Electronic Journal of Probability

Vol. 14 (2009), Paper no. 52, pages 1532–1540.

Journal URL
http://www.math.washington.edu/~ejpecp/

A Brownian sheet martingale with the same marginals as the arithmetic average of geometric Brownian motion.

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Abstract

We construct a martingale which has the same marginals as the arithmetic average of geometric Brownian motion. This provides a short proof of the recent result due to P. Carr et al [7] that the arithmetic average of geometric Brownian motion is increasing in the convex order. The Brownian sheet plays an essential role in the construction. Our method may also be applied when the Brownian motion is replaced by a stable subordinator.

Key words: Convex order, Brownian sheet, Asian option, Running average.

AMS 2000 Subject Classification: Primary 60G07, 60G60; Secondary: 62P05, 91B30.

Submitted to EJP on February 3, 2009, final version accepted May 19, 2009.

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1 Introduction and Main Result

To $(B_t, t \geqslant 0)$ a 1-dimensional Brownian motion, starting from 0, we associate the geometric Brownian motion:

$$\mathscr{E}_t = \exp(B_t - \frac{t}{2}), \qquad t \geqslant 0$$

and its arithmetic average:

$$\frac{1}{t}A_t = \frac{1}{t}\int_0^t ds\ \mathscr{E}_s, \qquad t \geqslant 0$$

A recent striking result by P. Carr et al [7] is the following:

Theorem 1. *i) The process* $(\frac{1}{t}A_t, t \geqslant 0)$ *is increasing in the convex order, that is: for every convex function* $g : \mathbb{R}_+ \to \mathbb{R}$*, such that* $\mathbb{E}\left[|g\left(\frac{1}{t}A_t\right)|\right] < \infty$ *for every* $t > 0$*, the function:*

$$t \to \mathbb{E}\left[g\left(\frac{1}{t}A_t\right)\right] \text{ is increasing}$$

ii) In particular, for any $K \geqslant 0$*, the call and put prices of the Asian option which we define as:*

$$C^+(t,K) = \mathbb{E}\left[\left(\frac{1}{t}A_t - K\right)^+\right] \text{ and } C^-(t,K) = \mathbb{E}\left[\left(K - \frac{1}{t}A_t\right)^+\right]$$

are increasing functions of $t \geqslant 0$.

Comments on Theorem 1

a) One of the difficulties inherent to the proof of ii), say, is that the law of A_t for fixed t, is complicated, as can be seen from the literature on Asian options.

b) A common belief among practitioners is that any "decent" option price should be increasing with maturity. But examples involving "strict local martingales" show that this need not be the case. See e.g. Pal-Protter [1], Delbaen-Schachermayer [2]. On the other hand Theorem 1 offers a proof of the increase in maturity for Asian options.

The proof of Theorem 1 as given in [7] (see also [8] for a slight variation) is not particularly easy, as it involves the use of either a maximum principle argument (in [7]) or a supermartingale argument (in [8]). We note that the proofs given in [7] and [8] show that for any individual convex function g, the associated function $G(t) = \mathbb{E}[g(\frac{1}{t}A_t)]$ is increasing. In contrast, in the present paper we obtain directly the result of Theorem 1 as a consequence of Jensen's inequality, thanks to the following

Theorem 2. *i)There exists a filtered probability space* $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbb{Q})$ *and a continuous martingale* $(M_t, t \geqslant 0)$ *on this space such that:*

for every fixed $t \geqslant 0$, $\frac{1}{t}A_t \overset{(law)}{=} M_t$

ii)More precisely, if $(W_{u,t}, u \geqslant 0, t \geqslant 0)$ *denotes the standard Brownian sheet and* $\mathscr{F}_{u,t} = \sigma\{W_{v,s}, v \leqslant u, s \leqslant t\}$ *its natural increasing family of* σ*-fields, one may choose:*

$$M_t = \int_0^1 du \exp(W_{u,t} - \frac{ut}{2}) \quad , \qquad t \geq 0$$

which is a continuous martingale with respect to $(\mathscr{F}_{\infty,t}, t \geqslant 0)$

We note that in [9] several methods have been developed to construct martingales with given marginals, an important problem considered by Strassen, Doob, Kellerer among others. See, e.g., references in [9]. Theorem 2 may also be considered in this light, providing a martingale whose one-dimensional marginals are those of $(\frac{1}{t}A_t, t \geqslant 0)$. In Section 2, we give our (very simple!) proof of Theorem 2, and we comment on how we arrived gradually at the formulation of Theorem 2. We also obtain a variant of Theorem 2 when $(\exp(B_t - \frac{t}{2}), t \geqslant 0)$ is replaced by $(\exp(B_t - at), t \geqslant 0)$ for any $a \in \mathbb{R}$.

In Section 3, we study various possible extensions of Theorem 2, i.e. : when the original Brownian motion $(B_t, t \geqslant 0)$ is replaced by certain Lévy processes, in particular stable subordinators and self-decomposable Lévy processes. In Section 4, we study some consequences of Theorem 1.

2 Proof of Theorem 2, and Comments

(2.1) We first make the change of variables: $u = vt$, in the integral

$$A_t = \int_0^t du \exp(B_u - \frac{u}{2})$$

We get: $\frac{1}{t}A_t = \int_0^1 dv \exp(B_{vt} - \frac{vt}{2})$
It is now immediate that since, for fixed t,

$$(B_{vt}, v \geqslant 0) \stackrel{(law)}{=} (W_{v,t}, v \geqslant 0), \text{ then:}$$

$$\text{for fixed } t, \qquad \frac{1}{t}A_t \stackrel{(law)}{=} \int_0^1 dv \exp(W_{v,t} - \frac{vt}{2})$$

Denoting by (M_t) the right-hand side, it remains to prove that it is a $(\mathscr{F}_{\infty,t}, t \geqslant 0)$ martingale. However, let $s < t$, then:

$$\mathbb{E}\left[M_t \middle| \mathscr{F}_{\infty,s}\right] = \int_0^1 dv\, \mathbb{E}\left[\exp(W_{v,t} - \frac{vt}{2}) \middle| \mathscr{F}_{\infty,s}\right].$$

Since $(W_{v,t} - W_{v,s})$ is independent from $\mathscr{F}_{\infty,s}$, we get:

$$\mathbb{E}\left[\exp(W_{v,t} - \frac{vt}{2}) \middle| \mathscr{F}_{\infty,s}\right] = \exp(W_{v,s} - \frac{vs}{2})$$

so that, finally: $\mathbb{E}\left[M_t \middle| \mathscr{F}_{\infty,s}\right] = M_s$.
This ends the proof of Theorem 2.

Remark: The same argument of independence allows to show more generally that, if $f : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is space-time harmonic, i.e. $(f(B_t, t), t \geqslant 0)$ is a martingale, then:

$$M_t^{(f)} \stackrel{def}{=} \int_0^1 du\, f(W_{u,t}, ut)$$

is a $(\mathscr{F}_{\infty,t}, t \geqslant 0)$ martingale. Thus in particular, for any $n \in \mathbb{N}$, one gets:

$$\text{for fixed } t, \qquad \frac{1}{t}\int_0^t du\, H_n(B_u, u) \overset{(law)}{=} M_t^{(n)}$$

$$\text{where: } M_t^{(n)} = \int_0^1 du\, H_n(W_{u,t}, ut)$$

and $H_n(x,t) = t^{n/2} h_n(\frac{x}{\sqrt{t}})$ denotes the n^{th} Hermite polynomial in the two variables $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ Consequently, in that generality,

$$(\frac{1}{t}\int_0^t du\, f(B_u, u), t \geqslant 0)$$

is increasing in the convex order sense.

(2.2) At this point, we feel that a few words of comments on how we arrived gradually at the statement of Theorem 2 may not be useless.
(2.2.1) We first recall the basic result of Rothschild and Stiglitz [5]. The notation $\leqslant_{cv}$ means domination in the convex order sense; see [3], [4].

Proposition 1. *Two variables X and Y on a probability space satisfy:*
$X \leqslant_{cv} Y$ if and only if on some (other) probability space, there exists $\hat{X}$ and $\hat{Y}$ such that:

$$\text{(i) } X \overset{(law)}{=} \hat{X} \qquad \text{(ii) } Y \overset{(law)}{=} \hat{Y} \qquad \text{(iii) } \mathbb{E}\left[\hat{Y}\middle|\hat{X}\right] = \hat{X}$$

For discussions, variants, amplifications of the RS result, we refer the reader to the books of Shaked-Shantikumar ([3], [4]). Thus in order to show that a process $(H_t, t \geqslant 0)$ is increasing in the convex order sense, one is led naturally to look for a martingale $(M_t^H, t \geqslant 0)$ such that:

$$\text{for fixed } t, \qquad H_t \overset{(law)}{=} M_t^H$$

In fact the papers of Strassen, Doob and Kellerer, refered in [9], show that there exists such a martingale $(M_t^H, t \geq 0)$.
(2.2.2) The following variants of Proposition 1 shall lead us to consider properties of the process:

$$\frac{1}{t}A_t^{(a)} \equiv \frac{1}{t}\int_0^t ds \exp(B_s - as)$$

for any $a \in \mathbb{R}$.
The notation [icv], resp. [dcv] used below indicates the notion of "increasing convex ", resp. "decreasing convex" order. (See e.g. [3], [4] for details; in particular, Theorem 2.A.3 in [3] and Theorem 3.A.4 in [4])

Proposition 2. *Two variables X and Y on a probability space satisfy:*
$X \leqslant_{[icv]} Y$ if and only if there exists on some (other) probability space, a pair $(\hat{X}, \hat{Y})$ such that:

$$\text{(i) } X \overset{(law)}{=} \hat{X} \qquad \text{(ii) } Y \overset{(law)}{=} \hat{Y} \qquad (iii)_{\uparrow} \quad \hat{X} \leqslant \mathbb{E}\left[\hat{Y}\middle|\hat{X}\right]$$

Proposition 3. *Same as Proposition 2, but where [icv] is replaced by [dcv], and* $(iii)_\uparrow$ *by:* $(iii)_\downarrow$ $\hat{X} \geqslant \mathbb{E}\left[\hat{Y}\middle|\hat{X}\right]$

We now apply Propositions 2 and 3 to the process $(\frac{1}{t}A_t^{(a)}, t \geqslant 0)$

Theorem 3. *1) Let* $a \leqslant \frac{1}{2}$*. Then the process* $(\frac{1}{t}A_t^{(a)}, t \geqslant 0)$ *increases in the [icv] sense*
2) Let $a \geqslant \frac{1}{2}$*. Then, the process* $(\frac{1}{t}A_t^{(a)}, t \geqslant 0)$ *increases in the [dcv] sense.*

We leave the details of the proof of Theorem 3 to the reader as it is extremely similar to that of Theorem 2.

(2.2.3) The following statement is presented here in order to help with our explanation of how we arrived gradually at the statement of Theorem 2.

Proposition 4. *Let* (Z_u) *and* $(Z_u^{'})$ *denote two processes. Then under obvious adequate integrability assumptions, we have:*

$$\int_0^1 du\ Z_u\ \mathbb{E}\left[Z_u^{'}\middle|Z\right] \leqslant_{cv} \int_0^1 du\ Z_u\ Z_u^{'}$$

Again, the proof is an immediate application of Jensen's inequality.
We now explain how we arrived at Theorem 2:
we first showed that, for $0 < \sigma^{'} < \sigma$, there is the inequality:

$$I_{\sigma'} \equiv \int_0^1 du \exp(\sigma^{'} B_u - \frac{\sigma^{'2}}{2} u) \leqslant_{cv} \int_0^1 du \exp(\sigma B_u - \frac{\sigma^2 u}{2}) \equiv I_\sigma \qquad (2)$$

Indeed, to obtain (2) as a consequence of Proposition 4, it suffices to write: $(\sigma B_u, u \geqslant 0) \overset{(law)}{=} (\sigma^{'} B_u + \gamma \beta_u, u \geqslant 0)$ where $(\beta_u, u \geqslant 0)$, is a BM independent from $(B_u, u \geqslant 0)$
and $\sigma^2 = (\sigma^{'})^2 + \gamma^2$, i.e. $\gamma = \sqrt{\sigma^2 - (\sigma^{'})^2}$
Once we had made this remark, it seemed natural to look for a "process" argument (with respect to the parameter σ), and this is how the Brownian sheet comes naturally into the picture.

3 Variants involving stable subordinators and self-decomposable Lévy processes

(3.1) Here is an analogue of Theorem 1 when we replace Brownian motion by a (α)-stable subordinator (T_t), for $0 < \alpha < 1$, whose law is characterized by:

$$\mathbb{E}\left[\exp(-\lambda T_t)\right] = \exp(-t\lambda^\alpha) \quad , \quad t \geqslant 0,\ \lambda \geqslant 0$$

Theorem 4. *The process* $\frac{1}{t}A_t^{(\alpha)} \overset{def}{=} \frac{1}{t}\int_0^t ds \exp(-\lambda T_s + s\lambda^\alpha)$ *is increasing for the convex order.*

We prove Theorem 4 quite similarly to the way we proved Theorem 1, namely: there exists a α-stable sheet $(T_{s,t}, s \geqslant 0, t \geqslant 0)$ which may be described as follows:

$(T(A), A \in \mathscr{B}(\mathbb{R}_+^2), |A| < \infty)$ is a random measure such that:
i) for all $A_1, ..., A_k$ disjoint Borel sets with $|A_i| < \infty$,
$T(A_1), .., T(A_k)$ are independent random variables,
ii) $\mathbb{E}\left[\exp(-\lambda T(A_i))\right] = \exp(-|A_i|\lambda^\alpha), \lambda \geq 0$.
($T(A_i)$ is an α-stable random variable)

Then we denote $T_{s,t} = T(R_{s,t})$, with $R_{s,t} \equiv [0,s] \times [0,t]$
See, e.g., [6] for the existence of such measures. The result of Theorem 4 is a consequence of:

Theorem 5. *The process* $M_t^{(\alpha)} = \int_0^1 du \exp(-\lambda T_{u,t} + ut\lambda^\alpha)$ *is a* $\mathscr{F}_{\infty,t}^{(\alpha)} \equiv \sigma\{T_{h,k}, h \geqslant 0, k \leqslant t\}$ *martingale, and for fixed t:*

$$\frac{1}{t} A_t^{(\alpha)} \overset{(law)}{=} M_t^{(\alpha)}$$

(3.2) We now consider a self-decomposable Lévy process.
(See e.g., Jeanblanc-Pitman-Yor [11] for a number of properties of these processes.)
Assuming that: $\forall \alpha > 0$, $\mathbb{E}\left[\exp(\alpha X_u)\right] < \infty$, then:

$$\mathbb{E}\left[\exp(\alpha X_u)\right] = \exp(u\varphi(\alpha)), \text{ for some function } \varphi.$$

In this framework, we show the following.

Theorem 6. *The process* $(I_\alpha = \int_0^1 du \exp(\alpha X_u - u\varphi(\alpha)),\ \alpha \geqslant 0)$ *is increasing in the convex order.*

Proof. Since $(X_u, u \geqslant 0)$ is self-decomposable, there exists, for any $c \in (0,1)$, another Lévy process $(\eta_u^{(c)}, u \geqslant 0)$ such that:
$(X_u, u \geqslant 0) \overset{(law)}{=} (cX_u + \eta_u^{(c)}, u \geqslant 0)$, with independence of X and $\eta^{(c)}$. Consequently, we obtain, for any $(\alpha, c) \in (0,\infty) \times (0,1)$

$$I_\alpha \overset{(law)}{=} \int_0^1 du \exp(\alpha c X_u - u\varphi(\alpha c)) \exp(\alpha \eta_u^{(c)} - u\varphi_c(\alpha)) \qquad (3)$$

where on the RHS of (3), X and $\eta^{(c)}$ are assumed to be independent.
Denote by $I_\alpha^{'}$ the RHS of (3), then :

$$\mathbb{E}\left[I_\alpha^{'} \middle| X\right] = \int_0^1 du \exp(\alpha c X_u - u\varphi(\alpha c)) = I_{\alpha c}$$

which implies, from Jensen's inequality: for every convex function g,

$$\mathbb{E}\left[g(I_{\alpha c})\right] \leqslant \mathbb{E}\left[g(I_\alpha)\right]$$

□

However we have not found, in this case, a martingale $(\mu_\alpha,\ \alpha \geqslant 0)$ such that:

$$\text{for every fixed } \alpha, \qquad I_\alpha \overset{(law)}{=} \mu_\alpha$$

Remark: We note that the above argument is a particular case of the argument presented in Proposition 4, which involves two processes Z and $Z^{'}$.

4 Some consequences

Since the process $(\frac{1}{t}A_t, t \geqslant 0)$ is increasing in the convex order, we find, by differentiating the increasing function of t: $\mathbb{E}[(K - \frac{1}{t}A_t)^+]$

$$\text{for every } K \geqslant 0 \text{ and } t \geqslant 0, \qquad \mathbb{E}\left[\mathbf{1}(\frac{1}{t}A_t < K)\ (\mathscr{E}_t - \frac{1}{t}A_t)\right] \geqslant 0,$$

although, it is not true that: $\mathbb{E}\left[\mathscr{E}_t \;\middle|\; \frac{1}{t}A_t\right]$ is greater than or equal to $\frac{1}{t}A_t$, since this would imply that: $\frac{1}{t}A_t = \mathscr{E}_t$, as the common expectation of both quantities is 1.

(4.1) More generally, the following proposition presents a remarkable consequence of the increasing property of the process $(\frac{1}{t}A_t, t \geq 0)$ in the convex order sense.

Proposition 5. *For every increasing Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ there is the inequality:*

$$\mathbb{E}\left[\varphi\left(\frac{1}{t}A_t\right)\left(\frac{1}{t}A_t\right)\right] \leq \mathbb{E}\left[\varphi\left(\frac{1}{t}A_t\right)\mathscr{E}_t\right]. \qquad (\star)$$

Equivalently,

$$\mathbb{E}\left[\varphi\left(\frac{1}{t}A_t\right)\left(\frac{1}{t}A_t\right)\right] \leq \mathbb{E}\left[\varphi\left(\frac{1}{t}\tilde{A}_t\right)\right], \qquad (\star\star)$$

where $\tilde{A}_t = \int_0^t du \exp(B_u + \frac{u}{2})$

Proof. We may assume φ bounded. Then, $g(x) = \int_0^x dy\, \varphi(y)$ is convex (its derivative is increasing), and formula $(\star)$ follows by differentiating the increasing function:

$$t \to \mathbb{E}\left[g\left(\frac{1}{t}A_t\right)\right].$$

Formula $(\star\star)$ follows from $(\star)$ by using the Cameron-Martin relationship between $(B_u, u \leq t)$ and $(B_u + u, u \leq t)$

□

(4.2) As a partial check on the previous result $(\star)$, we now prove directly that, for every integer $n \geq 1, t \to \mathbb{E}[(\frac{1}{t}A_t)^n]$ is increasing and that: $\mathbb{E}[(\frac{1}{t}A_t)^n] \leq \mathbb{E}[(\frac{1}{t}A_t)^{n-1}\mathscr{E}_t]$

Here are two explicit formulae for: $\alpha_n(t) = \mathbb{E}[(\frac{1}{t}A_t)^n]$, and $\beta_n(t) = \mathbb{E}[(\frac{1}{t}A_t)^{n-1}\mathscr{E}_t]$.

$$\begin{aligned}\alpha_n(t) &= \frac{n!}{t^n}\mathbb{E}[\int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{n-1}}^t ds_n \exp((B_{s_1} + ... + B_{s_n}) - \frac{1}{2}(s_1 + .. + s_n))] \\ &= \frac{n!}{t^n}\int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{n-1}}^t ds_n \exp(\frac{1}{2}C(s_1, .., s_n))\end{aligned}$$

$$\begin{aligned}\text{where } C(s_1, .., s_n) &= \mathbb{E}[(B_{s_1} + B_{s_2} + .. + B_{s_n})^2] - (s_1 + ... + s_n) \\ &= 2\sum_{1 \leq i \leq n} s_i(n-i) \qquad (\geqslant 0)\end{aligned}$$

Consequently:

$$\alpha_n(t) = n! \int_0^1 du_1 ... \int_{u_{n-1}}^1 du_n \exp(\frac{t}{2}C(u_1, ..., u_n)) \qquad (3)$$

from which it follows that $\alpha_n(t)$ is increasing in t.

$$\begin{aligned}&\text{Now } \beta_n(t) = \frac{(n-1)!}{t^{n-1}} \times \\ &\int_0^t ds_1 ... \int_{s_{n-1}}^t ds_{n-1} \mathbb{E}\left[\exp((B_{s_1} + ... + B_{s_{n-1}} + B_t) - \frac{1}{2}(s_1 + ... + s_{n-1} + t))\right] \\ &= (n-1)! \int_0^1 du_1 ... \int_{u_{n-2}}^1 du_{n-1} \exp(\frac{t}{2}C(u_1, ..., u_{n-1}, 1)) \qquad (4)\end{aligned}$$

We have already seen from formula (3), that $\alpha_n(t)$ is increasing in t; consequently: $\alpha_n^{'}(t) \geq 0$ and by definition of α_n:

$$\begin{aligned}\alpha_n^{'}(t) &= n\,\mathbb{E}\left[\left(\frac{1}{t}A_t\right)^{n-1}\left(-\frac{1}{t^2}A_t + \frac{\mathscr{E}_t}{t}\right)\right] \\ &= \frac{n}{t}\{\beta_n(t) - \alpha_n(t)\}\end{aligned}$$

Hence: $\beta_n(t) \geq \alpha_n(t)$.

(4.3) To conclude this paper, let us connect the properties of increase of the functions α_n and β_n with our method of proving Theorem 1 using the Wiener sheet, as performed in Theorem 2. Indeed, the same argument as in Theorem 2 shows that for any positive measure $\mu(du_1, ..., du_n)$ on $[0,1]^n$ the process:

$$\int \mu(du_1, .., du_n) \prod_{i=1}^n \mathscr{E}_{(u_i t)} \qquad (5)$$

admits the same one-dimensional marginals as the $(\mathscr{W}_t)$ submartingale

$$\int \mu(du_1,..,du_n)\prod_{i=1}^{n}\mathscr{E}_t^{(u_i)}(W) \qquad (6)$$

where $\mathscr{E}_t^{(u)}(W)=\exp(W_{u,t}-\frac{ut}{2})$.
Hence, the common expectation of (5) and (6) increases with t; $\alpha_n(t)$ and $\beta_n(t)$ constitute particular examples of this.
A final Note: Pushing further the use of the Brownian sheet and a variation from the construction of the Ornstein-Uhlenbeck process on the canonical path-space $C([0,1];\mathbb{R})$ in terms of that sheet, Hirsch-Yor [10] obtain a large class of processes, adapted to the brownian filtration, which admit the one-dimensional marginals of a martingale.

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