lambda t)$.

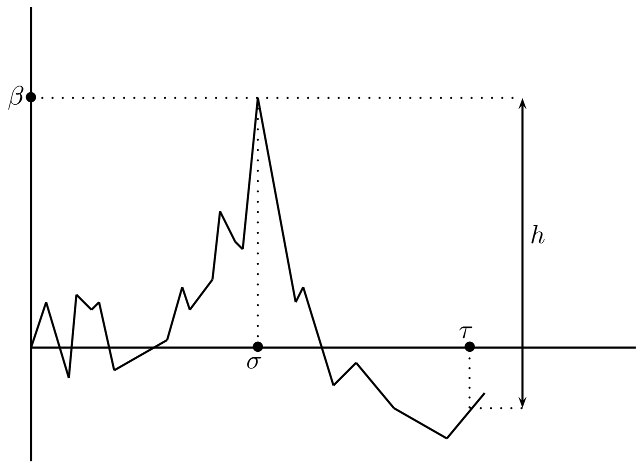

FIG. 1 *Definition of β, σ, τ.*

Further Neveu and Pitman proved that (β, σ) and $\tau - \sigma$ are independent and give the corresponding Laplace transforms. In particular one has

$$I\!E[e^{-\lambda\sigma}] = (h\sqrt{2\lambda})^{-1}\tanh(h\sqrt{2\lambda}). \tag{5.9}$$

Now call $\tau_0 = \tau$, $\beta_0 = \beta$, $\sigma_0 = \sigma$ and define recursively $\tau_n, \beta_n, \sigma_n$ $(n \geq 1)$, so that $(\tau_{n+1} - \tau_n, \beta_{n+1}, \sigma_{n+1} - \tau_n)$ is the (τ, β, σ)–triplet associated to the Brownian motion $\big((-1)^{n-1}(W_{\tau_n + t} - W_{\tau_n}),\, t \geq 0\big)$. By construction, for $n \geq 1$, σ_{2n} is the time of an h-maximum and for $n \geq 0$, σ_{2n+1} is the time of a h-minimum. Note that since we have considered just the part on the right of the origin, in general σ_0 is not an h maximum. The definition only requires $W_t \leq W_{\sigma_0}$ for $t \in [0, \sigma_0)$, therefore $W_{\sigma_0} = W_{\sigma_0} - B_0$ could be smaller than h. The trajectory of the BBM on the left of the origin will determine whether σ_0 is or is not an h–maximum. ¿From the above mentioned fact that (β, σ) and $\tau - \sigma$ are independent, it follows that the variables $\sigma_{n+1} - \sigma_n$ for $n \geq 1$ are independent with Laplace transform $(\cosh(h\sqrt{2\lambda}))^{-1}$. In this way Neveu and Pitman define a renewal process on $\mathbb{R}_+$, with a delay distribution, *i.e.* the one of σ_0, that have Laplace transform (5.9).

Since the times of h-extrema for the BBM depend only on its increments, these times should form a stationary process on $\mathbb{R}$. The above one side construction does not provide stationary on the positive real axis $\mathbb{R}^+$ since the delay distribution is not the one of the limiting distribution of the residual life as it should be, see [1] Theorem 3.1. In fact the Laplace transform of limiting distribution of the residual life is given by (2.40) which is different from (5.9).

There is a standard way to get a stationary renewal process. Pick up an $r_0 > 0$, translate the origin to $-r_0$ and repeat for $(W_{t+r_0}, t > -r_0)$ the above construction. One gets $\sigma_0(r_0)$ and the sequence of point of h-extrema $(\sigma_n(r_0), n \geq 1)$. Let $\nu(r_0) \equiv \inf(n > 0 : \sigma_n(r_0) > 0)$ be the number of renewals up to time 0 (starting at $-r_0$). In this way, $\sigma_{\nu(r_0)}(r_0)$ is the residual life at "time" zero for the Brownian motion starting at $-r_0$. So taking $r_0 \uparrow \infty$, the distribution of $\sigma_{\nu(r_0)}(r_0)$ will converge to the one of the residual life and using [1], Theorem 3.1, one gets a stationary renewal process on $\mathbb{R}^+$. So conditionally on $\sigma_1(r_0) < 0$, define $S_i(r_0) = \sigma_{\nu(r_0)+i-1}(r_0)$ for all $i \geq 1$. Then since the event $\{\sigma_1(r_0) < 0\}$ has a probability that goes to 1 as $r_0 \uparrow \infty$, one gets, as $r_0 \uparrow \infty$, a stationary renewal process on $\mathbb{R}_+$ as well on $\mathbb{R}$. Since the Laplace transform of the inter–arrival time distribution is $(\cosh(h\sqrt{2\lambda}))^{-1}$, one gets easily that the Laplace transform of the distribution of S_1 (and also of S_0) is (2.40).

With this in mind we start the construction for the random walk $\{\hat{W}^\epsilon\}$. Denote $V^\epsilon(t) = \hat{W}^\epsilon(t)\mathbb{1}_{\{t \geq 0\}}$ and $\hat{\mathcal{F}}_t^+$, $t \geq 0$ the associated σ- algebra. Define the rescaled running maximum for $V^\epsilon(t)$, $t \geq 0$

$$\sqrt{\epsilon}\hat{M}(n) = \max_{0 \leq k \leq n} V^\epsilon(k\epsilon). \tag{5.10}$$

The $\sqrt{\epsilon}$ multiplying $\hat{M}(n)$ comes from $\mathbb{E}\left[\left(\frac{1}{\sqrt{\epsilon}}V^\epsilon(k\epsilon)\right)^2\right] = k$, see (3.20). For any $h > 0$, define the $\hat{\mathcal{F}}_t^+$ stopping time

$$\hat{\tau}_0(\epsilon) \equiv \hat{\tau}_0 = \min\{n \geq 0 : \sqrt{\epsilon}\hat{M}(n) - V^\epsilon(n\epsilon) \geq h\}, \tag{5.11}$$

$$\sqrt{\epsilon}\hat{\beta}_0(\epsilon) \equiv \sqrt{\epsilon}\hat{\beta}_0 = \max\{V^\epsilon(k\epsilon) : 1 \leq k \leq \hat{\tau}_0\} \tag{5.12}$$

and

$$\hat{\sigma}_0(\epsilon) \equiv \hat{\sigma}_0 = \max\{k : 1 \leq k \leq \hat{\tau}_0; V^\epsilon(k\epsilon) = \sqrt{\epsilon}\hat{\beta}_0\}. \tag{5.13}$$

By construction

$$\sqrt{\epsilon}\hat{\beta}_0 \equiv \sqrt{\epsilon}\hat{M}(\hat{\tau}_0) = \max_{0 \leq k \leq \hat{\tau}_0} V^\epsilon(k\epsilon) = V^\epsilon(\hat{\sigma}_0\epsilon) \geq V^\epsilon(\hat{\tau}_0\epsilon) + h. \tag{5.14}$$

Since $\hat{\tau}_0$ is a $\hat{\mathcal{F}}_t^+$ stopping time for $(V^\epsilon(t), t \geq 0)$, the translated and reflected motion $(-1)[V^\epsilon(\epsilon\tau_0 + t) - V^\epsilon(\epsilon\tau_0)]$, for $t \geq 0$, is a new random walk independent of $(V^\epsilon(t), 0 \leq t \leq \epsilon\tau_0)$ on which we will iterate the previous construction. It follows from the Donsker invariance principle and the continuous mapping theorem, Theorem 5.2 of [2], that

$$\lim_{\epsilon \downarrow 0}\left[\epsilon\hat{\tau}_i(\epsilon), \sqrt{\epsilon}\hat{\beta}_i(\epsilon), \epsilon\hat{\sigma}_i(\epsilon), i \geq 0, i \in \mathbb{N}\right] \overset{\text{Law}}{=} [\tau_i, \beta_i, \sigma_i, i \geq 0, i \in \mathbb{N}], \tag{5.15}$$

where $\tau_i, \beta_i, \sigma_i, i \geq 0, i \in I\!N$ are the quantities defined by Neveu Pitman, see [13], for a Brownian motion. By construction the random walk satisfies the following :

Property (5.A) In the interval $[\hat{\sigma}_{2i}, \hat{\sigma}_{2i+1}]$, $i \geq 0$, we have

$$V^{\epsilon}(\hat{\sigma}_{2i+1}\epsilon) - V^{\epsilon}(\hat{\sigma}_{2i}\epsilon) \leq -h, \qquad V^{\epsilon}(k\epsilon) - V^{\epsilon}(k'\epsilon) < h \qquad \forall k' < k \in [\hat{\sigma}_{2i}, \hat{\sigma}_{2i+1}], \tag{5.16}$$

$$V^{\epsilon}(\hat{\sigma}_{2i+1}\epsilon) \leq V^{\epsilon}(k\epsilon) \leq V^{\epsilon}(\hat{\sigma}_{2i}\epsilon) \qquad \hat{\sigma}_{2i} < k < \hat{\sigma}_{2i+1}. \tag{5.17}$$

Property (5.B) In the interval $[\hat{\sigma}_{2i-1}, \hat{\sigma}_{2i}]$, $i \geq 1$, we have

$$V^{\epsilon}(\hat{\sigma}_{2i}\epsilon) - V^{\epsilon}(\hat{\sigma}_{2i-1}\epsilon) \geq h, \qquad V^{\epsilon}(k\epsilon) - V^{\epsilon}(k'\epsilon) > -h \qquad \forall k' < k \in [\hat{\sigma}_{2i-1}, \hat{\sigma}_{2i}], \tag{5.18}$$

$$V^{\epsilon}(\hat{\sigma}_{2i-1}\epsilon) \leq V^{\epsilon}(k\epsilon) \leq V^{\epsilon}(\hat{\sigma}_{2i}\epsilon) \qquad \hat{\sigma}_{2i-1} < k < \hat{\sigma}_{2i}. \tag{5.19}$$

Following the Neveu–Pitman construction, we set $V^{\epsilon}_{r_0}(s) = V^{\epsilon}(s+r_0)$, $s \geq -r_0$, $r_0 = r_0(\gamma)$ positive (diverging when $\gamma \downarrow 0$) and repeat the previous construction. We denote by $(\hat{\sigma}_i(r_0) = \hat{\sigma}_i(\epsilon, r_0), i \geq 1, i \in I\!N)$ the points of h–extrema for $V^{\epsilon}_{r_0}(\cdot)$.

5.3. The maximal b elongations with excess f as defined in [6]

In this subsection we recall the definition of the maximal elongations from [6]. We extract it from the first 5 pages of Section 5 of [6], with different conventions that will be pointed out.

Definition 5.1 (The maximal b–elongations with excess f). *Given $b > f$ positive real numbers, the $\mathcal{Y}(\alpha)$, $\alpha \in Z\!\!\!Z$, have maximal b–elongations with excess f if there exists an increasing sequence $\{\alpha^*_i, i \in Z\!\!\!Z\}$ such that in each of the intervals $[\alpha^*_i, \alpha^*_{i+1}]$ we have either (1) or (2) below:*

(1) In the interval $[\alpha^*_i, \alpha^*_{i+1}]$ (negative maximal elongation):

$$\mathcal{Y}(\alpha^*_{i+1}) - \mathcal{Y}(\alpha^*_i) \leq -b - f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) < b - f, \quad \forall x < y \in [\alpha^*_i, \alpha^*_{i+1}]; \tag{5.20}$$

$$\mathcal{Y}(\alpha^*_{i+1}) \leq \mathcal{Y}(\alpha) \leq \mathcal{Y}(\alpha^*_i), \quad \alpha^*_i \leq \alpha \leq \alpha^*_{i+1}. \tag{5.21}$$

(2) In the interval $[\alpha^*_i, \alpha^*_{i+1}]$ (positive maximal elongation):

$$\mathcal{Y}(\alpha^*_{i+1}) - \mathcal{Y}(\alpha^*_i) \geq b + f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) > -b + f, \quad \forall x < y \in [\alpha^*_i, \alpha^*_{i+1}]; \tag{5.22}$$

$$\mathcal{Y}(\alpha^*_i) \leq \mathcal{Y}(\alpha) \leq \mathcal{Y}(\alpha^*_{i+1}), \quad \alpha^*_i \leq \alpha \leq \alpha^*_{i+1}. \tag{5.23}$$

Moreover, if in the interval $[\alpha^*_i, \alpha^*_{i+1}]$ we have (5.20) and (5.21) (resp. (5.22) and (5.23)) then in the adjacent interval $[\alpha^*_{i+1}, \alpha^*_{i+2}]$ we have (5.22) and (5.23) (resp. (5.20) and (5.21)). Furthermore we make the convention

$$\alpha^*_0 \leq 0 < \alpha^*_1. \tag{5.24}$$

Remark 5.2 . In [6] the convention $\alpha^*_{-1} \leq 0 < \alpha^*_0$ was assumed.

We say that the interval $[\alpha^*_i, \alpha^*_{i+1}]$ gives rise to a *negative maximal* b elongation with excess f in the first case and to a *positive maximal* b elongation with excess f in the second case.

Note that $\alpha^*_i \equiv \alpha^*_i(\gamma, \epsilon, b, f, \omega, 0)$, where 0 is to recall that $\mathcal{Y}(0) = 0$. We will write explicitly the dependence on one, some or all the parameters only when needed. Since the α^*_i are points of local extrema, see (5.21) and (5.23), for a given realization of the random walk, various sequences $\{\alpha^*_i, i \in Z\!\!\!Z\}$ could have the properties listed above. This because a random walk can have locally and globally multiple maximizers or minimizers. In [6], we have chosen to take the first minimum time or the first maximum time instead of the last one. However we could have taken the last minimum time or the last maximum time without any substantial change. From now on, we make this last choice. With this choice and the convention (5.24) the points α_i^* are unambiguously defined.

In [6] we determined the maximal b– elongation with excess f, $[\alpha_0^*, \alpha_1^*]$ containing the origin and estimated the $I\!P$-probability of the occurrence of $[\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon]$ taking care of ambiguities mentioned above. Applying 5.8, 5.9 and Corollary 5.2 of [6], choosing δ^*, Q and ϵ as in Subsection 2.5, $b = 2\mathcal{F}^*$, and see (5.30) in [6], $f = 5/g(\delta^*/\gamma)$, we proved that, for $a > 0$,

$$I\!P\left[([\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon])^c\right] \le \epsilon^{\frac{a}{32(2+a)}}. \tag{5.25}$$

Here we have a slightly different point of view. We want to construct *all* the maximal b–elongations with excess f that are within $[-Q/\epsilon, Q/\epsilon]$. Denote $\Omega_L^+(Q, f, b, 0)$ (resp.$\Omega_L^-(Q, f, b, 0)$) the event that on $[Q/\epsilon, (Q+L)/\epsilon]$ (resp. $[(Q-L)/\epsilon, Q/\epsilon]$) there are two disjoint random intervals so that (5.20) holds in the first interval and (5.22) in the second, or (5.22) holds in the first interval and (5.20) in the second for the process $\mathcal{Y}(\cdot)$. The 0 in the argument of $\Omega_L(\cdot)$ is to recall that $\mathcal{Y}(0) = 0$. After a moment of reflection, one realizes that the occurrence of the event $\Omega_L^+(Q, f, b, 0) \cap \Omega_L^-(Q, f, b, 0)$ should allow to construct all the maximal b–elongations with excess f that are within $[-Q/\epsilon, Q/\epsilon]$. Moreover, for reasons which will be clear soon, we want also to construct the process (5.1) for $\alpha \in [-4Q/\epsilon, 8Q/\epsilon]$. This can be done on a probability subset Ω_p, $I\!P(\Omega_p) \ge 1 - e^{-\frac{1}{32}\left(\frac{\delta^*}{2\gamma}\right)^{\frac{1}{2}}}$, provided $|I| = 12\frac{Q}{\gamma}$ (macroscale) satisfies (3.23), a condition satisfied by the choice done of the parameters, see (2.59). Denote

$$\Omega_L([-Q, Q], f, b, 0) \equiv \Omega_L^-(Q, f, b, 0) \cap \{[\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon]\} \cap \Omega_L^+(Q, f, b, 0) \cap \Omega_p. \tag{5.26}$$

Applying Lemma 5.9 of [6], setting $L = cte \log(Q^2 g(1/\gamma))$ one gets the estimate:

$$I\!P[\Omega_L([-Q, Q], f, b, 0)] \ge 1 - 12\epsilon^{\frac{a}{32(2+a)}}. \tag{5.27}$$

On $\Omega_L([-Q, Q], f, b, 0) \cap \Omega_{\rm urt}$ where $\Omega_{\rm urt}$ is defined in Lemma 5.3, the $\kappa^*(\pm Q)$ defined in (2.36) can be estimated as in (5.65) and

$$-\frac{Q}{\epsilon} < \alpha^*_{\kappa^*(-Q)+1} \le \ldots \le \alpha_0^* < 0 < \alpha_1^* \le \ldots \alpha^*_{\kappa^*(Q)-1} < \frac{Q}{\epsilon}. \tag{5.28}$$

Set $r_0 = 4Q$ and consider $\mathcal{Y}_{r_0}(\cdot) \equiv \mathcal{Y}(\cdot + [\frac{r_0}{\epsilon}])$. Similarly to what done before, we construct in the interval $[-Q/\epsilon, +Q/\epsilon]$ the maximal b–elongations with excess f for the process $(\mathcal{Y}_{r_0}(\alpha), \alpha \in Z\!\!\!Z)$. Note that $\mathcal{Y}_{r_0}(0)$ is not necessarily equal to zero, but the construction of maximal b–elongations with excess f given in [6], depends only on the increments of $\mathcal{Y}_{r_0}(\cdot)$. We can define, as in (5.26), a probability space on which we have $0 \in [\alpha_0^*(r_0), \alpha_1^*(r_0)] \subset [-Q/\epsilon, +Q/\epsilon]$. As before we can determine $\Omega_L^+(Q, f, b, r_0)$ ($\Omega_L^-(Q, f, b, r_0)$) for the process $\mathcal{Y}_{r_0}(\cdot)$ and hence the space $\Omega_L([-Q, Q], f, b, r_0)$. Similarly to (5.28) we have on $\Omega_L([-Q, Q], f, b, r_0) \cap \Omega_{\rm urt}$

$$-\frac{Q}{\epsilon} < \alpha^*_{\kappa^*(-Q,r_0)}(r_0) + 1 \le \ldots \le \alpha_0^*(r_0) < 0 < \alpha_1^*(r_0) \le \ldots \alpha^*_{\kappa^*(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon} \tag{5.29}$$

where

$$\kappa^*(-Q, r_0) = \sup(i \le 0 \,:\, \epsilon\alpha_i^*(r_0) < -Q) > -\infty \tag{5.30}$$

and

$$\kappa^*(Q, r_0) = \inf(i \ge 1 \,:\, \epsilon\alpha_i^*(r_0) > Q) < \infty. \tag{5.31}$$

By translational invariance and (5.27), we have

$$\mathbb{P}[\Omega_L(Q,f,b,r_0)] = \mathbb{P}[\Omega_L(Q,f,b,0)] \geq 1 - 4\epsilon^{\frac{a}{32(2+a)}}, \tag{5.32}$$

and therefore

$$\begin{aligned}&(\alpha_i^*(r_0),\, \forall i \in \mathbb{Z} \,:\, \kappa^*(-Q,r_0) < i < \kappa^*(Q,r_0)) \text{ on } \Omega_L([-Q,+Q],f,b,r_0) \cap \Omega_{\text{urt}} \\ &\stackrel{\text{Law}}{=} (\alpha_i^*,\, \forall i \in \mathbb{Z} \,:\, \kappa^*(-Q) < i < \kappa^*(Q)) \text{ on } \Omega_L([-Q,+Q],f,b,0) \cap \Omega_{\text{urt}}.\end{aligned} \tag{5.33}$$

Here X on $\Omega_1 =^{\text{Law}} Y$ on Ω_2 means that the respective conditional distributions are the same.

5.4. Relation between h–extrema and maximal b–elongation with excess f

Consider the h–extrema for the random walk $V^\epsilon_{r_0}$ defined at the end of Subsection 5.2. Define

$$\hat{\kappa}(-Q,r_0) = \sup\,(i \geq 1 : \epsilon\hat{\sigma}_i(r_0) < -Q)\,; \tag{5.34}$$

$$\hat{\nu}(r_0) = \inf\,(i \geq \hat{\kappa}(-Q,r_0) : \epsilon\hat{\sigma}_i(r_0) > 0)\,; \tag{5.35}$$

$$\hat{\kappa}(Q,r_0) = \inf\,(i \geq \hat{\nu}(r_0) : \epsilon\hat{\sigma}_i(r_0) > Q)\,. \tag{5.36}$$

On $\{\hat{\kappa}(Q,r_0) < \infty\}$ there are $\hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) + 1$ points of h–extrema within $[-Q,+Q]$. So let

$$\Omega_0(Q,r_0) \equiv \{\omega \in \Omega,\, \hat{\kappa}(-Q,r_0) < \nu(r_0) < \hat{\kappa}(Q,r_0) < \infty, \hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) \geq 1\} \tag{5.37}$$

be the set of realizations such that there exists at least one interval $[\epsilon\hat{\sigma}_i(r_0), \epsilon\hat{\sigma}_{i+1}(r_0)] \subset [-Q,Q]$, for some $i \in \mathbb{Z}$ with $\hat{\sigma}_i(r_0)$ and $\hat{\sigma}_{i+1}(r_0)$ that are h–extrema of $V^\epsilon_{r_0}(\cdot)$. On $\Omega_0(Q,r_0)$ we have

$$-\frac{Q}{\epsilon} < \hat{\sigma}_{\hat{\kappa}(-Q,r_0)+1}(r_0) < \dots < \hat{\sigma}_{\hat{\nu}(r_0)-1}(r_0) < 0 < \hat{\sigma}_{\hat{\nu}(r_0)}(r_0) < \dots < \hat{\sigma}_{\hat{\kappa}(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon}. \tag{5.38}$$

Note that $\Omega_0(Q,r_0) \supset \Omega_L([-Q,+Q],b,f,r_0) \cap \{\hat{\kappa}(Q,r_0) < \infty\}$. Namely, if $[\epsilon\alpha_i^*(f,r_0), \epsilon\alpha_{i+1}^*(f,r_0)) \subset [-Q,Q]$ gives rise to a maximal b–elongation with excess f for $\mathcal{Y}_{r_0}$, then for the same process it gives rise to a maximal b–elongation with excess $f=0$, see Definition 5.1. Therefore $\epsilon\alpha_i^*(f,r_0) = \epsilon\alpha_i^*(0,r_0)$ and $\epsilon\alpha_{i+1}^*(f,r_0) = \epsilon\alpha_{i+1}^*(0,r_0)$. Furthermore, since $\mathcal{Y}_{r_0}(\alpha) = \sqrt{c(\beta,\theta,\gamma/\delta^*)}\hat{V}^\epsilon_{r_0}(\alpha\epsilon)$, for $\alpha\epsilon \geq -r_0$, $\epsilon\alpha_i^*(0,r_0)$ and $\epsilon\alpha_{i+1}^*(0,r_0)$ are points of $h = b/\sqrt{c(\beta,\theta,\delta^*/\gamma)}$–extrema for $\hat{V}^\epsilon_{r_0}$, see Property (5.A) and (5.B).

Next, we show that the probability to have points which are $h-$extrema but do not give rise to maximal b-elongations with excess f is small.

Lemma 5.3 *Set $b = 2\mathcal{F}^*$, $h = \frac{2\mathcal{F}^*}{\sqrt{c(\beta,\theta,\delta^*/\gamma)}}$, all the remaining parameters as in Subsection 2.5, $L = cte\log(Q^2 g(\frac{\delta^*}{\gamma}))$ and $f = \frac{5}{g(\frac{\delta^*}{\gamma})}$. Set*

$$\Omega(f,r_0) = \Omega_L([-Q,+Q],b,f,r_0) \cap \{\hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) > \kappa^*(Q,r_0) - \kappa^*(-Q,r_0)\} \cap \{\hat{\kappa}(Q,r_0) < \infty\}. \tag{5.39}$$

We have

$$\mathbb{P}[\Omega(f,r_0)] \leq 200\epsilon^{\frac{a}{32(2+a)}}. \tag{5.40}$$

where a is given in (2.56).

Proof: Denote

$$\Omega' = \Big\{\omega : -\frac{Q}{\epsilon} < \hat\sigma_{\hat\kappa(-Q,r_0)+1}(r_0) < \ldots < \hat\sigma_{\hat\nu(r_0)-1}(r_0) < 0 < \hat\sigma_{\nu(r_0)}(r_0) < \ldots < \hat\sigma_{\hat\kappa(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon};\ \exists i,\ \hat\kappa(-Q,r_0)+1 \le i \le \hat\kappa(Q,r_0)-2 \text{ such that } [\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)) \text{ does not satisfy (1) and (2) of Definition 5.1 but does satisfy (5.16) and (5.17) or (5.18) and (5.19)} \Big\} \cap \{\hat\kappa(Q,r_0) < \infty\}. \tag{5.41}$$

Note that

$$\Omega(f, r_0) \subset \Omega' \cap \Omega_L(Q, f, b, r_0). \tag{5.42}$$

To estimate the $I\!\!P$–probability of the event in the right hand side of (5.42), let i, $\hat\kappa(-Q,r_0)+1 \le i \le \hat\kappa(Q,r_0)-2$ be such that $[\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)]$ does not satisfy (1) and (2) of Definition 5.1 but does satisfy (5.16) and (5.17) or (5.18) and (5.19). It is enough to consider the case where $[\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)]$ does not satisfy (1) of Definition 5.1 but does satisfy (5.16) and (5.17). There are two cases:

- first case

$$-b-f \le \mathcal{Y}(\hat\sigma_{i+1}(r_0)) - \mathcal{Y}(\hat\sigma_i(r_0)) \le -b, \quad \mathcal{Y}(y) - \mathcal{Y}(x) \le b-f \quad \forall x, y : x < y \in [\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)] \tag{5.43}$$

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) < \mathcal{Y}(\alpha) \le \mathcal{Y}(\hat\sigma_i(r_0)) \qquad \forall \alpha : \hat\sigma_i(r_0) < \alpha \le \hat\sigma_{i+1}(r_0) \tag{5.44}$$

- second case

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) - \mathcal{Y}(\hat\sigma_i(r_0)) \le -b-f,\ \exists x_0, y_0,\ x_0 < y_0 \in [\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)] :\ b \ge \mathcal{Y}(y_0) - \mathcal{Y}(x_0) \ge b-f, \tag{5.45}$$

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) < \mathcal{Y}(\alpha) \le \mathcal{Y}(\hat\sigma_i(r_0)) \qquad \hat\sigma_i(r_0) < \alpha \le \hat\sigma_{i+1}(r_0). \tag{5.46}$$

Let us denote

$$\mathcal{Y}^*(\underline{\alpha}, \alpha_1, \alpha_2) \equiv \max_{\alpha_1 \le \tilde\alpha \le \alpha_2} \sum_{\alpha=\underline{\alpha}}^{\tilde\alpha} \chi(\alpha) \tag{5.47}$$

and

$$\mathcal{Y}_*(\underline{\alpha}, \alpha_1, \alpha_2) \equiv \min_{\alpha_1 \le \tilde\alpha \le \alpha_2} \sum_{\alpha=\underline{\alpha}}^{\tilde\alpha} \chi(\alpha) \tag{5.48}$$

where $\epsilon\underline{\alpha} = -4Q$. To get the estimates for both the cases we follow an argument already used in the proof of Theorem 5.1 in [6]. Take $\rho' = (9f)^{1/(2+a)}$, for some $a > 0$. Divide the interval $[-Q, Q]$ into blocks of length ρ' and consider the event

$$\tilde{\mathcal{D}}(Q, \rho', \epsilon) \equiv \Big\{\exists \ell, \ell',\ -Q/\rho' \le \ell < \ell' \le (Q-1)/\rho';\ |\mathcal{Y}^*(\underline{\alpha}, \tfrac{\rho'\ell}{\epsilon}, \tfrac{\rho'(\ell+1)}{\epsilon}) - \mathcal{Y}_*(\underline{\alpha}, \tfrac{\rho'\ell'}{\epsilon}, \tfrac{\rho'(\ell'+1)}{\epsilon}) - b| \le 9f\Big\}.$$

Simple observations show that those ω that belong to $\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\}$ and are such that there exists i, $\hat\kappa(-Q,r_0)+1 \le i \le \hat\kappa(Q,r_0)-2$ such that (5.43) and (5.44) hold, belong also to $\tilde{\mathcal{D}}(Q, \rho', \epsilon)$.

For the second case, we can assume that x_0 is a local minimum and y_0 a local maximum, therefore those ω that belong to $\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\}$ and are such that there exists i, $\hat\kappa(-Q,r_0)+1 \le i \le \hat\kappa(Q,r_0)-2$ such that (5.45) and (5.46) hold, belong also to $\tilde{\mathcal{D}}(Q, \rho', \epsilon)$. Therefore we obtain that

$$\Omega' \cap \{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\} \subset \tilde{\mathcal{D}}(Q, \rho', \epsilon).$$

The estimate of $I\!P\left[\tilde{\mathcal{D}}(Q,\rho',\epsilon)\cap\Omega_L(Q,f,b,r_0)\right]$ is done in [6] where a similar set $\tilde{\mathcal{D}}(Q,\rho',\epsilon)$, see pag 834 there, was considered. It is based on Lemma 5.11 and Lemma 5.12 of [6]. Here we recall the final estimate

$$
\begin{aligned}
&I\!P\left[\tilde{\mathcal{D}}(Q,\rho',\epsilon)\cap\Omega_L(Q,f,b,r_0)\right]\le\\
&8(2(Q+L)+1)^2\frac{2\sqrt{2\pi}}{V(\beta,\theta)}(9f)^{a/(2+a)}+(2(Q+L)+1)\frac{1296}{V(\beta,\theta)}\frac{9f+(2+V(\beta,\theta))\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{(9f)^{3/(4+2a)}} \qquad (5.49)\\
&\quad+\frac{4(Q+L)}{\epsilon}e^{-\frac{f}{4\epsilon V^2(\beta,\theta)}}.
\end{aligned}
$$

Furthermore, by Chebyshev inequality, we obtain that

$$
I\!P\left[\{\max_{\alpha\in[-Q/\epsilon,Q/\epsilon]}|\chi(\alpha)|\ge f\}\right]\le\frac{I\!E\left[\{\max_{\alpha\in[-Q/\epsilon,Q/\epsilon]}|\chi(\alpha)|\}\right]}{f}\le 2\left(\epsilon V_+^2\log\{\frac{2Q}{\epsilon}\}\right)^{\frac{1}{2}}(1+\frac{1}{\log\{\frac{2Q}{\epsilon}\}}).
$$

For the last inequality, see formula 5.38 in [6]. Choosing the parameters as in Subsection 2.5 we obtain the thesis. ■

Denote $\tilde{\Omega}_L(Q,f,b,r_0)\equiv\Omega_{\rm urt}\cap\Omega'_{\rm urt}\cap\Omega_L(Q,f,b,r_0)\setminus\Omega(f,r_0)$, where $\Omega_{\rm urt}$ is defined in Lemma 5.7 and $\Omega'_{\rm urt}$ in Lemma 5.8. Obviously $\{\hat\kappa(Q,r_0)<\infty\}\supset\Omega'_{\rm urt}$ and (5.29) and (5.38) hold: a point is a beginning or an ending of an interval of maximal b–elongations with excess $f>0$ for $\mathcal{Y}_{r_0}$ if and only if it is a point of h–extremum for $V^\epsilon_{r_0}$ and $b=h\sqrt{c(\beta,\theta,\frac{\gamma}{\delta^*})}$. Relabel the variables $\hat\sigma_i(r_0)$ in (5.38) as in Neveu and Pitman, that is define

$$
\hat S_i(r_0)=\hat\sigma_{\hat\nu(r_0)+i-1}(r_0),\ \forall i\in Z\!\!\!Z\ :\ \hat\kappa(-Q,r_0)\le\hat\nu(r_0)+i-1<\hat\kappa(Q,r_0). \qquad (5.50)
$$

Therefore, on $\tilde{\Omega}_L(Q,f,b,r_0)$, we have

$$
\hat S_i(r_0)=\alpha^*_i(r_0),\ \forall i\in Z\!\!\!Z\ :\ -\frac{Q}{\epsilon}\le\hat S_i(r_0)\le\frac{Q}{\epsilon}. \qquad (5.51)
$$

Lemma 5.4 *Take $b=2\mathcal{F}^*$, $h=\frac{2\mathcal{F}^*}{V(\beta,\theta)}$, all the remaining parameters as in Subsection 2.5, $f=\frac{5}{g(\frac{\delta^*}{\gamma})}$ and $L=cte\log(Q^2g(\frac{\delta^*}{\gamma}))$. Let $\Omega_L(Q,f,b,0)$ be the probability space defined in (5.26) and $\Omega_{\rm urt}$ defined in lemma 5.7 with $I\!P[\Omega_L(Q,f,b,0)\cap\Omega_{\rm urt}]\ge 1-200\epsilon^{\frac{a}{32(2+a)}}$ for some $a>0$. Let*

$$
-\frac{Q}{\epsilon}<\alpha^*_{\kappa^*(-Q)+1}<....<\alpha^*_{-1}<\alpha^*_0<0<\alpha^*_1<...<\alpha^*_{\kappa^*(Q)-1}<\frac{Q}{\epsilon}
$$

be the maximal b–elongations with excess f, see (5.28), and $\{S_i,i\in Z\!\!\!Z\}$ the point process of h-extrema of the BBM defined in Neveu-Pitman [13]. We have

$$
\lim_{\gamma\to 0}\epsilon(\gamma)\alpha^*_i(\epsilon(\gamma),f(\gamma))\overset{\rm Law}{=}S_i\qquad i\in Z\!\!\!Z. \qquad (5.52)
$$

Proof: This is an immediate consequence of (5.33), Lemma 5.3, (5.51), (3.21) and the continuous mapping theorem, Theorem 5.2 of [2]. ■

5.5. Probability estimates

Lemma 5.6, stated below, gives lower and upper bound on the $\alpha_i^*(b,f,0)$, $i \in \mathbb{Z}$, in term of suitable stopping times. This is a device constantly used in [6] even if it was not formulated in its whole generality. We set $\hat{T}_0 = 0$, and define, for $k \geq 1$:

$$\begin{aligned} \hat{T}_k &= \inf\{t > \hat{T}_{k-1}: |\sum_{\alpha=\hat{T}_{k-1}+1}^{t} \chi(\alpha)| \geq \frac{b+f}{2}\}, \\ \hat{T}_{-k} &= \sup\{t < \hat{T}_{-(k-1)}: |\sum_{\alpha=t+1}^{\hat{T}_{-(k-1)}} \chi(\alpha)| \geq \frac{b+f}{2}\}. \end{aligned} \tag{5.53}$$

The random variables $\Delta\hat{T}_{k+1} := \hat{T}_{k+1} - \hat{T}_k$, $k \in \mathbb{Z}$, are independent and identically distributed.
Remark: The $(\hat{T}_i, i \in \mathbb{Z})$ were denoted $(\tau_i, i \in \mathbb{Z})$ in [6].
Define

$$\tilde{S}_k = \operatorname{sgn}\Big(\sum_{j=\hat{T}_{k-1}+1}^{\hat{T}_k} \chi(j)\Big); \qquad \tilde{S}_{-k} = \operatorname{sgn}\Big(\sum_{j=\hat{T}_{-k}+1}^{\hat{T}_{-k+1}} \chi(j)\Big) \qquad \text{for} \quad k \geq 1. \tag{5.54}$$

To detect elongations with alternating sign, we introduce on the right of the origin

$$\begin{aligned} i_1^* &\equiv \inf\left\{i \geq 1 : \tilde{S}_i = \tilde{S}_{i+1}\right\} \\ i_{j+1}^* &\equiv \inf\left\{i \geq (i_j^*+2) : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_j^*}\right\} \qquad j \geq 1, \end{aligned} \tag{5.55}$$

and on the left

$$i_{-1}^* \equiv \begin{cases} -1 & \text{if } \tilde{S}_{-1} = \tilde{S}_1 = -\tilde{S}_{i_1^*}, \\ \sup\left\{i \leq -2 : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_1^*}\right\} & \text{if } \tilde{S}_{-1} \neq \tilde{S}_1 \text{ or } \tilde{S}_1 = -\tilde{S}_{i_1^*}, \end{cases}$$

$$i_{-j-1}^* \equiv \sup\left\{i \leq i_j^* - 2 : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_j^*}\right\} \qquad j \geq 1. \tag{5.56}$$

The corresponding estimates are given by the following Lemma which was proved in [6], see Lemma 5.9 there.
Lemma 5.5 *There exists an ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, all k and L positive integers, L even, (just for simplicity of writing) and all $s > 0$ we have:*

$$I\!P\left[\hat{T}_{kL-1} \leq \frac{(kL-1)(s+\log 2)C_1}{\epsilon}, \forall_{1\leq j\leq k}\ i_j^* < jL\right] \geq \left(1 - e^{-s(kL-1)}\right)\left(1 - \tfrac{1}{2^{L-1}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{L/2}\right)^{k-1} \tag{5.57}$$

and

$$\begin{aligned} &I\!P\left[\hat{T}_{-kL} \geq \frac{-kL(s+\log 2)C_1}{\epsilon}, \hat{T}_{L-1} \leq \frac{(L-1)(s+\log 2)C_1}{\epsilon}, i_1^* < L, \forall_{1\leq j\leq k}\ i_{-j}^* > -jL\right] \\ &\geq \left(1 - e^{-s(kL-1)}\right)\left(1 - \tfrac{1}{2^{L-1}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{L/2}\right)^{k}. \end{aligned} \tag{5.58}$$

where $C_1 = C_1(\beta,\theta)$ is a constant.

Applying Lemma 5.5 with $L = cte \log(Q^2 g(\frac{\delta^*}{\gamma}))$, taking the parameters as in Subsection 2.5, one gets (5.27) by a short computation.

Lemma 5.6 *On $\Omega_L([-Q,+Q],f,b,0)$, see (5.26), we have*

$$\hat{T}_i \leq \alpha^*_{i+1}, \tag{5.59}$$

and

$$\alpha^*_i \leq \hat{T}_{i^*_{i+1}}, \quad \forall i : 1 \leq i < \kappa^*(Q), \tag{5.60}$$

where $\kappa^*(Q)$ is defined in (2.36).

Proof: Recall that on $\Omega_L([-Q,+Q],f,b,0)$ we have assumed that $\alpha^*_0 \leq 0 < \alpha^*_1$. To prove (5.59) we start proving that $\hat{T}_1 \leq \alpha^*_2$. Suppose that $\alpha^*_2 < \hat{T}_1$. Then, from (5.53), since $\alpha^*_1 < \alpha^*_2 < \hat{T}_1$ we have

$$|\mathcal{Y}(\alpha^*_1)| < \frac{b+f}{2} \qquad \text{and} \qquad |\mathcal{Y}(\alpha^*_2)| < \frac{b+f}{2} \tag{5.61}$$

which is a contradiction since by assumption $[\epsilon\alpha^*_1, \epsilon\alpha^*_2]$ is a maximal b elongation with excess f, see Definition 5.1. Similar arguments apply for $i \geq 2$. To prove the second inequality in (5.59), we assume that $[\alpha^*_0, \alpha^*_1]$ gives rise to a positive maximal elongation. The case of a negative elongation is similar. We show that $\alpha^*_1 \leq \hat{T}_{i^*_2}$. By definition of i^*_1, i^*_2 we have that $[\hat{T}_{i^*_1-1}, \hat{T}_{i^*_1+1}]$ is within an elongation with a sign, say $\hat{S}_{i^*_1}$ and $[\hat{T}_{i^*_2-1}, \hat{T}_{i^*_2+1}]$ is within an elongation with opposite sign, $\hat{S}_{i^*_2} = -\hat{S}_{i^*_1}$. Therefore, either $\hat{S}_{i^*_1}$ or $\hat{S}_{i^*_2}$ is negative, which implies that $\alpha^*_1 \leq \hat{T}_{i^*_2}$. The general case is done similarly. ■

Given an integer $R > 0$, we denote as in (2.36) $\kappa^*(R) = \inf\{i \geq 1 : \epsilon\alpha^*_i \geq R\}$. We define the stopping time $\tilde{k}(R) = \inf\{i \geq 0 : \epsilon\hat{T}_i \geq R\}$. By definition

$$\epsilon\hat{T}_{\tilde{k}(R)-1} < R \leq \epsilon\hat{T}_{\tilde{k}(R)} \tag{5.62}$$

Using the left part of (5.59), we get that

$$R \leq \epsilon\hat{T}_{\tilde{k}(R)} \leq \epsilon\alpha^*_{\tilde{k}(R)+1} \tag{5.63}$$

therefore

$$\kappa^*(R) \leq 1 + \tilde{k}(R). \tag{5.64}$$

Lemma 5.7 *For all $b > 0$, there exists $\Omega_{urt} \equiv \Omega_{urt}(b)$, $I\!P[\Omega_{urt}] \geq 1 - 145\epsilon^{\frac{a}{32(2+a)}}$, where $a > 0$, so that for $f = \frac{5}{g(\frac{\delta^*}{\gamma})}$, for all $1 < R \leq 12Q$*

$$\kappa^*(R) \leq 1 + \tilde{k}(R) \leq 2 + 4\frac{V_+^2}{b^2} R \log\left[R^2 g(\delta^*/\gamma)\right] \tag{5.65}$$

and

$$\epsilon\alpha^*_{\kappa^*(R)+1}(b) \leq \frac{6C_1 V_+^2 \log 2}{b^2 \log(4/3)} R \left[\log(R^2 g(\delta^*/\gamma))\right]^2, \tag{5.66}$$

where $V_+ = V(\beta,\theta)\left[1 + (\gamma/\delta^*)^{\frac{1}{5}}\right]$ and $C_1 = C_1(\beta,\theta)$ is a positive constant. Furthermore on Ω_{urt}, $|\kappa^*(-R)|$ and $|\epsilon\alpha^*_{\kappa^*(-R)}|$ satisfy (5.65) and (5.66).

Remark: It is well known that, almost surely, $\lim_{R\uparrow\infty} \tilde{k}(R)/R = (I\!E[\hat{T}_1])^{-1}$, see [1] Proposition 4.1.4. The estimate (5.65) allows us to have an upper bound valid uniformly with respect to $R \geq 1$ with an explicit bound on the probability. This is the main reason to have a $\log[R^2 g(d^*/\gamma)]$ in the right hand side of (5.65).

Proof: The proof of (5.65) is based on two ingredients: Assume that we are on $\Omega_L([-12Q,+12Q],f,b,0)$. Suppose first that $\tilde{k}(R)>1$ then

$$\frac{\epsilon\hat{T}_{\tilde{k}(R)-1}}{\tilde{k}(R)-1}<\frac{R}{\tilde{k}(R)-1}\le\frac{\epsilon\hat{T}_{\tilde{k}(R)}}{\tilde{k}(R)-1}. \tag{5.67}$$

The second ingredient is the estimate derived in Lemma 5.7 of [6], which holds for any choice $b>0$, (in [6] a specific choice was done). We obtain that for all positive integer n and s, $0<s<(\frac{b+f}{2})^2[4(\log 2)V_+^2]^{-1}$,

$$I\!P\left[\epsilon\hat{T}_n\le ns\right]\le e^{-n\frac{b^2}{16sV_+^2}}. \tag{5.68}$$

Therefore

$$I\!P\left[\exists n\ge 1:\frac{\epsilon\hat{T}_n}{n}\le s\right]\le\frac{e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}{1-e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}. \tag{5.69}$$

Applying (5.67), we get that for $\tilde{k}(R)>1$

$$I\!P\left[\tilde{k}(R)\le 1+\frac{R}{s}\right]\ge\frac{1-2e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}{1-e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}. \tag{5.70}$$

When $\tilde{k}(R)=0$ or $\tilde{k}(R)=1$, (5.70) is certainly true, therefore (5.70) holds for all $\tilde{k}(R)\ge 0$. Choosing in (5.70)

$$s_0^{-1}=\frac{4V_+^2}{(\mathcal{F}^*)^2}\left[\log R^2 g(\delta^*/\gamma)\right] \tag{5.71}$$

we get

$$I\!P\left[\forall R\ge 1,\,\tilde{k}(R)\le 1+\frac{R}{s_0}\right]\ge 1-\sum_{R\ge 1}\frac{\frac{2}{g(\delta^*/\gamma)R^2}}{1-\frac{2}{g(\delta^*/\gamma)R^2}}\ge 1-\frac{3}{g(\delta^*/\gamma)}. \tag{5.72}$$

Recalling (5.64), for all $R\ge 1$,

$$\kappa^*(R)\le 1+\tilde{k}(R)\le 2+\frac{4V_+^2}{(\mathcal{F}^*)^2}R\left[\log R^2 g(\delta^*/\gamma)\right] \tag{5.73}$$

which is (5.65). Next we prove (5.66). Applying (5.60) and (5.64) we have

$$\epsilon\alpha^*_{k(L)+1}\le\epsilon\hat{T}_{i^*_{\tilde{k}(L)+2}}. \tag{5.74}$$

Using (5.57) with

$$\begin{aligned} L=L_0&=1+3\frac{\log(R^2 g(\delta^*/\gamma))}{\log(4/3)}\\ k=k_0&=2+\frac{4V_+^2}{(\mathcal{F}^*)^2}R\log[R^2 g(\delta^*/\gamma)]. \end{aligned} \tag{5.75}$$

After an easy computation, given $R\ge 1$ with a $I\!P$–probability greater than $1-c(\beta,\theta)\frac{\log(R^2 g(\delta^*/\gamma))}{g(\delta^*/\gamma)^{3/2}R^2}$ we have

$$\epsilon\hat{T}_{(2+k_0)L_0}\le\frac{24C_1V_+^2\log 2}{(\mathcal{F}^*)^2\log(4/3)}R\left[\log(R^2 g(\delta^*/\gamma))\right]^2,\quad\forall j:\,1\le j\le k_0,\ i^*_j<jL_0. \tag{5.76}$$

Therefore, with a $I\!P$–probability greater than

$$1 - c(\beta,\theta)\frac{\log g(\delta^*/\gamma)}{g(\delta^*/\gamma)^{3/2}} \geq 1 - \frac{1}{g(\delta^*/\gamma)} \tag{5.77}$$

for all $R \geq 1$, (5.76) holds. Using (5.73) we have, for all $R \geq 1$,

$$1 + \kappa^*(R) < \tilde{k}(R) + 2 \leq 3 + \frac{4V_+^2}{(\mathcal{F}^*)^2} R \left[\log R^2 g(\delta^*/\gamma)\right]. \tag{5.78}$$

Therefore collecting (5.76) and (5.78) we obtain that for all $R \geq 1$

$$i^*_{\tilde{k}(R)+2} \leq (2 + k_0)L_0. \tag{5.79}$$

From which using again (5.76) and recalling (5.74), we get that for all $R \geq 1$

$$\begin{aligned} \epsilon\alpha^*_{\kappa^*(R)+1} &\leq \epsilon\hat{T}_{i^*_{\tilde{k}(R)+2}} \leq \epsilon\hat{T}_{(2+k_0)L-0} \\ &\leq \frac{24C_1 V_+^2 \log 2}{(\mathcal{F}^*)^2 \log(4/3)} R \left[\log(R^2 g(\delta^*/\gamma))\right]^2 \end{aligned} \tag{5.80}$$

which is (5.66). Denote by Ω_{urt} the intersection of $\Omega_L([-Q,+Q], f, b, 0)$ with the probability subsets in (5.72) and (5.77). Recalling (5.27) and the choice of ϵ, see (2.57), we get the Lemma. ■

Lemma 5.8 *There exists Ω'_{urt}, $I\!P[\Omega'_{urt}] \geq 1 - 153\epsilon^{\frac{a}{32(2+a)}}$, $a > 0$, so that for $r_0 \leq 4Q$, see (5.36),*

$$\hat{\kappa}(Q, r_0) \leq 4 + \frac{24V_+^2}{b^2} Q \log\left[16Q^2 g(\delta^*/\gamma)\right]. \tag{5.81}$$

Proof: We construct in the interval $[-4Q, 2Q]$ all the maximal $\frac{b}{2}$ elongations with excess f, see Definition 5.1, for the process $\mathcal{Y}_{r_0}$ defined after (5.28). We do this repeating step by step the construction done in [6] and recalled in (5.26), (5.27) replacing the interval $[-Q, Q]$ with $[-4Q, 2Q]$ and b with $\frac{b}{2}$. On $\Omega([-4Q, 2Q], f, \frac{b}{2}, r_0) \cap \Omega_{\rm urt}(b/2)$, $r_0 \leq 4Q$, see Lemma 5.7, we have

$$|\kappa^*(-4Q, r_0)| \leq 2 + 16(V_+^2/b^2)Q\log(16Q^2 g(\delta^*/\gamma)) \,\text{and}\, \kappa^*(2Q, r_0) \leq 2 + 8(V_+^2/b^2)Q\log(4Qg(\delta^*/\gamma)).$$

Furthermore if $\{\hat{\sigma}_i(r_0)\}$ are $h-$extrema, then within the interval $[\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0))$ there is at least one maximal $\frac{b}{2}$–elongation with excess $f = 5/g(\delta^*/\gamma)$ for the process $\mathcal{Y}_{r_0}$. Therefore one gets (5.81). ■

Proof of Theorem 2.4:

We need to exhibit a probability subspace Ω_1 in which the minimal distance between two points of jump of u^*_γ is larger than $8\rho + 8\delta$. Define

$$\Omega_{1,1} = \left\{\omega \in \Omega_{urt} : \forall i,\ -Q \leq \epsilon\alpha^*_i \leq Q;\ \epsilon\alpha^*_{i+1} - \epsilon\alpha^*_i \geq 8\rho + 8\delta\right\}. \tag{5.82}$$

where Ω_{urt} is the probability subspace in Lemma 5.7. There the total number of jumps of u^*_γ within $[-Q, +Q]$ is bounded by $2K(Q) + 1$ with $K(Q)$ given in (2.31). Since the points of jumps of u^*_γ are the $\epsilon\alpha^*_i, i \in Z\!\!Z$, from Proposition 5.3 in [6] we have that for all $i \in Z\!\!Z$, for all $0 \leq x \leq (\mathcal{F}^*)^2/(V^2(\beta,\theta)18\log 2)$

$$I\!P[\epsilon\alpha^*_{i+1} - \epsilon\alpha^*_i < x] \leq 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}}. \tag{5.83}$$

Then one one gets

$$\mathbb{P}\left[\Omega_{1,1}\right] \geq 1 - \left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{8(2+a)}} - 2K(Q)e^{-\frac{(\mathcal{F}^*)^2}{18(8\rho+8\delta)V^2(\beta,\theta)}}. \tag{5.84}$$

Recalling (2.56), (2.59) and (2.55) one gets

$$\mathbb{P}\left[\Omega_{1,1}\right] \geq 1 - \left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{10(2+a)}}. \tag{5.85}$$

Denote by

$$\Omega_1 = \Omega_{\gamma,K(Q)} \cap \Omega_{1,1} \tag{5.86}$$

where $\Omega_{\gamma,K(Q)}$ is the probability subspace in Theorem 2.2 of [6] and $K(Q)$ is given in (2.31). From the results stated in Theorem 2.1, 2.2 and 2.4 of [6] we obtain (2.30) and (2.33).

6 Proof of the results

Proof of Theorem 2.5 The (2.37) is proved in Lemma 5.4.

Proof of Corollary 2.6 Since we already proved the convergence of finite dimensional distributions see (2.37), to get (2.43) it is enough to prove that for any subsequence $\{u^*_\gamma, 0<\gamma<\gamma_0\} \in BV_{\rm loc}(\mathbb{R}, \{m_\beta, Tm_\beta\})$, with $\gamma \downarrow 0$, one can extract a subsequence $\{u^*_{\gamma_n}, 0<\gamma_n<\gamma_0\}$ that convergences in Law. In fact, since $BV_{\rm loc}(\mathbb{R}, \{m_\beta, Tm_\beta\})$ is endowed with the topology induced by the metric $d(\cdot,\cdot)$ defined in (5.5), this implies that the points of jumps of $\{u^*_{\gamma_n}, 0<\gamma_n<\gamma_0\}$ will converge in Law to some points that by (2.37) are necessarily the $(S_i, i \in \mathbb{Z})$, this will imply (2.43).

So let $\gamma \downarrow 0$ be any subsequence that goes to 0. We will prove that for any chosen ϵ_1, it is possible to extract a subsequence $\gamma_n \downarrow 0$ and to construct a probability subset $\mathcal{K}_\epsilon \subset \Omega$ with

$$\mathbb{P}[\mathcal{K}_{\epsilon_1}] \geq 1 - \epsilon_1 \tag{6.1}$$

so that on $\mathcal{K}_{\epsilon_1}$, the subsequence $\{u^*_{\gamma_n},\ 0<\gamma_n \leq \gamma_0\}$ is a compact subset of $BV_{\rm loc}(\mathbb{R}, \{m_\beta, Tm_\beta\})$ for the topology induced by the metric (5.5).

To construct $\mathcal{K}_{\epsilon_1}$ and the subsequence γ_n, we use the following probability estimates. Let $b = 2\mathcal{F}^*$ and $\Omega_L([-Q,+Q], f, b, 0)$ the probability subspace defined in (5.26), $\mathbb{P}[\Omega_L([-Q,+Q], f, b, 0)] \geq 1 - 3\epsilon^{\frac{a}{32(2+a)}}$, see (5.27). On $\Omega_L([-Q,+Q], f, b, 0)$ $u^*_\gamma(\cdot)$ jumps at the points $\{\epsilon\alpha^*_i, \kappa^*(-Q)+1 \leq i \leq \kappa^*(Q)-1\}$. By Lemma 5.7, on the probability subspace $\Omega_{\rm urt}$, with $P[\Omega_{\rm urt}] \geq 1 - (\frac{5}{g(\delta^*/\gamma)})^{\frac{a}{8(2+a)}}$ for some $a>0$, the number of jumps within $[-Q,+Q]$ is smaller than $4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q \log\left[Q^2 g(\delta^*/\gamma)\right]$. Therefore, calling

$$\Omega_Q(x,\gamma) \equiv \left\{\omega \in \Omega_{\rm urt}; \forall i: \epsilon\alpha^*_i \in [-Q,+Q], \epsilon\alpha^*_{i+1} - \epsilon\alpha^*_i > x\right\}$$

one has, see (5.83), that

$$\mathbb{P}[\Omega_Q(x,\gamma)] \geq 1 - 4(\frac{5}{g(\delta^*/\gamma)})^{\frac{a}{32(2+a)}} - \left(4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q \log\left[Q^2 g(\delta^*/\gamma)\right]\right) 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}}. \tag{6.2}$$

For any subsequence $\gamma \downarrow 0$, one can pick up a subsequence $\{\gamma_n\}$ such that

$$\sum_{n\geq 1} \left(\frac{5}{g(\delta^*(\gamma_n)/\gamma_n)}\right)^{\frac{a}{32(2+a)}} < \infty \tag{6.3}$$

and recalling that $Q = Q(\gamma) \uparrow \infty$ when $\gamma \downarrow 0$, one can take $x = x(\gamma_n) > 0$ such that

$$\sum_{n\geq 1} \left(4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q(\gamma_n) \log\left[Q^2(\gamma_n) g(\delta^*(\gamma_n)/\gamma_n)\right] \right) 2e^{-\frac{(\mathcal{F}^*)^2}{18x(\gamma_n)V^2(\beta,\theta)}} < \infty. \tag{6.4}$$

Now using (6.2), (6.3) and (6.4), given $\epsilon_1 > 0$, one can choose $n_0 = n_0(\epsilon_1)$ such that

$$I\!P[\bigcap_{n\geq n_0} \Omega_{Q(\gamma_n)}(x(\gamma_n), \gamma_n)] \geq 1 - \epsilon_1. \tag{6.5}$$

Denote $\mathcal{K}_{\epsilon_1} \equiv \bigcap_{n\geq n_0} \Omega_{Q(\gamma_n)}(x(\gamma_n), \gamma_n)$ and we have proven (6.1).

Let $\omega \in \mathcal{K}_\epsilon$ and $\{u^*_{\gamma_n} = u^*_{\gamma_n}(\omega), n \geq n_0\}$ the above constructed subsequence. Necessary and sufficient conditions for the compactness of $\{u^*_{\gamma_n}, n \geq n_0\}$ is to exhibit for all $\tilde\epsilon$ say, $\tilde\epsilon < 1/2$ and for some numerical constant c a *finite* $c\tilde\epsilon$–net for $\{u^*_{\gamma_n}, n \geq n_0(\epsilon)\}$, see [2] pg. 217. One can also assume that $n_0 = n_0(\epsilon, \tilde\epsilon)$ is such that

$$e^{-Q(\frac{\delta^*(\gamma_{n_0})}{\gamma_{n_0}})} \leq \tilde\epsilon \tag{6.6}$$

Set $y^2 \equiv y^2_{\gamma_n} = \frac{\tilde\epsilon x(\gamma_n)}{4(1+\tilde\epsilon)}$, let $k_Q \in Z\!\!\!Z$ and $k_{-Q} \in Z\!\!\!Z$ so that $k_Q y_n^2 \leq Q < (k_Q+1)y^2$ and respectively $k_{-Q}y^2 \leq -Q < (k_{-Q}+1)y^2$. Denote $\mathcal{B}(y^2, Q) \subset BV_{\rm loc}$ the finite subset

$$\mathcal{B}(y^2, Q) = \left\{ \begin{array}{l} u_0 \in BV_{\rm loc} : u_0 \quad \text{constant on} \quad [ky^2, (k+1)y^2), k \in [k_{-Q}, k_Q] \cap Z\!\!\!Z, \\ \forall r \geq Q, u_0(r) = u_0(k_Q); \quad \forall r \leq -Q, u_0(r) = u_0(k_{-Q}) \end{array} \right\}$$

Let $\omega \in \mathcal{K}_\epsilon$ and $k_i^* \equiv k_i^*(\omega, \gamma_n) \in Z\!\!\!Z$ such that $k_i^* y^2 \leq \epsilon(\gamma_n)\alpha_i^*(\omega, \gamma_n) < (k_i^*+1)y^2$, for all i such that $\epsilon\alpha^*_{i-1} \in [-Q, +Q]$. Let $u_0 \in \mathcal{B}(y^2, Q)$ such that $u_0(k_i^* y^2) = u^*_{\gamma_n}(\epsilon\alpha_i^*)$. It remains to check that $d(u^*_{\gamma_n}, u_0) \leq c\tilde\epsilon$ for some numerical constant c, where $d(\cdot,\cdot)$ is defined in (5.5). Let us define the following $\lambda_{\gamma_n}(.) \in \Lambda_{\rm Lip}$ by $\lambda_{\gamma_n}(k_i^* y^2) = \epsilon\alpha_i^*$ and linear between $k_i^* y^2$ and $(k_i^*+1)y^2$ for $r > Q$ take $\lambda_{\gamma_n}(r) = \lambda_{\gamma_n}(Q) + t - Q$ and for $r \leq -Q$ take $\lambda_{\gamma_n}(r) = \lambda_{\gamma_n}(-Q) + t + Q$. For all i such that $\epsilon\alpha^*_{i-1} \in [-Q, +Q]$, one has

$$\left|\lambda_{\gamma_n}(k_i^* y^2) - \lambda_{\gamma_n}(k^*_{i-1}) - (k_i^* - k^*_{i-1})y^2\right| = \left|\epsilon\alpha^*_{\ell+1} - \epsilon\alpha^*_\ell - (k_i^* - k^*_{i-1})y^2\right| \leq 2y^2. \tag{6.7}$$

On the other hand on $\mathcal{K}_\epsilon$ one has $\epsilon\alpha_i^* - \epsilon\alpha^*_{i-1} \geq x(\gamma_n)$ and therefore $(k_i^* - k^*_{i-1})y^2 > x(\gamma_n) - 2y^2$. Using $2y^2 \leq \tilde\epsilon(x(\gamma_n) - 2y^2)$ and (6.7), one gets

$$\left|\lambda_{\gamma_n}(k_i^* y^2) - \lambda_{\gamma_n}(k^*_{i-1}) - (k_i^* - k^*_{i-1})y^2\right| \leq 2y^2 \leq \tilde\epsilon(x(\gamma_n) - 2y^2) \leq \tilde\epsilon(k_i^* y^2 - k^*_{i-1}y^2). \tag{6.8}$$

Since λ is piecewise linear one has also, for $s < t \in [k^*_{i-1}y^2, k_i^* y^2)$

$$|\lambda_{\gamma_n}(t) - \lambda_{\gamma_n}(s) - (t-s)| \leq \tilde\epsilon(t-s). \tag{6.9}$$

Since λ_{γ_n} has a slope 1 outside $[-Q, +Q]$, one gets for all $s < t \in I\!R$

$$\log(1-\tilde\epsilon) \leq \log\frac{\lambda_{\gamma_n}(t) - \lambda_{\gamma_n}(s)}{t-s} \leq \log(1+\tilde\epsilon). \tag{6.10}$$

Therefore, recalling (5.3), (6.10) entails $\|\lambda_{\gamma_n}\| \leq 4\frac{\tilde\epsilon}{3}$ and using (6.6) to control $\int_Q^\infty e^{-T}\,dT$ in (5.5) , one gets after an easy computation $d(u^*_{\gamma_n}, u_0) \leq 3\tilde\epsilon$. ■

Proof of Theorem 2.7

Let $\{W(r), r \in I\!R\}$ be a realization of the BBM and $u^*(W)$ the random function defined in (2.42). We need to show that for $v \in BV_{\rm loc}$, $\mathcal{P}$ a.s. $\Gamma(v|u^*, W) \geq 0$, the equality holding only when $v = u^*$. Let S_0 be a point of $h-$ minimum, $h = \frac{2\mathcal{F}^*}{V(\beta,\theta)}$. By definition, this implies that in the interval $[S_0, S_1)$

$$W(S_1) - W(S_0) \geq \frac{2\mathcal{F}^*}{V(\beta,\theta)}, \qquad W(y) - W(x) > -\frac{2\mathcal{F}^*}{V(\beta,\theta)}, \qquad \forall x < y \in [S_0, S_1); \tag{6.11}$$

$$W(S_0) \leq W(x) \leq W(S_1) \qquad S_0 \leq x \leq S_1. \tag{6.12}$$

Suppose first that v differs from $u^*(W)$ only in intervals contained in $[S_0, S_1)$. Since $u^*(r) = m_\beta$, for $r \in [S_0, S_1)$, see (2.42), set $v(r) = Tm_\beta 1\!\!1_{[r_1,r_2)}$ for $[r_1, r_2) \subset [S_0, S_1)$ and $v(r) = u^*(r)$ for $r \notin [r_1, r_2)$. When the interval $[r_1, r_2)$ is strictly contained in $[S_0, S_1)$ the function v has two jumps more than u^*. Then the value of $\Gamma(v|u^*, W)$, see (2.45), is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) = 2\mathcal{F}^* + V(\beta,\theta)[W(r_2) - W(r_1)] > 0, \tag{6.13}$$

which is strictly positive using the second property in (6.11). When $[r_1, r_2) \equiv [S_0, S_1)$ then the function v has two jumps less than u^*. Namely u^* jumps in S_0 and in S_1 and u does not. In such a case is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) + \Gamma_{[S_1,S_2)}(v|u^*, W) = -2\mathcal{F}^* + V(\beta,\theta)[W(S_1) - W(S_0)] \geq 0. \tag{6.14}$$

The last inequality holds since the first property in (6.11). When $[r_1, r_2) \subset [S_0, S_1)$, $r_1 = S_0$, $r_2 < S_1$, the function v has the same number of jumps as u^*. The value of $\Gamma(v|u^*, W)$ is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) = V(\beta,\theta)[W(r_2) - W(S_0)] \geq 0 \tag{6.15}$$

which is still positive because of (6.12). When $[r_1, r_2) \subset [S_0, S_1)$, $r_1 > S_0$, $r_2 = S_1$ then, as in the previous case, the function v has the same number of jumps as u^* and again by (6.12),

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) + \Gamma_{[S_1,S_2)}(v|u^*, W) = V(\beta,\theta)[W(S_1) - W(r_1)] \geq 0. \tag{6.16}$$

The case when v differs from u^*, still only in $[S_0, S_1)$, but in more than one interval can be reduced to the previous cases. By assumption $v \in BV_{\rm loc}$ and then the number of intervals in $[S_i, S_{i+1})$ where v might differ from u^* is $\mathcal{P}$ a.s finite. The conclusion is therefore that if $v \neq u^*$ in $[S_0, S_1)$, $\Gamma(v|u^*, W) \geq 0$. When v differs from u^* in $[S_1, S_2)$, S_1 is an $h-$ maximum and $u^*(r) = Tm_\beta$, see (2.42), one repeats the previous arguments recalling that by definition in $[S_1, S_2)$

$$W(S_2) - W(S_1) \leq -h \qquad W(y) - W(x) \leq h \qquad \forall x < y \in [S_1, S_2) \tag{6.17}$$

$$W(S_2) \leq W(x) < W(S_1) \qquad S_1 \leq x < S_2. \tag{6.18}$$

In the general case one therefore obtains

$$\Gamma(v|u^*, W) = \sum_{i \in Z\!\!\!Z} \Gamma_{[S_i,S_{i+1})}(v|u^*, W) \geq 0. \tag{6.19}$$

To prove that u^* is $\mathcal{P}$ a.s. the unique minimizer of $\Gamma(\cdot|u^*, W)$ it is enough to show that each term among (6.14), (6.15) and (6.16) is strictly positive, so that we get a strict inequality in (6.19). Since, see [15], page 108, exercise (3.26),

$$\mathcal{P}[\exists r \in [S_0, S_1] : [W(r) - W(S_1)] = 0] = 0,$$

we obtain that (6.16), (6.14) and by a simple argument (6.15) are strictly positive. ■

Proof of Theorem 2.9 The proof of (2.50) is an immediate consequence of Proposition 4.1 and Theorem 2.5. ■

References

[1] S. Asmussen. (1987).*Applied Probability and Queues*, J. Wiley and Sons .

[2] P. Billingsley (1968)*Convergence of probability measures.* Wiley and Sons, NY .

[3] M. Cassandro, E. Orlandi, and E. Presutti. (1993). Interfaces and typical Gibbs configurations for one-dimensional Kac potentials. *Prob. Theor. Rel. Fields* **96**, 57-96.

[4] M. Cassandro, E. Orlandi, and P.Picco. (1999). Typical configurations for one-dimensional random field Kac model. *Ann. Prob.* **27**, No 3, 1414-1467.

[5] M. Cassandro, E.Orlandi, and P. Picco. (2002). Uniqueness and global stability of the interface in a model of phase separation. *Nonlinearity* **15** No. 5, 1621-1651.

[6] M. Cassandro, E. Orlandi, P. Picco and M.E. Vares (2005). One-dimensional random field Kac's model: Localization of the Phases *Electron. J. Probab.*, **10**, 786-864.

[7] S.N. Ethier and T.G. Kurtz. (1986). *Markov Processes Characterization and convergence.* Wiley and Sons

[8] A.O. Golosov. (1986). On limiting distribution for a random walk in random environment. *Russian Math. Surveys* **41**, 199–200.

[9] M. Kac, G. Uhlenbeck, and P.C. Hemmer. (1963) On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model. *J. Math. Phys.* **4**, 216–228; (1963) II. Discussion of the distribution functions. *J. Math. Phys.* **4**, 229–247; (1964) III. Discussion of the critical region. *J. Math. Phys.* **5**, 60–74.

[10] I. Karatzas and S. Shreve (1991) *Brownian Motion and Stochastic Calculus* Second edition, Grduate text in Mathematics, Springer.

[11] H. Kesten. (1986) The limit distribution of Sinai's random walk in random environment. *Physica* 138A, 299-309.

[12] J. Lebowitz, and O. Penrose. (1966). Rigorous treatment of the Van der Waals Maxwell theory of the liquid-vapour transition. *J. Math. Phys.* **7**, 98–113.

[13] J. Neveu and J. Pitman. (1989). Renewal property of the extrema and tree property of the excursion of a one dimensional Brownian motion Séminaire de Probabilité XXlll, Lect. Notes in Math. **1372** 239–247, Springer Berlin-Heidelberg-New York.

[14] O. Penrose and J.L. Lebowitz. (1987). Towards a rigorous molecular theory of metastability. Fluctuation Phenomena (W. Montroll and J.L. Lebowitz ed) North-Holland Physics Publishing.

[15] D. Revuz and M. Yor. (1991) *Continuous Martingales and Brownian Motion.* Springer Verlag, Berlin-Heidelberg-New York.

[16] Y. Sinai. (1982). The limiting behavior of a one-dimensional random walk in random environment *Theory Probab. Appl.* **27**, 256–268.

[17] A.V. Skorohod. (1956) Limit Theorems for stochastic processes *Theory Probab. Appl.* **1** 261-290.

[18] H. M. Taylor. (1975) A stopped Brownian Motion Formula *Ann. Prob.* **3** 234–236.

[19] D. Willians. (1975). On a stopped Brownian Motion Formula of H.M. Taylor Sem. Prob. Strasb. X, Springer Lectures Note in Math. **511** 235–239.