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SPDES IN $L_Q((0,\tau]], L_P)$ SPACES

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Abstract Existence and uniqueness theorems are presented for evolutional stochastic partial differential equations of second order in L_p-spaces with weights allowing derivatives of solutions to blow up near the boundary. It is allowed for the powers of summability with respect to space and time variables to be different.

Keywords Stochastic partial differential equations, Sobolev spaces with weights.

AMS subject classification 60H15, 35R60.

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1. Introduction

The main goal of this article is to extend results of [4] and [10] to the case of different powers of summability with respect to time and space variables. We are dealing with the equation

$$du = (a^{ij}u_{x^ix^j} + f)\,dt + (\sigma^{ik}u_{x^i} + g^k)\,dw^k_t \tag{1.1}$$

given for $t \geq 0$ and $x \in \mathbb{R}^d$ (in Sec. 2) or $x \in \mathbb{R}^d_+ := \{x = (x^1, x') : x^1 > 0, x' \in \mathbb{R}^{d-1}\}$ (in Sec. 3). Here w^k_t are independent one-dimensional Wiener processes, i and j run from 1 to d, k runs through $\{1, 2, ...\}$ with the summation convention being enforced, and f and g^k are some given functions of (ω, t, x) defined for $k \geq 1$. The functions a^{ij} and σ^{ik} are assumed to depend only on ω and t, and in this sense we consider equations with "constant" coefficients. Without loss of generality we also assume that $a^{ij} = a^{ji}$. Equation (1.1) is assumed to be parabolic in an appropriate sense.

As in [10], let us mention that such equations with finite number of the processes w^k_t appear, for instance, in nonlinear filtering problem for partially observable diffusions (see [13]). Considering infinitely many w^k_t allows us to treat equations for measure-valued processes, for instance, driven by space-time white noise (see [4]).

As in [4], [10], and [12] we are dealing with Sobolev space theory of (1.1), so that the derivatives are understood as generalized functions, "the number" of derivatives can be fractional or negative and the underlying power of summability in x is $p \in [2, \infty)$. The reader is referred to [4], [10], and [12] for motivation of considering such wide range of "derivatives" and $p \geq 2$. In contrast with these articles, here the power of summability in time is allowed to be $q \geq p$.

Challengingly enough, our results and methods of proofs in the case $q = p$ and equations in $\mathbb{R}^d_+$ do not allow us to cover the results of [10] which are obtained for wider range of weights. Yet we still get an additional information on solutions of SPDEs if $q > p$, which is discussed after Theorem 1.1 below and in Sec. 3. It is also worth noting that, if there are no stochastic terms in (1.1), the corresponding $L_q(L_p)$-theory is developed in [8] for any $q, p \in (1, \infty)$ and, in the case of $\mathbb{R}^d_+$, the range of weights turns out to be just natural, which in terms of certain parameter θ measuring the weights and introduced later is written as $d - 1 < \theta < d - 1 + p$.

Apart from the sections we have mentioned above, there is also Sec. 5, where we prove our main results for $\mathbb{R}^d_+$ stated as Theorems 3.1 and 3.2. These proofs are based on some auxiliary facts collected in Sec. 4 and Sec. 6. The reason for one of them to be deferred until the last section is that it bears on purely analytic properties of a barrier function.

To give the reader a flavor of our results we state a particular case of Theorem 3.2 along with its corollary for $\theta = d$, $\gamma = 1$, and $u_0 = 0$. At this moment we do not make precise what we mean by "vanishing for $t = 0$ and for $x^1 = 0$ in a natural sense" in Theorem 1.1. Actual meaning is that the solution belongs to a function space to be specified later.

Theorem 1.1. *Let $q \geq p \geq 2$, $T \in (0, \infty)$, w_t be a one-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ and let $g(t, x) = g(\omega, t, x)$ be nonanticipating as a function of (ω, t) and such that*

$$E\int_0^T \Big(\int_{\mathbb{R}^d_+} |g(t, x)|^p\,dx\Big)^{q/p}\,dt < \infty.$$

Then

(i) on $\Omega \times [0, T] \times \bar{\mathbb{R}}^d_+$, there is a unique up to a.e. function u satisfying the equation

$$du = \Delta u\,dt + g\,dw_t \quad in \quad (0, T) \times \mathbb{R}^d_+, \tag{1.2}$$

vanishing for $t=0$ and for $x^1=0$ in a natural sense and such that

$$E\int_0^T\Big(\int_{\mathbb{R}^d_+}(|u(t,x)/x^1|^p+|u_x(t,x)|^p)\,dx\Big)^{q/p}\,dt<\infty.$$

(ii) In addition, if $2/q+d/p<1$, then, for any $\beta\in(2/q,1-d/p)$ $(\neq\emptyset)$ and $\varepsilon:=1-\beta-d/p$ (>0), we have

$$E\sup_{t\le T,x\in\mathbb{R}^d_+}|(x^1)^{-\varepsilon}u(t,x)|^q<\infty. \tag{1.3}$$

Notice that assertion (ii) of Theorem 1.1 immediately follows from (i) and Theorem 4.7 of [7]. In particular, (1.3) holds with any q if $p>d$ and

$$||g(t,\cdot)||_{L_p(\mathbb{R}^d_+)} \tag{1.4}$$

is a bounded function of (ω,t). In that case, we basically rediscover one of the statements of [11] under weaker asumptions. Our improvements are that one need not the boundedness of (1.4) at the same time obtaining control on u_x and on the behavior of u near the boundary $x^1=0$. Also, in contrast with [11], Theorem 3.2 bears on equations with random and time-dependent coefficients and with infinitely many Wiener processes, the latter allowing one to treat, for instance, one-dimensional equations driven by space-time white noise.

In conclusion we introduce some notation. We are working on a complete probability space $(\Omega,\mathcal{F},P)$ with an increasing filtration $(\mathcal{F}_t,t\ge0)$ of complete σ-fields $\mathcal{F}_t$. The predictable σ-field generated by $(\mathcal{F}_t,t\ge0)$ is denoted $\mathcal{P}$. The coefficients a and σ of (1.1) are assumed to be predictable. We are also given independent one-dimensional Wiener processes w^k_t which are Wiener with respect to $(\mathcal{F}_t,t\ge0)$.

In the whole article p,q are some numbers satisfying $q\ge p\ge2$. We are also using the notation introduced in [7]. In particular, by M^α we denote the operator of multiplying by $|x^1|^\alpha$, $M=M^1$. The function spaces H^γ_p are usual spaces of Bessel potentials on $\mathbb{R}^d$. As in [7], we use the spaces $H^\gamma_{p,\theta}$ (the formal definition of which can be seen from (4.2)) and, for any stopping time τ, we define

$$\mathbb{H}^{\gamma,q}_p(\tau)=L_q((0,\tau]\!],\mathcal{P},H^\gamma_p),\quad \mathbb{H}^{\gamma,q}_{p,\theta}(\tau)=L_q((0,\tau]\!],\mathcal{P},H^\gamma_{p,\theta}).$$

Also from [7] we take the spaces $\mathcal{H}^{\gamma,q}_p(\tau)$, $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, and $\mathcal{H}^\gamma_p(\tau)=\mathcal{H}^{\gamma,p}_p(\tau)$, $\mathfrak{H}^\gamma_{p,\theta}(\tau)=\mathfrak{H}^{\gamma,p}_{p,\theta}(\tau)$. Without going into detail, we only mention that γ is the "number of derivatives", the spaces with θ are the ones of functions on $\mathbb{R}^d_+$, θ is responsible for the rate with which "the derivatives" are allowed to "blow up" near $x^1=0$. If $\gamma=0$ we use L and $\mathbb{L}$ instead of H^γ and $\mathbb{H}^\gamma$, so that $L_{p,\theta}=H^0_{p,\theta}$. It is helpful to remember that $L_{p,\theta}=L_p(\mathbb{R}^d_+,(x^1)^{\theta-d}\,dx)$.

The letter τ indicates that we are dealing with functions depending on t (and ω). The spaces $\mathbb{H}$ are just L_p-type spaces with no control on the continuity of functions with respect to t. The spaces $\mathcal{H}^{\gamma,q}_p(\tau)$ and $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ are Banach spaces of functions having stochastic differential with respect to t:

$$du=f\,dt+g^k\,dw^k_t=:\mathbb{D}u\,dt+\mathbb{S}^ku\,dw^k_t.$$

It is important to keep in mind that, if $u\in\mathcal{H}^{\gamma,q}_p(\tau)$ ($u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$), then $u\in\mathbb{H}^{\gamma,q}_p(\tau)$, $f\in\mathbb{H}^{\gamma-2,q}_p(\tau)$, and the l_2-valued function $g\in\mathbb{H}^{\gamma-1,q}_p(\tau)$ (respectively, $M^{-1}u\in\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$, $Mf\in\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)$, and $g\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$).

Finally, the norms in $L_p(\mathbb{R}^d)$ and in $L_p(\mathbb{R}^d_+)$ are denoted by $||\cdot||_p$.

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2. Main results for SPDEs in $\mathbb{R}^d$

Here we deal with equation (1.1) given for $(\omega, t, x) \in (\!(0, \tau]\!] \times \mathbb{R}^d$. We assume that a^{ij}, σ^{ik} are predictable functions of (ω, t) (independent of x) and, for a constant $\delta \in (0,1)$, they satisfy $a^{mn} = a^{nm}$ and

$$\delta^{-1}|\lambda|^2 \geq a^{ij}\lambda^i\lambda^j \geq \delta|\lambda|^2, \quad (1-\delta)a^{ij}\lambda^i\lambda^j \geq \alpha^{ij}\lambda^i\lambda^j, \tag{2.1}$$

for all $\lambda \in \mathbb{R}^d$ and n, m, ω, t, where

$$\alpha^{ij} = (1/2)\sigma^{ik}\sigma^{jk}.$$

Our main result for $\mathbb{R}^d$ is as follows.

Theorem 2.1. *Let $T \in (0, \infty)$, $\tau \leq T$, $\gamma \in \mathbb{R}$, $q \geq p \geq 2$, $f \in \mathbb{H}_p^{\gamma,q}(\tau)$, and $g \in \mathbb{H}_p^{\gamma+1,q}(\tau)$. Then,*

(i) in $\mathcal{H}_p^{\gamma+2,q}(\tau)$, equation (1.1) with zero initial condition has a unique solution. For this solution we have

$$||u_{xx}||_{\mathbb{H}_p^{\gamma,q}(\tau)} \leq N(||f||_{\mathbb{H}_p^{\gamma,q}(\tau)} + ||g_x||_{\mathbb{H}_p^{\gamma,q}(\tau)}), \tag{2.2}$$

where $N = N(d, \delta, p, q)$;

(ii) if in addition we are given a function $u_0 \in L_q(\Omega, \mathcal{F}_0, H_p^{\gamma+2-2/q+\varepsilon})$, where $\varepsilon > 0$, then in $\mathcal{H}_p^{\gamma+2,q}(\tau)$, equation (1.1) with initial condition u_0 has a unique solution. For this solution we have

$$||u||^q_{\mathbb{H}_p^{\gamma+2,q}(\tau)} \leq N(||f||^q_{\mathbb{H}_p^{\gamma,q}(\tau)} + ||g||^q_{\mathbb{H}_p^{\gamma+1,q}(\tau)} + E||u_0||^q_{H_p^{\gamma+2-2/q+\varepsilon}}), \tag{2.3}$$

where $N = N(d, \delta, p, q, \varepsilon, T)$. Moreover if $q = p$, one can take $\varepsilon = 0$.

We give the proof of this theorem later in this section after we prepare some auxiliary results.

Lemma 2.2. *Let $p \geq 2$, $u \in \mathcal{H}_p^0(\tau)$ be a solution of the equation*

$$du = (a^{ij}u_{x^ix^j} + f^{ij}_{x^ix^j})\,dt + (\sigma^{ik}u_{x^i} + g^{ik}_{x^i})\,dw_t^k$$

with zero initial data and with $f^{ij}, g^{i\cdot} \in \mathbb{L}_p(\tau)$. Then

$$||u||_{\mathbb{L}_p(\tau)} \leq N(||f||_{\mathbb{L}_p(\tau)} + ||g||_{\mathbb{L}_p(\tau)}),$$

where $N = N(d, \delta, p)$.

This basic apriori estimate follows from Theorem 5.1 of [4] up to the assertion that N is independent of T. The later is obtained in a standard way by using self similarity.

In the next lemma we do the first step to considering the power of summability in t equal to multiples of p.

Lemma 2.3. *Let $p \geq 2$, $T \in (0, \infty)$, $\tau \leq T$, $n \in \{1, 2, ...\}$, and, for $i = 1, ..., n$,*

$$\lambda_i \in (0, \infty), \quad \gamma_i \in \mathbb{R}, \quad u^{(i)} \in \mathcal{H}_p^{\gamma_i+2}(\tau), \quad u^{(i)}(0) = 0.$$

Denote $\Lambda_i = (\lambda_i - \Delta)^{\gamma_i/2}$. Then (i)

$$E\int_0^\tau \prod_{i=1}^n ||\Lambda_i \Delta u^{(i)}(t)||_p^p \, dt$$

$$\leq N\sum_{i=1}^n E\int_0^\tau (||\Lambda_i f^{(i)}(t)||_p^p + ||\Lambda_i g_x^{(i)}(t)||_p^p)\prod_{j\neq i}||\Lambda_j \Delta u^{(j)}(t)||_p^p \, dt$$

$$+ N\sum_{1\leq i<j\leq n} E\int_0^\tau ||\Lambda_i g_x^{(i)}(t)||_p^p ||\Lambda_j g_x^{(j)}(t)||_p^p \prod_{k\neq i,j}||\Lambda_k \Delta u^{(k)}(t)||_p^p \, dt, \quad (2.4)$$

where $N = N(n, d, p, \delta)$ and

$$f^{(i)} = \mathbb{D}u^{(i)} - a^{rs}u^{(i)}_{x^r x^s}, \quad g^{(i)k} = \mathbb{S}^k u^{(i)} - \sigma^{rk}u^{(i)}_{x^r}. \quad (2.5)$$

(ii) Replace $\Delta u^{(i)}$ and $||\Lambda_i g_x^{(i)}||_p$ with $u^{(i)}$ and $||\Lambda_i g^{(i)}||_{H_p^1}$, respectively, everywhere in (2.4). Then the estimate will be true with $N = N(n, d, p, \delta, T)$.

Proof. (i) By considering $\Lambda_i u^{(i)}$ instead of $u^{(i)}$, we see that without loss of generality we may assume $\gamma_i = 0$. In this case first let $\sigma^{ik} = 0$ and define $v^{(i)} = \Delta u^{(i)}$. Furthermore, for $X = (x_1, ..., x_n) \in \mathbb{R}^{nd}$ with $x_i \in \mathbb{R}^d$, define

$$V(t, X) = v^{(1)}(t, x_1) \cdot ... \cdot v^{(n)}(t, x_n).$$

Observe that

$$dv^{(i)} = (a^{rs}v^{(i)}_{x^r x^s} + \Delta f^{(i)})\, dt + \Delta g^{(i)k}\, dw_t^k$$

and by Itô's formula

$$dV(t, X) = (\boldsymbol{L}V(t, X) + F(t, X) + H(t, X))\, dt + G^k(t, X)\, dw_t^k,$$

where $\boldsymbol{L}V = a^{rs}(V_{x_1^r x_1^s} + ... + V_{x_n^r x_n^s})$,

$$F(t, X) = \Delta_{x_i}\bar{F}^i(t, X), \quad G^k(t, X) = (\bar{G}^k_{ir}(t, X))_{x_i^r}$$

$$H(t, X) = \sum_{1\leq i<j\leq n}(\bar{H}^{ij}_{rs}(t, X))_{x_i^r x_j^s}, \quad \bar{F}^i(t, X) = f^{(i)}(t, x_i)\prod_{j\neq i}v^{(j)}(t, x_j),$$

$$\bar{G}^k_{ir}(t, X) = g^{(i)k}_{x_i^r}(t, x_i)\prod_{j\neq i}v^{(j)}(t, x_j),$$

$$\bar{H}^{ij}_{rs}(t, X) = g^{(i)k}_{x_i^r}(t, x_i)g^{(j)k}_{x_j^s}(t, x_j)\prod_{m\neq i,j}v^{(m)}(t, x_m).$$

Hence by Lemma 2.2

$$||V||_{\mathbb{L}_p(\tau)} \leq N(\sum_i ||\bar{F}^i||_{\mathbb{L}_p(\tau)} + \sum_{i<j}\sum_{r,s}||\bar{H}^{ij}_{rs}||_{\mathbb{L}_p(\tau)} + \sum_{i,r}||\bar{G}_{ir}||_{\mathbb{L}_p(\tau)}). \quad (2.6)$$

Here

$$||V||^p_{\mathbb{L}_p(\tau)} = E\int_0^\tau \int_{\mathbb{R}^{nd}} |v^{(1)}(t, x_1) \cdot ... \cdot v^{(n)}(t, x_n)|^p \, dx_1 \cdot ... \cdot dx_n dt$$

$$= E\int_0^\tau ||v^{(1)}(t, \cdot)||_p^p \cdot ... \cdot ||v^{(n)}(t, \cdot)||_p^p \, dt$$

$$||\bar{G}_{ir}||^p_{\mathbb{L}_p(\tau)} = E\int_0^\tau ||g^{(i)}_{x^r}(t, \cdot)||_p^p \prod_{j\neq i}||v^{(j)}(t, \cdot)||_p^p \, dt,$$

$$||\bar{F}^i||^p_{\mathbb{L}_p(\tau)} = E\int_0^\tau ||f^{(i)}(t,\cdot)||_p^p \prod_{j\neq i} ||v^{(j)}(t,\cdot)||_p^p \,dt,$$

$$||\bar{H}^{ij}_{rs}||^p_{\mathbb{L}_p(\tau)} = E\int_0^\tau h^{ij}_{rs}(t) \prod_{m\neq i,j} ||v^{(j)}(t,\cdot)||_p^p \,dt,$$

where

$$h^{ij}_{rs}(t) := \int_{\mathbb{R}^{2d}} \big(\sum_k g^{(i)k}_{x_1^r}(t,x_1) g^{(j)k}_{x_2^s}(t,x_2)\big)^p \,dx_1 dx_2$$
$$\leq ||g^{(i)}_{x^r}(t,\cdot)||_p^p ||g^{(j)}_{x^s}(t,\cdot)||_p^p.$$

Therefore, (2.6) implies (2.4).

To consider the case of general σ^{ik} observe that, if $u \in \mathcal{H}^{\gamma+2}_p(\tau)$, then $\bar{u} \in \mathcal{H}^{\gamma+2}_p(\tau)$, where

$$\bar{u}(t,x) = u(t,x-\xi_t), \quad \xi^i_t = \int_0^t \sigma^{ik}(s)\,dw^k_s.$$

Adding to this that,

$$\mathbb{S}^k\bar{u}(t,x) = \mathbb{S}^k u(t,x-\xi_t) - \sigma^{ik}(t)u_{x^i}(t,x-\xi_t),$$
$$\mathbb{D}\bar{u}(t,x) = \mathbb{D}u(t,x-\xi_t) + \alpha^{ij}(t)u_{x^ix^j}(t,x-\xi_t) - (\mathbb{S}^k u)_{x^i}(t,x-\xi_t)\sigma^{ik}(t)$$
$$= \mathbb{D}u(t,x-\xi_t) - \alpha^{ij}(t)u_{x^ix^j}(t,x-\xi_t) - (\mathbb{S}^k \bar{u})_{x^i}(t,x-\xi_t)\sigma^{ik}(t)$$

and using the translation invariance of L_p-norms one easily reduces the general case to the particular one. This proves assertion (i).

The proof of assertion (ii) follows the same lines and is much simpler. Again we may take $\gamma_i = 0$. Then in the case $\sigma \equiv 0$ it suffices to write down the equation for $u^{(1)}(t,x_1)\cdot ... \cdot u^{(n)}(t,x_n)$ and apply Theorem 4.10 (iii) of [4]. In this case we get even stronger estimate with $||g||_p$ instead of $||g||_{H^1_p}$. The case of general σ is treated in the same way as above. The lemma is proved.

Remark 2.4. Obviously, in (2.5) one could use a and σ depending on i.

Proof of Theorem 2.1. Bearing in mind that one can apply the operator $(1-\Delta)^{\gamma/2}$ to both parts of (1.1), we see that we only have to consider the case $\gamma = 0$. Also notice that in the case $q = p$ our result is known from [4] and, for $q > p$, the uniqueness of solutions follows from that for $q = p$ and thus follows from [4].

To prove the theorem for $q > p$ we first relate (ii) to (i). We know from [14] that, for any ω, there exists a continuation of $u_0(\omega,x)$ to a function $\bar{u}(\omega,t,x)$ defined for $t \geq 0$ and $x \in \mathbb{R}^d$ such that $\bar{u}(0,x) = u_0$ and

$$||\bar{u}_t||_{L_q([0,T],L_p)} + ||\bar{u}||_{L_q([0,T],L_p)} + ||\bar{u}_x||_{L_q([0,T],L_p)}$$
$$+||\bar{u}_{xx}||_{L_q([0,T],L_p)} \leq N||u_0||_{H_p^{2-2/q+\varepsilon}}.$$

In addition the continuation operator is deterministic and linear. Therefore, $\bar{u} \in \mathcal{H}^{2,q}_p(T)$.

If we change the unknown function u by subtracting off $\bar{u}$, we will come to the situation with zero initial data. It follows that we only need to prove (i) and (ii) for $u_0 = 0$. Now the only difference between (i) and (ii) is in the estimates (2.2) and (2.3).

First assume that $q = np$, where $n = 1, 2....$ Take $f \in \mathbb{L}^q_p(\tau)$ and $g \in \mathbb{H}^{1,q}_p(\tau)$. Then, by Hölder's inequality, $f \in \mathbb{L}_p(\tau)$ and $g \in \mathbb{H}^1_p(\tau)$. Now remember that, as we know from [4], if $f, g \in \mathbb{H}^m_p(\tau)$, then equation (1.1) with zero initial condition has a unique solution $u \in \mathcal{H}^{m+1}_p(\tau)$ and that if m

is large enough, the norm $||u(t)||_{H^2_p}$ is a continuous function of t. In that case, for the stopping times

$$\tau_r := \tau \wedge \inf\{t : ||u(t)||_{H^2_p} \geq r\}, \quad r > 0,$$

we have $u \in \mathcal{H}^{2,q}_p(\tau_r)$ and $\tau_r = \tau$ if r is large enough depending on ω. By adding to this that $\mathbb{H}^m_p(\tau) \cap \mathbb{H}^{k,q}_p(\tau)$ is everywhere dense in $\mathbb{H}^{k,q}_p(\tau)$ for any m and k, we easily see that, to prove both existence and uniqueness in assertions (i) and (ii), it only remains to prove the apriori estimates (2.2) and (2.3) with $\gamma = 0$ and $u_0 = 0$ assuming that the solution $u \in \mathcal{H}^{2,q}_p(\tau)$ exists already.

By Lemma 2.3 (i) applied to $u^{(i)} = u$ we have

$$||u_{xx}||^{np}_{\mathbb{L}^{np}_p(\tau)} \leq N(E\int_0^\tau (||f(t)||^p_p + ||g_x(t)||^p_p)||u_{xx}(t)||^{(n-1)p}_p \, dt$$
$$+ NE\int_0^\tau ||g_x(t)||^{2p}_p ||u_{xx}(t)||^{(n-2)p}_p \, dt. \quad (2.7)$$

Owing to Young's inequality, for any $\varepsilon > 0$, we have

$$(||f(t)||^p_p + ||g_x(t)||^p_p)||u_{xx}(t)||^{(n-1)p}_p$$
$$\leq \varepsilon ||u_{xx}(t)||^{np}_p + N(\varepsilon)(||f(t)||^{np}_p + ||g_x(t)||^{np}_p),$$
$$||g_x(t)||^{2p}_p ||u_{xx}(t)||^{(n-2)p}_p \leq \varepsilon ||u_{xx}(t)||^{np}_p + N(\varepsilon)||g_x(t)||^{np}_p,$$

where $N(\varepsilon)$ is independent of u, f, g. This and (2.7) yield (2.2) for $q = np$. In the same way from Lemma 2.3 (ii) we get an estimate for $||u||_{\mathbb{L}^{np}_p(\tau)}$ which combined with the above estimate leads to (2.3).

To treat general $q \geq p$, we use the Marcinkiewicz interpolation theorem. Mapping $(f, g) \to u$ is a linear operator, say R acting on $L_q((0, \tau]], \mathcal{P}, H^0_p \times H^1_p)$ with values in $L_q((0, \tau]], \mathcal{P}, H^2_p)$ defined for $q = np$. From uniqueness, it follows that R does not depend on n. In addition, R is bounded for $q = np$. Let N_q be the norm of R. By the Marcinkiewicz interpolation theorem R is bounded for any $q \geq p$ and $N_q \leq N(N_1 + N_{np})$, where n is such that $q \leq np$ and $N = N(q, p, n)$. This is equivalent to our assertions. The theorem is proved.

Corollary 2.5. *Let $T \in (0, \infty)$, $\tau \leq T$, $q \geq p \geq 2$, $f = (f^1, ..., f^d) \in \mathbb{L}^q_p(\tau)$, and $g \in \mathbb{L}^q_p(\tau)$. Then, in $\mathcal{H}^{1,q}_p(\tau)$, the equation*

$$du = (a^{ij} u_{x^i x^j} + f^i_{x^i}) \, dt + (\sigma^{ik} u_{x^i} + g^k) \, dw^k_t \quad (2.8)$$

with zero initial condition has a unique solution. For this solution we have

$$||u_x||_{\mathbb{L}^q_p(\tau)} \leq N(||f||_{\mathbb{L}^q_p(\tau)} + ||g||_{\mathbb{L}^q_p(\tau)}), \quad (2.9)$$

where $N = N(d, p, q, \delta)$.

Indeed,

$$||f^i_{x^i}||_{\mathbb{H}^{-1,q}_p(\tau)} \leq N||f||_{\mathbb{L}^q_p(\tau)}, \quad ||g_x||_{\mathbb{H}^{-1,q}_p(\tau)} \leq N||g||_{\mathbb{L}^q_p(\tau)}.$$

Therefore, Theorem 2.1 is applicable with $\gamma = -1$. By observing that $||u_x||_p \leq N(||u_{xx}||_{-1,p} + ||u||_p)$, from (2.2), we find that

$$||u_x||_{\mathbb{L}^q_p(\tau)} \leq N(||f||_{\mathbb{L}^q_p(\tau)} + ||g||_{\mathbb{L}^q_p(\tau)} + ||u||_{\mathbb{L}^q_p(\tau)}). \quad (2.10)$$

Next notice that, for any constant $c>0$, the function $u(c^2t,cx)$ satisfies an equation similar to (2.8) with $cf^i(c^2t,cx)$ and $cg(c^2t,cx)$ in place of f and g, respectively. For this function (2.10) with $c^{-2}\tau$ in place of τ becomes

$$||u_x||_{\mathbb{L}_p^q(\tau)} \le N(||f||_{\mathbb{L}_p^q(\tau)} + ||g||_{\mathbb{L}_p^q(\tau)} + c^{-1}||u||_{\mathbb{L}_p^q(\tau)}).$$

By letting $c\to\infty$, we arrive at (2.9).

Corollary 2.6. *Let $T\in(0,\infty)$, $\tau\le T$, $q\ge p\ge 2$, $Mf\in\mathbb{L}_p^q(\tau)$, and $g\in\mathbb{L}_p^q(\tau)$. Assume that f is an odd function with respect to x^1. Then, in $\mathcal{H}_p^{1,q}(\tau)$, equation (1.1) with zero initial condition has a unique solution. For this solution we have*

$$||u_x||_{\mathbb{L}_p^q(\tau)} \le N(||Mf||_{\mathbb{L}_p^q(\tau)} + ||g||_{\mathbb{L}_p^q(\tau)}), \tag{2.11}$$

where $N=N(d,p,q,\delta)$.

To deduce this from Corollary 2.5, it suffices to notice that there exists a function F, which is even with respect to x^1, satisfies $F_{x^1}=f$, and (Hardy's inequality)

$$||F(t,\cdot)||_p \le p||Mf(t,\cdot)||_p.$$

Corollary 2.6, which hold for any σ, plays an important role in our treatment of SPDEs in $\mathbb{R}^d_+$ in the case $\sigma^{1k}\equiv 0$, $k=1,2,....$

3. SPDEs in $\mathbb{R}^d_+$

In this section we consider equation (1.1) for $x\in\mathbb{R}^d_+$ assuming that its coefficients satisfy the conditions listed in the beginning of Sec. 2. It is well known that (1.1) in $\mathbb{R}^d_+$ is solvable in Sobolev spaces only if they are provided with weights. Therefore, we use the spaces $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ introduced in [7] and recalled in Introduction.

Before stating our main results we point out that the conditions (3.1) and (3.2) below play an important role. Therefore, it is worth noting that, if $q=np$ with $n=1,2,...$, they become

$$d-1+p>\theta\ge d-1+p-1/n-\chi,$$
$$(1-\delta)a^{ij}(t)\lambda^i\lambda^j \ge \alpha^{ij}(t)\lambda^i\lambda^j + \alpha^{11}(t)(\lambda^1)^2(q-p)/(p-1)$$

respectively. Notice that these conditions are much more restrictive than $d-1+p>\theta>d-1$, which we have in the deterministic case (see [8]).

Theorem 3.1. *Let $q\ge p\ge 2$, $\varepsilon\in(0,2/q)$, $T\in(0,\infty)$, $\tau\le T$, $\gamma\in\mathbb{R}$. Let $Mf\in\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)$, $g\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$, $M^{2/q-1-\varepsilon}u_0\in L_q(\Omega,\mathcal{F}_0,H^{\gamma-2/q+\varepsilon}_{p,\theta})$ and one of the following conditions hold*

$$d-1+p>\theta\ge d-1+p-\chi-1/\lceil q/p\rceil, \tag{3.1}$$

where $\chi=\chi(d,p,q,\delta)>0$ is a (small) constant to be specified in the proof,

$$\text{or}\quad \begin{cases} d-1+p>\theta>d-2+p \quad \text{and} \\ (1-\delta)a^{ij}(t)\lambda^i\lambda^j \ge \alpha^{ij}(t)\lambda^i\lambda^j + \alpha^{11}(t)(\lambda^1)^2(\lceil q/p\rceil-1)p/(p-1) \end{cases} \tag{3.2}$$

for all $\lambda\in\mathbb{R}^d$ and $t\ge 0$, where $\lceil r\rceil$ is the smallest integer $\ge r$. Then in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ there is a unique solution of equation (1.1) with initial condition $u(0)=u_0$. For this solution, we have

$$||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)} \le N(||Mf||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)} + ||g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)} + E||M^{2/q-1-\varepsilon}u_0||^q_{H^{\gamma-2/q+\varepsilon}_{p,\theta}}), \tag{3.3}$$

where the constant $N=N(\gamma,d,p,q,\delta,\varepsilon,T)$. In addition, if $u_0=0$, then estimate (3.3) holds with $N=N(\gamma,d,p,q,\delta)$ and if $q=p$, one can take $\varepsilon=0$ and $N=N(\gamma,d,p,q,\delta)$.

Theorem 3.2. *Let $q \geq p \geq 2$, $\varepsilon \in (0, 2/q)$, $T \in (0, \infty)$, $\tau \leq T$, $\gamma \in \mathbb{R}$,*

$$d - 1 + p > \theta \geq d - \chi, \quad \sigma^{1k} \equiv 0 \quad \forall k,$$

$Mf \in \mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$, $M^{2/q-1-\varepsilon}u_0 \in L_q(\Omega, \mathcal{F}_0, H^{\gamma-2/q+\varepsilon}_{p,\theta})$. Then all the assertions of Theorem 3.1 hold true again.

Remark 3.3. If $q = p$, (3.1) becomes $d - 1 + p > \theta \geq d - 2 + p - \chi$. We know from [10] that, if $p = q$, then the assertions of Theorems 3.1 and 3.2 hold with $\varepsilon = 0$ and N independent of T under somewhat weaker assumptions on θ:

$$d - 1 + p\big(1 - \frac{1}{p(1-\delta_1) + \delta_1}\big) < \theta < d - 1 + p, \tag{3.4}$$

where δ_1 is any constant satisfying $(1 - \delta_1)a \geq \alpha$. Also observe that, generally for $q \geq p$, the uniqueness of solutions in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ follows from that in $\mathfrak{H}^{\gamma}_{p,\theta}(\tau) = \mathfrak{H}^{\gamma,p}_{p,\theta}(\tau)$ and hence follows from [10].

The proofs of Theorems 3.1 and 3.2 are given in Sec. 5. In this section we discuss the result of Theorem 3.1 in the sequence of remark bearing on the following example.

Example 3.4. For $d = 1$, consider the equation

$$du = u_{xx}\,dt + \sigma u_x\,dw_t \tag{3.5}$$

with nonrandom initial data $u_0 \in C_0^\infty(\mathbb{R}_+)$, $u_0 \geq 0$, $u_0 \not\equiv 0$. Here σ is a nonnegative number satisfying $\sigma^2 < 2$ (in accordance with (2.1)) and w_t is a one-dimensional Wiener process.

Remark 3.5. Condition (3.4) and, for $q = p$, even its slightly stronger version (3.1) look quite reasonable. To see this in Example 3.4, take $m = 1, 2, ...$, $q = mp$, and $\theta = p - 1/m - \chi$. Then condition (3.1) is satisfied and by Theorem 3.1 we get that $u \in \mathfrak{H}^{\gamma,mp}_{p,\theta}(1)$ for any γ. By embedding theorems (see Theorem 3.11 of [7])

$$E\int_0^1 \sup_{x>0} x^{-1-m\chi}|u(t,x)|^{mp}\,dt < \infty. \tag{3.6}$$

It turns out that, for any $n \in \{2, 3, ...\}$ and $\varepsilon > 0$, one can find $\sigma^2 < 2$ such that

$$\lim_{x\downarrow 0}\inf_{t\in[1/2,1]} x^{-1-\varepsilon}Eu^n(t,x) = \infty, \tag{3.7}$$

so that (3.6) cannot be substantially improved. We will not use this result and only give an explanation why this happens. We also notice that (3.7) contradicts Theorem 1.1 of [3] (the fact that there is a gap in the proof of this theorem is noticed in [9]).

By the Itô-Wentzell formula, the function $\bar{u}(t,x) := u(t, x - \sigma w_t)$ satisfies the deterministic equation

$$d\bar{u}(t,x) = (1 - \sigma^2/2)\bar{u}_{xx}(t,x)\,dt \tag{3.8}$$

in the random domain $t > 0, x > \sigma w_t$ with boundary conditions $\bar{u}|_{t=0} = u_0$, $\bar{u}(t, \sigma w_t) = 0$. It follows that $\bar{u} \geq 0$, $u \geq 0$, and $u(t,x) > 0$ for $t > 0, x > 0$.

Next, owing to Itô's formula, for $x = (x_1, ..., x_n)$, the function $v(t,x) := u(t,x_1) \cdot ... \cdot u(t,x_n)$ satisfies

$$dv = (\Delta v + \beta\sum_{i\neq j} v_{x_i x_j})\,dt + \sigma(v_{x_1} + ... + v_{x_n})\,dw_t,$$

where $\beta := \sigma^2/2$. By taking expectations, for $V = Ev$ we find

$$\frac{\partial V}{\partial t} = \Delta V + \beta\sum_{i\neq j} V_{x_i x_j} =: L_{n,\beta}V, \quad x_1, ..., x_n > 0.$$

Also $V = 0$ for $\min_i x_i = 0$, $V(0,x) \geq 0$, and $V(t,x) > 0$ for $t, x_1, ..., x_n > 0$.

Now let P be a positive solution of $L_{n,\beta}P = 0$ in

$$G := \{\min_i x_i > 0\} \cap \{|x| < 1\}$$

with zero boundary condition on $\min_i x_i = 0$ and some nonzero nonnegative condition on $|x| = 1$. It follows from the theory of parabolic equations (see, for instance, Theorem 4.3 of [1]) that, for $t \in [1/2, 1]$ and $|x| \leq 1/2$, we have $V(t,x) \geq \eta P(x)$, where the constant $\eta > 0$.

To estimate P from below, define κ by formula (6.9) of Sec. 6 and define $s_0 \in (0,1)$ and $\gamma > 1$ by

$$s_0^2 = \frac{(n-1)(1-\kappa)^2}{n(1-\kappa)^2 + 2\kappa(1-\kappa) + \kappa^2},$$

$$\gamma(\gamma + n - 2) = n - 1 + (n^2 - 1)s_0.$$

Observe that, if $\sigma^2 \uparrow 2$, then $\beta \uparrow 1$, $\kappa \uparrow 1$, $s_0 \to 0$, and $\gamma \to 1$. Finally let $e := (1, ..., 1)/\sqrt{n}$, $y = x - \kappa(x,e)e$, $s = (y,e)/|y|$, $f(s) = (s - s_0)^{1+(n-1)s_0}$, and

$$P_0(x) := |y|^\gamma f(s) =: R_0(y).$$

One can check that, in the domain $\{s > s_0\}$, we have

$$\frac{n-1}{ns_0^2}|y|^{2-\gamma}L_{n,\beta}P_0(x) = |y|^{2-\gamma}\Delta_y R_0(y)$$

$$= (1-s^2)f''(s) - (n-1)sf'(s) + \gamma(\gamma+n-2)f(s)$$

$$= s_0(n-1)[s^2(1-(n-1)s_0^2) - (1+(n+3)s_0)s + 1 + ns_0 + (n+1)^2s_0^2].$$

If $s_0 = 0$, the expression in the brackets becomes $s^2 - s + 1$, which is strictly positive. Therefore, there is a small $\bar{s}_0 \in (0,1)$ such that, for any $s_0 \in (0, \bar{s}_0)$, we have $L_{n,\beta}P_0 \geq 0$ for $s \in (s_0, 1]$. It is easy to check that $\{s_0 < s \leq 1, |x| < 1\} \subset G$. Also $P_0 = 0 \leq P$ for $s = s_0$. Therefore, if we take $P = P_0$ for $|x| = 1$, then by the maximum principle $P \geq P_0$ in $\{s_0 \leq s \leq 1, |x| \leq 1\}$. In particular, if $s = 1$ and $t \in [1/2, 1]$ and $x \in [0, 1/d]$, then we have

$$V(t, x, ..., x) \geq \eta P(x, ..., x) = \eta(1-s_0)^{1+(n-1)s_0}x^\gamma(1-\kappa)^\gamma. \tag{3.9}$$

As we have pointed out above, if σ^2 is close to 2, then $\gamma < 1 + \varepsilon$ and (3.9) implies (3.7) owing to $Eu^n(t,x) = V(t, x, ..., x)$.

Remark 3.6. It turns out that Theorem 3.1 yields better properties of t-traces of u than the ones following from [10]. For instance, in Example 3.4 let $p \geq 2$, $n \in \{1, 2, ...\}$, $q = np$, and $\theta = p - 1/n - \chi$. Then condition (3.1) is satisfied and by Theorem 3.1 we get that $u \in \mathfrak{H}^{\gamma,np}_{p,\theta}(1)$ for any γ. By Theorem 4.1 of [7] this implies

$$E\sup_{t\leq 1}\Big(\int_0^\infty x^{1/n-1}|u(t,x)|^p\,dx\Big)^n < \infty$$

for any $p \geq 2$ and $n \geq 1$. Observe that the bigger n the better information about the behavior of u at $x = 0$ we get.

Remark 3.7. Estimate (3.6) shows that the function u from Example 3.4 vanishes at $x = 0$ in certain integral sense with respect to t. An interesting issue is whether u vanishes at $x = 0$ for all $t > 0$ at once (a.s.). One of approaches to resolve this issue could be using equation (3.8). However we do not know how to do this and instead we are again going to use embedding theorems.

Let $p \geq 2$, $n \in \{1, 2, ...\}$, $q := np$ and assume

$$\sigma^2 < 2\,\frac{p-1}{q-1}, \tag{3.10}$$

which holds, for instance, if $n = 1$. Then condition (3.2) is satisfied and by Theorem 3.1 we get that $u \in \mathfrak{H}^{\gamma,q}_{p,\theta}(1)$ for all γ if $\theta = p - 1 + \varepsilon$, where ε is any number in $(0, 1)$. By Theorem 4.1 of [7] this implies

$$E \sup_{t \leq 1} ||M^{\eta-1} u(t)||^q_{H^\gamma_{p,\theta}} < \infty$$

if $2/q < \eta \leq 1$. This and Lemma 2.2 of [7] yield

$$E \sup_{t \leq T, x > 0} (x^{\eta-(1-\varepsilon)/p} |u(t, x)|)^q < \infty. \tag{3.11}$$

One can get more specific results if $n_0 := 2/\sigma$ is an integer ≥ 3, so that $\sigma \leq 2/3$. In that case (3.10) with $n = n_0$ is satisfied if

$$p > \frac{2 - \sigma^2}{2 - \sigma^2 n_0} = \frac{2 - \sigma^2}{2 - 2\sigma} =: p_0.$$

Notice that for p sufficiently close to p_0 and η to $2/(n_0 p_0)$

$$\eta - (1 - \varepsilon)/p = 2/(n_0 p_0) - 1/p_0 + \varepsilon_1,$$

where $\varepsilon_1 > 0$ is as small as we like. By Hölder's inequality and (3.11) we get

$$E \sup_{t \leq T, x > 0} (x^{\varepsilon_1 - 2\xi_0} |u(t, x)|)^{r_0} < \infty, \tag{3.12}$$

where

$$\xi_0 = \frac{1 - \sigma^2}{2 - \sigma^2}\,, \quad r_0 = \frac{2 - \sigma^2}{\sigma - \sigma^2}\,.$$

It is interesting to compare this result with what can be obtained from Theorem 3.2 of [10], where $q = p$. There the restriction on θ and p is (3.4), that is

$$p \geq 2, \quad 1 - \frac{2}{\sigma^2(p-1) + 2} < \theta/p < 1.$$

If these inequalities hold, the results of [10] and [7] imply that

$$E \sup_{t \leq T, x > 0} (x^{\eta - 1 + \theta/p} |u(t, x)|)^p < \infty$$

if $1 \geq \eta > 2/p$. The freedom of choice of θ and η leads to

$$E \sup_{t \leq T, x > 0} (x^{\varepsilon - 2\xi} |u(t, x)|)^p < \infty, \tag{3.13}$$

where ε is any small strictly positive number and

$$\xi = \frac{1}{\sigma^2(p-1) + 2} - \frac{1}{p}\,.$$

We want to have ξ positive and as big as possible. The inequality $\xi > 0$ is equivalent to $\sigma^2 < (p-2)/(p-1)$, so that we need $\sigma < 1$ instead of $\sigma < 2/3$ above. It turns out that the largest value of ξ occurs for $p = r_0$ (which is compatible with $\sigma^2 < (p-2)/(p-1)$). For $p = r_0$ we have $\xi = \xi_0$, so that inequality (3.13) yields (3.12) under less restrictive assumption on σ.

We want to mention several more features of Example 3.4. The fact that we only allow n to be an integer makes (3.12) available only if $2/\sigma$ is an integer, whereas there is no such restriction in (3.13). We do not know how to properly interpolate our results between integers. If we knew, we would probably be able to get (3.12) for any $\sigma < 1$ from $L_q(L_p)$-theory.

By using the results of [10], we have proved above that u vanishes for $x = 0$ for all t (a.s.) only if $\sigma^2 < 1$. We do not know if the same is true for $1 \leq \sigma^2 < 2$. Although Theorem 3.1 is still available for θ satisfying (3.1), this range of θ does not allow to conclude that u is continuous up to the boundary for almost all ω.

4. Auxiliary results

To prove Theorems 3.1 and 3.2 we need several auxiliary results, which we prove in this section, and a barrier function, which is constructed in Sec. 6. First we prove a regularizing property for solutions of (1.1). Notice that there is no restriction on θ in Lemma 4.1.

Lemma 4.1. *Let $p \geq 2$, $n \in \{1, 2, ...\}$, $\gamma \geq \nu$, $\theta \in \mathbb{R}$, $u \in \mathfrak{H}^{\nu,np}_{p,\theta}(\tau)$, $u(0) = 0$, and u satisfy (1.1) in $(0, \tau]\!] \times \mathbb{R}^d_+$ with $Mf \in \mathbb{H}^{\gamma-2,np}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma-1,np}_{p,\theta}(\tau)$. Then $u \in \mathfrak{H}^{\gamma,np}_{p,\theta}(\tau)$ and*

$$||M^{-1}u||_{\mathbb{H}^{\gamma,np}_{p,\theta}(\tau)} \leq N(||Mf||_{\mathbb{H}^{\gamma-2,np}_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{\gamma-1,np}_{p,\theta}(\tau)} + ||M^{-1}u||_{\mathbb{H}^{\nu,np}_{p,\theta}(\tau)}), \tag{4.1}$$

where $N = N(\gamma, d, p, n, \delta)$.

Proof. Clearly (4.1) becomes stronger if ν decreases. Therefore we may assume that $\nu = \gamma - k$, where k is an integer, and bearing in mind an obvious induction, we see that, without loss of generality, we may let $\nu = \gamma - 1$.

Now notice that, for a function $\zeta \in C^\infty_0(\mathbb{R}_+)$ and $\zeta(x) := \zeta(x^1)$, we have

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}(\tau)} = E\int_0^\tau ||M^{-1}u(t)||^{np}_{H^\gamma_{p,\theta}}\,dt \leq NE\int_0^\tau ||u(t)||^{np}_{H^\gamma_{p,\theta-p}}\,dt$$

$$= N\sum_{m_1,...,m_n=-\infty}^{\infty} e^{(\theta-p)\bar{m}}E\int_0^\tau \prod_{i=1}^n ||u(t, e^{m_i}\,\cdot)\zeta||^p_{H^\gamma_p}\,dt, \tag{4.2}$$

with $\bar{m} := m_1 + ... + m_n$. Here

$$||u(t, e^m\,\cdot)\zeta||^p_{H^\gamma_p} \sim ||\Delta(u(t, e^m\,\cdot)\zeta)||^p_{H^{\gamma-2}_p}$$

$$= ||(1-\Delta)^{\gamma/2-1}\Delta(u(t, e^m\,\cdot)\zeta)||^p_p = e^{m(p\gamma-d)}||(\lambda_m - \Delta)^{\gamma/2-1}\Delta(u(t)\zeta_m)||^p_p,$$

where $\lambda_m = e^{-2m}$ and $\zeta_m(x) = \zeta(e^{-m}x)$. Furthermore,

$$d(u\zeta_m) = (a^{ij}(u\zeta_m)_{x^ix^j} + \bar{f}_m)\,dt + (\sigma^{ik}(u\zeta_m)_{x^i} + \bar{g}^k_m)\,dw^k_t,$$

where,

$$\bar{f}_m = f\zeta_m - 2a^{1j}\zeta_{mx^1}u_{x^j} - a^{11}u\zeta_{mx^1x^1}, \quad \bar{g}^k_m = g^k\zeta_m - \sigma^{1k}u\zeta_{mx^1}.$$

Similarly to the above computation

$$||(\lambda_m - \Delta)^{\gamma/2-1}\bar{f}_m(t)||^p_p = e^{-m(p\gamma-2p-d)}||\bar{f}_m(t, e^m\,\cdot)||^p_{H^{\gamma-2}_p},$$

$$||(\lambda_m - \Delta)^{\gamma/2-1}\bar{g}_{mx}(t)||^p_p = e^{-m(p\gamma-p-d)}||(\bar{g}_m(t, e^m\,\cdot))_x||^p_{H^{\gamma-2}_p}$$

$$\leq e^{-m(p\gamma-p-d)}N||\bar{g}_m(t, e^m\,\cdot)||^p_{H^{\gamma-1}_p}.$$

Therefore, by Lemma 2.3, for any $m_1, ..., m_n$, we have

$$E\int_0^\tau \prod_{i=1}^n ||u(t,e^{m_i}\cdot)\zeta||^p_{\gamma,p}\,dt \le NE\int_0^\tau \big(\sum_{i=1}^n e^{m_i p}J_i(t) + \sum_{1\le i<j\le n} e^{(m_i+m_j)p}J_{ij}(t)\big)\,dt,$$

where

$$J_i(t) := (e^{m_i p}||\bar f_{m_i}(t,e^{m_i}\cdot)||^p_{H^{\gamma-2}_p} + ||\bar g_{m_i}(t,e^{m_i}\cdot)||^p_{H^{\gamma-1}_p})\prod_{j\ne i}||u(t,e^{m_j}\cdot)\zeta||^p_{H^\gamma_p},$$

$$J_{ij}(t) := ||\bar g_{m_i}(t,e^{m_i}\cdot)||^p_{H^{\gamma-1}_p}||\bar g_{m_j}(t,e^{m_j}\cdot)||^p_{H^{\gamma-1}_p}\prod_{k\ne i,j}||u(t,e^{m_k}\cdot)\zeta||^p_{H^\gamma_p}.$$

Coming back to (4.2), we conclude

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}(\tau)} \le NE\int_0^\tau (F(t)+G(t))||u(t)||^{(n-1)p}_{H^\gamma_{p,\theta-p}}\,dt$$

$$+NE\int_0^\tau G^2(t)||u(t)||^{(n-2)p}_{H^\gamma_{p,\theta-p}}\,dt,$$

where

$$G(t) := \sum_{m=-\infty}^\infty e^{m\theta}||\bar g_m(t,e^m\cdot)||^p_{H^{\gamma-1}_p} \le N\sum_{m=-\infty}^\infty e^{m\theta}||g(t,e^m\cdot)\zeta||^p_{H^{\gamma-1}_p}$$

$$+N\sum_{m=-\infty}^\infty e^{m(\theta-p)}||u(t,e^m\cdot)\zeta'||^p_{H^{\gamma-1}_p} \le N(||g(t)||^p_{H^{\gamma-1}_{p,\theta}} + ||M^{-1}u(t)||^p_{H^{\gamma-1}_{p,\theta}}),$$

$$F(t) := \sum_{m=-\infty}^\infty e^{m(\theta+p)}||\bar f_m(t,e^m\cdot)||^p_{H^{\gamma-2}_p} \le N\sum_{m=-\infty}^\infty e^{m(\theta+p)}||f(t,e^m\cdot)\zeta||^p_{H^{\gamma-2}_p}$$

$$+N\sum_{m=-\infty}^\infty e^{m\theta}||u_x(t,e^m\cdot)\zeta'||^p_{H^{\gamma-2}_p} + N\sum_{m=-\infty}^\infty e^{m(\theta-p)}||u(t,e^m\cdot)\zeta''||^p_{H^{\gamma-2}_p}$$

$$\le N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||Mu_x(t)||^p_{H^{\gamma-2}_{p,\theta-p}} + ||M^{-1}u(t)||^p_{H^{\gamma-2}_{p,\theta}})$$

$$\le N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||u(t)||^p_{H^{\gamma-1}_{p,\theta-p}})$$

$$\le N(||Mf(t)||^p_{H^{\gamma-2}_{p,\theta}} + ||M^{-1}u(t)||^p_{H^{\gamma-1}_{p,\theta}}).$$

As in the end of the proof of Theorem 2.1, this yields

$$||M^{-1}u||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}(\tau)} \le NE\int_0^\tau (F^n(t)+G^n(t))\,dt$$

$$\le N(||Mf||^{np}_{\mathbb{H}^{\gamma-2,np}_{p,\theta}(\tau)} + ||g||^{np}_{\mathbb{H}^{\gamma-1,np}_{p,\theta}(\tau)} + ||M^{-1}u||^{np}_{\mathbb{H}^{\gamma-1,np}_{p,\theta}(\tau)}).$$

The lemma is proved.

This lemma reduces obtaining a priori estimates of higher order derivatives to estimating any lower order norm. In the next lemma we show that if we have an estimate of the zeroth order norm, then we have the solvability in all spaces $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$. First we give a definition.

Definition 4.2. For a function $u = u(\omega, t, x)$ we write $u \in \mathfrak{C}$ if

(i) the function u is defined on $\Omega \times [0, \infty) \times \mathbb{R}^d_+$, is predictable as a function of (ω, t), and is of class $C_0^\infty(\mathbb{R}^d_+)$ as a function of x with support belonging to the same compact subset of $\mathbb{R}^d_+$ for all (ω, t) and with the norms $||u(t, \cdot)||_{C^n(\mathbb{R}^d_+)}$ bounded on $\Omega \times [0, \infty)$ for any $n = 1, 2, ...$;

(ii) the function u is continuous as a function of t and there exists an integer $k < \infty$ such that, on $\Omega \times [0, \infty) \times \mathbb{R}^d_+$,

$$u(t, x) = \int_0^t f(s, x)\, ds + \sum_{i=1}^k \int_0^t g^i(s, x)\, dw^i_s,$$

where $f, g^1, ..., g^k$ are certain functions enjoying the properties listed in (i) for u and such that, for any multi-index α, the function

$$\sum_{i=1}^k \mathrm{Var}_{[0,\infty)}\, D^\alpha g^i(\cdot, x) + \mathrm{Var}_{[0,\infty)}\, D^\alpha f(\cdot, x) \tag{4.3}$$

is a bounded function of (ω, x). In that case, as usual, we denote $f = \mathbb{D}u,\ g = \mathbb{S}u$.

Lemma 4.3. *Let $p \geq 2$, $n \in \{1, 2, ...\}$, $\varepsilon \in (0, 2/(np))$, $\gamma \in \mathbb{R}$, $T \in (0, \infty)$, $\tau \leq T$,*

$$d - 1 + p > \theta > d - 1. \tag{4.4}$$

Define $\bar{a}^{11} = a^{11}$, $\bar{a}^{ij} = a^{ij}$ if $i \geq 2$ and $j \geq 2$, and $\bar{a}^{ij} = 0$ in all remaining cases. Assume that there is a constant N_0 such that, for any $u \in \mathfrak{C}$ and $\lambda \in [0, 1]$,

$$||M^{-1}u||_{\mathbb{L}^{np}_{p,\theta}(\tau)} \leq N_0(||M\big(\mathbb{D}u - (\lambda a^{ij} + (1 - \lambda)\bar{a}^{ij})u_{x^i x^j}\big)||_{\mathbb{L}^{np}_{p,\theta}(\tau)} + ||\mathbb{S}u - \lambda\sigma^{i\cdot}u_{x^i}||_{\mathbb{H}^{1,np}_{p,\theta}(\tau)}). \tag{4.5}$$

Then, whenever $Mf \in \mathbb{H}^{\gamma-2,np}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma-1,np}_{p,\theta}(\tau)$, and

$$M^{2/(np)-1-\varepsilon}u_0 \in L_{np}(\Omega, \mathcal{F}_0, H^{\gamma-2/(np)+\varepsilon}_{p,\theta}),$$

equation (1.1) with initial condition $u(0) = u_0$ has a unique solution in $\mathfrak{H}^{\gamma,np}_{p,\theta}(\tau)$. In addition, for this solution, estimate (3.3) holds for $q = np$ with $N = N(N_0, \gamma, d, p, n, \delta, T)$, whereas, if $u_0 = 0$, then $N = N(N_0, \gamma, d, p, n, \delta)$.

Proof. As we have already pointed out, uniqueness follows from [10]. While proving the existence of solutions, we can assume that $u_0 = 0$. Indeed, by Theorem 8.6 of [7], there exists a continuation operator, that is, for each ω, there exists a function $\bar{u} \in H^{\gamma,np}_{p,\theta}(T)$ such that $u(0) = u_0$ and

$$||M^{-1}\bar{u}||^{np}_{\mathbb{H}^{\gamma,np}_{p,\theta}(T)} + ||M\bar{u}_t||^{np}_{\mathbb{H}^{\gamma-2,np}_{p,\theta}(T)} \leq N||M^{2/(np)-1-\varepsilon}u_0||^{np}_{H^{\gamma-2/(np)+\varepsilon}_{p,\theta}},$$

where N depends only on $d, p, n, \varepsilon, \gamma, \theta$, and T. If we subtract $\bar{u}$ from u, we will reduce the problem to the one with zero initial condition and $\bar{f} = f + a^{ij}\bar{u}_{x^i x^j} - \bar{u}_t$ and $\bar{g}^k = g^k + \sigma^{ik}\bar{u}_{x^i}$ in place of f and g^k, respectively. In addition (see [5])

$$||M\bar{u}_{xx}(t)||_{H^{\gamma-2}_{p,\theta}} \leq N||\bar{u}_x(t)||_{H^{\gamma-1}_{p,\theta}} \leq N||M^{-1}\bar{u}(t)||_{H^{\gamma}_{p,\theta}}$$

with N depending only on d, p, γ, and θ, where condition (4.4) is not used. This implies that $\bar{f}$ and $\bar{g}$ are functions of the same classes as f and g, respectively. Therefore, in the rest of the proof we assume $u_0 = 0$.

The above argument is absolutely standard. The rest of the proof of existence on the basis of the apriori estimate (4.5) and the method of continuity is also standard and we only give the most important details. By the way, notice that Lemma 4.1 and assumption (4.5) along with the denseness of $\mathfrak{C}$ in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)\cap\{u(0)=0\}$ imply that, for $q=np$ and $\gamma\geq 2$, the apriori estimate (3.3) holds for any $u\in\mathfrak{H}^{\gamma,np}_{p,\theta}(\tau)$ with $u(0)=0$.

Case $\gamma=2$ and $a^{1j}\equiv 0$ for $j\geq 2$ and $\sigma^{ik}\equiv 0$. In that case first we take sufficiently smooth functions f and g vanishing in a neighborhood of $x^1=0$ and for large x^1 and continue them for $x^1<0$ as odd functions with respect to x^1. Then we get a solution $u\in\mathcal{H}^{2,np}_p(\tau)$ by Theorem 2.1. Since in a neighborhood of $x^1=0$ the function u satisfies a deterministic equation, as in Lemma 4.2 of [9], we get that $u\in\mathfrak{H}^{2,np}_{p,\theta}(\tau)$. Actually, here we only need $d+p\geq\theta>d-1$ instead of (4.4) but estimate (4.5) cannot hold for $\theta\geq d-1+p$ anyway. This gives the solvability of (1.1) for particular f, g. For general f and g we easily get our assertions by using approximations and the apriori estimate (3.3).

Case $\gamma\geq 2$ and general a and σ. If $\gamma=2$, general a and σ are considered on the basis of the method of continuity and the apriori estimate (3.3). After that, our theorem in the case $\gamma\geq 2$ follows directly from Lemma 4.1.

Case $\gamma<2$. One can just repeat the proof of Theorem 3.2 of [10]. The lemma is proved.

The following lemma will allow us to assume that α^{11} is constant. This will be done after enlarging our initial probability space. Suppose that we are given a complete probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde P)$ and an increasing family of complete σ-algebras $(\tilde{\mathcal{F}}_t,t\geq 0)$, $\tilde{\mathcal{F}}_t\subset\tilde{\mathcal{F}}$. Suppose that we are also given independent one-dimensional processes $\tilde w^k_t$ which are Wiener with respect to $(\tilde{\mathcal{F}}_t,t\geq 0)$. Define

$$(\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{F}}_t,\bar P)=(\Omega,\mathcal{F},\mathcal{F}_t,P)\times(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde P),$$

define $\bar{\mathcal{P}}$ as the predictable σ-algebra relative to $(\bar{\mathcal{F}}_t,t\geq 0)$ and introduce the spaces $\bar{\mathbb{H}}^{\gamma,q}_{p,\theta}(\tau)$ and $\bar{\mathfrak{H}}^{\gamma,q}_{p,\theta}(\tau)$ on the basis of $\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{F}}_t,\bar P$ and the Wiener processes $w^k_t,\tilde w^r_t,k,r=1,2,....$

Lemma 4.4. *In the above notation, the operator $\Pi:u\to\Pi u$ with*

$$\Pi u(\omega,t,x):=\int_{\tilde\Omega}u(\omega,\tilde\omega,t,x)\,\tilde P(d\tilde\omega)$$

is a bounded operator with unit norm mapping the spaces $\bar{\mathbb{H}}^{\gamma,q}_{p,\theta}(\tau)$ and $\bar{\mathfrak{H}}^{\gamma,q}_{p,\theta}(\tau)$ onto $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, respectively. In addition, $\mathbb{D}\Pi=\Pi\mathbb{D}$ and $\mathbb{S}\Pi=\Pi\mathbb{S}$. Moreover, $\mathbb{S}^k\Pi=0$ if k corresponds to the additional Wiener processes $\tilde w^k_t$.

Proof. Integration theory shows that Π preserves measurability and integrability properties involved in the definitions of $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$. Also, obviously, $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ are subspaces of $\bar{\mathbb{H}}^{\gamma,q}_{p,\theta}(\tau)$ and $\bar{\mathfrak{H}}^{\gamma,q}_{p,\theta}(\tau)$, respectively, and $\Pi\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)=\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$, $\Pi\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)=\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$. Therefore, mapping Π is onto.

The fact that Π maps $\bar{\mathbb{H}}^{\gamma,q}_{p,\theta}(\tau)$ to $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ with unit norm follows easily from Minkowski's inequality.

Since the set $\bar{\mathfrak{C}}$, constructed as $\mathfrak{C}$ on the basis of $\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{F}}_t,w^k_t,\tilde w^r_t$, and $\bar P$, is everywhere dense in $\bar{\mathfrak{H}}^{\gamma,q}_{p,\theta}(\tau)$ and the operators $\mathbb{D}$ and $\mathbb{S}$ are bounded, to prove our assertions concerning $\bar{\mathfrak{H}}^{\gamma,q}_{p,\theta}(\tau)$ it suffices to concentrate on elements of $\bar{\mathfrak{C}}$. For $u\in\bar{\mathfrak{C}}$, the equalities $\Pi\mathbb{D}u=\mathbb{D}\Pi u$ and $\Pi\mathbb{S}u=\mathbb{S}\Pi u$ follow immediately from Fubini's theorem in its usual version. Its stochastic version is not needed since under (4.3) stochastic integrals of $\mathbb{S}u$ are rewritten as usual integrals using integration by parts. Finally, for each k,

$$\int_{\tilde\Omega}\Big(\int_0^t g(\tilde\omega,s)\,d\tilde w^k_s(\tilde\omega)\Big)\,\tilde P(d\tilde\omega)=\tilde E\int_0^t g(s)\,d\tilde w^k_s=0,$$

where $\tilde E$ is the symbol of expectation relative to $\tilde P$, if g is appropriately measurable and $\tilde E\int_0^t|g(s)|^2\,ds<\infty$. This easily implies that $\Pi\mathbb{S}^k u=0$ if the term $\mathbb{S}^k u\,d\tilde w^k_t$ is part of the stochastic differential of u. The lemma is proved.

The following two lemmas contain the most crucial information needed for proving Theorems 3.1 and 3.2. In the proofs of these lemmas we use the barrier constructed in Sec. 6.

Lemma 4.5. *Let $p\geq 2$, $n\geq 1$ be an integer, and*

$$d-1+p>\theta\geq d-1+p-1/n.$$

Assume that $a^{11}\equiv 1$ and α^{11} is a constant. Then there exists a constant $N_0=N_0(d,p,n,\theta,\delta)<\infty$ such that (4.5) holds for any $u\in\mathfrak{C}$, $\lambda\in[0,1]$, and stopping time τ with $\mathbb{L}^{np}_{p,\theta}(\tau)$ in place of $\mathbb{H}^{1,np}_{p,\theta}(\tau)$.

Proof. Notice at once that, if $n=1$, the statement of the lemma is known from [10] even for a wider range of θ (see (3.4)). Therefore, we assume that $n\geq 2$. Furthermore, upon taking a $u\in\mathfrak{C}$ and using standard stopping times one easily reduces the situation of general τ to the one in which all terms in (4.5) are finite and τ is bounded. Finally obviously, assuming $\lambda=1$ does not restrict generality.

Denote $f=\mathbb{D}u-a^{ij}u_{x^ix^j}$ and $g=\mathbb{S}u-\sigma^{i\cdot}u_{x^i}$, so that

$$u(t,x)=\int_0^t(a^{ij}u_{x^ix^j}(s,x)+f(s,x))\,ds+\int_0^t(\sigma^{ik}u_{x^i}(s,x)+g^k(s,x))\,dw^k_s.$$

Next, for $X=(x_1,...,x_n)\in(\mathbb{R}^d_+)^n$ with $x_i\in\mathbb{R}^d_+$, define

$$U(t,X)=u(t,x_1)\cdot...\cdot u(t,x_n).$$

Observe that by Itô's formula

$$dU(t,X)=(\boldsymbol{L}U(t,X)+F(t,X)+H(t,X))\,dt+G^k(t,X)\,dw^k_t, \tag{4.6}$$

where $\boldsymbol{L}U=a^{rs}(U_{x^r_1x^s_1}+...+U_{x^r_nx^s_n})$,

$$F(t,X)=\sum_{i=1}^n f(t,x_i)\prod_{j\neq i}u(t,x_j)=U(t,X)\sum_{i=1}^n(u^{-1}f)(t,x_i),$$

$$G^k=\sum_{i=1}^n G^k_i,\quad G^k_i(t,X)=\sigma^{rk}U_{x^r_i}(t,X)+U(t,X)(u^{-1}g^k)(t,x_i),$$

$$H=U^{-1}\sum_{1\leq i<j\leq n}G^k_iG^k_j.$$

Finally, fix a $\beta\in[0,1)$ which will be specified later, and take the function $\Phi(x)$ from Theorem 6.1 corresponding to this β and $\alpha=d+p-\theta$, which is possible since in notation of Theorem 6.1 we have $1<d+p-\theta<\bar\alpha(n,\beta)$. Define $\Phi(X)=\Phi(x^1_1,...,x^1_n)$, and apply Itô's formula to $\Phi|U|^p$. Then from (4.6) we find that (a.s.) for all $X\in(\mathbb{R}^d_+)^n$

$$\Phi(X)|U(\tau,X)|^p=\int_0^\tau\Phi(X)\big[p|U|^{p-2}U(\boldsymbol{L}U+F+H)(s,X)$$

$$+\tfrac{1}{2}p(p-1)|U(s,X)|^{p-2}\sum_k (G^k(s,X))^2\big]\,ds+\int_0^\tau p\Phi(X)|U|^{p-2}UG^k(s,X)\,dw_s^k. \tag{4.7}$$

Since $u\in\mathfrak{C}$, we can take the expectations of both parts of this equality, drop the expectation of the stochastic integral, integrate with respect to X over $(\mathbb{R}^d_+)^n$, and use Fubini's theorem to integrate first with respect to X. While integrating with respect to X, we integrate by parts using the fact that Φ depends only on $x_1^1,...,x_n^1$. First we find that

$$\int_{(\mathbb{R}^d_+)^n}\Phi(X)p|U|^{p-2}U\boldsymbol{L}U(s,X)\,dX=-I(s)-p(p-1)\delta^{ij}a^{rt}J^{rt}_{ij}(s),$$

where,

$$I(s):=\int_{(\mathbb{R}^d_+)^n}a^{rt}\Phi_{x_i^t}(X)(|U(s,X)|^p)_{x_i^r}\,dX,$$

$$J^{rt}_{ij}(s):=\int_{(\mathbb{R}^d_+)^n}\Phi(X)|U|^{p-2}U_{x_i^t}(s,X)U_{x_j^r}(s,X)\,dX.$$

By denoting $\boldsymbol{M}$ the operator of multiplying by $x_1^1\cdot...\cdot x_n^1$, remembering that $a^{11}=1$, and using (6.1), we further find

$$I(s)=-\int_{(\mathbb{R}^d_+)^n}\delta^{ij}\Phi_{x_i^1x_j^1}(X)|U(s,X)|^p\,dX=\beta K(s)$$

$$+\int_{(\mathbb{R}^d_+)^n}\boldsymbol{M}^{\theta-d-p}|U(s,X)|^p\,dX=\beta K(s)+||M^{-1}u(s,\cdot)||^{np}_{L_{p,\theta}}, \tag{4.8}$$

where

$$K(s):=\sum_{i\neq j}\int_{(\mathbb{R}^d_+)^n}\Phi_{x_i^1x_j^1}(X)|U(s,X)|^p\,dX.$$

Thus,

$$\int_{(\mathbb{R}^d_+)^n}\Phi(X)p|U|^{p-2}U\boldsymbol{L}U(s,X)\,dX$$

$$=-\beta K(s)-||M^{-1}u(s,\cdot)||^{np}_{L_{p,\theta}}-p(p-1)\delta^{ij}a^{rt}J^{rt}_{ij}(s). \tag{4.9}$$

More terms J^{rt}_{ij} come from the part in (4.7) containing H and $(G^k)^2$. To estimate these terms we notice that

$$|U(s,X)(u^{-1}g^k)(s,x_i)\sigma^{rk}U_{x_j^r}(s,X)|$$

$$\le|U(s,X)u^{-1}g(s,x_i)|(2\alpha^{rt}U_{x_j^r}(s,X)U_{x_j^t}(s,X))^{1/2},$$

so that

$$|U(s,X)(u^{-1}g^k)(s,x_i)\sigma^{rk}U_{x_j^r}(s,X)|$$

$$\le\varepsilon\alpha^{rt}U_{x_j^r}(s,X)U_{x_j^t}(s,X)+NU^2(s,X)\sum_i|u^{-1}g(s,x_i)|^2,$$

where $\varepsilon>0$ is arbitrary and $N=N(\varepsilon)$. This shows

$$p|U|^{p-2}UH+\frac{1}{2}p(p-1)|U|^{p-2}\sum_k|G^k|^2=p|U|^{p-2}\big(\sum_{1\le i<j\le n}G_i^kG_j^k+\frac{1}{2}(p-1)\sum_{1\le i,j\le n}G_i^kG_j^k\big)$$

$$\le p(p-1)(1+\varepsilon)|U|^{p-2}\delta^{ij}\alpha^{rt}U_{x_i^r}U_{x_j^t}+p^2|U|^{p-2}\sum_{i\neq j}\alpha^{rt}U_{x_i^r}U_{x_j^t}$$

$$+N|U|^p \sum_i |u^{-1}g(s,x_i)|^2.$$

Also notice that due to (6.2), for each i,

$$\int_{\mathbb{R}^d_+} \Phi(X)|U(s,X)|^p|u^{-1}g(s,x_i)|^2\,dx_i$$

$$\leq N\int_{\mathbb{R}^d_+}(x_i^1)^{\theta-d}|M^{-1}u(s,x_i)|^{p-2}|g(s,x_i)|^2\,dx_i\prod_{j\neq i}(x_j^1)^{\theta-d}|(M^{-1}u)(s,x_j)|^p$$

$$\leq N||M^{-1}u(s,\cdot)||^{p-2}_{L_{p,\theta}}||g(s,\cdot)||^2_{L_{p,\theta}}\prod_{j\neq i}(x_j^1)^{\theta-d}|(M^{-1}u)(s,x_j)|^p,$$

which implies that

$$\int_{(\mathbb{R}^d_+)^n} \Phi(X)|U(s,X)|^p|u^{-1}g(s,x_i)|^2\,dX$$

$$\leq N||M^{-1}u(s,\cdot)||^{np-2}_{L_{p,\theta}}||g(s,\cdot)||^2_{L_{p,\theta}}$$

$$\leq \varepsilon||M^{-1}u(s,\cdot)||^{np}_{L_{p,\theta}}+N||g(s,\cdot)||^{np}_{L_{p,\theta}}.$$

Therefore,

$$\int_{(\mathbb{R}^d_+)^n}\Phi(X)\big[p|U|^{p-2}UH+\frac{1}{2}p(p-1)|U|^{p-2}\sum_k|G^k|^2(s,X)\big]\,dX$$

$$\leq p(p-1)(1+\varepsilon)\delta^{ij}\alpha^{rt}(s)J^{rt}_{ij}(s)+p^2\sum_{i\neq j}\alpha^{rt}(s)J^{rt}_{ij}(s)$$

$$+\varepsilon||M^{-1}u(s,\cdot)||^{np}_{L_{p,\theta}}+N||g(s,\cdot)||^{np}_{L_{p,\theta}} \quad (4.10)$$

To estimate the last term in (4.7) containing F, notice that by (6.2), for each i,

$$\int_{\mathbb{R}^d_+}\Phi(X)|U(s,X)|^{p-1}|f(s,x_i)\prod_{j\neq i}u(s,x_j)|\,dx_i$$

$$=\int_{\mathbb{R}^d_+}\Phi(X)|u(s,x_i)|^{p-1}|f(s,x_i)|\,dx_i\prod_{j\neq i}|u(s,x_j)|^p$$

$$\leq NJ_i(s)\prod_{j\neq i}(x_j^1)^{\theta-d}|(M^{-1}u)(s,x_j)|^p,$$

where

$$J_i(s):=\int_{\mathbb{R}^d_+}(x_i^1)^{\theta-d}|(M^{-1}u)(s,x_i)|^{p-1}|(Mf)(s,x_i)|\,dx_i,$$

which by Hölder's inequality is less than $||M^{-1}u(s,\cdot)||^{p-1}_{L_{p,\theta}}||Mf(s,\cdot)||_{L_{p,\theta}}$. It follows that

$$\int_{(\mathbb{R}^d_+)^n}\Phi(X)|U(s,X)|^{p-1}|f(s,x_i)\prod_{j\neq i}u(s,x_j)|\,dX$$

$$\leq N||M^{-1}u(s,\cdot)||^{(n-1)p+p-1}_{L_{p,\theta}}||Mf(s,\cdot)||_{L_{p,\theta}},$$

$$\int_{(\mathbb{R}^d_+)^n}\Phi(X)p|U|^{p-1}|F|(s,X)\,dX\leq\varepsilon||M^{-1}u(s,\cdot)||^{np}_{L_{p,\theta}}+N||Mf(s,\cdot)||^{np}_{L_{p,\theta}}.$$

Upon combining this estimate with (4.10) and (4.9) and coming back to (4.7) we get

$$0 \le -\beta E \int_0^\tau K(s)\,ds - ||M^{-1}u||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} - p(p-1)\delta^{ij} E\int_0^\tau a^{rt} J^{rt}_{ij}(s)\,ds$$

$$+p(p-1)(1+\varepsilon)\delta^{ij}E\int_0^\tau \alpha^{rt}J^{rt}_{ij}(s)\,ds + p^2\sum_{i\neq j}E\int_0^\tau \alpha^{rt}J^{rt}_{ij}(s)\,ds$$

$$+\varepsilon||M^{-1}u||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} + N||Mf||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} + N||g||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)}.$$

Hence, if $\varepsilon \le 1/2$,

$$\frac{1}{2}\,||M^{-1}u||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} \le -\delta^{ij}E\int_0^\tau \{p(p-1)[a^{rt}-\alpha^{rt}] - \varepsilon\alpha^{rt}\}J^{rt}_{ij}(s)\,ds \tag{4.11}$$

$$+N(||Mf||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} + ||g||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)}) + R,$$

where

$$R = p^2\sum_{i\neq j}E\int_0^\tau \alpha^{rt}J^{rt}_{ij}(s)\,ds - \beta E\int_0^\tau K(s)\,ds$$

$$= p^2\sum_{i\neq j}E\int_0^\tau \alpha^{rt}J^{rt}_{ij}(s)\,ds - \beta E\int_0^\tau \sum_{i\neq j}\int_{(\mathbb{R}^d_+)^n}\Phi_{x^1_i x^1_j}(X)|U(s,X)|^p\,dXds.$$

The matrix $p(p-1)(a^{rt}-\alpha^{rt}) - \varepsilon(\alpha^{rt})$ is nonnegative (actually, strictly positive) if ε is small enough. Therefore by (4.11)

$$||M^{-1}u||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} \le N(||Mf||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)} + ||g||^{np}_{\mathbb{L}^{np}_{p,\theta}(\tau)}) + 2R, \tag{4.12}$$

Now we specify the choice of β. We take $\beta = \alpha^{11}$ and then prove that $R=0$. To do this take $i\neq j$ and integrate by parts using the fact that Φ depends only on $x^1_1,...,x^1_n$ and that

$$U_{x^r_i x^t_j} = U^{-1}U_{x^r_i}U_{x^t_j}$$

if $i\neq j$. For such i and j, we find

$$(p-1)J^{rt}_{ij}(s) = \int_{(\mathbb{R}^d_+)^n}\Phi(X)(|U|^{p-2}U)_{x^r_i}U_{x^t_j}(s,X)\,dX$$

$$= -\int_{(\mathbb{R}^d_+)^n}\Phi(X)|U|^{p-2}UU_{x^r_i x^t_j}(s,X)\,dX$$

$$-p^{-1}\delta^{r1}\int_{(\mathbb{R}^d_+)^n}\Phi_{x^1_i}(X)(|U|^p)_{x^t_j}(s,X)\,dX$$

$$= -J^{rt}_{ij}(s) + p^{-1}\delta^{r1}\delta^{t1}\int_{(\mathbb{R}^d_+)^n}\Phi_{x^1_i x^1_j}(X)|U(s,X)|^p\,dX.$$

By collecting like terms in the above computations, we obtain

$$p^2J^{rt}_{ij}(s) = \delta^{r1}\delta^{t1}\int_{(\mathbb{R}^d_+)^n}\Phi_{x^1_i x^1_j}(X)|U(s,X)|^p\,dX. \tag{4.13}$$

This and the equality $\alpha^{11}=\beta$ imply that $R=0$ indeed. Now (4.12) becomes (4.5) and the lemma is proved.

Next we give a version of Lemma 4.5 for a wider range of θ but with a nontrivial restriction on α^{11}. In the following lemma we make the second condition in (2.1) stronger.

Lemma 4.6. *Let $p \geq 2$, $n \geq 1$ be an integer, and*
$$d-1+p > \theta > d-2+p.$$
Assume that, for any $\lambda \in \mathbb{R}^d$ and $s \geq 0$,
$$(1-\delta)a^{rt}(s)\lambda^r\lambda^t \geq \alpha^{rt}(s)\lambda^r\lambda^t + \alpha^{11}(s)(\lambda^1)^2(n-1)p/(p-1). \tag{4.14}$$
Then the assertion of Lemma 4.5 holds true again.

Proof. Standard random time change immediately reduces the situation to the one in which $a^{11} \equiv 1$. Then we take the function $\Phi(X)$ from Theorem 6.1 corresponding to $\alpha = d+p-\theta$ and $\beta = 0$. After that we repeat the proof of Lemma 4.5 word for word up to formula (4.11). However, this time generally $\alpha^{11} \neq 0 = \beta$ and we have to deal with R from (4.11) differently. Formula (4.13) shows that $J^{rt}_{ij}(s) = \delta^{r1}\delta^{t1}J^{11}_{ij}(s)$. By using the fact that, for any $\lambda \in \mathbb{R}^n$,
$$\sum_{i\neq j}\lambda_i\lambda_j \leq (n-1)\sum_i \lambda_i^2,$$
we get
$$R = p^2\sum_{i\neq j}E\int_0^\tau \alpha^{11}J^{11}_{ij}(s)\,ds \leq p^2(n-1)\sum_i E\int_0^\tau \alpha^{11}J^{11}_{ii}(s)\,ds.$$
We combine this term with the first term on the right in (4.11) and notice that, due to (4.14),
$$p(p-1)[a^{rt}-\alpha^{rt}]J^{rt}_{ii}(s) - p^2(n-1)\alpha^{11}J^{rt}_{ii} \geq \varepsilon\alpha^{rt}J^{rt}_{ii}(s)$$
if ε is small enough. Then we see that (4.11) implies (4.5). The lemma is proved.

5. Proofs of Theorems 3.1 and 3.2

First we show how to take care of χ in Theorems 3.1 and 3.2.

Lemma 5.1. *Let $p,q \in (1,\infty)$, $\theta_0 \in \mathbb{R}$, τ be a stopping time. Assume that there exists a constant N_0 such that, for any $u \in \mathfrak{H}^{2,q}_{p,\theta_0}(\tau)$ satisfying $u(0)=0$, we have*
$$||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leq N(||M(\mathbb{D}u - a^{ij}u_{x^ix^j})||_{\mathbb{L}^q_{p,\theta}(\tau)} + ||\mathbb{S}u - \sigma^{i\cdot}u_{x^i}||_{\mathbb{H}^{1,q}_{p,\theta}(\tau)}), \tag{5.1}$$
where $\theta = \theta_0$ and $N = N_0$. Then there exists $\chi = \chi(d,p,\delta,\theta_0,N_0) > 0$ such that, for any $\theta \in (\theta_0-\chi, \theta_0+\chi)$, estimate (5.1) holds with $N = N(d,p,\delta,\theta_0)N_0$ whenever $u \in \mathfrak{H}^{2,q}_{p,\theta}(\tau)$ and $u(0)=0$.

Proof. We use a simple perturbation argument. Remember (see [5]) that
$$H^2_{p,\theta} = \{v : v, Mv_x, M^2v_{xx} \in L_p(\mathbb{R}^d_+, (x^1)^{\theta-d}\,dx)\}.$$
It follows easily that if $|\theta-\theta_0| \leq 1$, then
$$||v||_{H^2_{p,\theta}} \leq N_1(d,p,\theta_0)||M^{(\theta-\theta_0)/p}v||_{H^2_{p,\theta_0}} \leq N_2(d,p,\theta_0)||v||_{H^2_{p,\theta}}.$$
(Actually, we used this fact from [5] in a more general setting in the proof of Lemma 4.1.) Therefore, $u \in \mathfrak{H}^{2,q}_{p,\theta}(\tau)$ if and only if $M^{(\theta-\theta_0)/p}u \in \mathfrak{H}^{2,q}_{p,\theta_0}(\tau)$ and the norms $||u||_{\mathfrak{H}^{2,q}_{p,\theta}(\tau)}$ and $||M^{(\theta-\theta_0)/p}u||_{\mathfrak{H}^{2,q}_{p,\theta_0}(\tau)}$ are equivalent. Also obviously $\mathbb{D}M^{(\theta-\theta_0)/p}u = M^{(\theta-\theta_0)/p}\mathbb{D}u$ and $\mathbb{S}M^{(\theta-\theta_0)/p}u = M^{(\theta-\theta_0)/p}\mathbb{S}u$. Therefore, by denoting N various constants depending only on d, p, δ, and θ_0, from (5.1) with $\theta=\theta_0$, we get that, for $v = M^{(\theta-\theta_0)/p}u$,
$$||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leq N||M^{-1}v||_{\mathbb{H}^{2,q}_{p,\theta_0}(\tau)}$$

$$\leq NN_0(||M(\mathbb{D}v - a^{ij}v_{x^ix^j})||_{\mathbb{L}^q_{p,\theta_0}(\tau)} + ||\mathbb{S}v - \sigma^{i\cdot}v_{x^i}||_{\mathbb{H}^{1,q}_{p,\theta_0}(\tau)}) =: NN_0(I+J).$$

We denote $\varepsilon = (\theta - \theta_0)/p$ and observe that

$$M(\mathbb{D}v - a^{ij}v_{x^ix^j}) = M^{(\theta-\theta_0)/p}M(\mathbb{D}u - a^{ij}u_{x^ix^j})$$
$$-2\varepsilon M^{(\theta-\theta_0)/p}a^{1j}u_{x^j} - \varepsilon(\varepsilon-1)M^{(\theta-\theta_0)/p}a^{11}M^{-1}u,$$

which implies

$$I \leq N||M(\mathbb{D}u - a^{ij}u_{x^ix^j})||_{\mathbb{L}^q_{p,\theta}(\tau)} + \varepsilon N(||u_x||_{\mathbb{L}^q_{p,\theta}(\tau)} + ||M^{-1}u||_{\mathbb{L}^q_{p,\theta}(\tau)})$$
$$\leq N||M(\mathbb{D}u - a^{ij}u_{x^ix^j})||_{\mathbb{L}^q_{p,\theta}(\tau)} + \varepsilon N||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)}.$$

Similarly, $J \leq N||\mathbb{S}u - \sigma^{i\cdot}u_{x^i}||_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} + \varepsilon N||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)}$, so that

$$||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leq NN_0(||M(\mathbb{D}u - a^{ij}u_{x^ix^j})||_{\mathbb{L}^q_{p,\theta}(\tau)}$$
$$+||\mathbb{S}u - \sigma^{i\cdot}u_{x^i}||_{\mathbb{H}^{1,q}_{p,\theta}(\tau)})) + \varepsilon N_1N_0||M^{-1}u||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)}.$$

It only remains to choose ε so small that $\varepsilon N_1N_0 \leq 1/2$. The lemma is proved.

Proof of Theorem 3.1. We start with the case in which condition (3.1) holds and $q = np$, where $n = 1, 2, \ldots$. If $n = 1$ the assertion of the theorem is known from [10]. For general n, by Lemma 4.3 we only need to prove (4.5) for $u \in \mathfrak{C}$ and $\lambda \in [0,1]$. Next, if (4.5) holds with $\theta = d - 1 + p - 1/n$, Lemmas 4.3 and 5.1 imply that (4.5) also holds with θ close to $d-1+p-1/n$, which takes care of χ. Therefore, we may and will only concentrate on the case $d-1+p > \theta \geq d-1+p-1/n$. Finally obviously, assuming $\lambda = 1$ in (4.5) does not restrict generality. A random time change shows that we may also assume $a^{11} \equiv 1$.

Now take $u \in \mathfrak{C}$, define $f = \mathbb{D}u - a^{ij}u_{x^ix^j}$, $g^k = \mathbb{S}^ku - \sigma^{ik}u_{x^k}$, take a one-dimensional Wiener process b_t independent of $\{\mathcal{F}_t\}$, and consider the equation

$$dv = (a^{ij}v_{x^ix^j} + f)\,dt + (\sigma^{ik}v_{x^k} + g^k)\,dw^k_t + \beta v_{x^1}\,db_t, \tag{5.2}$$

with zero initial condition, where $\beta = \sqrt{2(1-\delta-\alpha^{11})}$. The term α^{11} corresponding to (5.2) is $\alpha^{11} + \beta^2/2 = 1 - \delta = \text{const}$. Therefore, by Lemmas 4.5 and 4.3, for any bounded τ, equation (5.2) with zero initial condition has a unique solution $v \in \mathfrak{H}^{2,np}_{p,\theta}(\tau)$ satisfying

$$||M^{-1}v||_{\mathbb{L}^{np}_{p,\theta}(\tau)} \leq N_0(||Mf||_{\mathbb{L}^{np}_{p,\theta}(\tau)} + ||g||_{\mathbb{L}^{np}_{p,\theta}(\tau)}). \tag{5.3}$$

By Lemma 4.4, there is a solution $\bar{u} \in \mathfrak{H}^{2,np}_{p,\theta}(\tau)$ of the equation

$$d\bar{u} = (a^{ij}\bar{u}_{x^ix^j} + f)\,dt + (\sigma^{ik}\bar{u}_{x^k} + g^k)\,dw^k_t$$

with zero initial data and with $||M^{-1}\bar{u}||_{\mathbb{L}^{np}_{p,\theta}(\tau)} \leq ||M^{-1}v||_{\mathbb{L}^{np}_{p,\theta}(\tau)}$. By uniqueness, we have $\bar{u} = u$, which after combining the above estimates leads to (4.5) and finishes the proof of the theorem in the case $q = np$ under condition (3.1).

If condition (3.2) is satisfied and $q = np$, the proof goes the same way apart from using Lemma 4.4, which is not needed in this case since Lemma 4.6 provides the necessary apriori estimate without any additional assumptions on α^{11}.

To consider the case of general $q \geq p$, we use the Marcinkiewicz interpolation theorem while interpolating with respect to q for fixed values of other parameters. Let $n = \lceil q/p \rceil$, $q_1 = p$, and $q_2 = np$. Then $q_1 \leq q \leq q_2$ and one of the conditions (3.1) and (3.2) is satisfied for q_1 and q_2 in place of q. Therefore, we can apply the result proved for the case $q = np$ and solve (1.1) in $\mathfrak{H}^{\gamma,q_i}_{p,\theta}(\tau)$ with zero initial data.

The mapping $(f,g) \to u$ is a linear bounded operator, say R acting on

$$L_q((0,\tau]],\mathcal{P},(M^{-1}H^{\gamma-2}_{p,\theta})\times H^{\gamma-1}_{p,\theta})$$

with values in $L_q((0,\tau]],\mathcal{P},MH^{\gamma+2}_{p,\theta})$ defined for $q=q_1,q_2$. Let N_q be the norm of R. By the Marcinkiewicz interpolation theorem R is bounded for any $q\in[q_1,q_2]$ and $N_q\le N(N_{q_1}+N_{q_2})$. Finally, for

$$(f,g)\in L_{np}((0,\tau]],\mathcal{P},(M^{-1}H^{\gamma-2}_{p,\theta})\times H^{\gamma-1}_{p,\theta}),$$

the function $R(f,g)$ belongs to $\mathfrak{H}^{\gamma,np}_{p,\theta}(\tau)\subset\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ and solves equation (1.1) with zero initial condition. The estimate $N_q\le N(N_{q_1}+N_{q_2})$ and the completeness of $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ implies that $R(f,g)$ belongs to $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ and solves equation (1.1) with zero initial condition for any

$$(f,g)\in L_q((0,\tau]],\mathcal{P},(M^{-1}H^{\gamma-2}_{p,\theta})\times H^{\gamma-1}_{p,\theta}).$$

This proves the theorem.

Proof of Theorem 3.2. The same argument as in the above proof of Theorem 3.1 shows that we only need to prove (4.5) for $n=1,2,...$, $u\in\mathfrak{C}$, $\lambda=1$, and $d-1+p>\theta\ge d$.

Case $a^{1j}\equiv 0$ *for* $j\ge 2$. Observe that, for $d-1+p>\theta\ge d-1-1/n+p=:\bar\theta$, the assertions of the theorem and, in particular, estimate (4.5) follow from Theorem 3.1. To cover $\theta\in[d,\bar\theta]$, we use complex interpolation.

If $\theta=d$, to get (4.5) for $u\in\mathfrak{C}$, we continue u in an odd way across $x^1=0$. Let T be the operator of odd continuation. Observe that, due to $a^{1j}=\sigma^{1k}=0$ for $j\ge 2$ and $k\ge 1$, we have $a^{ij}(Tu)_{x^ix^j}=Ta^{ij}u_{x^ix^j}$ and $\sigma^{ik}(Tu)_{x^i}=T\sigma^{ik}u_{x^i}$. Also obviously $\mathbb{D}T=T\mathbb{D}$ and $\mathbb{S}T=T\mathbb{S}$. Therefore, $\mathbb{D}Tu-a^{ij}(Tu)_{x^ix^j}$ is an odd function with respect to x^1 and by Corollary 2.6, for any $q\ge p$, we have that

$$||M^{-1}u||_{\mathbb{L}^q_{p,d}(\tau)}\le N(||M(\mathbb{D}u-a^{ij}u_{x^ix^j}||_{\mathbb{L}^q_{p,d}(\tau)}+||\mathbb{S}u-\sigma^{i\cdot}u_{x^i}||_{\mathbb{L}^q_{p,d}(\tau)}).$$

Of course, we take $q=np$ and then the above estimate becomes (4.5) with $\theta=d$. Now Lemma 4.3 allows us to define an operator

$$R:(M^{-1}\mathbb{H}^{-1,q}_{p,d}(\tau))\times\mathbb{L}^q_{p,d}(\tau)\to\mathfrak{H}^{1,q}_{p,d}(\tau)$$

as the operator solving equation (1.1) with zero initial data in the space $\mathfrak{H}^{1,q}_{p,d}(\tau)$ for each $(f,g)\in(M^{-1}\mathbb{H}^{-1,q}_{p,d}(\tau))\times\mathbb{L}^q_{p,d}(\tau)$. This operator is bounded. In addition, from uniqueness and the above result for $d-1+p>\theta\ge\bar\theta$ it follows easily that R is also a bounded operator from $(M^{-1}\mathbb{H}^{-1,q}_{p,\bar\theta}(\tau))\times\mathbb{L}^q_{p,\bar\theta}(\tau)$ to $\mathfrak{H}^{1,q}_{p,\bar\theta}(\tau)$.

In particular, for $\theta=d,\bar\theta$, we have

$$||M^{-1}R(f,g)||_{\mathbb{L}^q_{p,\theta}(\tau)}\le N(||Mf||_{\mathbb{L}^q_{p,\theta}(\tau)}+||g||_{\mathbb{L}^q_{p,\theta}(\tau)}),\tag{5.4}$$

which we choose to rewrite as

$$||M^{(z-d)/p-1}RM^{(d-z)/p}(f,g)||_{\mathbb{L}^q_{p,d}(\tau)}\le N(||Mf||_{\mathbb{L}^q_{p,d}(\tau)}+||g||_{\mathbb{L}^q_{p,d}(\tau)})\tag{5.5}$$

for $z=d,\bar\theta$. *Obviously, (5.5) also holds for complex z such that $\Re z=d,\bar\theta$.*

Next take some

$$(f,g)\in(M^{-1}\mathbb{L}^q_{p,d}(\tau))\times\mathbb{L}^q_{p,d}(\tau)$$

vanishing for $x^1 \notin (\varepsilon, \varepsilon^{-1})$, where ε is a constant, and for complex $z \in \Gamma := \{z : d \leq \Re z \leq \bar{\theta}\}$ consider the function

$$v(z) = M^{(z-d)/p-1} R M^{(d-z)/p}(f, g).$$

Observe that $M^{(d-z)/p}(f, g) = M^{(d-\theta)/p}(M^{(\theta-z)/p} f, M^{(\theta-z)/p} g)$ with the norm of $(M^{(\theta-z)/p} f, M^{(\theta-z)/p} g)$ in $(M^{-1}\mathbb{L}^q_{p,d}(\tau)) \times \mathbb{L}^q_{p,d}(\tau)$ bounded for $z \in \Gamma$ if $\theta = d, \bar{\theta}$. It follows from (5.5) that the norm

$$||M^{(\theta-d)/p-1} R M^{(d-z)/p}(f, g)||_{\mathbb{L}^q_{p,d}(\tau)}$$

is bounded for $z \in \Gamma$ if $\theta = d, \bar{\theta}$. The estimate $|M^{(z-d)/p} h| \leq |h| + |M^{(\bar{\theta}-d)/p} h|$ for $z \in \Gamma$ shows that the norm of $v(z)$ in $\mathbb{L}^q_{p,d}(\tau)$ is bounded on Γ. A small modification of this argument taking into account the dominated convergence theorem proves that $v(z)$ is a continuous $\mathbb{L}^q_{p,d}(\tau)$-valued function on Γ. Finally, by using the fact that, for any $\alpha > 0$, we have $|\log x^1| \leq N((x^1)^\alpha + (x^1)^{-\alpha})$, one easily proves that the derivative $dv(z)/dz$ exists as an $\mathbb{L}^q_{p,d}(\tau)$-valued function and $v(z)$ is an analytic $\mathbb{L}^q_{p,d}(\tau)$-valued function in the interior of Γ.

Now by the maximum principle and (5.5), for any $\theta \in [d, \bar{\theta}]$,

$$\begin{aligned}
||M^{(\theta-d)/p-1} R M^{(d-\theta)/p}(f, g)||_{\mathbb{L}^q_{p,d}(\tau)} &= ||v(\theta)||_{\mathbb{L}^q_{p,d}(\tau)} \\
&\leq \sup_{z \in \partial\Gamma} ||M^{(z-d)/p-1} R M^{(d-z)/p}(f, g)||_{\mathbb{L}^q_{p,d}(\tau)} \\
&\leq N(||Mf||_{\mathbb{L}^q_{p,d}(\tau)} + ||g||_{\mathbb{L}^q_{p,d}(\tau)}).
\end{aligned}$$

The inequality between the extreme terms means that we have (5.4) for any $\theta \in [d, \bar{\theta}]$. In particular, (4.5) holds indeed.

General case. Let $\bar{a}^{ij} = a^{ij}$ if either $i \geq 2$ or $j \geq 2$ or else $i = j = 1$ and let $\bar{a}^{ij} = 0$ in the remaining cases. Take $u \in \mathfrak{C}$ and define $f = \mathbb{D}u - a^{ij} u_{x^i x^j}$, $g = \mathbb{S}u - \sigma^{i \cdot} u_{x^i}$. By the above case, there exists $\bar{u} \in \mathfrak{H}^{2,q}_{p,\theta}(\tau)$ with $\bar{u}(0) = 0$ such that

$$d\bar{u} = (\bar{a}^{ij} \bar{u}_{x^i x^j} + f)\,dt + (\sigma^{ik} \bar{u}_{x^i} + g^k)\,dw^k_t,$$

$$||M^{-1}\bar{u}||_{\mathbb{L}^q_{p,\theta}(\tau)} \leq ||M^{-1}\bar{u}||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leq N(||Mf||_{\mathbb{L}^q_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{1,q}_{p,\theta}(\tau)}). \qquad (5.6)$$

Furthermore, for $\xi^i_t = \int_0^t \sigma^{ik}(s)\,dw^k_s$ and the functions $v = u - \bar{u}$,

$$\bar{v}(t, x) = v(t, x - \xi_t), \quad \bar{f}(t, x) = (a^{ij} - \bar{a}^{ij})(t) \bar{u}_{x^i x^j}(t, x - \xi_t),$$

we have

$$\begin{aligned}
dv &= (a^{ij} u_{x^i x^j} - \bar{a}^{ij} \bar{u}_{x^i x^j})\,dt + \sigma^{ik}(u_{x^i} - \bar{u}_{x^i})\,dw^k_t \\
&= (a^{ij} v_{x^i x^j} + (a^{ij} - \bar{a}^{ij}) \bar{u}_{x^i x^j})\,dt + \sigma^{ik} v_{x^i}\,dw^k_t, \\
d\bar{v} &= ((a^{ij} - \alpha^{ij}) \bar{v}_{x^i x^j} + \bar{f})\,dt.
\end{aligned}$$

The latter is a deterministic equation and by Theorem 3.2 of [8] (even for $d - 1 + p > \theta > d - 1$) we have $||M^{-1}\bar{v}||_{\mathbb{L}^q_{p,\theta}(\tau)} \leq N||M\bar{f}||_{\mathbb{L}^q_{p,\theta}(\tau)}$. Now to get (4.5), it only remains to add that $||M^{-1}v||_{\mathbb{L}^q_{p,\theta}(\tau)} = ||M^{-1}\bar{v}||_{\mathbb{L}^q_{p,\theta}(\tau)}$,

$$||M\bar{f}||_{\mathbb{L}^q_{p,\theta}(\tau)} = ||M(a^{ij} - \bar{a}^{ij})(t) \bar{u}_{x^i x^j}||_{\mathbb{L}^q_{p,\theta}(\tau)} \leq N||M^{-1}\bar{u}||_{\mathbb{H}^{2,q}_{p,\theta}(\tau)}$$

and use (5.6) along with $||M^{-1}u||_{\mathbb{L}^q_{p,\theta}(\tau)} \leq ||M^{-1}\bar{u}||_{\mathbb{L}^q_{p,\theta}(\tau)} + ||M^{-1}v||_{\mathbb{L}^q_{p,\theta}(\tau)}$.

The theorem is proved.

6. A barrier function

In this section we construct the function which is used in Sec. 4.

Theorem 6.1. *Let n be an integer ≥ 1 and β be a number in $[0,1)$. Then there exists a constant $\bar{\alpha} = \bar{\alpha}(n,\beta) > 1 + 1/n$ if $n \geq 2$, $\bar{\alpha} = 2$ if $n = 1$, and $\bar{\alpha}(n,0) = 2$, such that for any $\alpha \in (1,\bar{\alpha})$ there exists a nonnegative function $\Phi(x_1,...,x_n)$ defined on*

$$Q := (\mathbb{R}_+)^n$$

and possessing the following properties

(i) the function Φ is infinitely differentiable on Q and satisfies the equation

$$L_{n,\beta}\Phi := \Delta\Phi(x) + \beta\sum_{i\neq j}\Phi_{x_i x_j}(x) = -\frac{1}{(x_1\cdot ...\cdot x_n)^\alpha} =: -h_{n,\alpha}(x); \tag{6.1}$$

(ii) for a constant N, independent of x, and $m(x) := \min_i x_i$, we have

$$\Phi(x) \leq Nm^2(x)h_{n,\alpha}(x); \tag{6.2}$$

(iii) for any constant $c > 0$, $\Phi(cx) = c^{2-n\alpha}\Phi(x)$.

Remark 6.2. Theorem 6.1 allows us to get some nontrivial estimates on solutions of $L_{n,\beta}u = f$. Indeed, for $\beta \in [0,1)$, the operator $L_{n,\beta}$ is a uniformly elliptic operator with constant coefficients. In the domain Q with zero boundary condition it has a Green's function which we denote by $g(x,y)$. Since $L_{n,\beta}$ is formally self adjoint, we have $g(x,y) = g(y,x)$. Introduce an operator $G : f \to Gf$ with

$$Gf(x) = \int_Q g(x,y)f(y)\,dy.$$

By the maximum principle, the smallest nonnegative function satisfying (6.1) (if at least one Φ exists) is

$$\Phi(x) = \int_Q g(x,y)h_{n,\alpha}(y)\,dy. \tag{6.3}$$

Therefore, under the conditions of Theorem 6.1 we get that $Gf(x)$

$$\leq Gh_{n,\alpha}(x)\sup_Q(h_{n,\alpha}^{-1}f) = \Phi(x)\sup_Q(h_{n,\alpha}^{-1}f), \quad \sup_Q(\Phi^{-1}Gf) \leq \sup_Q(h_{n,\alpha}^{-1}f).$$

For the operator $T : f \to \Phi^{-1}G(h_{n,\alpha}f)$ this means

$$||Tf||_{L_\infty(Q)} \leq ||f||_{L_\infty(Q)}. \tag{6.4}$$

By duality, we have

$$||h_{n,\alpha}G(\Phi^{-1}g)||_{L_1(Q)} \leq ||g||_{L_1(Q)},$$
$$||(\Phi h_{n,\alpha})\Phi^{-1}G(h_{n,\alpha}g)||_{L_1(Q)} \leq ||\Phi h_{n,\alpha}g||_{L_1(Q)},$$
$$||Tg||_{L_1(Q,\Phi h_{n,\alpha}\,dx)} \leq ||g||_{L_1(Q,\Phi h_{n,\alpha}\,dx)}.$$

Interpolating between the last estimate and (6.4) yields

$$||Tf||_{L_p(Q,\Phi h_{n,\alpha}\,dx)} \leq ||f||_{L_p(Q,\Phi h_{n,\alpha}\,dx)} \tag{6.5}$$

for any $p \in [1,\infty]$.

Furthermore, by definition, for any $u \in C_0^\infty(Q)$ and $x \in Q$, we have

$$u(x) = -\int_Q g(x,y)L_{n,\beta}u(y)\,dy = -\Phi(x)T(h_{n,\alpha}^{-1}L_{n,\beta}u)(x),$$

so that (6.5) implies that under the conditions of Theorem 6.1, for any $u \in C_0^\infty(Q)$,

$$\int_Q \Phi^{1-p} h_{n,\alpha} |u|^p(y)\, dy \le \int_Q \Phi h_{n,\alpha}^{1-p} |L_{n,\beta} u|^p(y)\, dy.$$

One can simplify the last estimate, if one notices that $m^2 h_{n,\alpha} \le N\Phi$, the proof of which is not too hard but goes beyond the scope of this article. Then

$$\int_Q m^{2-p} h_{n,\alpha}^{2-p} |u/m|^p\, dx \le N \int_Q m^{2-p} h_{n,\alpha}^{2-p} |m L_{n,\beta} u|^p\, dx.$$

It is also worth noting that, since $g \ge 0$, for any $u \in C_0^\infty(Q)$, we have $u \le G(L_{n,\beta} u(x))_-$ and

$$\int_Q u(x) h_{n,\alpha}(x)\, dx \le N \int_Q m^2(x) h_{n,\alpha}(x) (L_{n,\beta} u(x))_-\, dx.$$

To prove the theorem we need the following lemma.

Lemma 6.3. *Under the assumptions of Theorem 6.1, let $n \ge 2$, Γ be a closed bounded subset of $\bar{Q}$, and h be a nonnegative function on Γ. Define*

$$u(x) := \int_\Gamma g(x,y) h(y)\, dy$$

and assume that u is finite on $Q \setminus \Gamma$. Then

(i) the function u is bounded in $Q \setminus G$ for any open G such that $\Gamma \subset G$;

(ii) if $\Gamma \subset \{x_n > 2\varepsilon\}$, where the constant $\varepsilon > 0$, then

$$u(x) \le N m(x) \quad \text{for} \quad x \in Q \cap \{x_n \le \varepsilon\},$$

where N is independent of x;

(iii) there are numbers $\bar{\gamma} = \bar{\gamma}(n,\beta) > 1$, with $\bar{\gamma}(n,0) = n$, and $N < \infty$ such that, for $x \in Q \cap \{|x| \ge N\}$,

$$u(x) \le N |x|^{1-n-\bar{\gamma}} m(x). \tag{6.6}$$

Proof. (i) Let B be a large ball centered at origin. The fact that u is bounded near the part of the boundary of $B \cap (Q \setminus G)$ belonging to ∂Q follows from Theorem 4.3 of [1]. Then its boundedness in $B \cap (Q \setminus G)$ follows by Harnack's inequality and its boundedness in $Q \setminus G$ is obtained by the maximum principle.

Assertion (ii) follows from the boundedness of u and the condition $u = 0$ on $\{x_n < 2\varepsilon\} \cap \partial Q$ in a standard way on the basis of considering simple barriers. For instance, fix $x_0 \in Q$ with $x_{0n} \le \varepsilon$ and define $v(x) := N_0[x_1 - N_0 x_1^2 + |\Pi(x - x^0)|^2]$, where N_0 is a large constant and Π is the orthogonal projection operator on $\{x_1 = 0\}$. Then v satisfies $L_{n,\beta} v \le 0$ and $v \ge u$ on the boundary of

$$D := Q \cap \{0 < x_n < \varepsilon, |\Pi(x - x^0)| < 1, x_1 < (2N_0)^{-1}\}.$$

By the maximum principle $v \ge u$ in $\bar{D}$. In particular, $u(x_1, \Pi x^0) \le N_0 x_1 - N_0 x_1^2 \le N_0 x_1$ if $0 < x_1 < (2N_0)^{-1}$. In addition, the restriction $0 < x_1 < (2N_0)^{-1}$ is irrelevant, since u is bounded, so that $u \le N x_1$ for $x_1 \ge (2N_0)^{-1}$. By taking $x_1 = x_1^0$, we get the result.

(iii) First, we deal with the case $\beta = 0$. Define $w(x) = (x_1 \cdot \ldots \cdot x_n)|x|^{2-3n}$. It is not hard to check that $L_{n,\beta} w = \Delta w = 0$, so that

$$L_{n,\beta} w \le 0 \quad \text{in} \quad Q, \quad w \ge 0 \quad \text{on} \quad \partial Q. \tag{6.7}$$

From the boundedness of u for $|x|$ large and from the maximum principle, it follows that there exist constants N, N_0 such that

$$u(x) \leq Nw(x) \leq N_0|x|^{2-n-\bar{\gamma}} \tag{6.8}$$

if $|x| \geq N_0$, where $\bar{\gamma} = n$.

Now we want to prove this intermediate estimate in the case of general β. Let

$$\kappa = 1 - \left(\frac{1-\beta}{(n-1)\beta+1}\right)^{1/2} \tag{6.9}$$

$(0 < \kappa < 1)$. Notice that, if a function $w(x)$ satisfies $L_{n,\beta}w = 0$ in a domain of x's, then the function $v(y) = w(x)$ with $y = x - \kappa(x,e)e$, $e := (1, ..., 1)/\sqrt{n}$ satisfies $\Delta v = 0$ in the corresponding domain of y's. Furthermore, $(y,e) = (1-\kappa)(x,e)$, so that if $x_1, .., x_n > 0$, then $(y,e) > 0$. In that case also $|x| \leq \sqrt{n}(x,e)$, so that

$$\begin{aligned} |y|^2 &= |x|^2 - \kappa(2-\kappa)(x,e)^2 \leq (n - \kappa(2-\kappa))(x,e)^2 \\ &= \frac{n-\kappa(2-\kappa)}{(1-\kappa)^2}(y,e)^2 = \frac{n-1+(1-\kappa)^2}{(1-\kappa)^2}(y,e)^2, \end{aligned}$$

$$\frac{(y,e)^2}{|y|^2} \geq \frac{(1-\kappa)^2}{n-1+(1-\kappa)^2} = \frac{1}{n}\,\frac{1-\beta}{(n-2)\beta+1} =: t_0^2,$$

where in the last definition we assume $t_0 > 0$.

Looking for solutions of $\Delta v \leq 0$ in the form

$$v(y) = \frac{1}{|y|^{n-2+\gamma}}f(t), \quad t = \frac{(y,e)}{|y|}, \quad \gamma > 0$$

we compute and find

$$|y|^{\gamma+2}\Delta v = (1-t^2)f'' - (n-1)tf' + \gamma(n-2+\gamma)f.$$

It is not hard to check that for any number $\rho \geq 0$ the function $g(t) = \cos(\rho \arccos t)$ is twice continuously differentiable on $[t_0, 1]$ and satisfies

$$(1-t^2)g'' - tg' + \rho^2 g = 0.$$

In addition, $tg'(t) \geq g(t) \geq 0$ on $[t_0, 1]$ provided $\rho \arccos t_0 \leq \pi/2$ and $\rho \geq 1$. Indeed, the inequality $tg'(t) \geq g(t)$ is equivalent to the inequality $\tan\phi \geq \rho^{-1}\tan(\phi\rho^{-1})$, where $\phi = \rho \arccos t$, and the latter inequality is true for any $\phi < \pi/2$ if $\rho \geq 1$.

Letting $\rho = \pi/(2\arccos t_0)$, we get that

$$(1-t^2)g'' - (n-1)tg' + \gamma(n-2+\gamma)g \leq [\gamma(n-2+\gamma) - (n-2) - \rho^2]g = 0$$

on $[t_0, 1]$ provided that $\gamma = \bar{\gamma} = \bar{\gamma}(n,\beta)$, where $\bar{\gamma}$ is defined as the positive solution of

$$\bar{\gamma}(n-2+\bar{\gamma}) = \Big(\frac{\pi}{2\arccos t_0}\Big)^2 + n - 2. \tag{6.10}$$

Observe that $t_0 > 0$ and the right-hand side of (6.10) is bigger than $n-1$. It follows that $\bar{\gamma}(n,\beta) > 1$.

Now, for

$$w(x) = v(y) = \frac{1}{|y|^{n-2+\bar{\gamma}}}\cos\Big(\frac{\pi}{2\arccos t_0}\arccos\frac{(y,e)}{|y|}\Big),$$

we easily get (6.7) and (6.8) again.

This result allows us to get (6.6) by using the same barriers as in the proof of (ii). Indeed, take N_0 from (6.8) and let $|x^0| \geq 4N_0$ and $x_1^0 = \min_i x_i^0$. Define $r = |x^0|$, $\bar{x}^0 = x^0/r$, and $\bar{u}(x) = u(rx)$. Also let Π again be the orthogonal projection operator on $\{x_1 = 0\}$. Notice that $r \leq 2|\Pi x^0|$, which implies that, if $|\Pi x^0 - \Pi x| \leq r/4$, then $|x| \geq |\Pi x| \geq r/4 \geq N_0$. It follows easily that $L_{n,\beta}\bar{u} = 0$ in

$$D := \{0 < x_1 < 2, |\Pi x - \Pi \bar{x}_0| < 1/4\} \cap Q,$$

$\bar{u}$ is bounded in $\bar{D}$ by $N_0 r^{2-n-\bar{\gamma}}|x|^{2-n-\bar{\gamma}} \leq N r^{2-n-\bar{\gamma}}$ and $\bar{u}$ vanishes for $x^1 = 0$. As in (ii) we see that $\bar{u}(x_1, \Pi\bar{x}^0) \leq N r^{2-n-\bar{\gamma}} x_1$ for $0 < x_1 < 2$. Upon substituting here $x_1 = \bar{x}_1^0$, we get (6.6). The lemma is proved.

Proof of Theorem 6.1. We are going to use the induction on n fixing β and denoting by $\Phi_n(x)$ the function we are looking for. If $n = 1$ and $\bar{\alpha} = 2$, one can take $\Phi_1(x) = (2-\alpha)^{-1}(\alpha-1)^{-1}x^{2-\alpha}$.

Upon assuming that $n \geq 2$ and that the function Φ_{n-1} exists for $n-1$, we construct Φ_n. Notice that our assertions just mean that the function Φ defined in (6.3) enjoys all the properties listed in the theorem. By the way, the fact that for this Φ property (iii) holds follows trivially from a standard scaling argument showing that $g(cx, cy) = c^{2-n}g(x, y)$ for any $c > 0$. Also if we knew that Φ is locally bounded, then (6.1) and the fact that Φ is infinitely differentiable would follow from standard results from the theory of elliptic equations (with constant coefficients). Therefore, one only needs to prove assertion (ii).

Take $\bar{\gamma} = \bar{\gamma}(n, \beta)$ from Lemma 6.3 and define

$$\bar{\alpha}(n, \beta) = \bar{\alpha}(n-1, \beta) \wedge (1 + \bar{\gamma}/n).$$

Then for $\alpha \in (1, \bar{\alpha}(n, \beta))$ we can use the induction assumption. Also by induction and using $\bar{\gamma} > 1$ one sees that $\bar{\alpha}(n, \beta) > 1 + 1/n$ for any $\beta \in [0, 1)$, whereas $\bar{\gamma}(n, 0) = n$ and $\bar{\alpha}(n, 0) = 2$.

Denote

$$\Gamma = \{x \in Q : 2 \leq \max_i x_i \leq 4n\}, \quad \Gamma_i = \Gamma \cap \{x_i \geq 2\},$$

$$u_i(x) = \int_{\Gamma_i} g(x, y)h_{n,\alpha}(y)\, dy, \quad u(x) = \int_{\Gamma} g(x, y)h_{n,\alpha}(y)\, dy.$$

Observe that, for $x \in Q$ and $\bar{x} = (x_1, ..., x_{n-1})$,

$$L_{n,\beta}\Phi_{n-1}(\bar{x}) = \sum_{i=1}^{n} \Phi_{n-1,x_i x_i}(\bar{x}) + 2\beta \sum_{1 \leq i < j \leq n} \Phi_{n-1,x_i x_j}(\bar{x}) = -h_{n-1,\alpha}(\bar{x}).$$

Since $h_{n-1,\alpha}(\bar{x}) \geq h_{n,\alpha}(x)I_{x_n \geq 2}$ and $\Phi_{n-1} \geq 0$, it easily follows by the maximum principle that

$$u_n(x) \leq \int_{Q, y_n \geq 2} g(x, y)h_{n,\alpha}(y)\, dy \leq \Phi_{n-1}(\bar{x}). \tag{6.11}$$

Here the right-hand side is finite. Therefore, we can apply Lemma 6.3 to u_n. Similar argument holds for any u_k, so that by Lemma 6.3

$$u_k(x) \leq N_0 m(x) \quad \text{if} \quad 0 < x_k \leq 1, \tag{6.12}$$

$$u_k(x) \leq N_0|x|^{1-n-\bar{\gamma}}m(x) \quad \text{for} \quad |x| \geq N_0, \tag{6.13}$$

where $N_0 \geq 1$ is a constant.

Our next step is to define

$$\Psi_k(x) = \int_0^\infty t^{n\alpha-2}u_k(tx)\, \frac{dt}{t}\,,$$

and prove that Ψ_k's satisfy estimate (6.2). Obviously, for any constant $c > 0$, we have $\Psi_k(cx) = c^{2-n\alpha}\Psi_k(x)$. Therefore we only need to be concerned with $|x| = 1$ in (6.2).

Observe that

$$\Psi_k(x) = \int_0^1 + \int_1^{N_0} + \int_{N_0}^\infty =: J_{k1}(x) + J_{k2}(x) + J_{k3}(x).$$

If $|x| = 1$, then $\max x_i \le 1$, and (6.12) along with $n\alpha \ge 2$ imply $J_{k1}(x) \le Nm(x)$. Also for $|x| = 1$, owing to (6.13) and $\alpha < 1 + \bar\gamma/n$,

$$J_{k3}(x) \le Nm(x) \int_{N_0}^\infty t^{n\alpha-n-\bar\gamma}\,\frac{dt}{t} = Nm(x).$$

Hence, for $|x| = 1$,

$$J_{k1}(x) + J_{k3}(x) \le Nm(x) \le N\frac{m^2(x)}{(x_1\cdot\ldots\cdot x_n)^\alpha},$$

where the last inequality is due to $x_i \le x_i^{2-\alpha} \le x_i^2/(x_1\cdot\ldots\cdot x_n)^\alpha$ (remember $\alpha > 1$).

Finally, we show how to estimate J_{k2}. It suffices to consider $k = n$. In the same way as above, for $|x| = 1$, $x_n \le 1/N_0$, and $1 \le t \le N_0$, it holds that $tx_n \le 1$ and we get from (6.12) that

$$J_{n2}(x) \le Nm(x) \le N\frac{m^2(x)}{(x_1\cdot\ldots\cdot x_n)^\alpha}.$$

On the other hand, if $|x| = 1$ and $x_n \ge 1/N_0$, by (6.11) and by the induction hypothesis,

$$J_{n2}(x) \le N\Phi_{n-1}(\bar x) \le N\frac{m^2(x)}{(x_1\cdot\ldots\cdot x_n)^\alpha}.$$

Thus, Ψ_k's indeed satisfy estimate (6.2). The same holds for their sum $\Psi := \Psi_1 + \ldots + \Psi_n$ and for a smaller function

$$\begin{aligned}\int_0^\infty t^{n\alpha-2}u(tx)\,\frac{dt}{t} &= \int_0^\infty t^{n\alpha-2}\Big(\int_Q g(tx,y)h_{n,\alpha}(y)I_\Gamma(y)\,dy\Big)\,\frac{dt}{t}\\ &= \int_0^\infty t^{n(\alpha-1)}\Big(\int_Q g(x,t^{-1}y)h_{n,\alpha}(y)I_\Gamma(y)\,dy\Big)\,\frac{dt}{t}\\ &= \int_0^\infty t^{n\alpha}\Big(\int_Q g(x,y)h_{n,\alpha}(ty)I_\Gamma(ty)\,dy\Big)\,\frac{dt}{t}\\ &= \int_Q g(x,y)h_{n,\alpha}(y)\Big(\int_0^\infty I_\Gamma(ty)\,\frac{dt}{t}\Big)\,dy.\end{aligned}$$

Upon noticing that the last integral with respect to t will become smaller if we replace Γ with a smaller set $Q \cap \{y : 2n < |y| < 3n\}$ and with such a replacement the integral is independent of y, we conclude that Φ satisfies (6.2). The theorem is proved.

Remark 6.4. If $n = 2$, we have

$$\bar\alpha(2,\beta) = 1 + \bar\gamma(2,\beta)/2 = 1 + \pi/(4\arccos t_0),$$

where $t_0^2 = (1-\beta)/2$. It turns out that, for $n = 2$, the restriction $\alpha \in (1, \bar\alpha(2,\beta))$ is sharp as long as the existence of Φ is concerned. One can prove this by finding $g(x,y)$ explicitly by means of changing coordinates and using complex variables. It is also worth noting that $\bar\gamma(2,\beta) \downarrow 1$ and $\bar\alpha(2,\beta) \downarrow 3/2$ as $\beta \uparrow 1$.

Generally, we have

$$\inf_{\beta\in[0,1)} \bar\alpha(n,\beta) = 1 + 1/n \quad \forall n = 2, 3, \ldots.$$

Indeed, if we had $\bar{\alpha}(n,\beta) > 1 + (1+\varepsilon)/n$ for all $\beta \in [0,1)$ with a constant $\varepsilon > 0$, then Lemma 4.5 and Theorem 3.1 would hold for $q = np$ and $d-1+p > \theta \geq d-1+p-(1+\varepsilon)/n$. But then, estimate (3.6) would hold with $n = m$ and $\chi = \varepsilon/n$ for the function u from in Example 3.4 and this is impossible by Remark 3.5.

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