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Differentiability of stochastic flow of reflected Brownian motions

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Abstract

We prove that a stochastic flow of reflected Brownian motions in a smooth multidimensional domain is differentiable with respect to its initial position. The derivative is a linear map represented by a multiplicative functional for reflected Brownian motion. The method of proof is based on excursion theory and analysis of the deterministic Skorokhod equation.

Key words: Reflected Brownian motion, multiplicative functional.

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1 Introduction

This article contains a result on a stochastic flow X_t^x of reflected Brownian motions in a smooth bounded domain $D \subset \mathbf{R}^n$, $n \geq 2$. We will prove that for some stopping times σ_r defined later in the introduction, the mapping $x \to X_{\sigma_r}^x$ is differentiable a.s., and we will identify the derivative with a mapping already known in the literature.

We start with an informal overview of our research project. We call a pair of reflected Brownian motions X_t and Y_t in D a *synchronous coupling* if they are both driven by the same Brownian motion. To make things interesting, we assume that $X_0 \neq Y_0$. The ultimate goal of the research project of which this paper is a part, is to understand the long time behavior of $V_t := X_t - Y_t$ in smooth domains. This project was started in [BCJ], where synchronous couplings in 2-dimensional smooth domains were analyzed. An even earlier paper [BC] was devoted to synchronous couplings in some classes of planar non-smooth domains. Multidimensional domains present new challenges due to the fact that the curvature of ∂D is not a scalar quantity and it has a significant influence on V_t. Eventually, we would like to be able to prove a theorem analogous to the main result of [BCJ], Theorem 1.2. That theorem shows that $|V_t|$ goes to 0 exponentially fast as t goes to infinity, provided a certain parameter $\Lambda(D)$ characterizing the domain D is strictly positive. The exponential rate at which $|V_t|$ goes to 0 is equal to $\Lambda(D)$. The proof of Theorem 1.2 in [BCJ] is extremely long and we expect that an analogous result in higher dimensions will not be easier to prove. This article and its predecessor [BL] are devoted to results providing technical background for the multidimensional analogue of Theorem 1.2 in [BCJ].

Suppose that $|V_t|$ is very small for a very long time. Then we can think about the evolution of V_t as the evolution of an infinitesimally small vector, or a differential form, associated to X_t. This idea is not new—in fact it appeared in somewhat different but essentially equivalent ways in [A; IW1; IW2; H]. The main theorem of [BL] showed existence of a multiplicative functional governing the evolution of V_t, using semi-discrete approximations. The result does not seem to be known in this form, although it is close to theorems in [A; IW1; H]. However, the main point of [BL] was not to give a new proof to a slightly different version of a known result but to develop estimates using excursion techniques that are analogous to those in [BCJ], and that can be applied to study V_t.

Suppose that for every $x \in \overline{D}$ we have a reflecting Brownian motion X_t^x in $\overline{D}$ starting from $X_0^x = x$, and all processes $X_t^x, x \in \overline{D}$, are driven by the same Brownian motion. For a fixed $x_0 \in D$, let σ_r be the first time t when the local time of X^{x_0} on ∂D reaches the value r. The main result of the present article, Theorem 3.1, says that for every $r > 0$, the mapping $x \to X_{\sigma_r}^x$ is differentiable at $x = x_0$ a.s., and the derivative is a linear mapping defined in Theorem 3.2 of [BL].

The differentiability in the initial data was proved in [DZ] for a stochastic flow of reflected diffusions. The main difference between our result and that in [DZ] is that that paper was concerned with diffusions in $(0, \infty)^n$, and our main goal is to study the effect of the curvature of ∂D. The results in [DZ] have been transferred to SDEs in a convex polyhedron with possibly oblique reflection—see a paper by Andres [An1]. A new preprint by Andres [An2] goes much further, proving differentiability in the initial condition for a large class of solutions to reflecting SDE's in smooth domains, generalizing the results of this paper. An effective representation of the derivative is a subtle issue; it is tackled in different ways in the present paper and in [An2]. The author has recently learned about a series of papers by Pilipenko [P1; P2; P3]. They discuss differentiability of stochastic flows in initial data in a generalized sense. The article [P4] is posted on Math ArXiv; it is a review and discussion of Pilipenko's previously published results. Differentiability of a stochastic flow of diffusions (without reflection) in the initial condition is a classical topic, see, e.g., [K], Chap. II, Thm. 3.1.

Our main result can be considered a pathwise version of theorems proved in [A; H; IW1] and [IW2], Section V.6 (see also references therein). In a sense, we exchange the operations of taking the derivative with respect to the initial condition and the operation of transporting a (non-zero) vector along the trajectories of the process. To be more precise, the publications cited above are concerned with the motion of differentiable forms—this can be interpreted as taking the limit in the first place, so that the difference in the initial condition is infinitesimally small. A similar approach was taken in [BL]. In this paper, the derivative in the initial condition is taken at a (random) time greater than zero. Hence, our main theorem is closer in spirit to the results in [LS; S; DI; DR]. There is a difference, though. The articles [LS; S; DI; DR] are concerned with the transformation of the whole driving path into a reflected path (the "Skorokhod map"). At this level of generality, the Skorokhod map was proved to be Hölder with exponent $1/2$ in Theorems 1.1 an 2.2 of [LS] and Lipschitz in Proposition 4.1 in [S]. See [S] for further references and history of the problem. Under some other assumptions, the Skorokhod map was proved to have the Lipschitz property in [DI; DR]. Articles [MM] (Lemma 5.2) and [MR] contain results about directional derivatives of the Skorokhod map in an orthant, without and with oblique reflection, respectively. The first theorems on existence and uniqueness of solutions to the stochastic differential equation representing reflected Brownian motion were given in [T]. Some results on stochastic flows of reflected Brownian motions were proved in an unpublished thesis [W]. Synchronous couplings in convex domains were studied in [CLJ1; CLJ2], where it was proved that under mild assumptions, V_t is not 0 at any finite time.

The proof of the main result depends in a crucial way on ideas developed in a joint project with Jack Lee ([BL]). I am indebted to him for his implicit contributions to this paper. I am grateful to Sebastian Andres, Peter Baxendale, Elton Hsu and Kavita Ramanan for very helpful advice. I would like to thank the referee for many helpful suggestions.

2 Preliminaries

2.1 General notation

All constants are assumed to be strictly positive and finite, unless stated otherwise. The open ball in $\mathbf{R}^n$ with center x and radius r will be denoted $\mathscr{B}(x,r)$. We will use $\mathbf{d}(\cdot,\cdot)$ to denote the distance between a point and a set.

2.2 Differential geometry

We will review some notation and results from [BL]. We will be concerned with a bounded domain $D \subset \mathbf{R}^n$, $n \geq 2$, with a C^2 boundary ∂D. We may consider $M := \partial D$ to be a smooth, properly embedded, orientable hypersurface (i.e., submanifold of codimension 1) in $\mathbf{R}^n$, endowed with a smooth unit normal inward vector field $\mathbf{n}$. We consider M as a Riemannian manifold with the induced metric. We use the notation $\langle\cdot,\cdot\rangle$ for both the Euclidean inner product on $\mathbf{R}^n$ and its restriction to the tangent space $\mathscr{T}_x M$ for any $x \in M$, and $|\cdot|$ for the associated norm. For any $x \in M$, let $\pi_x : \mathbf{R}^n \to \mathscr{T}_x M$ denote the orthogonal projection onto the tangent space $\mathscr{T}_x M$, so

$$\pi_x \mathbf{z} = \mathbf{z} - \langle \mathbf{z}, \mathbf{n}(x)\rangle \mathbf{n}(x), \tag{2.1}$$

and let $\mathscr{S}(x)\colon \mathscr{T}_x M \to \mathscr{T}_x M$ denote the *shape operator* (also known as the *Weingarten map*), which is the symmetric linear endomorphism of $\mathscr{T}_x M$ associated with the second fundamental form. It is characterized by

$$\mathscr{S}(x)\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}(x), \qquad \mathbf{v} \in \mathscr{T}_x M, \tag{2.2}$$

where $\partial_{\mathbf{v}}$ denotes the ordinary Euclidean directional derivative in the direction of $\mathbf{v}$. If $\gamma\colon [0,T] \to M$ is a smooth curve in M, a *vector field along* γ is a smooth map $\mathbf{v}\colon [0,T] \to M$ such that $\mathbf{v}(t) \in \mathscr{T}_{\gamma(t)}M$ for each t. The *covariant derivative of* $\mathbf{v}$ *along* γ is given by

$$\begin{aligned}\mathscr{D}_t\mathbf{v}(t) &:= \mathbf{v}'(t) - \langle \mathbf{v}(t), \mathscr{S}(\gamma(t))\gamma'(t)\rangle \mathbf{n}(\gamma(t)) \\ &= \mathbf{v}'(t) + \langle \mathbf{v}(t), \partial_t(\mathbf{n}\circ\gamma)(t)\rangle \mathbf{n}(\gamma(t)).\end{aligned}$$

The eigenvalues of $\mathscr{S}(x)$ are the principal curvatures of M at x, and its determinant is the Gaussian curvature. We extend $\mathscr{S}(x)$ to an endomorphism of $\mathbf{R}^n$ by defining $\mathscr{S}(x)\mathbf{n}(x) = 0$. It is easy to check that $\mathscr{S}(x)$ and π_x commute, by evaluating separately on $\mathbf{n}(x)$ and on $\mathbf{v} \in \mathscr{T}_x M$.

For any linear map $\mathscr{A} : \mathbf{R}^n \to \mathbf{R}^n$, we let $\|\mathscr{A}\|$ denote the operator norm.

We recall two lemmas from [BL].

Lemma 2.1. *For any bounded C^2 domain $D \subset \mathbf{R}^n$ and c_1, there exists c_2 such that the following estimates hold for all $x, y \in \partial D$, $0 \le l, r \le c_1$, $b \ge 0$ and $\mathbf{z} \in \mathbf{R}^n$:*

$$\|e^{b\mathscr{S}(x)}\| \le e^{c_2 b}. \tag{2.3}$$

$$\|e^{l\mathscr{S}(x)} - \mathrm{Id}\,\|_{\mathscr{T}_x} \le c_2 l. \tag{2.4}$$

$$\|e^{l\mathscr{S}(x)} - e^{l\mathscr{S}(y)}\| \le c_2 l\,|x-y|. \tag{2.5}$$

$$\|e^{l\mathscr{S}(x)} - e^{r\mathscr{S}(x)}\| \le c_2|l-r|. \tag{2.6}$$

$$|\mathbf{n}(x) - \mathbf{n}(y)| \le c_2|x-y|. \tag{2.7}$$

Lemma 2.2. *For any bounded C^2 domain $D \subset \mathbf{R}^n$, there exists a constant c_1 such that for all $w, x, y, z \in \partial D$, the following operator-norm estimate holds:*

$$\left\|\pi_z \circ \left(\pi_y - \pi_x\right) \circ \pi_w\right\| \le c_1\left(\left|w-y\right|\,\left|y-z\right| + |w-x|\,|x-z|\right).$$

Remark 2.3. Since ∂D is C^2, it is elementary to see that there exist $r > 0$ and $\nu \in (1, \infty)$ with the following properties. For all $x, y \in \partial D$, $z \in \overline{D}$, with $|x-y| \le r$ and $|x-z| \le r$,

$$1 - \nu|x-y|^2 \le \langle \mathbf{n}(x), \mathbf{n}(y)\rangle \le 1, \tag{2.8}$$

$$\left|\langle x-y, \mathbf{n}(x)\rangle\right| \le \nu|x-y|^2, \tag{2.9}$$

$$\langle x-z, \mathbf{n}(x)\rangle \le \nu|x-z|^2, \tag{2.10}$$

$$\langle x-z, \mathbf{n}(y)\rangle \le \nu|x-y|\,|x-z|, \tag{2.11}$$

$$|\pi_y(\mathbf{n}(x))| \le \nu|x-y|. \tag{2.12}$$

If $x, y \in \partial D$, $z \in \overline{D}$ and $|\pi_x(z-y)| \le |\pi_x(x-y)| \le r$ then

$$\langle z-y, \mathbf{n}(x)\rangle \ge -\nu|\pi_x(x-y)|\,|\pi_x(z-y)|. \tag{2.13}$$

2.3 Probability

Recall that $D \subset \mathbf{R}^n$, $n \geq 2$, is an open connected bounded set with C^2 boundary and $\mathbf{n}(x)$ denotes the unit inward normal vector at $x \in \partial D$. Let B be standard d-dimensional Brownian motion and consider the following Skorokhod equation,

$$X_t^x = x + B_t + \int_0^t \mathbf{n}(X_s^x) dL_s^x, \qquad \text{for } t \geq 0. \tag{2.14}$$

Here $x \in \overline{D}$ and L^x is the local time of X^x on ∂D. In other words, L^x is a non-decreasing continuous process which does not increase when X^x is in D, i.e., $\int_0^\infty \mathbf{1}_D(X_t^x) dL_t^x = 0$, a.s. Equation (2.14) has a unique pathwise solution (X^x, L^x) such that $X_t^x \in \overline{D}$ for all $t \geq 0$ (see [LS]). The reflected Brownian motion X^x is a strong Markov process. The results in [LS] are deterministic in nature, so with probability 1, for all $x \in \overline{D}$ simultaneously, (2.14) has a unique pathwise solution (X^x, L^x). In other words, there exists a stochastic flow $(x,t) \to X_t^x$, in which all reflected Brownian motions X^x are driven by the same Brownian motion B.

We fix a point $z_0 \in D$. We will abbreviate (X^{z_0}, L^{z_0}) by writing (X, L).

We need an extra "cemetery point" Δ outside $\mathbf{R}^n$, so that we can send processes killed at a finite time to Δ. For $s \geq 0$ such that $X_s \in \partial D$ we let $\zeta(e_s) = \inf\{t > 0 : X_{s+t} \in \partial D\}$. Here e_s is an excursion starting at time s, i.e., $e_s = \{e_s(t) = X_{t+s},\ t \in [0, \zeta(e_s))\}$. We let $e_s(t) = \Delta$ for $t \geq \zeta(e_s)$, so $e_t \equiv \Delta$ if $\zeta(e_s) = 0$.

Let σ be the inverse of local time L, i.e., $\sigma_t = \inf\{s \geq 0 : L_s \geq t\}$, and $\mathscr{E}_r = \{e_s : s < \sigma_r\}$. Fix some $r, \varepsilon > 0$ and let $\{e_{u_1}, e_{u_2}, \ldots, e_{u_m}\}$ be the set of all excursions $e \in \mathscr{E}_r$ with $|e(0) - e(\zeta-)| \geq \varepsilon$. We assume that excursions are labeled so that $u_k < u_{k+1}$ for all k and we let $\ell_k = L_{u_k}$ for $k = 1, \ldots, m$. We also let $u_0 = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0 = 0$, $\ell_{m+1} = r$, and $\Delta\ell_k = \ell_{k+1} - \ell_k$. Let $x_k = e_{u_k}(\zeta-)$ be the right endpoint of excursion e_{u_k} for $k = 1, \ldots, m$, and $x_0 = X_{u_0}$.

Recall from Section 2.2 that $\mathscr{S}$ denotes the shape operator and π_x is the orthogonal projection on the tangent space $\mathscr{T}_x \partial D$, for $x \in \partial D$. For $\mathbf{v}_0 \in \mathbf{R}^n$, let

$$\mathbf{v}_r = \exp(\Delta\ell_m \mathscr{S}(x_m))\pi_{x_m} \cdots \exp(\Delta\ell_1 \mathscr{S}(x_1))\pi_{x_1} \exp(\Delta\ell_0 \mathscr{S}(x_0))\pi_{x_0} \mathbf{v}_0. \tag{2.15}$$

Note that all concepts based on excursions e_{u_k} depend implicitly on $\varepsilon > 0$, which is often suppressed in the notation. Let $\mathscr{A}_r^\varepsilon$ denote the linear mapping $\mathbf{v}_0 \to \mathbf{v}_r$.

We will impose a geometric condition on ∂D. To explain its significance, we consider D such that ∂D contains n non-degenerate $(n-1)$-dimensional balls, such that vectors orthogonal to these balls are orthogonal to each other. If the trajectory $\{X_t, 0 \leq t \leq r\}$ visits the n balls and no other part of ∂D, then it is easy to see that $\mathscr{A}_r^\varepsilon = 0$. To avoid this uninteresting situation, we impose the following assumption on D.

Assumption 2.4. *For every $x \in \partial D$, the $(n-1)$-dimensional surface area measure of $\{y \in \partial D : \langle \mathbf{n}(y), \mathbf{n}(x) \rangle = 0\}$ is zero.*

The following theorem has been proved in [BL].

Theorem 2.5. *Suppose that D satisfies all assumption listed so far in Section 2. Then for every $r > 0$, a.s., the limit $\mathscr{A}_r := \lim_{\varepsilon \to 0} \mathscr{A}_r^\varepsilon$ exists and it is a linear mapping of rank $n-1$. For any $\mathbf{v}_0$, with probability 1, $\mathscr{A}_r^\varepsilon \mathbf{v}_0 \to \mathscr{A}_r \mathbf{v}_0$ as $\varepsilon \to 0$, uniformly in r on compact sets.*

Let $t_0 = \inf\{t \geq 0 : X_t \in \partial D\}$ and $z_1 = X_{t_0}$. Intuitively speaking, $\mathscr{A}_r$ is defined by $\mathbf{v}(r) = \mathscr{A}_r \mathbf{v}_0$, where $\mathbf{v}(t)$ represents the solution to the following ODE,

$$\mathscr{D}\mathbf{v} = (\mathscr{S} \circ X(\sigma_t))\mathbf{v}\,dt, \qquad \mathbf{v}(0) = \pi_{z_1}\mathbf{v}_0.$$

In the 2-dimensional case, and only in the 2-dimensional case, we have an alternative intuitive representation of $|\mathscr{A}_r \mathbf{v}_0|$. If $\mathbf{v}_0 = (v_0^1, v_0^2)$ then we write $\widehat{\mathbf{v}}_0 = (-v_0^2, v_0^1)$. Let $\mu(x)$ be the curvature at $x \in \partial D$, that is, the eigenvalue of $\mathscr{S}(x)$. Then

$$|\mathscr{A}_r \mathbf{v}_0| = \exp\left(\int_0^r \mu(X_{\sigma_t})dL_t\right) |\langle \mathbf{n}(z_1), \widehat{\mathbf{v}}_0\rangle| \prod_{e_s \in \mathscr{E}_r} \left|\langle \mathbf{n}(e_s(0)), \mathbf{n}(e_s(\zeta-))\rangle\right|.$$

The remaining part of this section is a short review of the excursion theory. See, e.g., [M] for the foundations of the excursion theory in the abstract setting and [Bu] for the special case of excursions of Brownian motion. Although [Bu] does not discuss reflected Brownian motion, all results we need from that book readily apply in the present context.

An "exit system" for excursions of the reflected Brownian motion X from ∂D is a pair (L_t^*, H^x) consisting of a positive continuous additive functional L_t^* and a family of "excursion laws" $\{H^x\}_{x\in\partial D}$. In fact, $L_t^* = L_t$; see, e.g., [BCJ]. Recall that Δ denotes the "cemetery" point outside $\mathbf{R}^n$ and let $\mathscr{C}$ be the space of all functions $f : [0,\infty) \to \mathbf{R}^n \cup \{\Delta\}$ which are continuous and take values in $\mathbf{R}^n$ on some interval $[0,\zeta)$, and are equal to Δ on $[\zeta,\infty)$. For $x \in \partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathscr{C}$, such that the canonical process is strong Markov on (t_0,∞), for every $t_0 > 0$, with transition probabilities of Brownian motion killed upon hitting ∂D. Moreover, H^x gives zero mass to paths which do not start from x. We will be concerned only with "standard" excursion laws; see Definition 3.2 of [Bu]. For every $x \in \partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Recall that excursions of X from ∂D are denoted e_s and $\sigma_t = \inf\{s \geq 0 : L_s \geq t\}$. Let I be the set of left endpoints of all connected components of $(0,\infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of [M],

$$\mathbf{E}\left[\sum_{t\in I} W_t \cdot f(e_t)\right] = \mathbf{E}\int_0^\infty W_{\sigma_s} H^{X(\sigma_s)}(f)ds = \mathbf{E}\int_0^\infty W_t H^{X_t}(f)dL_t, \tag{2.16}$$

where W_t is a predictable process and $f : \mathscr{C} \to [0,\infty)$ is a universally measurable non-negative function which vanishes on excursions e_t identically equal to Δ. Here $H^x(f) = \int_{\mathscr{C}} f\,dH^x$.

The normalization of the exit system is somewhat arbitrary, for example, if (L_t, H^x) is an exit system and $c \in (0,\infty)$ is a constant then $(cL_t, (1/c)H^x)$ is also an exit system. Let $\mathbf{P}_D^y$ denote the distribution of Brownian motion starting from y and killed upon exiting D. Theorem 7.2 of [Bu] shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to the reflected Brownian motion in $\mathbf{R}^n$. According to that result, we can take L_t to be the continuous additive functional whose Revuz measure is a constant multiple of the surface area measure on ∂D and H^x's to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta\downarrow 0} \frac{1}{\delta} \mathbf{P}_D^{x+\delta\mathbf{n}(x)}(A), \tag{2.17}$$

for any event A in a σ-field generated by the process on an interval $[t_0, \infty)$, for any $t_0 > 0$. The Revuz measure of L is the measure $dx/(2|D|)$ on ∂D where dx represents the surface area measure. In other words, if the initial distribution of X is the uniform probability measure μ in D then $\mathbf{E}^\mu \int_0^1 \mathbf{1}_A(X_s) dL_s = \int_A dx/(2|D|)$ for any Borel set $A \subset \partial D$, see Example 5.2.2 of [FOT]. It has been shown in [BCJ] that (L_t, H^x) is an exit system for X in D, assuming the above normalization.

3 Differentiability of the stochastic flow in the initial parameter

Recall that $z_0 \in D$ is a fixed point. Our main result is the following theorem.

Theorem 3.1. *Suppose that D satisfies all assumptions of Section 2. Then for every $r > 0$ and compact set $K \subset \mathbf{R}^n$, we have $\lim_{\varepsilon\to 0} \sup_{\mathbf{v}\in K} \left| (X^{z_0+\varepsilon\mathbf{v}}_{\sigma_r} - X^{z_0}_{\sigma_r})/\varepsilon - \mathscr{A}_r \mathbf{v} \right| = 0$, a.s.*

Note that in the above theorem, both processes are observed at the same random time σ_r, the inverse local time for the process X^{z_0}. In other words, we do *not* consider

$$(X^{z_0+\varepsilon\mathbf{v}}_{\sigma_r^{z_0+\varepsilon\mathbf{v}}} - X^{z_0}_{\sigma_r^{z_0}})/\varepsilon.$$

Corollary 3.2. *Suppose that D satisfies all assumptions of Section 2. Then for every $t > 0$ and compact set $K \subset \mathbf{R}^n$, we have $\lim_{\varepsilon\to 0} \sup_{\mathbf{v}\in K} \left| (X^{z_0+\varepsilon\mathbf{v}}_t - X^{z_0}_t)/\varepsilon - \mathscr{A}_{L_t} \mathbf{v} \right| = 0$, a.s.*

The proof of Theorem 3.1 will consist of several lemmas. We start by introducing some notation.

We will prove the theorem only for $r = 1$, and we will suppress r in the notation from now on. It is clear that the same proof applies to any other value of r.

It follows from Lemma 3.3 below that we can find a constant c_* and a sequence of stopping times $\widetilde{T}_k$ such that $\widetilde{T}_k \to \infty$, a.s., and $\sup_{z\in\overline{D}} L^z_{\widetilde{T}_k} \le k c_*$ for all k. We fix some integer $k_* \ge 1$ and let $\sigma_* = \sigma_1 \wedge \widetilde{T}_{k_*}$. The dependence of σ_* on k_* and c_* will be suppressed in the notation.

In much of the paper, we will consider "fixed" starting points z_0 and y. We will write $X_t = X^{z_0}_t$ and $Y_t = X^y_t$, so that $X_0 = z_0$ and $Y_0 = y$. Later in this section, we will often take $\varepsilon = |X_0 - Y_0|$. Let $\tau^+_\delta = \tau^+(\delta) = \inf\{t > 0 : |X_t - Y_t| \ge \delta\}$.

We fix some (small) $a_1, a_2 > 0$. We will impose some conditions on the values of a_1 and a_2 later on. Let $S_0 = U_0 = \inf\{t \ge 0 : X_t \in \partial D\}$ and for $k \ge 1$ define

$$\begin{aligned} S_k &= \inf\left\{t \ge U_{k-1} : \mathbf{d}(X_t, \partial D) \vee \mathbf{d}(Y_t, \partial D) \le a_2 |X_t - Y_t|^2\right\} \wedge \sigma_*, \\ U_k &= \inf\left\{t \ge S_k : |X_t - X_{S_k}| \vee |Y_t - Y_{S_k}| \ge a_1 |X_{S_k} - Y_{S_k}|\right\} \wedge \sigma_*. \end{aligned} \tag{3.1}$$

The filtration generated by the driving Brownian motion will be denoted $\mathscr{F}_t$. As usual, for a stopping time T, $\mathscr{F}_T$ will denote the σ-field of events preceding T.

Since D is bounded and ∂D is C^2, there exists $\delta_0 > 0$ such that if $x \in \overline{D}$ and $\mathbf{d}(x, \partial D) < \delta_0$ then there is only one point $y \in \partial D$ with $|x - y| = \mathbf{d}(x, \partial D)$. We will call this point $\Pi_x = \Pi(x)$. For all other points, we let $\Pi_x = z_*$, where $z_* \in \partial D$ is a fixed reference point. We define (random) linear operators,

$$\begin{aligned} \mathscr{G}_k &= \exp\left((L_{U_k} - L_{S_k}) \mathscr{S}(\Pi(X_{S_k}))\right) \pi_{\Pi(X_{S_k})}, \\ \mathscr{H}_k &= \exp\left((L_{S_{k+1}} - L_{S_k}) \mathscr{S}(\Pi(X_{S_k}))\right) \pi_{\Pi(X_{S_k})}. \end{aligned} \tag{3.2}$$

Recall the notation for excursions from Section 2.3. For $\varepsilon_* > 0$, let

$$\left\{e_{t_1^*}, e_{t_2^*}, \dots, e_{t_{m^*}^*}\right\} = \{e_t \in \mathscr{E}_1 : |e_t(0) - e_t(\zeta-)| \geq \varepsilon_*, t < \sigma_*\}.$$

We label the excursions so that $t_k^* < t_{k+1}^*$ for all k and we let $\ell_k^* = L_{t_k^*}$ for $k = 1, \dots, m^*$. We also let $t_0^* = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0^* = 0$, $\ell_{m^*+1}^* = L_{\sigma_*}$, and $\Delta \ell_k^* = \ell_{k+1}^* - \ell_k^*$. Let $x_k^* = e_{t_k^*}(\zeta-)$ for $k = 1, \dots, m^*$, and $x_0^* = X_{t_0^*}$. Let $\gamma^*(s) = x_k^*$ for $s \in [\ell_k^*, \ell_{k+1}^*)$ and $k = 0, 1, \dots, m^*$, and $\gamma^*(1) = \gamma^*(\ell_{m^*}^*)$. Let

$$\mathscr{I}_k = \exp(\Delta \ell_k^* \mathscr{S}(x_k^*)) \pi_{x_k^*}. \tag{3.3}$$

Let $\xi_k = t_k^* + \zeta(e_{t_k^*})$ for $k = 1, \dots, m^*$, and $\xi_0 = 0$.

Let m' be the largest integer such that $S_{m'} \leq \sigma_*$. We let $\ell_k' = L_{S_k}$ for $k = 1, \dots, m'$. We also let $t_0' = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0' = 0$, $\ell_{m_j'+1}' = L_{\sigma_*}$, and $\Delta \ell_k' = \ell_{k+1}' - \ell_k'$. Note that we may have $\Delta \ell_k' = 0$ for some k, with positive probability. Let $x_k' = \Pi(X_{S_k})$ for $k = 1, \dots, m'$, and $x_0' = X_{t_0'}$. Let $\gamma'(s) = x_k'$ for $s \in [\ell_k', \ell_{k+1}')$ and $k = 0, 1, \dots, m'$, and $\gamma'(1) = \gamma'(\ell_{m'}')$.

Let $\lambda : [0,1] \to [0,1]$ be an increasing homeomorphism with the following properties. If $t_j^* = \sigma_{\ell_j^*} \in (U_k, S_{k+1}]$ for some j and k then we let $\lambda(\ell_j^*) = \ell_{k+1}'$. For all other j, $\lambda(\ell_j^*) = \ell_j^*$. Let $\ell_k'' = \lambda(\ell_k^*)$ for $k = 1, \dots, m'' := m^*$. We also let $t_k'' = t_k^*$ for $k = 0, 1, \dots, m''$, $\ell_0'' = 0$, $\ell_{m_j''+1}'' = L_{\sigma_*}$, and $\Delta \ell_k'' = \ell_{k+1}'' - \ell_k''$. Let $x_k'' = x_k^*$ for $k = 0, 1, \dots, m''$. Let $\gamma''(s) = x_k''$ for $s \in [\ell_k'', \ell_{k+1}'')$ and $k = 0, 1, \dots, m''$, and $\gamma''(1) = \gamma''(\ell_{m''}'')$. Let

$$\mathscr{J}_k = \exp(\Delta \ell_k'' \mathscr{S}(x_k'')) \pi_{x_k''}.$$

Note that $\xi_k = t_k'' + \zeta(e_{t_k''})$.

Lemma 3.3. *There exists c_1 and c_2, depending only on D, such that if for some integer $m < \infty$ and a sequence $0 = s_0 < s_1 < \cdots < s_m$ we have $\sup_{s_k \leq s,t \leq s_{k+1}} |B_t - B_s| \leq c_1$ for $k = 0, 1, \dots, m-1$, then $\sup_{z \in \overline{D}} L_{s_m}^z \leq m c_2$. Therefore, for every $u < \infty$, we have $\sup_{z \in \overline{D}} L_u^z < \infty$, a.s.*

Proof. Let $\nu > 1$ and r be as in Remark 2.3. We can suppose without loss of generality that $1/(2\nu) < r$. Let $r_1 = 1/(64\nu)$. Then, by (2.8), for $|x - y| \leq r_1$, $x, y \in \partial D$, we have $|\langle \mathbf{n}(x), \mathbf{n}(y) \rangle - 1| \leq \nu r_1^2 < 1/2$, and, therefore, $\langle \mathbf{n}(x), \mathbf{n}(y) \rangle \geq 1/2$. Suppose that for some t_1 and ω, $\sup_{0 \leq s,t \leq t_1} |B_t - B_s| \leq r_1/64$. Consider any $z \in \overline{D}$ and let $t_2 = \inf\{t \geq 0 : X_t^z \in \partial D\} \wedge t_1$ and $y_1 = X_{t_2}^z$. If $t_2 = t_1$ then $L_{t_1}^z = 0$.

Suppose that $t_2 < t_1$. Let $t_3 = \inf\{t \geq t_2 : |X_t^z - y_1| \geq r_1\} \wedge t_1$, $t_4 = \sup\{t \leq t_3 : X_t^z \in \partial D\}$ and $z_1 = X_{t_4}^z$. Then $|z_1 - y_1| \leq 1/(64\nu)$, so, by (2.10), $\left|\langle z_1 - y_1, \mathbf{n}(y_1) \rangle\right| \leq \nu/(64^2\nu^2) = 1/(64^2\nu) = r_1/64$. We have $X_t^z - X_{t_4}^z = B_t - B_{t_4}$ for $t \in [t_4, t_1]$, so $\sup_{t_4 \leq s,t \leq t_1} |X_t^z - X_s^z| \leq r_1/64$. This implies that

$$\begin{aligned} \left\langle X_{t_3}^z - X_{t_2}^z, \mathbf{n}(y_1) \right\rangle &= \left\langle X_{t_3}^z - y_1, \mathbf{n}(y_1) \right\rangle \\ &= \left\langle X_{t_3}^z - z_1, \mathbf{n}(y_1) \right\rangle + \langle z_1 - y_1, \mathbf{n}(y_1) \rangle \\ &= \left\langle X_{t_3}^z - X_{t_4}^z, \mathbf{n}(y_1) \right\rangle + \langle z_1 - y_1, \mathbf{n}(y_1) \rangle \\ &\leq r_1/64 + r_1/64 = r_1/32. \end{aligned} \tag{3.4}$$

This implies that

$$(1/2)(L^z_{t_3} - L^z_{t_2}) \le \left\langle \int_{t_2}^{t_3} \mathbf{n}(X^z_t) dL^z_t, \mathbf{n}(y_1) \right\rangle \tag{3.5}$$
$$= \left\langle X^z_{t_3} - X^z_{t_2} - (B_{t_3} - B_{t_2}), \mathbf{n}(y_1) \right\rangle$$
$$= \left\langle X^z_{t_3} - X^z_{t_2}, \mathbf{n}(y_1) \right\rangle - \left\langle (B_{t_3} - B_{t_2}), \mathbf{n}(y_1) \right\rangle$$
$$\le r_1/32 + r_1/64 < r_1/16.$$

Thus

$$\left|\pi_{y_1}\left(X^z_{t_3} - X^z_{t_2}\right)\right| = \left|\pi_{y_1}\left(B_{t_3} - B_{t_2} + \int_{t_2}^{t_3} \mathbf{n}(X^z_t) dL^z_t\right)\right|$$
$$\le |B_{t_3} - B_{t_2}| + (L^z_{t_3} - L^z_{t_2}) \le r_1/64 + r_1/8 < r_1/4.$$

This and (3.4) imply that

$$|X^z_{t_3} - y_1| = |X^z_{t_3} - X^z_{t_2}| \le ((r_1/32)^2 + (r_1/4)^2)^{1/2} < r_1/2.$$

In view of the definition of t_3, we see that $t_1 = t_3$. Hence, (3.5) shows that $L^z_{t_1} = L^z_{t_1} - L^z_{t_2} \le r_1/8$. For a fixed ω, the above argument applies to all $z \in \overline{D}$ simultaneously, so $\sup_{z\in\overline{D}} L^z_{t_1} \le r_1/8$.

Suppose that for some integer $m < \infty$ and a sequence $0 = s_0 < s_1 < \cdots < s_m$, we have $\sup_{s_k \le s,t \le s_{k+1}} |B_t - B_s| \le r_1/64$ for $k = 0, 1, \ldots, m-1$. We can repeat the above argument on each interval $[s_k, s_{k+1}]$ to obtain $\sup_{z\in\overline{D}} L^z_{s_{k+1}} - L^z_{s_k} \le r_1/8$, and, consequently, $\sup_{z\in\overline{D}} L^z_{s_m} \le mr_1/8$. This proves the first assertion of the lemma.

By continuity of Brownian motion, for any fixed u, with probability 1, one can find a (random) integer $m < \infty$ and a sequence $0 = s_0 < s_1 < \cdots < s_m = u$ such that $\sup_{s_k \le s,t \le s_{k+1}} |B_t - B_s| \le r_1/64$ for $k = 0, 1, \ldots, m-1$. The second assertion of the lemma follows from this and the first part of the lemma. □

Recall σ_* defined at the beginning of this section.

Lemma 3.4. *There exists c_1 such that a.s., for all $t \le \sigma_*$ and $y, z \in \overline{D}$, we have $|X^y_t - X^z_t| < c_1|y - z|$.*

Proof. Fix any $y, z \in \overline{D}$, let $L^*_t = L^y_t + L^z_t$, and $\sigma^*_t = \inf\{s \ge 0 : L^*_s \ge t\}$. It follows from (2.10) that $\langle x - y, \mathbf{n}(x)\rangle \le c_2|x - y|^2$ for all $x \in \partial D$ and $y \in \overline{D}$. This and (2.14) imply that,

$$\frac{d}{dr}|X^z_{\sigma^*_r} - X^y_{\sigma^*_r}| = \left\langle \mathbf{n}(X^z_{\sigma^*_r}), \frac{X^z_{\sigma^*_r} - X^y_{\sigma^*_r}}{|X^z_{\sigma^*_r} - X^y_{\sigma^*_r}|} \right\rangle \mathbf{1}_{\{X^z_{\sigma^*_r}\in\partial D\}} + \left\langle \mathbf{n}(X^y_{\sigma^*_r}), \frac{X^y_{\sigma^*_r} - X^z_{\sigma^*_r}}{|X^y_{\sigma^*_r} - X^z_{\sigma^*_r}|} \right\rangle \mathbf{1}_{\{X^y_{\sigma^*_r}\in\partial D\}}$$
$$\le c_2|X^z_{\sigma^*_r} - X^y_{\sigma^*_r}|\mathbf{1}_{\{X^z_{\sigma^*_r}\in\partial D\}} + c_2|X^y_{\sigma^*_r} - X^z_{\sigma^*_r}|\mathbf{1}_{\{X^y_{\sigma^*_r}\in\partial D\}} \le 2c_2|X^z_{\sigma^*_r} - X^y_{\sigma^*_r}|.$$

By Gronwall's inequality,

$$|X^z_{\sigma^*_r} - X^y_{\sigma^*_r}| \le |X^z_{\sigma^*_0} - X^y_{\sigma^*_0}|e^{2c_2 r} = |y - z|e^{2c_2 r}.$$

Recall from the beginning of this section that $\sup_{z\in\overline{D}} L^z_{\sigma_*} \le k_* c_* < \infty$. This and the definitions of σ_* and σ^*_r imply that $\sigma_* \le \sigma^*_{2k_*c_*}$. Hence, $|X^z_t - X^y_t| < e^{4k_*c_*c_2}|y - z|$ for all $t \le \sigma_*$. □

Lemma 3.5. *Let $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ and $\tau_{\mathscr{B}(x,r)} = \inf\{t \geq 0 : X_t \notin \mathscr{B}(x,r)\}$.*
(i) There exists c_1 such that if $X_0 = z_0 \in D$ and $\mathbf{d}(z_0, \partial D) \leq r$ then,

$$\mathbf{P}(\tau_{\mathscr{B}(z_0,r)} \leq \tau_D) \leq c_1 \mathbf{d}(z_0, \partial D)/r.$$

(ii) Suppose $\mathbf{d}(X_0, \partial D) = b$. Then $\mathbf{E} \sup_{0 \leq t \leq \tau_D} \left| X_0 - X_t \right| \leq c_2 b |\log b|$.

Proof. (i) See Lemma 3.2 in [BCJ].
(ii) By part (i),

$$\begin{aligned}
\mathbf{E} \left| \sup_{0 \leq t \leq \tau_D} X_0 - X_t \right| &\leq \sum_{b \leq 2^j \leq \operatorname{diam}(D)} 2^{j+1} \mathbf{P}\left(\left| \sup_{0 \leq t \leq \tau_D} X_0 - X_t \right| \in [2^j, 2^{j+1}] \right) \\
&\leq \sum_{b \leq 2^j \leq \operatorname{diam}(D)} 2^{j+1} c_1 b 2^{-j} \leq c_2 b |\log b|.
\end{aligned}$$

□

Recall the notation from the beginning of this section. In particular, $\varepsilon = |X_0 - Y_0|$.

Lemma 3.6. *For some c_1,*

$$\mathbf{E}\left(\max_{0 \leq k \leq m^*} \sup_{\xi_k \leq t \leq t^*_{k+1}} |x^*_k - X_t| \right) \leq c_1 \varepsilon_*^{1/3}. \tag{3.6}$$

Proof. It follows from (3.19) in [BL] that, for any $\beta < 1$, some c_2, and all $\varepsilon_* > 0$,

$$\mathbf{E}\left(\max_{0 \leq k \leq m^*} \sup_{\xi_k \leq t \leq t^*_{k+1}, X_t \in \partial D} |x^*_k - X_t| \right) \leq c_2 \varepsilon_*^{\beta}. \tag{3.7}$$

The main difference between (3.6) and (3.7) is the presence of the condition $X_t \in \partial D$ in the supremum. Let

$$\widehat{\mathscr{E}}_1 = \{e \in \mathscr{E}_1 : |e(0) - e(\zeta-)| < \varepsilon_*, \sup_{0 \leq t < \zeta} |e(0) - e(t)| \geq \varepsilon_*\}.$$

Then

$$\begin{aligned}
\max_{0 \leq k \leq m^*} \sup_{\xi_k \leq t \leq t^*_{k+1}} |x^*_k - X_t| \leq{}& \max_{0 \leq k \leq m^*} \sup_{\xi_k \leq t \leq t^*_{k+1}, X_t \in \partial D} |x^*_k - X_t| \\
&+ \sup_{e \in \widehat{\mathscr{E}}_1} \sup_{0 \leq t < \zeta(e)} |e(0) - e(t)|.
\end{aligned} \tag{3.8}$$

Recall that $n \geq 2$ is the dimension of the space $\mathbf{R}^n$ into which D is embedded. Standard estimates show that if $T_{\partial D} = \inf\{t \geq 0 : X_t \in \partial D\}$, $x \in \partial D$, $y \in \partial \mathscr{B}(x,r) \cap D$, $r > \rho$, and $X_0 = y$, then

$$\mathbf{P}(X_{T_{\partial D}} \in \mathscr{B}(x,\rho) \cap \partial D) \leq c_3 (\rho/r)^{n-1}. \tag{3.9}$$

We have for every $x \in \partial D$ and $b > 0$,

$$c_4/b \le H^x \left(\sup_{0 \le t < \zeta(e)} |e(0) - e(t)| \ge b \right) \le c_5/b. \tag{3.10}$$

The upper bound in the last estimate follows from (2.17) and Lemma 3.5 (i). The lower bound can be proved in a similar way.

We combine (3.9) and (3.10) using the strong Markov property of the measure H^x applied at the hitting time of $\mathcal{B}(x,r)$ to obtain,

$$H^x \left(\sup_{0 \le t < \zeta(e)} |e(0) - e(t)| \ge \varepsilon_*^{1/3}, |e(0) - e(\zeta -)| < \varepsilon_* \right) \le c_5 \varepsilon_*^{-1/3} c_3 (\varepsilon_* / \varepsilon_*^{1/3})^{n-1} = c_6 \varepsilon_*^{(2/3)n-1}.$$

By the exit system formula (2.16),

$$\mathbf{P} \left(\exists e \in \widehat{\mathcal{E}}_1 : \sup_{0 \le t < \zeta} |e(0) - e(t)| \ge \varepsilon_*^{1/3} \right) \le c_7 \varepsilon_*^{(2/3)n-1}.$$

So

$$\begin{aligned} \mathbf{E} \left(\sup_{e \in \widehat{\mathcal{E}}_1} \sup_{0 \le t < \zeta(e)} |e(0) - e(t)| \right) &\le \varepsilon_*^{1/3} + \mathrm{diam}(D) \mathbf{P} \left(\exists e \in \widehat{\mathcal{E}}_1 : \sup_{0 \le t < \zeta} |e(0) - e(t)| \ge \varepsilon_*^{1/3} \right) \\ &\le \varepsilon_*^{1/3} + \mathrm{diam}(D) c_7 \varepsilon_*^{(2/3)n-1} \le c_8 \varepsilon_*^{1/3}. \end{aligned}$$

The lemma follows by combining this estimate with (3.7) and (3.8). □

Lemma 3.7. *There exists c_1 such that if $X_0 \in \partial D$ then,*

$$\mathbf{E} \left(\sup_{0 \le t \le \xi_1} \left| X_t - X_{\xi_1} \right| \right) \le c_1 \varepsilon_*^{1/3}.$$

Proof. We have

$$\sup_{0 \le t \le \xi_1} \left| X_t - X_{\xi_1} \right| \le \max_{0 \le k \le m^*} \sup_{\xi_k < t < t_{k+1}^*} |x_k^* - X_t| + \sup_{0 \le t \le \zeta(e_{t_1^*})} \left| e_{t_1^*}(0) - e_{t_1^*}(t) \right|. \tag{3.11}$$

It follows from Lemma 3.6 that, for some c_2,

$$\mathbf{E} \left(\max_{0 \le k \le m_*} \sup_{\xi_k < t < t_{k+1}^*} |x_k^* - X_t| \right) \le c_2 \varepsilon_*^{1/3}. \tag{3.12}$$

Estimate (3.10) and the exit system formula (2.16) imply that

$$\begin{aligned} \mathbf{E} \left(\sup_{0 \le t \le \zeta(e_{t_1^*})} \left| e_{t_1^*}(0) - e_{t_1^*}(t) \right| \right) &\le \varepsilon_* + \sum_{\varepsilon_* \le 2^j \le \mathrm{diam}(D)} 2^{j+1} \mathbf{P} \left(\sup_{0 \le t \le \zeta(e_{t_1^*})} \left| e_{t_1^*}(0) - e_{t_1^*}(t) \right| \ge 2^j \right) \\ &\le \varepsilon_* + \sum_{\varepsilon_* \le 2^j \le \mathrm{diam}(D)} 2^{j+1} c_3 \frac{2^{-j}}{1/\varepsilon_*} \le c_4 \varepsilon_* |\log \varepsilon_*|. \end{aligned}$$

The lemma follows by combining the last estimate with (3.11) and (3.12). □

Recall that $\tau_\delta^+ = \tau^+(\delta) = \inf\{t > 0 : |X_t - Y_t| \geq \delta\}$. Recall also that ε_* is the parameter used in the definition of ξ_j and x_j^* at the beginning of this section.

Lemma 3.8. *There exist $c_1, \dots, c_5$ and $\varepsilon_0, r_0, p_0 > 0$ with the following properties. Let $\varepsilon_2 = \varepsilon_0 \wedge r_0$. Assume that $X_0 \in \partial D$, $|X_0 - Y_0| = \varepsilon_1$, $\mathbf{d}(Y_0, \partial D) = r$ and let*

$$T_1 = \inf\{t \geq 0 : |X_t - X_0| \vee |Y_t - Y_0| \geq c_1 r\},$$
$$T_4 = \inf\{t \geq 0 : Y_t \in \partial D\}.$$

(T_2 and T_3 will be defined in the proof.)

(i) If $\varepsilon_1 \leq \varepsilon_0$ and $r \leq r_0$ then $\mathbf{P}(S_1 \leq T_1 \wedge T_4, L_{S_1} - L_0 \leq c_2 r) \geq p_0$.

(ii) If $\varepsilon_1 \leq \varepsilon_2$ then $\mathbf{E}(L_{S_1 \wedge \tau^+(\varepsilon_2)} - L_0) \leq c_3(r + \varepsilon_2^3)$.

(iii) If $\varepsilon_1 \leq \varepsilon_2$ then $\mathbf{E}(\sup_{0 \leq t \leq S_1 \wedge \tau^+(\varepsilon_2)} |X_t - X_0|) \leq c_4 |\log r|(r + \varepsilon_2^3)$.

(iv) If $\varepsilon_1 \leq \varepsilon_2$ and $\varepsilon_* \geq c_1 \varepsilon_2$ then for any $\beta_1 < 1$ and all k,

$$\mathbf{E}\left(\sum_{S_k \leq \xi_j \leq S_{k+1}} (L_{S_{k+1}} - L_{\xi_j})|x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{S_k}\right) \leq c_5 |X_{S_k} - Y_{S_k}|^{2+\beta_1}.$$

Remark 3.9. (i) Typically, we will be interested in small values of $\varepsilon_1 = |X_0 - Y_0|$. In view of Lemma 3.4, $|X_t - Y_t| \leq c_0 \varepsilon_1$ for all $t \leq \sigma_*$. Hence, $S_1 \wedge \tau^+(\varepsilon_2) = S_1$ for ε_1 much smaller than ε_2. It follows that parts (ii) and (iii) of Lemma 3.8 can be applied with S_1 in place of $S_1 \wedge \tau^+(\varepsilon_2)$, assuming small ε_1.

(ii) The following remark applies to Lemma 3.8 and all other lemmas. Typically, their proofs require that we assume that $|X_0 - Y_0|$ is bounded above. However, in many cases, the quantity that is being estimated is bounded above by a universal constant, for trivial reasons. Hence, by adjusting the constant appearing in the estimate, we can easily extend the lemmas to all values of $|X_0 - Y_0|$.

Proof of Lemma 3.8. (i) Recall ν defined in Remark 2.3. Assume that $r_0 < \varepsilon_0 < 1/(200\nu)$. Let $c_6 \in (0, 1/12)$ be a small constant whose value will be chosen later. Let

$$T_2 = \inf\{t \geq 0 : \langle Y_t - Y_0, \mathbf{n}(X_0)\rangle \geq 2r\},$$
$$T_3 = \inf\{t \geq 0 : |\pi_{X_0}(Y_t - Y_0)| \geq c_6 r\},$$
$$A_1 = \{T_4 \leq T_2 \wedge T_3\},$$
$$T_5 = \inf\{t \geq 0 : |\pi_{X_0}(X_t - X_0)| \geq 2c_6 r\}.$$

First we will assume that $r \leq \varepsilon_1/2$. We will show that $T_5 \geq T_2 \wedge T_3 \wedge T_4$ if A_1 holds. We will argue by contradiction. Assume that A_1 holds and $T_5 < T_2 \wedge T_3 \wedge T_4$. Then $\pi_{X_0}(B_t - B_0) = \pi_{X_0}(Y_t - Y_0)$ for $t \in [0, T_5]$ so $|\pi_{X_0}(B_t - B_0)| \leq c_6 r$ for the same range of t's. We have

$$\pi_{X_0}(X_{T_5} - X_0) = \pi_{X_0}(B_{T_5} - B_0) + \int_0^{T_5} \pi_{X_0}(\mathbf{n}(X_t)) dL_t,$$

so $\left|\int_0^{T_5} \pi_{X_0}(\mathbf{n}(X_t)) dL_t\right| \geq c_6 r$. By (2.12), we may assume that $\varepsilon_0 > 0$ is so small that for $r \leq r_0 < \varepsilon_0$ and $x \in \mathscr{B}(X_0, 2c_6 r)$, we have $|\pi_{X_0}(\mathbf{n}(x))| \leq 4\nu c_6 r$. This and the estimate $\left|\int_0^{T_5} \pi_{X_0}(\mathbf{n}(X_t)) dL_t\right| \geq$

$c_6 r$ imply that $L_{T_5} - L_0 \geq c_6 r/(4\nu c_6 r) = 1/(4\nu)$. By (2.8), we may choose ε_0 so small that for $r \leq r_0 < \varepsilon_0$ and $x \in \mathscr{B}(X_0, 2c_6 r) \cap \partial D$, $\langle \mathbf{n}(X_0), \mathbf{n}(x)\rangle \geq 1/2$. It follows that

$$\left\langle \mathbf{n}(X_0), \int_0^{T_5} \mathbf{n}(X_t) dL_t \right\rangle \geq 1/(8\nu). \tag{3.13}$$

By (2.9), we can assume that r_0 and ε_0 are so small that if for some $y \in \partial D$ we have $|\pi_{X_0}(y - X_0)| \leq 2c_6 r$ then

$$|\langle y - X_0, \mathbf{n}(X_0)\rangle| \leq r \leq \varepsilon_1 \leq \varepsilon_0. \tag{3.14}$$

Since $\mathbf{d}(Y_0, \partial D) = r$, it is easy to see that if $r_0 > 0$ is sufficiently small then for $r \leq r_0$ and $t \leq T_2 \wedge T_3 \wedge T_4$, we have $\langle Y_t - Y_0, \mathbf{n}(X_0)\rangle \geq -2r$, and, therefore,

$$|\langle Y_t - Y_0, \mathbf{n}(X_0)\rangle| \leq 2r. \tag{3.15}$$

Note that $\langle B_t - B_s, \mathbf{n}(X_0)\rangle = \langle Y_t - Y_s, \mathbf{n}(X_0)\rangle$ for $s, t \in [0, T_4]$. Since we have assumed that $T_5 < T_2 \wedge T_3 \wedge T_4$, it follows that for $s, t \in [0, T_5]$,

$$|\langle B_t - B_s, \mathbf{n}(X_0)\rangle| = |\langle Y_t - Y_s, \mathbf{n}(X_0)\rangle| \leq |\langle Y_t - Y_0, \mathbf{n}(X_0)\rangle| + |\langle Y_s - Y_0, \mathbf{n}(X_0)\rangle| \leq 4r. \tag{3.16}$$

This, (2.14) and (3.13) imply that

$$\begin{aligned}\langle X_{T_5} - X_0, \mathbf{n}(X_0)\rangle &\geq -|\langle B_{T_5} - B_0, \mathbf{n}(X_0)\rangle| + \left\langle \int_0^{T_5} \mathbf{n}(X_t) dL_t, \mathbf{n}(X_0) \right\rangle \\ &\geq -4r + 1/(8\nu) \geq -2\varepsilon_0 + 1/(8\nu) \geq 23\varepsilon_0.\end{aligned}$$

Let $T_6 = \sup\{t \leq T_5 : X_t \in \partial D\}$. The last estimate and (3.14) yield

$$\begin{aligned}\langle B_{T_5} - B_{T_6}, \mathbf{n}(X_0)\rangle = \langle X_{T_5} - X_{T_6}, \mathbf{n}(X_0)\rangle &= \langle X_{T_5} - X_0, \mathbf{n}(X_0)\rangle + \langle X_0 - X_{T_6}, \mathbf{n}(X_0)\rangle \\ &\geq 23\varepsilon_0 - \varepsilon_0 = 22\varepsilon_0,\end{aligned}$$

a contradiction with (3.16). This proves that $T_5 \geq T_2 \wedge T_3 \wedge T_4$ if A_1 holds. This and the definition of A_1 imply that if A_1 holds then $T_5 \geq T_4$.

We will next show that if A_1 holds then $S_1 \leq T_4$. Assume that A_1 holds and let $T_7 = \sup\{t \leq T_4 : X_t \in \partial D\}$. Note that neither X_t nor Y_t visit ∂D on the interval (T_7, T_4). Hence, $X_{T_7} - Y_{T_7} = X_{T_4} - Y_{T_4}$. If ε_0 and r_0 are sufficiently small then $|\pi_{X_0}(X_0 - Y_0)| \geq 3\varepsilon_1/8$ because $r \leq \varepsilon_1/2$ and $\mathbf{d}(Y_0, \partial D) = r$. We have assumed that A_1 holds so $|\pi_{X_0}(Y_{T_4} - Y_0)| \leq c_6 r$. We have proved that $T_5 \geq T_4$ on A_1, so $|\pi_{X_0}(X_{T_4} - X_0)| \leq 2c_6 r$. Recall that $c_6 \leq 1/12$ and $r \leq \varepsilon_1/2$. It follows that

$$\begin{aligned}|X_{T_7} - Y_{T_7}| = |X_{T_4} - Y_{T_4}| &\geq |\pi_{X_0}(X_{T_4} - Y_{T_4})| \\ &\geq |\pi_{X_0}(X_0 - Y_0)| - |\pi_{X_0}(Y_{T_4} - Y_0)| - |\pi_{X_0}(X_{T_4} - X_0)| \\ &\geq 3\varepsilon_1/8 - c_6 r - 2c_6 r \geq \varepsilon_1/4.\end{aligned} \tag{3.17}$$

We have from the definition of T_3 that

$$|\pi_{X_0}(Y_{T_4} - Y_{T_7})| = |\pi_{X_0}(Y_{T_4} - Y_0)| + |\pi_{X_0}(Y_0 - Y_{T_7})| \leq c_6 r + c_6 r = 2c_6 r. \tag{3.18}$$

The definition of T_3 and (3.15) imply that for $t \le T_2 \wedge T_3 \wedge T_4$,

$$|Y_0 - Y_t| \le 2r + c_6 r < 3r. \tag{3.19}$$

Hence,

$$|X_0 - Y_{T_7}| \le |X_0 - Y_0| + |Y_0 - Y_{T_7}| \le \varepsilon_1 + 3r \le 3\varepsilon_1. \tag{3.20}$$

We have proved that $T_5 \ge T_4$ on A_1, so

$$|\pi_{X_0}(X_{T_7} - X_0)| \le 2c_6 r \le \varepsilon_1. \tag{3.21}$$

Let $x_* \in \partial D$ be the point with the minimal distance to Y_{T_7} among points satisfying $\pi_{X_0}(x_*) = \pi_{X_0}(Y_{T_7})$. We use the definition of x_*, (3.18), (3.20) and (2.13) to see that

$$\langle Y_{T_4} - x_*, \mathbf{n}(X_0)\rangle \le \nu \cdot 2c_6 r \cdot 3\varepsilon_1 = 6c_6 \nu r \varepsilon_1. \tag{3.22}$$

We use the fact that $Y_{T_7} - Y_{T_4} = X_{T_7} - X_{T_4}$ and apply (2.13), (3.18) and (3.21), to obtain,

$$\langle Y_{T_7} - Y_{T_4}, \mathbf{n}(X_0)\rangle = \langle X_{T_7} - X_{T_4}, \mathbf{n}(X_0)\rangle \le \nu \cdot 2c_6 r \cdot \varepsilon_1 = 2c_6 \nu r \varepsilon_1.$$

We combine this estimate with (3.22) to see that

$$\begin{aligned} \mathbf{d}(Y_{T_7}, \partial D) &\le |Y_{T_7} - x_*| = \langle Y_{T_7} - x_*, \mathbf{n}(X_0)\rangle \\ &= \langle Y_{T_7} - Y_{T_4}, \mathbf{n}(X_0)\rangle + \langle Y_{T_4} - x_*, \mathbf{n}(X_0)\rangle \le 2c_6 \nu r \varepsilon_1 + 6c_6 \nu r \varepsilon_1 = 8c_6 \nu r \varepsilon_1. \end{aligned} \tag{3.23}$$

This bound and (3.17) yield

$$\frac{\mathbf{d}(Y_{T_7}, \partial D)}{|X_{T_7} - Y_{T_7}|} \le \frac{8c_6 \nu r \varepsilon_1}{\varepsilon_1/4} = 32c_6 r \nu \le 16c_6 \nu \varepsilon_1 \le 64c_6 \nu |X_{T_7} - Y_{T_7}|.$$

We make $c_6 > 0$ smaller, if necessary, so that $64c_6\nu \le a_2$. Then $\mathbf{d}(Y_{T_7}, \partial D) \le a_2 |X_{T_7} - Y_{T_7}|^2$. We obviously have $\mathbf{d}(X_{T_7}, \partial D) \le a_2 |X_{T_7} - Y_{T_7}|^2$ because $X_{T_7} \in \partial D$. This shows that $S_1 \le T_7$ and completes the proof that if A_1 holds then $S_1 \le T_4$.

Assume that A_1 holds and suppose that $\left\langle \mathbf{n}(X_0), \int_0^{T_4} \mathbf{n}(X_t) dL_t \right\rangle \ge 20r$. We will show that these assumptions lead to a contradiction. It follows from (3.15) that for $s, t \le T_2 \wedge T_3 \wedge T_4$,

$$|\langle Y_t - Y_s, \mathbf{n}(X_0)\rangle| \le 4r.$$

Since $Y_t - Y_s = B_t - B_s$ for the same range of s and t, we obtain

$$|\langle B_t - B_s, \mathbf{n}(X_0)\rangle| \le 4r. \tag{3.24}$$

This implies that

$$\langle \mathbf{n}(X_0), X_{T_4} - X_0\rangle \ge -|\langle \mathbf{n}(X_0), B_{T_4} - B_0\rangle| + \left\langle \mathbf{n}(X_0), \int_0^{T_4} \mathbf{n}(X_t) dL_t \right\rangle \ge -4r + 20r = 16r. \tag{3.25}$$

Recall that $T_7 = \sup\{t \le T_4 : X_t \in \partial D\}$. In view of the definition of T_5 and (3.14),

$$\langle \mathbf{n}(X_0), X_0 - X_{T_7} \rangle \ge -r. \tag{3.26}$$

We have $B_{T_4} - B_{T_7} = X_{T_4} - X_{T_7}$ so (3.25) and (3.26) give

$$\begin{aligned}\langle \mathbf{n}(X_0), B_{T_4} - B_{T_7} \rangle &= \langle \mathbf{n}(X_0), X_{T_4} - X_{T_7} \rangle \\ &= \langle \mathbf{n}(X_0), X_{T_4} - X_0 \rangle + \langle \mathbf{n}(X_0), X_0 - X_{T_7} \rangle \ge 16r - r = 15r.\end{aligned}$$

This contradicts (3.24) so we conclude that if A_1 holds then

$$\left\langle \mathbf{n}(X_0), \int_0^{T_4} \mathbf{n}(X_t) dL_t \right\rangle \le 20r. \tag{3.27}$$

Note that $\langle \mathbf{n}(X_0), \mathbf{n}(x) \rangle \ge 1/2$ for all $x \in \partial D \cap \mathscr{B}(X_0, 2c_6 r)$, assuming that $\varepsilon_0 > 0$ is small and $r \le r_0 < \varepsilon_0$. We have shown that if A_1 holds then $T_5 \ge T_4$, so $\langle \mathbf{n}(X_0), \mathbf{n}(X_t) \rangle \ge 1/2$ for $t \in [0, T_4]$ such that $X_t \in \partial D$. This and (3.27) imply that,

$$(1/2)(L_{S_1} - L_0) \le (1/2)(L_{T_4} - L_0) \le \left\langle \mathbf{n}(X_0), \int_0^{T_4} \mathbf{n}(X_t) dL_t \right\rangle \le 20r,$$

and, therefore, $L_{S_1} - L_0 \le 40r$.

By (3.24) and the fact that $L_{T_4} - L_0 \le 40r$, we have for $t \le T_4$,

$$|\langle \mathbf{n}(X_0), X_t - X_0 \rangle| \le |\langle \mathbf{n}(X_0), B_t - B_0 \rangle| + \left\langle \mathbf{n}(X_0), \int_0^{t} \mathbf{n}(X_t) dL_t \right\rangle \le 4r + 40r = 44r.$$

This, the definition of T_5 and the fact that $T_5 \ge T_4$ on A_1 imply that for $t \le T_4$, we have $|X_t - X_0| \le 45r$. If we take $c_1 = 45$ then this and (3.19) show that on A_1, $T_4 \le T_1$ and, therefore, $S_1 \le T_1 \wedge T_4$. We proved that $A_1 \subset \{S_1 \le T_1 \wedge T_4, L^X_{S_1} - L^X_0 \le 40r\}$. It is easy to see that $\mathbf{P}(A_1) > p_1$ for some $p_1 > 0$ which depends only on c_6. This completes the proof of part (i) in the case $r \le \varepsilon_1/2$, with $c_1 = 45$ and $c_2 = 40$.

Next consider the case when $r \ge \varepsilon_1/2$. Let

$$\begin{aligned}T_8 &= \inf\{t > 0 : |Y_t - X_0| \ge 2\varepsilon_1\}, \\ T_9 &= \inf\{t > 0 : X_t \in \partial D, \mathbf{d}(Y_t, \partial D) \le |X_t - Y_t|/2\}, \\ T_{10} &= \inf\{t > 0 : L_t - L_0 \ge 20\varepsilon_1\}, \\ A_2 &= \{T_4 \le T_8\}, \\ A_3 &= \{T_9 \le T_4 \wedge T_8 \wedge T_{10}\}.\end{aligned}$$

We will show that $A_2 \subset A_3$. Assume that A_2 holds. Let $T_{11} = \inf\{t \ge 0 : |\pi_{X_0}(X_t - X_0)| \ge 5\varepsilon_1\}$. We will show that $T_{11} \ge T_4$. We will argue by contradiction. Assume that $T_{11} < T_4$. We have assumed that A_2 holds, so $T_{11} < T_8$. Since $T_{11} < T_4$, we have $\pi_{X_0}(B_t - B_0) = \pi_{X_0}(Y_t - Y_0)$ and $\langle \mathbf{n}_{X_0}, B_t - B_0 \rangle = \langle \mathbf{n}_{X_0}, Y_t - Y_0 \rangle$ for $t \in [0, T_{11}]$, which implies in view of the definition of T_8 that for $s, t \in [0, T_{11}]$,

$$|\pi_{X_0}(B_t - B_0)| = |\pi_{X_0}(Y_t - Y_0)| \le |\pi_{X_0}(Y_t - X_0)| + |\pi_{X_0}(Y_0 - X_0)| \le 2\varepsilon_1 + \varepsilon_1 = 3\varepsilon_1, \tag{3.28}$$

$$|\langle \mathbf{n}_{X_0}, B_t - B_s \rangle| = |\langle \mathbf{n}_{X_0}, Y_t - Y_s \rangle| \le |\langle \mathbf{n}_{X_0}, Y_t - X_0 \rangle| + |\langle \mathbf{n}_{X_0}, X_0 - Y_s \rangle| \le 2\varepsilon_1 + 2\varepsilon_1 = 4\varepsilon_1. \tag{3.29}$$

We obtain from (3.28),

$$\left|\pi_{X_0}\left(\int_0^{T_{11}} \mathbf{n}(X_t)dL_t\right)\right| = |\pi_{X_0}(X_{T_{11}} - X_0) - \pi_{X_0}(B_{T_{11}} - B_0)| \tag{3.30}$$
$$\geq |\pi_{X_0}(X_{T_{11}} - X_0)| - |\pi_{X_0}(B_{T_{11}} - B_0)| \geq 5\varepsilon_1 - 3\varepsilon_1 = 2\varepsilon_1.$$

If $\varepsilon_0 > 0$ is sufficiently small and $\varepsilon_1 \leq \varepsilon_0$ then by (2.12), $|\pi_{X_0}(\mathbf{n}(x))| \leq 10\nu\varepsilon_1$ for $x \in \partial D \cap \mathscr{B}(X_0, 5\varepsilon_1)$. This and the estimate $\left|\int_0^{T_{11}} \pi_{X_0}(\mathbf{n}(X_t))dL_t\right| \geq 2\varepsilon_1$ imply that $L_{T_{11}} - L_0 \geq 2\varepsilon_1/(10\nu\varepsilon_1) = 1/(5\nu)$. By (2.8), we may choose ε_0 so small that for $\varepsilon_1 \leq \varepsilon_0$ and $x \in \mathscr{B}(X_0, 5\varepsilon_1) \cap \partial D$, $\langle \mathbf{n}(X_0), \mathbf{n}(x)\rangle \geq 1/2$. It follows that

$$\left\langle \mathbf{n}(X_0), \int_0^{T_{11}} \mathbf{n}(X_t)dL_t \right\rangle \geq 1/(10\nu).$$

Recall that $\varepsilon_1 < \varepsilon_0 < 1/(200\nu)$. We obtain from the last estimate and (3.29),

$$\langle \mathbf{n}_{X_0}, X_{T_{11}} - X_0\rangle \geq -|\langle \mathbf{n}_{X_0}, B_{T_{11}} - B_0\rangle| + \left\langle \mathbf{n}_{X_0}, \int_0^{T_{11}} \mathbf{n}(X_t)dL_t \right\rangle \geq -4\varepsilon_1 + 1/(10\nu) \geq 16\varepsilon_1.$$

Let $T_{12} = \sup\{t \leq T_{11} : X_t \in \partial D\}$ and note that, by (2.9), assuming ε_0 is small, we have

$$\langle \mathbf{n}_{X_0}, X_0 - X_t\rangle \geq -\varepsilon_1, \tag{3.31}$$

for $t \leq T_{11}$ such that $X_t \in \partial D$. Then

$$\langle \mathbf{n}_{X_0}, B_{T_{11}} - B_{T_{12}}\rangle = \langle \mathbf{n}_{X_0}, X_{T_{11}} - X_{T_{12}}\rangle$$
$$= \langle \mathbf{n}_{X_0}, X_{T_{11}} - X_0\rangle + \langle \mathbf{n}_{X_0}, X_0 - X_{T_{12}}\rangle \geq 16\varepsilon_1 - \varepsilon_1 = 15\varepsilon_1.$$

This contradicts (3.29) and, therefore, completes the proof that $T_{11} \geq T_4$.

Next we will prove that $L_{T_4} - L_0 \leq 20\varepsilon_1$. Suppose otherwise, i.e., $L_{T_4} - L_0 > 20\varepsilon_1$. We have $\langle \mathbf{n}_{X_0}, \mathbf{n}(x)\rangle \geq 1/2$ for $x \in \partial D \cap \mathscr{B}(0, 10\varepsilon_1)$, assuming $\varepsilon_0 > 0$ is small and $\varepsilon_1 \leq \varepsilon_0$. Since $T_{11} \geq T_4$, $\langle \mathbf{n}_{X_0}, \mathbf{n}(X_t)\rangle \geq 1/2$ for $t \leq T_4$ such that $X_t \in \partial D$, so, using (3.29),

$$\langle \mathbf{n}_{X_0}, X_{T_4} - X_0\rangle \geq -|\langle \mathbf{n}_{X_0}, B_{T_4} - B_0\rangle| + \left\langle \mathbf{n}_{X_0}, \int_0^{T_4} \mathbf{n}(X_t)dL_t \right\rangle \geq -4\varepsilon_1 + (1/2)(L_{T_4} - L_0)$$
$$\geq -4\varepsilon_1 + 10\varepsilon_1 = 6\varepsilon_1.$$

Recall that $T_7 = \sup\{t \leq T_4 : X_t \in \partial D\}$ and note that we can use (3.31) because $T_{11} \geq T_4$, so $\langle \mathbf{n}_{X_0}, X_0 - X_{T_7}\rangle \geq -\varepsilon_1$. Then

$$\langle \mathbf{n}_{X_0}, B_{T_4} - B_{T_7}\rangle = \langle \mathbf{n}_{X_0}, X_{T_4} - X_{T_7}\rangle = \langle \mathbf{n}_{X_0}, X_{T_4} - X_0\rangle + \langle \mathbf{n}_{X_0}, X_0 - X_{T_7}\rangle$$
$$\geq 6\varepsilon_1 - \varepsilon_1 = 5\varepsilon_1.$$

This contradicts (3.29) because $T_7 \leq T_4 \leq T_{11}$. This proves that if A_2 holds then

$$L_{T_4} - L_0 \leq 20\varepsilon_1 \leq 40r. \tag{3.32}$$

Recall the definition of T_{11} and the fact that $T_{11} \geq T_4$ to see that $|\pi_{X_0}(X_t - X_0)| \leq 5\varepsilon_1$ for $t \leq T_4$, assuming that A_2 holds. It follows from the definition of T_8 that $|Y_t - Y_0| \leq 4\varepsilon_1$ for $t \leq T_4$. Recall that $T_7 = \sup\{t \leq T_4 : X_t \in \partial D\}$. Note that $X_{T_4} - Y_{T_4} = X_{T_7} - Y_{T_7}$, $Y_{T_4}, X_{T_7} \in \partial D$, and $T_7 \leq T_4$. This and the bounds $|\pi_{X_0}(X_t - X_0)| \leq 5\varepsilon_1$ and $|Y_t - Y_0| \leq 4\varepsilon_1$ for $t \leq T_4$, easily imply that $\mathbf{d}(Y_{T_7}, \partial D) \leq |X_{T_7} - Y_{T_7}|/2$, assuming that ε_0 is small. Hence, $T_9 \leq T_4$. This fact combined with (3.32) shows that if A_2 occurs then $T_9 \leq T_4 \leq T_8 \wedge T_{10}$. This completes the proof that $A_2 \subset A_3$.

It is easy to see that $\mathbf{P}(A_2) > p_2$, for some $p_2 > 0$. It follows that $\mathbf{P}(A_3) > p_2$.

We may now apply the strong Markov property at the stopping time T_9 and repeat the argument given in the first part of the proof, which was devoted to the case $r \leq \varepsilon_1/2$. It is straightforward to complete the proof of part (i), adjusting the values of $c_1, c_2, \varepsilon_0, r_0$ and p_0, if necessary.

(ii) We will restart numbering of constants, i.e., we will use $c_6, c_7, \dots$, for constants unrelated to those with the same index in the earlier part of the proof.

Let c_1, c_2, ε_0 and r_0 be as in part (i) of the lemma, $\varepsilon_2 = \varepsilon_0 \wedge r_0$, and $\varepsilon_1 \leq \varepsilon_2$. Recall that $\tau^+(\varepsilon_2) = \inf\{t > 0 : |X_t - Y_t| \geq \varepsilon_2\}$. Let $T_5^0 = 0$, and for $k \geq 1$ let

$$T_1^k = \inf\{t \geq T_5^{k-1} : |X_{T_5^{k-1}} - X_t| \vee |Y_{T_5^{k-1}} - Y_t| \geq c_1 \mathbf{d}(Y_{T_5^{k-1}}, \partial D)\} \wedge \tau^+(\varepsilon_2), \tag{3.33}$$

$$T_2^k = \inf\{t \geq T_5^{k-1} : L_t - L_{T_5^{k-1}} \geq c_2 \mathbf{d}(Y_{T_5^{k-1}}, \partial D)\} \wedge \tau^+(\varepsilon_2), \tag{3.34}$$

$$T_3^k = \inf\{t \geq T_5^{k-1} : Y_t \in \partial D\} \wedge \tau^+(\varepsilon_2), \tag{3.35}$$

$$T_4^k = T_1^k \wedge T_2^k \wedge T_3^k, \tag{3.36}$$

$$T_5^k = \inf\{t \geq T_4^k : X_t \in \partial D\} \wedge \tau^+(\varepsilon_2). \tag{3.37}$$

We will estimate $\mathbf{E}\mathbf{d}(Y_{T_5^k}, \partial D)$. By Lemma 3.5 (i) and the definition of T_1^k, on the event $\{T_4^k < \tau^+(\varepsilon_2)\}$,

$$\begin{aligned}\mathbf{P}\left(\sup_{t \in [T_4^k, T_5^k]} |X_t - X_{T_4^k}| \in [2^{-j-1}, 2^{-j}] \mid \mathscr{F}_{T_4^k}\right) &\leq c_6 \mathbf{d}(X_{T_4^k}, \partial D)/2^{-j} \\ &\leq c_7 \mathbf{d}(Y_{T_5^{k-1}}, \partial D)/2^{-j}.\end{aligned} \tag{3.38}$$

Write $R = \mathbf{d}(Y_{T_5^{k-1}}, \partial D)$, assume that $T_4^k < \tau^+(\varepsilon_2)$, and let j be the largest integer such that $\sup_{t \in [T_4^k, T_5^k]} |X_t - X_{T_4^k}| \vee \varepsilon_2 \leq 2^{-j}$. We will show that $\mathbf{d}(Y_{T_5^k}, \partial D) \leq R + c_8 \varepsilon_2 2^{-j}$, a.s. Note that between times T_5^{k-1} and T_4^k, the process Y_t does not hit the boundary of D. Between times T_4^k and T_5^k, the process X_t does not hit ∂D. If Y_t does not hit the boundary on the same interval, it is elementary to see that $\mathbf{d}(Y_{T_5^k}, \partial D) \leq R + c_9 \varepsilon_2 2^{-j}$.

Suppose that $Y_{t_*} \in \partial D$ for some $t_* \in [T_4^k, T_5^k]$, and assume that t_* is the largest time with this property. If $t_* = T_5^k$ then $\mathbf{d}(Y_{T_5^k}, \partial D) = 0$. Otherwise we must have $\tau^+(\varepsilon_2) > T_5^k$, $X_{T_5^k} \in \partial D$, and $X_{T_5^k} - Y_{T_5^k} = X_{t_*} - Y_{t_*}$. Since both Y_{t_*} and $X_{T_5^k}$ belong to ∂D, easy geometry shows that in this case $\mathbf{d}(Y_{T_5^k}, \partial D) \leq c_{10} \varepsilon_2 2^{-j}$. This completes the proof that $\mathbf{d}(Y_{T_5^k}, \partial D) \leq R + c_8 \varepsilon_2 2^{-j}$, a.s.

Let j_0 be the smallest integer such that $2^{-j_0} \geq \mathrm{diam}(D)$ and let j_1 be the largest integer such that $2^{-j_1+1} \geq R$. The estimate $\mathbf{d}(Y_{T_5^k}, \partial D) \leq R + c_8 \varepsilon_2 2^{-j}$ and (3.38) imply that on the event $\{T_4^k <$

$\tau^+(\varepsilon_2)\}$,

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k},\partial D)\mid \mathscr{F}_{T_4^k})\\
&\le \sum_{j_0\le j\le j_1}(R+c_8\varepsilon_2 2^{-j})\mathbf{P}(\sup_{t\in[T_4^k,T_5^k]}|X_t-X_{T_4^k}|\in[2^{-j-1},2^{-j}]\mid \mathscr{F}_{T_4^k})\\
&\le R+\sum_{j_0\le j\le j_1}c_8\varepsilon_2 2^{-j}\mathbf{P}(\sup_{t\in[T_4^k,T_5^k]}|X_t-X_{T_4^k}|\in[2^{-j-1},2^{-j}]\mid \mathscr{F}_{T_4^k})\\
&\le R+\sum_{j_0\le j\le j_1}c_{11}\varepsilon_2 2^{-j}(R/2^{-j})\\
&\le R+c_{12}\varepsilon_2 R|\log R|\\
&=\mathbf{d}(Y_{T_5^{k-1}},\partial D)(1+c_{12}\varepsilon_2|\log \mathbf{d}(Y_{T_5^{k-1}},\partial D)|).
\end{aligned}\tag{3.39}$$

For $R\le\varepsilon_2^4$ we have $R(1+c_{12}\varepsilon_2|\log R|)\le c_{13}\varepsilon_2^3$, so $R(1+c_{12}\varepsilon_2|\log R|)\le R(1+4c_{12}\varepsilon_2|\log\varepsilon_2|)+c_{13}\varepsilon_2^3$. Thus, on the event $\{T_4^k<\tau^+(\varepsilon_2)\}$,

$$\mathbf{E}(\mathbf{d}(Y_{T_5^k},\partial D)\mid \mathscr{F}_{T_4^k})\le(1+c_{12}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{T_5^{k-1}},\partial D)+c_{13}\varepsilon_2^3.\tag{3.40}$$

Let $S_1^*=S_1\wedge\tau^+(\varepsilon_2)$. By the strong Markov property applied at T_5^{k-1} and part (i) of the lemma, on the event $\{S_1^*>T_5^{k-1}\}$,

$$\mathbf{P}(T_5^{k-1}<S_1^*\le T_5^k\mid \mathscr{F}_{T_5^{k-1}})\ge\mathbf{P}(T_5^{k-1}<S_1^*\le T_4^k\mid \mathscr{F}_{T_5^{k-1}})\ge p_0.\tag{3.41}$$

By the strong Markov property and induction,

$$\mathbf{P}(S_1^*>T_5^{k-1})\le c_{14}p_0^k.\tag{3.42}$$

This, (3.40) and (3.41) imply,

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{d}(Y_{T_5^k},\partial D)\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\right)\\
&=\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\mathbf{E}\left(\mathbf{d}(Y_{T_5^k},\partial D)\mid\mathscr{F}_{T_4^k}\right)\right)\\
&\le\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\left((1+c_{12}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{T_5^{k-1}},\partial D)+c_{13}\varepsilon_2^3\right)\right)\\
&=\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^{k-1}\}}\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\left((1+c_{12}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{T_5^{k-1}},\partial D)+c_{13}\varepsilon_2^3\right)\right)\\
&\le\mathbf{E}\Big(\mathbf{1}_{\{S_1^*>T_5^{k-1}\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\left((1+c_{12}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{T_5^{k-1}},\partial D)+c_{13}\varepsilon_2^3\right)\\
&\quad\times\mathbf{E}(\mathbf{1}_{\{S_1^*>T_5^k\}}\mid\mathscr{F}_{T_5^{k-1}})\Big)\\
&\le\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^{k-1}\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\left((1+c_{12}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{T_5^{k-1}},\partial D)+c_{13}\varepsilon_2^3\right)(1-p_0)\right)\\
&\le(1+c_{12}\varepsilon_2|\log\varepsilon_2|)(1-p_0)\mathbf{E}\left(\mathbf{d}(Y_{T_5^{k-1}},\partial D)\mathbf{1}_{\{S_1^*>T_5^{k-1}\}}\mathbf{1}_{\{T_5^{k-2}<\tau^+(\varepsilon_2)\}}\right)\\
&\quad+c_{13}(1-p_0)\varepsilon_2^3\mathbf{P}(S_1^*>T_5^{k-1})\\
&\le(1+c_{12}\varepsilon_2|\log\varepsilon_2|)(1-p_0)\mathbf{E}(\mathbf{d}(Y_{T_5^{k-1}},\partial D)\mathbf{1}_{\{S_1^*>T_5^{k-1}\}}\mathbf{1}_{\{T_5^{k-2}<\tau^+(\varepsilon_2)\}})\\
&\quad+c_{15}(1-p_0)\varepsilon_2^3p_0^k.
\end{aligned}$$

We assume without loss of generality that $p_0 > 0$ is so small that $(1-p_0)p_0^{-1} > 1$. We obtain by induction,

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k}, \partial D)\mathbf{1}_{\{S_1^* > T_5^k\}}\mathbf{1}_{\{T_5^{k-1} < \tau^+(\varepsilon_2)\}}) \\
&\quad \le (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k \mathbf{E}(\mathbf{d}(Y_{T_5^0}, \partial D)\mathbf{1}_{\{S_1^* > 0\}}\mathbf{1}_{\{T_5^0 < \tau^+(\varepsilon_2)\}}) \\
&\qquad + c_{15}(1-p_0)\varepsilon_2^3 \sum_{m=0}^{k-1}(1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^m (1-p_0)^m p_0^{k-m} \\
&\quad \le (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k r + c_{15}\varepsilon_2^3 p_0^k \sum_{m=0}^{k-1}(1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^m (1-p_0)^m p_0^{-m} \\
&\quad \le (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k r + c_{16}\varepsilon_2^3 p_0^k (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k p_0^{-k} \\
&\quad = (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k r + c_{16}\varepsilon_2^3 (1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k \\
&\quad \le c_{17}(1 + c_{12}\varepsilon_2|\log \varepsilon_2|)^k (1-p_0)^k (r + \varepsilon_2^3).
\end{aligned} \tag{3.43}$$

Note that, by (3.34) and (3.37),

$$\begin{aligned}
L_{T_2^{j+1}} - L_{T_5^j} &\le c_2 \mathbf{d}(Y_{T_5^j}, \partial D), \\
L_{T_5^{j+1}} - L_{T_2^{j+1}} &= 0.
\end{aligned}$$

Hence,

$$L_{T_5^{j+1}} - L_{T_5^j} \le c_2 \mathbf{d}(Y_{T_5^j}, \partial D). \tag{3.44}$$

It follows from this and (3.43) that

$$
\begin{aligned}
\mathbf{E}(L_{S_1\wedge\tau^+(\varepsilon_2)}-L_0) &= \mathbf{E}(L_{S_1^*}-L_0)\\
&= \sum_{k=0}^{\infty}\mathbf{E}\left((L_{S_1^*}-L_0)\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_2)\}}(L_{T_5^{j+1}}-L_{T_5^j})\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\right)\\
&= \mathbf{E}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\right)\\
&= \mathbf{E}\left(\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\right)\\
&= c_2\sum_{j=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^j\}}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^j},\partial D)\right)\\
&\le \sum_{j=0}^{\infty}c_{18}(1+c_{12}\varepsilon_2|\log\varepsilon_2|)^j(1-p_0)^j(r+\varepsilon_2^3).
\end{aligned}
$$

If we assume that $\varepsilon_2>0$ is sufficiently small, this is bounded by $c_{19}(r+\varepsilon_2^3)$.

(iii) We will restart numbering of constants, i.e., we will use $c_6, c_7, \dots$, for constants unrelated to those with the same index in the earlier part of the proof.

Recall that j_1 is the largest integer such that $2^{-j_1+1}\ge\mathbf{d}(Y_{T_5^{k-1}},\partial D)$. Let j_2 be the largest integer such that $2^{-j_2+1}\ge r$. By (3.33) and (3.38) we have for $j\le j_1$, on the event $\{T_5^{k-1}<\tau^+(\varepsilon_2)\}$,

$$
\begin{aligned}
&\mathbf{P}\left(\sup_{t\in[T_5^{k-1},T_5^k]}|X_t-X_{T_4^k}|\in[2^{-j-1},2^{-j}]\mid\mathscr{F}_{T_5^{k-1}}\right)\\
&\le \mathbf{P}\left(\sup_{t\in[T_5^{k-1},T_4^k]}|X_t-X_{T_5^{k-1}}|+\sup_{t\in[T_4^k,T_5^k]}|X_t-X_{T_4^k}|\in[2^{-j-1},2^{-j}]\mid\mathscr{F}_{T_5^{k-1}}\right)\\
&\le \mathbf{P}\left(c_1\mathbf{d}(Y_{T_5^{k-1}},\partial D)+\sup_{t\in[T_4^k,T_5^k]}|X_t-X_{T_4^k}|\in[2^{-j-1},2^{-j}]\mid\mathscr{F}_{T_5^{k-1}}\right)\\
&\le c_6\mathbf{d}(Y_{T_5^{k-1}},\partial D)/2^{-j}.
\end{aligned}
$$

We will also use the trivial estimate

$$\mathbf{P}\left(\sup_{t\in[T_5^{k-1},T_5^k]}|X_t - X_{T_4^k}| \le r \mid \mathscr{F}_{T_5^{k-1}}\right) \le 1.$$

We use the last two estimates, (3.42) and (3.43) to obtain

$$\begin{aligned}
&\mathbf{E}\left(\sup_{0\le t\le S_1\wedge\tau^+(\varepsilon_2)}|X_t - X_0|\right) = \mathbf{E}\left(\sup_{0\le t\le S_1^*}|X_t - X_0|\right)\\
&= \sum_{k=0}^{\infty}\mathbf{E}\left(\sup_{0\le t\le S_1^*}|X_t - X_0|\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_2)\}}\sup_{T_5^j\le t\le T_5^{j+1}}|X_t - X_0|\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\mathbf{E}\left(\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_2)\}}\sup_{T_5^j\le t\le T_5^{j+1}}|X_t - X_0| \mid \mathscr{F}_{T_5^{k-1}}\right)\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\left(r + \sum_{j_0\le i\le j_2}2^{-i}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}c_6\mathbf{d}(Y_{T_5^{j-1}},\partial D)/2^{-i}\right)\right)\\
&\le \sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\left(r + c_7|\log r|\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^{j-1}},\partial D)\right)\right)\\
&= \mathbf{E}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\left(r + c_7|\log r|\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^{j-1}},\partial D)\right)\right)\\
&= \mathbf{E}\left(\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\left(r + c_7|\log r|\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^{j-1}},\partial D)\right)\right)\\
&= \sum_{j=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^j\}}\left(r + c_7|\log r|\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^{j-1}},\partial D)\right)\right)\\
&= r\sum_{j=0}^{\infty}\mathbf{P}(S_1^* > T_5^j) + c_7|\log r|\sum_{j=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^j\}}\mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_2)\}}\mathbf{d}(Y_{T_5^{j-1}},\partial D)\right)\\
&\le r\sum_{j=0}^{\infty}c_8p_0^k + c_9|\log r|\sum_{j=0}^{\infty}(1+c_{10}\varepsilon_2|\log\varepsilon_2|)^j(1-p_0)^j(r+\varepsilon_2^3).
\end{aligned}$$

If we assume that $\varepsilon_2 > 0$ is sufficiently small, this is bounded by $c_{11}|\log r|(r+\varepsilon_2^3)$.

(iv) Once again, we will restart numbering of constants, i.e., we will use $c_6, c_7, \dots,$ for constants unrelated to those with the same index in the earlier part of the proof.

Recall that j_0 is the smallest integer such that $2^{-j_0} \ge \mathrm{diam}(D)$. Let j_3 be the smallest j with the property that $2^{-j} \le \mathbf{d}(Y_{T_5^k},\partial D)$. It follows from (3.38) that for any $\beta_2 < 1$, on the event $\{T_5^k <$

$\tau^+(\varepsilon_2)\}$,

$$\begin{aligned}
&\mathbf{E}\left(\sup_{T_5^k\le t\le T_5^{k+1}} |X_{T_5^k}-X_t| \mid \mathscr{F}_{T_5^k}\right)\\
&\le \mathbf{E}\left(\sup_{T_5^k\le t\le T_4^{k+1}} |X_{T_5^k}-X_t| \mid \mathscr{F}_{T_5^k}\right)+\mathbf{E}\left(\sup_{T_4^{k+1}\le t\le T_5^{k+1}} |X_{T_4^{k+1}}-X_t| \mid \mathscr{F}_{T_5^k}\right)\\
&\le c_1\mathbf{d}(Y_{T_5^k},\partial D)+\mathbf{E}\left(\sup_{T_4^{k+1}\le t\le T_5^{k+1}} |X_{T_4^{k+1}}-X_t| \mid \mathscr{F}_{T_5^k}\right)\\
&\le c_1\mathbf{d}(Y_{T_5^k},\partial D)+\sum_{j=j_0}^{j_3} c_6 2^{-j}\mathbf{d}(Y_{T_5^k},\partial D)/2^{-j}\\
&\le c_7\mathbf{d}(Y_{T_5^k},\partial D)(1+|\log \mathbf{d}(Y_{T_5^k},\partial D)|)\\
&\le c_8\mathbf{d}(Y_{T_5^k},\partial D)^{\beta_2}\le c_9\varepsilon_2^{\beta_2}.
\end{aligned}$$

This and (3.43) imply that

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{d}(Y_{T_5^k},\partial D)\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\sup_{T_5^k\le t\le T_5^{k+1}}|X_{T_5^k}-X_t|\right) \qquad (3.45)\\
&=\mathbf{E}\left(\mathbf{d}(Y_{T_5^k},\partial D)\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\mathbf{E}\left(\sup_{T_5^k\le t\le T_5^{k+1}}|X_{T_5^k}-X_t| \mid \mathscr{F}_{T_5^k}\right)\right)\\
&\le c_9\varepsilon_2^{\beta_2}\mathbf{E}\left(\mathbf{d}(Y_{T_5^k},\partial D)\mathbf{1}_{\{S_1^*>T_5^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_2)\}}\right)\\
&\le c_{10}\varepsilon_2^{\beta_2}(1+c_{11}\varepsilon_2|\log\varepsilon_2|)^k(1-p_0)^k(r+\varepsilon_2^3).
\end{aligned}$$

It follows from the definition of S_1 that $|\Pi(X_{S_1^*})-X_{S_1^*}|\le c_{11}\varepsilon_2^2$ if $S_1<\sigma_*\wedge\tau^+(\varepsilon_2)$. In the case when $S_1^*=\sigma_*\wedge\tau^+(\varepsilon_2)$, the distance between X and Y is increasing at this instance, so it is easy to see that the vector $X_{S_1^*}-Y_{S_1^*}$ must also have a position such that

$$|\Pi(X_{S_1^*})-X_{S_1^*}|\le c_{11}\varepsilon_2^2. \qquad (3.46)$$

Recall that we assume that $X_0\in\partial D$, $|X_0-Y_0|=\varepsilon_1$, $\mathbf{d}(Y_0,\partial D)=r$. Recall also that ε_* is the parameter used in the definition of ξ_j and x_j^* at the beginning of this section. It follows from (3.33)-(3.37) that if $\varepsilon_*\ge c_1\varepsilon_2$ then at most one ξ_i may belong to any given interval $(T_5^{k-1},T_5^k]$ and, moreover, if for some ξ_i we have $\xi_i\in(T_5^{k-1},T_5^k]$ then $\xi_i=T_5^k$. This, (3.43), (3.44), (3.45) and

(3.46) imply that,

$$
\begin{aligned}
&\mathbf{E}\left(\sum_{0\le\xi_i\le S_1^*}(L_{S_1^*}-L_{\xi_i})|x_i^*-\Pi(X_{S_1^*})|\right)\\
&=\sum_{k=0}^{\infty}\mathbf{E}\left(\sum_{0\le\xi_i\le S_1^*}(L_{S_1^*}-L_{\xi_i})|x_i^*-\Pi(X_{S_1^*})|\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\right)\\
&\le\sum_{k=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\sum_{j=0}^{k}\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}\mathbf{1}_{\{T_5^j\le\xi_i\le S_1^*\}}(L_{T_5^{j+1}}-L_{T_5^j})|x_i^*-\Pi(X_{S_1^*})|\right)\\
&\le\sum_{k=0}^{\infty}\mathbf{E}\Big(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\\
&\qquad\times\sum_{j=0}^{k}\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}\mathbf{1}_{\{T_5^j\le\xi_i\le S_1^*\}}(L_{T_5^{j+1}}-L_{T_5^j})\left(|X_{S_1^*}-\Pi(X_{S_1^*})|+|x_i^*-X_{S_1^*}|\right)\Big)\\
&\le\sum_{k=0}^{\infty}\mathbf{E}\Big(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\\
&\qquad\times\Big(\sum_{j=0}^{k}(j+1)\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}(L_{T_5^{j+1}}-L_{T_5^j})\Big(c_{11}\varepsilon_2^2+\sup_{T_5^j\le t\le T_5^{j+1}}|X_{T_5^j}-X_t|\Big)\Big)\Big)\\
&\le\sum_{k=0}^{\infty}\mathbf{E}\Big(\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}\\
&\qquad\times\Big(\sum_{j=0}^{k}(j+1)\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\Big(c_{11}\varepsilon_2^2+\sup_{T_5^j\le t\le T_5^{j+1}}|X_{T_5^j}-X_t|\Big)\Big)\Big)\\
&=\mathbf{E}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}(j+1)\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\Big(c_{11}\varepsilon_2^2+\sup_{T_5^j\le t\le T_5^{j+1}}|X_{T_5^j}-X_t|\Big)\right)\\
&=\mathbf{E}\left(\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\mathbf{1}_{\{S_1^*\in(T_5^k,T_5^{k+1}]\}}(j+1)\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\Big(c_{11}\varepsilon_2^2+\sup_{T_5^j\le t\le T_5^{j+1}}|X_{T_5^j}-X_t|\Big)\right)\\
&=\sum_{j=0}^{\infty}\mathbf{E}\left(\mathbf{1}_{\{S_1^*>T_5^j\}}(j+1)\mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}}c_2\mathbf{d}(Y_{T_5^j},\partial D)\Big(c_{11}\varepsilon_2^2+\sup_{T_5^j\le t\le T_5^{j+1}}|X_{T_5^j}-X_t|\Big)\right)\\
&\le\sum_{j=0}^{\infty}c_{12}(j+1)(1+c_{12}\varepsilon_2|\log\varepsilon_2|)^j(1-p_0)^j(\varepsilon_2^2+\varepsilon_2^{\beta_2})(r+\varepsilon_2^3).
\end{aligned}
$$

If we assume that $\varepsilon_2>0$ is sufficiently small, this is bounded by $c_{13}\varepsilon_2^{\beta_2}(r+\varepsilon_2^3)$.

Recall definitions of σ_* and S_1, and Lemma 3.4. There exists c_{14} such that if $\varepsilon_1\le c_{14}\varepsilon_2$ then

$\sigma_* < \tau^+(\varepsilon_2)$. Hence, if $\varepsilon_1 \le c_{14}\varepsilon_2$ then

$$\mathbf{E}\left(\sum_{0\le \xi_i\le S_1} (L_{S_1} - L_{\xi_i})|x_i^* - \Pi(X_{S_1})|\right) \le c_{13}\varepsilon_2^{\beta_2}(r+\varepsilon_2^3). \tag{3.47}$$

Let $\widehat{S}_k = \inf\{t \ge S_k : X_t \in \partial D\} \wedge \sigma_*$. The following estimate can be proved just like (3.39),

$$\mathbf{E}\left(\mathbf{d}(Y_{\widehat{S}_k}, \partial D) \mid \mathscr{F}_{S_k}\right) \le (1 + c_{14}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{S_k}, \partial D).$$

We use this estimate, (3.47), the strong Markov property at $\widehat{S}_k$, and the definition of S_k to see that

$$\begin{aligned}
&\mathbf{E}\left(\sum_{S_k\le \xi_j\le S_{k+1}} (L_{S_{k+1}} - L_{\xi_j})|x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{S_k}\right)\\
&= \mathbf{E}\left(\sum_{\widehat{S}_k\le \xi_j\le S_{k+1}} (L_{S_{k+1}} - L_{\xi_j})|x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{S_k}\right)\\
&= \mathbf{E}\left(\mathbf{E}\left(\sum_{\widehat{S}_k\le \xi_j\le S_{k+1}} (L_{S_{k+1}} - L_{\xi_j})|x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{\widehat{S}_k}\right) \mid \mathscr{F}_{S_k}\right)\\
&\le \mathbf{E}\left(c_{13}|X_{\widehat{S}_k} - Y_{\widehat{S}_k}|^{\beta_2}(\mathbf{d}(Y_{\widehat{S}_k}, \partial D) + |X_{\widehat{S}_k} - Y_{\widehat{S}_k}|^3) \mid \mathscr{F}_{S_k}\right)\\
&\le \mathbf{E}\left(c_{15}|X_{S_k} - Y_{S_k}|^{\beta_2}(\mathbf{d}(Y_{\widehat{S}_k}, \partial D) + |X_{S_k} - Y_{S_k}|^3) \mid \mathscr{F}_{S_k}\right)\\
&\le c_{15}|X_{S_k} - Y_{S_k}|^{\beta_2}\left((1 + c_{14}\varepsilon_2|\log\varepsilon_2|)\mathbf{d}(Y_{S_k}, \partial D)\right) + |X_{S_k} - Y_{S_k}|^3)\\
&\le c_{15}|X_{S_k} - Y_{S_k}|^{\beta_2}\left((1 + c_{14}\varepsilon_2|\log\varepsilon_2|)|X_{S_k} - Y_{S_k}|^2\right) + |X_{S_k} - Y_{S_k}|^3)\\
&\le c_{16}|X_{S_k} - Y_{S_k}|^{2+\beta_2}.
\end{aligned}$$

□

Lemma 3.10. *There exist c_1 and $a_0 > 0$ such that for $a_1, a_2 < a_0$, if $|X_0 - Y_0| = \varepsilon$ then a.s., for every $k \ge 1$, on the event $U_k < \sigma_*$,*

$$\left|\left\langle \mathbf{n}(\Pi(X_{U_k})), \frac{Y_{U_k} - X_{U_k}}{|Y_{U_k} - X_{U_k}|}\right\rangle\right| \le c_1\varepsilon.$$

Proof. First we will show that one can choose $c_1, a_0 > 0$ and $\varepsilon_0 > 0$ so that for $a_1 < a_0$, $\varepsilon \le \varepsilon_0$, $x \in \partial D$, $y \in \overline{D}$, $|x - y| \le \varepsilon$, $z \in \partial D$, $|x - z| \le 2a_1\varepsilon$ and $|y - z| \le 2a_1\varepsilon$, we have

$$\left\langle \mathbf{n}(z), \frac{y - x}{|y - x|}\right\rangle \ge -c_1\varepsilon/4. \tag{3.48}$$

First suppose that $y \in \partial D$. The assumptions that $x, y \in \partial D$, $|x - y| \le \varepsilon$, and ∂D is C^2 imply that the angle between $\mathbf{n}(x)$ and $y - x$ is in the range $[\pi/2 - c_2\varepsilon, \pi/2 + c_2\varepsilon]$ for some $c_2 < \infty$. It follows from the assumptions that $x, z \in \partial D$, $|x - z| \le 2a_1\varepsilon$, and ∂D is C^2 that the angle between $\mathbf{n}(x)$ and

$\mathbf{n}(z)$ is less than $c_3\varepsilon$ for some $c_3 < \infty$. Therefore, the angle between $\mathbf{n}(z)$ and $y - x$ is in the range $[\pi/2 - c_1\varepsilon/4, \pi/2 + c_1\varepsilon/4]$, where $c_1 = 4(c_2 + c_3)$. This implies that

$$\left|\langle \mathbf{n}(z), y - x\rangle\right| \le |y - x| \sin(c_1\varepsilon/4) \le c_1|y - x|\varepsilon/4.$$

Thus

$$\left|\left\langle \mathbf{n}(z), \frac{y-x}{|y-x|}\right\rangle\right| \le c_1\varepsilon/4,$$

and, therefore,

$$\left\langle \mathbf{n}(z), \frac{y-x}{|y-x|}\right\rangle \ge -c_1\varepsilon/4.$$

In other words, we proved that (3.48) in the special case when $y \in \partial D$. If $a_1 > 0$ and $\varepsilon_0 > 0$ are sufficiently small then $D \cap \mathscr{B}(z, 2a_1\varepsilon)$ lies above $\partial D \cap \mathscr{B}(z, 2a_1\varepsilon)$ in the coordinate system with the origin at $z = 0$ and the vertical axis containing $\mathbf{n}(z)$. This observation proves that (3.48) applies to all $y \in \overline{D}$ (satisfying all the remaining assumptions).

An argument based on similar ideas shows that if $x, y \in \overline{D}$, $w \in \partial D$, $|w - z| \le 2a_1\varepsilon$ and

$$\left|\left\langle \mathbf{n}(z), \frac{y-x}{|y-x|}\right\rangle\right| \le c_1\varepsilon/2,$$

then

$$\left|\left\langle \mathbf{n}(w), \frac{y-x}{|y-x|}\right\rangle\right| \le c_1\varepsilon. \tag{3.49}$$

If $|X_0 - Y_0| = \varepsilon$ then $|X_t - Y_t| \le c_4\varepsilon$ for all $t \le \sigma_*$, by Lemma 3.4. It follows easily from (3.1) that we can adjust the values of c_1 and ε_0 and choose $a_2 > 0$ so that if $|X_0 - Y_0| = \varepsilon \le \varepsilon_0$ then on the event $S_k < \sigma_*$,

$$\left|\left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_{S_k} - X_{S_k}}{|Y_{S_k} - X_{S_k}|}\right\rangle\right| \le c_1\varepsilon/2.$$

Let

$$A = \left\{ t \in [S_k, U_k] : \left|\left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_t - X_t}{|Y_t - X_t|}\right\rangle\right| > c_1\varepsilon/2 \right\}.$$

We will show that $A = \emptyset$. Suppose otherwise and let $T_1 = \inf A$. Then

$$\left|\left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right\rangle\right| = c_1\varepsilon/2.$$

We must have either $X_{T_1} \in \partial D$ or $Y_{T_1} \in \partial D$. It follows from (3.48) that either $X_{T_1} \notin \partial D$ or $Y_{T_1} \notin \partial D$. Suppose without loss of generality that $X_{T_1} \in \partial D$ and $Y_{T_1} \notin \partial D$. Then by (3.48),

$$\left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right\rangle = c_1\varepsilon/2.$$

By the definition of T_1, for every $\delta > 0$, L_t must increase on the interval $[T_1, T_1 + \delta]$. It is easy to see that this implies that the function

$$t \to \left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_t - X_t}{|Y_t - X_t|} \right\rangle$$

is decreasing on the interval $[T_1, T_1 + \delta_1]$, for some $\delta_1 > 0$. This contradicts the definition of T_1. Hence, for all $t \in [S_k, U_k]$,

$$\left| \left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_t - X_t}{|Y_t - X_t|} \right\rangle \right| \le c_1 \varepsilon / 2.$$

In particular,

$$\left| \left\langle \mathbf{n}(\Pi(X_{S_k})), \frac{Y_{U_k} - X_{U_k}}{|Y_{U_k} - X_{U_k}|} \right\rangle \right| \le c_1 \varepsilon / 2.$$

The lemma follows from the above estimate and (3.49). $\square$

Lemma 3.11. *There exists c_1 such that if $|X_0 - Y_0| \le \varepsilon$ then for every k,*

$$\mathbf{E} \sum_{U_k \le \sigma_* \wedge \tau^+(\varepsilon)} (L_{S_{k+1}} - L_{U_k}) \le c_1 \varepsilon |\log \varepsilon|.$$

Proof. We use the strong Markov property at the hitting time of ∂D by X and Lemma 3.8 (ii) to see that

$$\mathbf{E}(L_{S_1 \wedge \tau^+(\varepsilon)} - L_{U_0}) \le c_2 \varepsilon. \tag{3.50}$$

We will estimate $(L_{S_{k+1}} - L_{U_k}) \mathbf{1}_{\{U_k < \tau^+(\varepsilon)\}}$ for $k \ge 1$. Fix some $k \ge 1$ and assume that $U_k < \tau^+(\varepsilon)$. Note that $\mathbf{d}(X_{U_k}, \partial D) \le c_3 |X_{U_k} - Y_{U_k}|$. Let $T_1 = \inf\{t \ge U_k : X_t \in \partial D\} \wedge \sigma_* \wedge \tau^+(\varepsilon)$. Let j_0 be the greatest integer such that 2^{-j_0} is greater than the diameter of D and let j_1 be the least integer such that $2^{-j_1} \le |X_{U_k} - Y_{U_k}|$. By Lemma 3.5, for $j_0 \le j \le j_1$,

$$\mathbf{P}\left(|X_{U_k} - X_{T_1}| \in [2^{-j}, 2^{-j+1}] \mid \mathscr{F}_{U_k} \right) \le c_4 2^j |X_{U_k} - Y_{U_k}|. \tag{3.51}$$

Next we will estimate $\mathbf{d}(Y_{T_1}, \partial D)$. Between times U_k and T_1, the process X_t does not hit ∂D. If Y_t does not hit the boundary on the same interval, it is elementary to see from Lemma 3.10 that for $j_0 \le j \le j_1$,

$$\mathbf{d}(Y_{T_1}, \partial D) \le c_5 |X_{U_k} - Y_{U_k}|^2 + c_6 |X_{U_k} - Y_{U_k}| 2^{-j} \le c_7 |X_{U_k} - Y_{U_k}| 2^{-j}.$$

Suppose that for some $t_* \in [U_k, T_1]$ we have $Y_{t_*} \in \partial D$, and assume that t_* is the largest time with this property. If $t_* = T_1$ then $\mathbf{d}(Y_{T_1}, \partial D) = 0$. Otherwise we must have $\tau^+(\varepsilon) > t_*$, $X_{T_1} \in \partial D$, and $X_{T_1} - Y_{T_1} = X_{t_*} - Y_{t_*}$. Since both Y_{t_*} and X_{T_1} belong to ∂D, easy geometry shows that in this case $\mathbf{d}(Y_{T_1}, \partial D) \le c_8 |X_{U_k} - Y_{U_k}| 2^{-j}$. We conclude that $\mathbf{d}(Y_{T_1}, \partial D) \le c_9 |X_{U_k} - Y_{U_k}| 2^{-j}$, a.s. By Lemma 3.8 (ii) and the strong Markov property applied at U_k,

$$\mathbf{E}\left(L_{S_{k+1}} - L_{U_k} \mid U_k < \tau^+(\varepsilon), \mathscr{F}_{T_1} \right) \le c_{10}(|X_{U_k} - Y_{U_k}| 2^{-j} + |X_{U_k} - Y_{U_k}|^3) \le c_{11} |X_{U_k} - Y_{U_k}| 2^{-j}.$$

Hence, using (3.51),

$$\begin{aligned}\mathbf{E}\left(L_{S_{k+1}}-L_{U_k}\mid U_k<\tau^+(\varepsilon),\mathscr{F}_{U_k}\right)&=\mathbf{E}\left(\mathbf{E}\left(L_{S_{k+1}}-L_{U_k}\mid U_k<\tau^+(\varepsilon),\mathscr{F}_{T_1}\right)\mathscr{F}_{U_k}\right)\\&\le\sum_{j_0\le j\le j_1}c_4|X_{U_k}-Y_{U_k}|2^j c_{11}|X_{U_k}-Y_{U_k}|2^{-j}\\&\le c_{12}|X_{U_k}-Y_{U_k}|^2\,|\log|X_{U_k}-Y_{U_k}||.\end{aligned}$$

It is elementary to check that

$$\mathbf{E}\left(L_{U_k}-L_{S_k}\mid S_k<\tau^+(\varepsilon),\mathscr{F}_{S_k}\right)\ge c_{13}|X_{S_k}-Y_{S_k}|,$$

and the conditional distribution of $L_{U_k}-L_{S_k}$ given $\{S_k<\tau^+(\varepsilon)\}$ is stochastically bounded by an exponential random variable with mean $c_{14}|X_{S_k}-Y_{S_k}|$. Note that $|X_{U_k}-Y_{U_k}|\le c_{15}|X_{S_k}-Y_{S_k}|$. Thus,

$$\begin{aligned}&\mathbf{E}\left(L_{S_{k+1}}-L_{U_k}\mid U_k<\tau^+(\varepsilon),\mathscr{F}_{U_k}\right)\\&\le c_{16}|X_{U_k}-Y_{U_k}|\,|\log|X_{U_k}-Y_{U_k}||\,\mathbf{E}\left(L_{U_k}-L_{S_k}\mid S_k<\tau^+(\varepsilon),\mathscr{F}_{S_k}\right)\\&\le c_{17}\varepsilon|\log\varepsilon|\,\mathbf{E}\left(L_{U_k}-L_{S_k}\mid S_k<\tau^+(\varepsilon),\mathscr{F}_{S_k}\right).\end{aligned}$$

It follows that

$$N_m:=\sum_{k=1}^m c_{18}\varepsilon|\log\varepsilon|(L_{U_k}-L_{S_k})\mathbf{1}_{\{S_k<\tau^+(\varepsilon)\}}-(L_{S_{k+1}}-L_{U_k})\mathbf{1}_{\{U_k<\tau^+(\varepsilon)\}}$$

is a submartingale with respect to the filtration $\mathscr{F}^*_m=\mathscr{F}^{X,Y}_{S_{m+1}}$. If

$$M=\inf\{m:\sum_{k=1}^m(L_{U_k}-L_{S_k})\ge 1\}$$

and $M_i=M\wedge i$ then

$$\mathbf{E}\sum_{k=1}^{M_i}\left(c_{18}\varepsilon|\log\varepsilon|(L_{U_k}-L_{S_k})\mathbf{1}_{\{S_k<\tau^+(\varepsilon)\}}-(L_{S_{k+1}}-L_{U_k})\mathbf{1}_{\{U_k<\tau^+(\varepsilon)\}}\right)\ge 0,$$

and

$$\mathbf{E}\sum_{k=1}^{M_i}(L_{S_{k+1}}-L_{U_k})\mathbf{1}_{\{U_k<\tau^+(\varepsilon)\}}\le\mathbf{E}\sum_{k=1}^{M_i}c_{18}\varepsilon|\log\varepsilon|(L_{U_k}-L_{S_k})\mathbf{1}_{\{S_k<\tau^+(\varepsilon)\}}.$$

We let $i\to\infty$ and obtain by the monotone convergence

$$\begin{aligned}\mathbf{E}\sum_{k=1}^{M}(L_{S_{k+1}}-L_{U_k})\mathbf{1}_{\{U_k<\tau^+(\varepsilon)\}}&\le\mathbf{E}\sum_{k=1}^{M}c_{18}\varepsilon|\log\varepsilon|(L_{U_k}-L_{S_k})\mathbf{1}_{\{S_k<\tau^+(\varepsilon)\}}\\&\le c_{19}\varepsilon|\log\varepsilon|.\end{aligned}$$

Hence,

$$\mathbf{E}\sum_{k\ge1,U_k\le\sigma_*\wedge\tau^+(\varepsilon)}(L_{S_{k+1}}-L_{U_k})\le\mathbf{E}\sum_{k=1}^{M}(L_{S_{k+1}}-L_{U_k})\mathbf{1}_{\{U_k<\tau^+(\varepsilon)\}}\le c_{19}\varepsilon|\log\varepsilon|.$$

This and (3.50) imply the lemma. □

Recall parameters a_1 and a_2 and operator $\mathscr{G}_k$ defined in (3.2).

Lemma 3.12. *For any c_1 there exist $a_0, \varepsilon_0 > 0$ such that if $a_1, a_2 \in (0, a_0)$ and $|X_0 - Y_0| = \varepsilon \le \varepsilon_0$ then a.s., the following holds for all $k \ge 1$. Let*

$$\Theta = \left(\int_{S_k}^{U_k} \mathbf{n}(Y_t) dL_t^y - \int_{S_k}^{U_k} \mathbf{n}(\Pi(Y_{S_k})) dL_t^y \right) \left(|X_{S_k} - Y_{S_k}| \cdot |L_{U_k}^y - L_{S_k}^y| \right)^{-1},$$

with the convention that $b/0 = 0$. Then $|\Theta| \le c_1$ and

$$\Big| \mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k}) + \left(\mathbf{n}(\Pi(Y_{S_k})) + \Theta |X_{S_k} - Y_{S_k}| \right) \left((L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \right)$$
$$+ \pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k}) \Big| \le c_1 |L_{U_k} - L_{S_k}| \cdot |Y_{S_k} - X_{S_k}|.$$

Proof. By (2.2), for any c_2, we can find $\varepsilon_1 > 0$ so small that for any $x, y \in \partial D$ with $|x - y| \le 2\varepsilon_1$,

$$|\mathscr{S}(x)\pi_x(x - y) - (\mathbf{n}(y) - \mathbf{n}(x))| \le (c_2/2)|y - x|. \tag{3.52}$$

By Lemma 3.4, if we choose a sufficiently small $\varepsilon > 0$ then $|Y_t - X_t| \le 2\varepsilon_1$ for all $t \le \sigma_*$. Estimate (3.52) and C^2-smoothness of ∂D can be used to show that for any c_2 one can choose small $a_1, a_2 > 0$ and $\varepsilon_0 > 0$ so that for every $k \ge 1$ and all $t \in [S_k, U_k]$ such that $X_t \in \partial D$,

$$|\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k} - Y_{S_k}) - (\mathbf{n}(\Pi(Y_{S_k})) - \mathbf{n}(X_t))| \le c_2 |Y_{S_k} - X_{S_k}|. \tag{3.53}$$

We obtain from (2.14) and the triangle inequality,

$$\Big| (Y_{U_k} - X_{U_k}) - (Y_{S_k} - X_{S_k}) - \mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k} - Y_{S_k}) |L_{U_k} - L_{S_k}|$$
$$- \left(\mathbf{n}(\Pi(Y_{S_k})) + \Theta |X_{S_k} - Y_{S_k}| \right) \left((L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \right) \Big|$$
$$= \Big| \int_{S_k}^{U_k} \mathbf{n}(Y_t) dL_t^y - \int_{S_k}^{U_k} \mathbf{n}(X_t) dL_t - \mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k} - Y_{S_k}) |L_{U_k} - L_{S_k}|$$
$$- \left(\mathbf{n}(\Pi(Y_{S_k})) + \Theta |X_{S_k} - Y_{S_k}| \right) \left((L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \right) \Big|$$
$$\le \left| \int_{S_k}^{U_k} \mathbf{n}(Y_t) dL_t^y - \int_{S_k}^{U_k} \mathbf{n}(\Pi(Y_{S_k})) dL_t^y - \Theta |X_{S_k} - Y_{S_k}| (L_{U_k}^y - L_{S_k}^y) \right|$$
$$+ |\Theta| \, |X_{S_k} - Y_{S_k}| (L_{U_k} - L_{S_k})$$
$$+ \left| \int_{S_k}^{U_k} (\mathbf{n}(\Pi(Y_{S_k})) - \mathbf{n}(X_t)) dL_t - \mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k} - Y_{S_k}) |L_{U_k} - L_{S_k}| \right|$$
$$+ \left| \int_{S_k}^{U_k} \mathbf{n}(\Pi(Y_{S_k})) dL_t^y - \int_{S_k}^{U_k} \mathbf{n}(\Pi(Y_{S_k})) dL_t - \mathbf{n}(\Pi(Y_{S_k})) \left((L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \right) \right|.$$

The expression on the last line is equal to zero for elementary reasons, so

$$\begin{aligned}
&\Big|(Y_{U_k}-X_{U_k})-(Y_{S_k}-X_{S_k})-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}| \\
&\quad-\left(\mathbf{n}(\Pi(Y_{S_k}))+\Theta|X_{S_k}-Y_{S_k}|\right)\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)\Big| \\
&\le\left|\int_{S_k}^{U_k}\mathbf{n}(Y_t)dL^y_t-\int_{S_k}^{U_k}\mathbf{n}(\Pi(Y_{S_k}))dL^y_t-\Theta|X_{S_k}-Y_{S_k}|(L^y_{U_k}-L^y_{S_k})\right| \\
&\quad+|\Theta|\,|X_{S_k}-Y_{S_k}|(L_{U_k}-L_{S_k}) \\
&\quad+\left|\int_{S_k}^{U_k}(\mathbf{n}(\Pi(Y_{S_k}))-\mathbf{n}(X_t))dL_t-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}|\right|.
\end{aligned}$$

The first term on the right hand side is equal to 0 by the definition of Θ. It is easy to see that this claim holds even if the definition of Θ involves the division by 0. We have obtained

$$\begin{aligned}
&\Big|(Y_{U_k}-X_{U_k})-(Y_{S_k}-X_{S_k})-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}| \\
&\quad-\left(\mathbf{n}(\Pi(Y_{S_k}))+\Theta|X_{S_k}-Y_{S_k}|\right)\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)\Big| \\
&\le|\Theta|\,|X_{S_k}-Y_{S_k}|(L_{U_k}-L_{S_k}) \\
&\quad+\left|\int_{S_k}^{U_k}(\mathbf{n}(\Pi(Y_{S_k}))-\mathbf{n}(X_t))dL_t-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}|\right|.
\end{aligned}\tag{3.54}$$

It follows from the definitions of S_k, U_k and Π_x that for sufficiently small a_1 and a_2, we have for $t\in[S_k,U_k]$,

$$|Y_t-\Pi(Y_{S_k})|\le 2a_1|X_{S_k}-Y_{S_k}|,$$

and a similar formula holds for X in place of Y on the left hand side. Hence, by (2.7), for some c_3,

$$\begin{aligned}
\left|\int_{S_k}^{U_k}\mathbf{n}(Y_t)dL^y_t-\int_{S_k}^{U_k}\mathbf{n}(\Pi(Y_{S_k}))dL^y_t\right| &\le\int_{S_k}^{U_k}|\mathbf{n}(Y_t)-\mathbf{n}(\Pi(Y_{S_k}))|dL^y_t \\
&\le\int_{S_k}^{U_k}c_3|Y_t-\Pi(Y_{S_k})|dL^y_t \\
&\le\int_{S_k}^{U_k}c_3\cdot 2a_1|X_{S_k}-Y_{S_k}|dL^y_t \\
&\le 2a_1c_3|X_{S_k}-Y_{S_k}|\cdot|L^y_{U_k}-L^y_{S_k}|.
\end{aligned}$$

This shows that if we take a_1 sufficiently small then $|\Theta|\le c_1$.

We use (3.53) to derive the following estimate,

$$\begin{aligned}
&\left|\int_{S_k}^{U_k}(\mathbf{n}(\Pi(Y_{S_k}))-\mathbf{n}(X_t))dL_t-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}|\right| \\
&\le c_2|X_{S_k}-Y_{S_k}|\cdot|L_{U_k}-L_{S_k}|.
\end{aligned}\tag{3.55}$$

We combine (3.54)-(3.55) to see that

$$\Big|(Y_{U_k}-X_{U_k})-(Y_{S_k}-X_{S_k})-\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}| \tag{3.56}$$
$$-\left(\mathbf{n}(\Pi(Y_{S_k}))+\Theta|X_{S_k}-Y_{S_k}|\right)\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)\Big|$$
$$\leq (c_1/2+c_2)|X_{S_k}-Y_{S_k}|\cdot|L_{U_k}-L_{S_k}|.$$

For any c_2, we can choose small ε_0 so that

$$\Big|\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})+\mathscr{S}(\Pi(X_{S_k}))\pi_{\Pi(X_{S_k})}(X_{S_k}-Y_{S_k})|L_{U_k}-L_{S_k}|$$
$$-\exp((L_{U_k}-L_{S_k})\mathscr{S}(\Pi(X_{S_k})))\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})\Big|$$
$$\leq c_2|X_{S_k}-Y_{S_k}|\cdot|L_{U_k}-L_{S_k}|.$$

This and (3.56) imply that

$$\Big|Y_{U_k}-X_{U_k}-\mathscr{G}_k(Y_{S_k}-X_{S_k})-\left(\mathbf{n}(\Pi(Y_{S_k}))+\Theta|X_{S_k}-Y_{S_k}|\right)\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)$$
$$+\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})-(Y_{S_k}-X_{S_k})\Big|$$
$$=\Big|Y_{U_k}-X_{U_k}-\exp((L_{U_k}-L_{S_k})\mathscr{S}(\Pi(X_{S_k})))\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})$$
$$-\left(\mathbf{n}(\Pi(Y_{S_k}))+\Theta|X_{S_k}-Y_{S_k}|\right)\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)$$
$$+\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})-(Y_{S_k}-X_{S_k})\Big|$$
$$\leq (c_1/2+2c_2)|X_{S_k}-Y_{S_k}|\cdot|L_{U_k}-L_{S_k}|.$$

We obtain the lemma by choosing sufficiently small c_2. □

Lemma 3.13. *If a_1 is sufficiently small then for some $c_1,\varepsilon_0>0$ and all $\varepsilon<\varepsilon_0$, if $|X_0-Y_0|=\varepsilon$ then a.s., for all $k\geq 1$,*

$$|(L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})|\leq c_1|Y_{S_k}-X_{S_k}|^2.$$

Proof. Let $\mathbf{w}=\mathbf{n}(\Pi(X_{S_k}))$. It follows from the definition of U_k that

$$|\Pi(X_{S_k})-X_t|\vee|\Pi(X_{S_k})-Y_t|\leq c_2|Y_{S_k}-X_{S_k}|,$$

for $t\in[S_k,U_k]$. This and (2.8) imply that for some c_3 and $t\in[S_k,U_k]$,

$$1-c_3|Y_{S_k}-X_{S_k}|^2\leq\langle\mathbf{n}(X_t),\mathbf{w}\rangle\leq 1,\qquad \text{for } t \text{ such that } X_t\in\partial D, \tag{3.57}$$
$$1-c_3|Y_{S_k}-X_{S_k}|^2\leq\langle\mathbf{n}(Y_t),\mathbf{w}\rangle\leq 1,\qquad \text{for } t \text{ such that } Y_t\in\partial D. \tag{3.58}$$

We appeal to (2.13) to see that if a_1 is sufficiently small and $y\in\partial D$ and $z\in\overline{D}$ are such that

$$\max(|z-X_{S_k}|,|y-Y_{S_k}|)\leq a_1|X_{S_k}-Y_{S_k}|$$

then for some c_4,

$$|\langle y-z,\mathbf{w}\rangle|\leq c_4|Y_{S_k}-X_{S_k}|^2, \tag{3.59}$$

and

$$|\left\langle Y_{S_k} - X_{S_k}, \mathbf{w} \right\rangle| \leq c_4 |Y_{S_k} - X_{S_k}|^2. \tag{3.60}$$

Let $I = \{t \in [S_k, U_k] : \langle Y_t - X_t, \mathbf{w} \rangle \geq 2c_4 |Y_{S_k} - X_{S_k}|^2\}$. We claim that $I = \varnothing$. Suppose otherwise and let $t_1 = \inf I$ and $t_2 = \sup\{t \in [S_k, t_1] : Y_t \in \partial D\}$, with the convention that $\sup \varnothing = S_k$. By (3.57), (3.59) and (3.60),

$$\begin{aligned}
\left\langle Y_{t_1} - X_{t_1}, \mathbf{w} \right\rangle &= \left\langle Y_{t_2} - X_{t_2}, \mathbf{w} \right\rangle + \left\langle \int_{t_2}^{t_1} \mathbf{n}(Y_s) dL_s^y, \mathbf{w} \right\rangle - \left\langle \int_{t_2}^{t_1} \mathbf{n}(X_s) dL_s, \mathbf{w} \right\rangle \\
&\leq \left\langle Y_{t_2} - X_{t_2}, \mathbf{w} \right\rangle + \left\langle \int_{t_2}^{t_1} \mathbf{n}(Y_s) dL_s^y, \mathbf{w} \right\rangle \\
&= \left\langle Y_{t_2} - X_{t_2}, \mathbf{w} \right\rangle \leq c_4 |Y_{S_k} - X_{S_k}|^2.
\end{aligned}$$

This contradicts the definition of t_1, so we see that $I = \varnothing$. Similarly, one can prove that

$$\{t \in [S_k, U_k] : \langle X_t - Y_t, \mathbf{w} \rangle \geq 2c_4 |Y_{S_k} - X_{S_k}|^2\} = \varnothing.$$

Hence

$$\{t \in [S_k, U_k] : |\left\langle X_t - Y_t, \mathbf{w} \right\rangle| \geq 2c_4 |Y_{S_k} - X_{S_k}|^2\} = \varnothing.$$

This and (3.57)-(3.58) yield,

$$\begin{aligned}
&(1 + c_3 |Y_{S_k} - X_{S_k}|^2)(L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \\
&\leq \left\langle \int_{S_k}^{U_k} \mathbf{n}(Y_s) dL_s^y, \mathbf{w} \right\rangle - \left\langle \int_{S_k}^{U_k} \mathbf{n}(X_s) dL_s, \mathbf{w} \right\rangle \\
&= \left\langle (Y_{U_k} - Y_{S_k}) - (X_{U_k} - X_{S_k}), \mathbf{w} \right\rangle \\
&\leq 4c_4 |Y_{S_k} - X_{S_k}|^2.
\end{aligned}$$

By the definition of σ_*, $L_{U_k}^y - L_{S_k}^y \leq c_5$, so the above estimate implies

$$(L_{U_k}^y - L_{S_k}^y) - (L_{U_k} - L_{S_k}) \leq 4c_4 |Y_{S_k} - X_{S_k}|^2 + c_3 |Y_{S_k} - X_{S_k}|^2 (L_{U_k}^y - L_{S_k}^y) \leq c_6 |Y_{S_k} - X_{S_k}|^2.$$

An analogous argument gives

$$(L_{U_k} - L_{S_k}) - (L_{U_k}^y - L_{S_k}^y) \leq c_7 |Y_{S_k} - X_{S_k}|^2.$$

The lemma follows from the last two estimates. □

Lemma 3.14. *For some c_1 there exist $a_0, \varepsilon_0 > 0$ such that if $a_1, a_2 \in (0, a_0)$, $\varepsilon \leq \varepsilon_0$ and $|X_0 - Y_0| = \varepsilon$ then for all $k \geq 1$,*

$$\mathbf{E}\left(\left| \pi_{\Pi(X_{S_{k+1}})} \left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k}) \right) \right| \mid \mathscr{F}_{S_k} \right) \leq c_1 \varepsilon |\log \varepsilon|^2 |Y_{S_k} - X_{S_k}|^2.$$

Proof. The vector $\mathbf{w}_k := \pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})$ is parallel to $\mathbf{n}(\Pi(X_{S_k}))$. It is easy to check from the definition of S_k that $|\mathbf{w}_k| \leq c_2 |Y_{S_k} - X_{S_k}|^2$.

Let $T_1 = \inf\{t \geq U_k : X_t \in \partial D\}$. It follows from Lemma 3.4 and definition of U_k that $\mathbf{d}(X_{U_k}, \partial D) \leq c_3 \varepsilon$. Let j_0 be the smallest integer such that $\varepsilon 2^{j_0}$ is greater than the diameter of D. Lemma 3.5 (i) shows that for some c_4 and all $j = 1, 2, \dots, j_0$,

$$\mathbf{P}(|X_{T_1} - X_{U_k}| \geq \varepsilon 2^j \mid \mathscr{F}_{U_k}) \leq c_4 2^{-j}.$$

By Lemma 3.8 (iii), the strong Markov property applied at T_1, and Chebyshev's inequality,

$$\mathbf{P}(|X_{T_1} - X_{S_{k+1}}| \geq \varepsilon 2^j \mid \mathscr{F}_{T_1}) \leq c_5 \varepsilon |\log \varepsilon| / (\varepsilon 2^j) = c_5 2^{-j} |\log \varepsilon|.$$

The fact that $|X_{S_k} - X_{U_k}| \leq c_6 \varepsilon$ and the last two estimates show that

$$\mathbf{P}(|X_{S_k} - X_{S_{k+1}}| \geq \varepsilon 2^j \mid \mathscr{F}_{S_k}) \leq c_6 2^{-j} |\log \varepsilon|.$$

It is easy to see that $|\pi_{\Pi(X_{S_{k+1}})} \mathbf{w}_k| \leq c_7 \varepsilon 2^j |\mathbf{w}_k|$ if $|X_{S_k} - X_{S_{k+1}}| \leq \varepsilon 2^j$. It follows that

$$\begin{aligned}
&\mathbf{E}\left(\left| \pi_{\Pi(X_{S_{k+1}})} \left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k}) \right) \right| \mid \mathscr{F}_{S_k} \right) \\
&\leq c_7 \varepsilon |\mathbf{w}_k| + \sum_{j=1}^{j_0} c_7 \varepsilon 2^{j+1} |\mathbf{w}_k| \mathbf{P}(|X_{S_k} - X_{S_{k+1}}| \in [\varepsilon 2^j, \varepsilon 2^{j+1}] \mid \mathscr{F}_{S_k}) \\
&\leq c_7 \varepsilon c_2 |Y_{S_k} - X_{S_k}|^2 + \sum_{j=1}^{j_0} c_7 \varepsilon 2^{j+1} c_2 |Y_{S_k} - X_{S_k}|^2 c_6 2^{-j} |\log \varepsilon| \\
&\leq c_8 \varepsilon |\log \varepsilon|^2 \, |Y_{S_k} - X_{S_k}|^2.
\end{aligned}$$

□

Lemma 3.15. *For some c_1 there exist $a_0, \varepsilon_0 > 0$ such that if $a_1, a_2 \in (0, a_0)$, $\varepsilon \leq \varepsilon_0$ and $|X_0 - Y_0| = \varepsilon$ then for all $k \geq 1$,*

$$\mathbf{E}\left(\left| \pi_{\Pi(X_{S_{k+1}})} \left((Y_{U_k} - X_{U_k}) - (Y_{S_{k+1}} - X_{S_{k+1}}) \right) \right| \mid \mathscr{F}_{U_k} \right) \leq c_1 |Y_{U_k} - X_{U_k}|^3 |\log |Y_{U_k} - X_{U_k}||^2.$$

Proof. Fix some k and let

$$T_1 = \inf\{t \geq U_k : X_t \in \partial D \text{ or } Y_t \in \partial D\}$$

and $\varepsilon_1 = |X_{U_k} - Y_{U_k}|$. We will assume from now on that $X_{T_1} \in \partial D$. The rest of the argument is similar if $Y_{T_1} \in \partial D$.

It follows from Lemma 3.4 and definition of U_k that $\mathbf{d}(X_{U_k}, \partial D) \leq c_2 \varepsilon_1$. Let j_0 be the smallest integer such that $\varepsilon_1 2^{j_0}$ is greater than the diameter of D. Lemma 3.5 shows that for some c_3 and all $j = 1, 2, \dots, j_0$,

$$\mathbf{P}(|X_{T_1} - X_{U_k}| \geq \varepsilon_1 2^j) \leq c_3 2^{-j}. \tag{3.61}$$

By (2.9), we can choose c_4 so small that for $x \in \partial D \cap \mathscr{B}(X_{T_1}, 5 c_4 \varepsilon_1)$,

$$|\langle x - X_{T_1}, \mathbf{n}(X_{T_1}) \rangle| \leq a_2 \varepsilon_1^2 / 800. \tag{3.62}$$

By the definition of σ_*, $|Y_t - X_t| \le c_5\varepsilon_1$ for $t \le \sigma_*$. We make c_4 smaller, if necessary, so that, in view of (2.11),

$$|\langle y - x, \mathbf{n}(z)\rangle| \le a_2\varepsilon_1^2/400, \tag{3.63}$$

assuming that $x, y, z \in \partial D$, $|y - z| \le (c_5 + 5c_4)\varepsilon_1$ and $|x - y| \le 10c_4\varepsilon_1$.

The following definitions contain a parameter c_6, the value of which will be chosen later. Let

$$\begin{aligned} J &= \inf\{j \ge 1 : |X_{T_1} - X_{U_k}| \le \varepsilon_1 2^j\}, \\ T_2 &= \inf\{t \ge T_1 : |B_t - B_{T_1}| \ge c_4\varepsilon_1\}, \\ T_3 &= \inf\{t \ge T_1 : \langle \mathbf{n}(X_{T_1}), B_t - B_{T_1}\rangle \le -c_6\varepsilon_1^2 2^J\}, \\ A_1 &= \{T_3 \le T_2\}. \end{aligned}$$

Note that neither X nor Y touches the boundary of D between times U_k and T_1, so $Y_{T_1} - X_{T_1} = Y_{U_k} - X_{U_k}$. Hence, by Lemma 3.10 and the strong Markov property applied at S_k,

$$\left|\left\langle \mathbf{n}(\Pi(X_{U_k})), \frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right\rangle\right| \le c_7\varepsilon_1. \tag{3.64}$$

The angle between $\mathbf{n}(\Pi(X_{U_k}))$ and $\mathbf{n}(X_{T_1})$ is bounded by $c_8\varepsilon_1 2^J$ because ∂D is C^2. This and (3.64) imply that

$$\left|\left\langle \mathbf{n}(X_{T_1}), \frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right\rangle\right| \le c_9\varepsilon_1 2^J. \tag{3.65}$$

Let k_1 be such that $c_9\varepsilon_1 2^J \le 1/10$ if $J \le k_1$, and let $F_1 = \{J \le k_1\}$. If F_1 holds then (3.65) implies that,

$$\left|\pi_{X_{T_1}}\left(\frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right)\right| \ge 1/10. \tag{3.66}$$

Case(i). This case is devoted to an estimate of the random variable in the statement of the lemma assuming that $A_1 \cap F_1$ holds. Since $|Y_{T_1} - X_{T_1}| = \varepsilon_1$, (3.65) implies that

$$\mathbf{d}(Y_{T_1}, \partial D) \le c_{10}\varepsilon_1^2 2^J. \tag{3.67}$$

Let $c_{11} = 5c_4$ and

$$\begin{aligned} T_4 &= \inf\{t \ge T_1 : |X_t - X_{T_1}| \ge c_{11}\varepsilon_1\} \wedge T_2 \wedge T_3, \\ T_5 &= \sup\{t \le T_4 : X_t \in \partial D\}. \end{aligned}$$

We will show that $T_4 = T_2 \wedge T_3$, if ε (and, therefore, ε_1) is sufficiently small. By (2.11),

$$\langle x - y, \mathbf{n}(X_{T_1})\rangle \le c_{12}\varepsilon_1^2 \tag{3.68}$$

for all $x, y \in \mathscr{B}(X_{T_1}, c_{11}\varepsilon_1)$ such that $x \in \partial D$ and $y \in \overline{D}$. Since $T_5 \le T_3$, we have

$$\langle (B_{T_5} - B_{T_1}), \mathbf{n}(X_{T_1})\rangle \ge -c_6\varepsilon_1^2 2^J. \tag{3.69}$$

This and (3.68) imply that

$$\left\langle \int_{T_1}^{T_5} \mathbf{n}(X_s) dL_s, \mathbf{n}(X_{T_1}) \right\rangle = \left\langle (X_{T_5} - X_{T_1}) - (B_{T_5} - B_{T_1}), \mathbf{n}(X_{T_1}) \right\rangle \le c_{13} \varepsilon_1^2 2^J. \tag{3.70}$$

For $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{11}\varepsilon_1)$ we have by (2.8), for small ε_1,

$$\langle \mathbf{n}(x), \mathbf{n}(X_{T_1}) \rangle \ge 1 - c_{14}\varepsilon_1^2 \ge 1/2. \tag{3.71}$$

This and (3.70) show that

$$L_{T_5} - L_{T_1} \le 2 \left\langle \int_{T_1}^{T_5} \mathbf{n}(X_s) dL_s, \mathbf{n}(X_{T_1}) \right\rangle \le c_{15} \varepsilon_1^2 2^J. \tag{3.72}$$

For $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{11}\varepsilon_1)$,

$$\left| \pi_{X_{T_1}}(\mathbf{n}(x)) \right| \le c_{16}\varepsilon_1. \tag{3.73}$$

It follows from this and (3.72) that

$$\left| \pi_{X_{T_1}} \left(\int_{T_1}^{T_5} \mathbf{n}(X_s) dL_s \right) \right| \le c_{17} \varepsilon_1^3 2^J \le c_{18} \varepsilon_1^2. \tag{3.74}$$

We can assume that ε_1 is so small that for $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{11}\varepsilon_1)$,

$$|x - X_{T_1}| \le 2|\pi_{X_{T_1}}(x - X_{T_1})|. \tag{3.75}$$

Since $T_4 \le T_2 \wedge T_3$, we can use (3.74) and (3.75) to obtain,

$$\begin{aligned} \left| X_{T_4} - X_{T_1} \right| &\le \left| X_{T_4} - X_{T_5} \right| + \left| X_{T_5} - X_{T_1} \right| \le \left| X_{T_4} - X_{T_5} \right| + 2|\pi_{X_{T_1}}(X_{T_5} - X_{T_1})| \qquad (3.76) \\ &\le \left| B_{T_4} - B_{T_5} \right| + 2\left| \pi_{X_{T_1}}(B_{T_5} - B_{T_1}) \right| + 2\left| \pi_{X_{T_1}} \left(\int_{T_1}^{T_5} \mathbf{n}(X_s) dL_s \right) \right| \\ &\le \left| B_{T_4} - B_{T_1} \right| + \left| B_{T_1} - B_{T_5} \right| + 2\left| \pi_{X_{T_1}}(B_{T_5} - B_{T_1}) \right| + 2\left| \pi_{X_{T_1}} \left(\int_{T_1}^{T_5} \mathbf{n}(X_s) dL_s \right) \right| \\ &\le 4c_4\varepsilon_1 + 2c_{18}\varepsilon_1^2. \end{aligned}$$

Recall that $c_{11} = 5c_4$. Hence, the last estimate and the definition of T_4 show that $T_4 = T_2 \wedge T_3$, if ε_1 is sufficiently small.

Next we will estimate $\mathbf{d}(X_{T_3}, \partial D)$. Let $R_1 = \sup\{t \le T_3 : X_t \in \partial D\}$. By the definition of T_3,

$$\langle B_{T_3} - B_{R_1}, \mathbf{n}(X_{T_1}) \rangle \le 0.$$

This and the fact that $X_{T_3} - X_{R_1} = B_{T_3} - B_{R_1}$ imply that,

$$\langle X_{T_3} - X_{R_1}, \mathbf{n}(X_{T_1}) \rangle \le 0. \tag{3.77}$$

Since $X_{R_1} \in \partial D \cap \mathscr{B}(X_{T_1}, c_{11}\varepsilon_1)$, it follows from (3.62) and (3.77) that

$$\langle X_{T_3} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle = \langle X_{T_3} - X_{R_1}, \mathbf{n}(X_{T_1})\rangle + \langle X_{R_1} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle \le a_2\varepsilon_1^2/800.$$

This and (3.62) imply that

$$\mathbf{d}(X_{T_3}, \partial D) \le 2a_2\varepsilon_1^2/800 = a_2\varepsilon_1^2/400. \tag{3.78}$$

Our next goal is to estimate $\mathbf{d}(Y_{T_3}, \partial D)$. Recall that $|Y_t - X_t| \le c_5\varepsilon_1$ for $t \le \sigma_*$. Since $T_4 = T_2 \wedge T_3$, the definition of T_4 implies that for $t \in [T_1, T_2 \wedge T_3]$,

$$|Y_t - X_{T_1}| \le |Y_t - X_t| + |X_t - X_{T_1}| \le c_5\varepsilon_1 + c_{11}\varepsilon_1 = c_{19}\varepsilon_1. \tag{3.79}$$

Let $c_{20} = 5c_4$ and

$$T_6 = \inf\{t \ge T_1 : |Y_t - Y_{T_1}| \ge c_{20}\varepsilon_1\} \wedge T_2 \wedge T_3.$$

If $Y_t \notin \partial D$ for $t \in [T_1, T_6]$ then $L^y_{T_6} - L^y_{T_1} = 0$. Suppose that $Y_t \in \partial D$ for some $t \in [T_1, T_6]$ and let

$$T_7 = \sup\{t \le T_6 : Y_t \in \partial D\}.$$

We will show that $T_6 = T_2 \wedge T_3$, if ε (and, therefore, ε_1) is sufficiently small. By (2.11),

$$\langle x - y, \mathbf{n}(X_{T_1})\rangle \le c_{21}\varepsilon_1^2 \tag{3.80}$$

for all $x, y \in \mathscr{B}(X_{T_1}, c_{19}\varepsilon_1)$ such that $x \in \partial D$ and $y \in \overline{D}$. Since $T_7 \le T_3$, we have

$$\langle (B_{T_7} - B_{T_1}), \mathbf{n}(X_{T_1})\rangle \ge -c_6\varepsilon_1^2 2^J.$$

Since $T_7 \le T_2 \wedge T_3$, we can use (3.80) and the last estimate to see that

$$\left\langle \int_{T_1}^{T_7} \mathbf{n}(Y_s)dL^y_s, \mathbf{n}(X_{T_1}) \right\rangle = \left\langle (Y_{T_7} - Y_{T_1}) - (B_{T_7} - B_{T_1}), \mathbf{n}(X_{T_1}) \right\rangle \le c_{22}\varepsilon_1^2 2^J. \tag{3.81}$$

The above estimate is also valid in the case when $Y_t \notin \partial D$ for $t \in [T_1, T_6]$ because in this case $L^y_{T_6} - L^y_{T_1} = 0$.
For $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{19}\varepsilon_1)$ we have by (2.8), for small ε_1,

$$\langle \mathbf{n}(x), \mathbf{n}(X_{T_1})\rangle \ge 1 - c_{23}\varepsilon_1^2 \ge 1/2.$$

This and (3.81) show that

$$L^y_{T_7} - L^y_{T_1} \le 2\left\langle \int_{T_1}^{T_7} \mathbf{n}(Y_s)dL^y_s, \mathbf{n}(X_{T_1}) \right\rangle \le c_{24}\varepsilon_1^2 2^J. \tag{3.82}$$

For $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{19}\varepsilon_1)$, we have $\left|\pi_{X_{T_1}}(\mathbf{n}(x))\right| \le c_{25}\varepsilon_1$. It follows from this and (3.82) that

$$\left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_7} \mathbf{n}(Y_s)dL^y_s\right)\right| \le c_{26}\varepsilon_1^3 2^J \le c_{27}\varepsilon_1^2. \tag{3.83}$$

We can assume that ε_1 is so small that for $x \in \partial D \cap \mathscr{B}(X_{T_1}, c_{19}\varepsilon_1)$,

$$|x - X_{T_1}| \le 2|\pi_{X_{T_1}}(x - X_{T_1})|. \tag{3.84}$$

Since $T_6 \le T_2 \wedge T_3$, (3.83) and (3.84) imply that

$$\begin{aligned}\left|Y_{T_6} - Y_{T_1}\right| &\le \left|Y_{T_6} - Y_{T_7}\right| + \left|Y_{T_7} - Y_{T_1}\right| \le \left|Y_{T_6} - Y_{T_7}\right| + 2|\pi_{X_{T_1}}(Y_{T_7} - Y_{T_1})| \\ &\le \left|B_{T_6} - B_{T_7}\right| + 2\left|\pi_{X_{T_1}}(B_{T_7} - B_{T_1})\right| + 2\left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_7} \mathbf{n}(Y_s)dL_s^y\right)\right| \\ &\le \left|B_{T_6} - B_{T_1}\right| + \left|B_{T_1} - B_{T_7}\right| + 2\left|\pi_{X_{T_1}}(B_{T_7} - B_{T_1})\right| + 2\left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_7} \mathbf{n}(Y_s)dL_s^y\right)\right| \\ &\le 4c_4\varepsilon_1 + 2c_{27}\varepsilon_1^2.\end{aligned} \tag{3.85}$$

Recall that $c_{20} = 5c_4$. The last estimate and the definition of T_6 show that $T_6 = T_2 \wedge T_3$, if ε_1 is sufficiently small.

If ε_1 is small then, by (3.79), for $t \in [T_1, T_2 \wedge T_3]$,

$$|\Pi(Y_t) - X_{T_1}| \le 2|Y_t - X_{T_1}| \le 2c_{19}\varepsilon_1.$$

For $x \in \partial D \cap \mathscr{B}(X_{T_1}, 2c_{19}\varepsilon_1)$, by (2.9),

$$|\langle x - X_{T_1}, \mathbf{n}(X_{T_1})\rangle| \le c_{28}\varepsilon_1^2, \tag{3.86}$$

so, in particular,

$$|\langle \Pi(Y_{T_1}) - X_{T_1}, \mathbf{n}(X_{T_1})\rangle| \le c_{28}\varepsilon_1^2.$$

This and (3.67) imply that

$$\begin{aligned}|\langle Y_{T_1} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle| &\le |\langle \Pi(Y_{T_1}) - X_{T_1}, \mathbf{n}(X_{T_1})\rangle| + |\langle \Pi(Y_{T_1}) - Y_{T_1}, \mathbf{n}(X_{T_1})\rangle| \\ &\le c_{28}\varepsilon_1^2 + c_{10}\varepsilon_1^2 2^J \le c_{29}\varepsilon_1^2 2^J.\end{aligned} \tag{3.87}$$

Recall that we assume that A_1 holds so that $T_3 \le T_2$. By (2.10), for $x \in \overline{D} \cap \mathscr{B}(X_{T_1}, c_{19}\varepsilon_1)$,

$$\langle x - X_{T_1}, \mathbf{n}(X_{T_1})\rangle \ge -c_{30}\varepsilon_1^2,$$

so, in view of (3.79),

$$\langle Y_{T_3} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle \ge -c_{30}\varepsilon_1^2. \tag{3.88}$$

We now choose the parameter c_6 in the definition of T_3 so that $-c_6 + c_{29} \le -2c_{30}$. We will show that given this choice of c_6, we must have $Y_t \in \partial D$ for $t \in [T_1, T_3]$. Suppose that $Y_t \notin \partial D$ for $t \in [T_1, T_3]$. Then $Y_t - Y_{T_1} = B_t - B_{T_1}$ for the same range of t's. It follows from (3.87) and from the definition of T_3 that

$$\begin{aligned}\langle Y_{T_3} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle &= \langle Y_{T_3} - Y_{T_1}, \mathbf{n}(X_{T_1})\rangle + \langle Y_{T_1} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle \\ &= \langle B_{T_3} - B_{T_1}, \mathbf{n}(X_{T_1})\rangle + \langle Y_{T_1} - X_{T_1}, \mathbf{n}(X_{T_1})\rangle \\ &\le -c_6\varepsilon_1^2 2^J + c_{29}\varepsilon_1^2 2^J \le -2c_{30}\varepsilon_1^2.\end{aligned}$$

This contradicts (3.88), so we conclude that Y must cross ∂D between times T_1 and T_3. Hence, T_7 is well defined. Since we are assuming that A_1 holds, $T_7 \le T_3 = T_6$. Therefore,

$$|Y_{T_7} - Y_{T_3}| \le |Y_{T_7} - Y_{T_1}| + |Y_{T_1} - Y_{T_3}| \le 2c_{20}\varepsilon_1 = 10c_4\varepsilon_1. \tag{3.89}$$

By (3.79), $|Y_{T_7} - X_{T_1}| \le (c_5 + 5c_4)\varepsilon_1$. This and (3.89) imply that the following can be derived as a special case of (3.63),

$$|\langle Y_{T_7} - x, \mathbf{n}(X_{T_1})\rangle| \le a_2\varepsilon_1^2/400, \tag{3.90}$$

for $x \in \partial D \cap \mathscr{B}(Y_{T_7}, 2c_{20}\varepsilon_1)$. By the definition of T_3,

$$\langle B_{T_3} - B_{T_7}, \mathbf{n}(X_{T_1})\rangle \le 0.$$

This and the fact that $Y_{T_3} - Y_{T_7} = B_{T_3} - B_{T_7}$ imply that,

$$\langle Y_{T_3} - Y_{T_7}, \mathbf{n}(X_{T_1})\rangle \le 0.$$

We use this estimate and (3.90) to conclude that

$$\mathbf{d}(Y_{T_3}, \partial D) \le a_2\varepsilon_1^2/400. \tag{3.91}$$

Recall that we are assuming that F_1 holds. It follows from (3.66) that

$$\left|\pi_{X_{T_1}}\left(\frac{Y_{T_1} - X_{T_1}}{|Y_{T_1} - X_{T_1}|}\right)\right| \ge 1/10,$$

and, therefore,

$$\left|\pi_{X_{T_1}}\left(Y_{T_1} - X_{T_1}\right)\right| \ge \varepsilon_1/10.$$

By (3.74) and (3.83)

$$\begin{aligned}
\left|\pi_{X_{T_1}}\left(Y_{T_3} - X_{T_3}\right)\right| &\ge \left|\pi_{X_{T_1}}\left(Y_{T_1} - X_{T_1}\right)\right| - \left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_3} \mathbf{n}(X_s)dL_s\right)\right| - \left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_3} \mathbf{n}(Y_s)dL_s^y\right)\right| \\
&= \left|\pi_{X_{T_1}}\left(Y_{T_1} - X_{T_1}\right)\right| - \left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_5} \mathbf{n}(X_s)dL_s\right)\right| - \left|\pi_{X_{T_1}}\left(\int_{T_1}^{T_7} \mathbf{n}(Y_s)dL_s^y\right)\right| \\
&\ge \varepsilon_1/10 - c_{18}\varepsilon_1^2 - c_{27}\varepsilon_1^2.
\end{aligned}$$

For small ε_1, this is bounded below by $\varepsilon_1/20$. Hence,

$$|Y_{T_3} - X_{T_3}| \ge \left|\pi_{X_{T_1}}\left(Y_{T_3} - X_{T_3}\right)\right| \ge \varepsilon_1/20.$$

This, (3.78) and (3.91) imply that $S_{k+1} \le T_3$, assuming $A_1 \cap F_1$ holds.

It follows from the definition of T_4 and the fact that $S_{k+1} \le T_3 = T_4$ that $|X_{S_{k+1}} - X_{T_1}| \le c_{11}\varepsilon_1$. This implies that $|\Pi(X_{S_{k+1}}) - X_{T_1}| \le 2c_{11}\varepsilon_1$, assuming that ε_1 is sufficiently small. Let

$$T_8 = \sup\{t \in [T_1, S_{k+1}] : X_t \in \partial D\}.$$

It is routine to check that (3.68)-(3.73) hold with X_{T_1} replaced with $\Pi(X_{S_{k+1}})$, and T_5 replaced with T_8 (the values of the constants may have to be adjusted). Hence, we obtain as in (3.74) that

$$\left|\pi_{\Pi(X_{S_{k+1}})}\left(\int_{T_1}^{S_{k+1}} \mathbf{n}(X_s) dL_s\right)\right| = \left|\pi_{\Pi(X_{S_{k+1}})}\left(\int_{T_1}^{T_8} \mathbf{n}(X_s) dL_s\right)\right| \leq c_{31}\varepsilon_1^3 2^J. \tag{3.92}$$

Similarly, an argument analogous to that in (3.80)-(3.83) yields

$$\left|\pi_{\Pi(X_{S_{k+1}})}\left(\int_{T_1}^{S_{k+1}} \mathbf{n}(Y_s) dL_s^y\right)\right| \leq c_{32}\varepsilon_1^3 2^J.$$

This and (3.92) imply that

$$\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k} - X_{U_k}) - (Y_{S_{k+1}} - X_{S_{k+1}})\right)\right| \tag{3.93}$$

$$= \left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{T_1} - X_{T_1}) - (Y_{S_{k+1}} - X_{S_{k+1}})\right)\right| \tag{3.94}$$

$$= \left|\pi_{\Pi(X_{S_{k+1}})}\left(\int_{T_1}^{S_{k+1}} \mathbf{n}(X_s) dL_s - \int_{T_1}^{S_{k+1}} \mathbf{n}(Y_s) dL_s^y\right)\right|$$

$$\leq c_{33}\varepsilon_1^3 2^J.$$

We obtain from this and (3.61),

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k} - X_{U_k}) - (Y_{S_{k+1}} - X_{S_{k+1}})\right)\right| \mathbf{1}_{A_1 \cap F_1} \mid \mathscr{F}_{U_k}\right) \tag{3.95}$$

$$\leq \sum_{j=1}^{j_0} c_{34}\varepsilon_1^3 2^j 2^{-j} \leq c_{35}\varepsilon_1^3 |\log \varepsilon_1| = c_{35}\varepsilon_1^2 |\log \varepsilon| |Y_{U_k} - X_{U_k}|.$$

Case (ii). We will now analyze the case when A_1 does not occur. The rest of the proof is an outline only. Most steps are very similar to those in Case (i), so we omit details to save space.

Standard estimates show that

$$\mathbf{P}(A_1^c \mid \mathscr{F}_{T_1}) \leq c_{36}\varepsilon_1 2^J. \tag{3.96}$$

Recall that we have assumed that $X_{T_1} \in \partial D$. Let

$$T_9 = \inf\{t \geq T_2 : Y_t \in \partial D\}.$$

For some c_{37} and c_{38}, we let

$$K = \inf\{j \geq 1 : \sup_{t \in [T_2, T_9]} |Y_t - Y_{T_2}| \leq \varepsilon_1 2^j\},$$

$$T_8 = \inf\{t \geq T_7 : |B_t - B_{T_7}| \geq c_{37}\varepsilon_1\},$$

$$T_9 = \inf\{t \geq T_7 : \langle \mathbf{n}(Y_{T_7}), B_t - B_{T_7}\rangle \leq -c_{38}\varepsilon_1^2 2^K\},$$

$$A_2 = \{T_9 \leq T_8\}.$$

Let $T_{10} = \sup\{t \le T_9 : X_t \in \partial D\}$ and note that $X_{T_9} - Y_{T_9} = X_{T_{10}} - Y_{T_{10}}$. Using the fact that $X_{T_1} \in \partial D$ and definitions of T_1, T_2 and K, one can show that

$$\left| \left\langle \mathbf{n}(Y_{T_9}), \frac{Y_{T_9} - X_{T_9}}{|Y_{T_9} - X_{T_9}|} \right\rangle \right| = \left| \left\langle \mathbf{n}(Y_{T_9}), \frac{Y_{T_{10}} - X_{T_{10}}}{|Y_{T_{10}} - X_{T_{10}}|} \right\rangle \right| \le c_{39} \varepsilon_1 2^K. \tag{3.97}$$

This implies that $\mathbf{d}(X_{T_9}, \partial D) \le c_{40} \varepsilon_1^2 2^K$. We can repeat the argument proving (3.94), with the roles of X and Y interchanged and T_1 replaced by T_9, to see that if A_2 holds then $S_{k+1} \le T_9$ and

$$\left| \pi_{\Pi(X_{S_{k+1}})} \left((Y_{T_9} - X_{T_9}) - (Y_{S_{k+1}} - X_{S_{k+1}}) \right) \right| \le c_{41} \varepsilon_1^3 2^K. \tag{3.98}$$

The angle between $\mathbf{n}(Y_{T_9})$ and $\mathbf{n}(\Pi(X_{S_{k+1}}))$ is less than $c_{42}\varepsilon_1$. We know from (3.67) that $\mathbf{d}(Y_{T_1}, \partial D) \le c_{43} \varepsilon_1^2 2^J$. These facts and (3.97) imply that

$$\left| \left\langle \mathbf{n}(\Pi(X_{S_{k+1}})), \int_{T_2}^{T_9} \mathbf{n}(X_s) dL_s \right\rangle \right| = \left| \left\langle \mathbf{n}(\Pi(X_{S_{k+1}})), (Y_{T_9} - X_{T_9}) - (Y_{T_2} - X_{T_2}) \right\rangle \right| \le c_{44} \varepsilon_1^2 2^{J \vee K}.$$

Let k_2 be the largest integer such that if $K \le k_2$ then for $x \in \partial D \cap \mathscr{B}(Y_{T_2}, 2\varepsilon_1 2^K)$ we have $\langle \mathbf{n}(x), \mathbf{n}(\Pi(X_{S_{k+1}})) \rangle \ge 1/2$. Assume that $F_2 := \{K \le k_2\}$ holds. It follows that

$$L_{T_9} - L_{T_2} \le 2 \left\langle \int_{T_2}^{T_9} \mathbf{n}(X_s) dL_s, \mathbf{n}(\Pi(X_{S_{k+1}})) \right\rangle \le c_{45} \varepsilon_1^2 2^{J \vee K}.$$

We also have $L_{T_2} - L_{T_1} \le c_{46} \varepsilon_1^2 2^J$ by (3.72). Hence, $L_{T_9} - L_{T_1} \le c_{47} \varepsilon_1^2 2^{J \vee K}$.
For $x \in \partial D \cap \mathscr{B}(Y_{T_2}, 2\varepsilon_1 2^K)$, we have $|\pi_{\Pi(X_{S_{k+1}})}(\mathbf{n}(x))| \le c_{48} \varepsilon_1 2^K$, so

$$\left| \pi_{\Pi(X_{S_{k+1}})} \left(\int_{T_1}^{T_9} \mathbf{n}(X_s) dL_s \right) \right| \le c_{49} \varepsilon_1^3 2^{(J \vee K)+K}.$$

By (3.82), $L^y_{T_2} - L^y_{T_1} \le c_{50} \varepsilon_1^2 2^J$, so

$$\left| \pi_{\Pi(X_{S_{k+1}})} \left(\int_{T_1}^{T_9} \mathbf{n}(Y_s) dL^y_s \right) \right| = \left| \pi_{\Pi(X_{S_{k+1}})} \left(\int_{T_1}^{T_2} \mathbf{n}(Y_s) dL^y_s \right) \right| \le c_{51} \varepsilon_1^3 2^{J+K}.$$

Combining the last two estimates with (3.98), we obtain,

$$\begin{aligned} &\left| \pi_{\Pi(X_{S_{k+1}})} \left((Y_{S_{k+1}} - X_{S_{k+1}}) - (Y_{T_1} - X_{T_1}) \right) \right| \\ &= \left| \pi_{\Pi(X_{S_{k+1}})} \left((Y_{S_{k+1}} - X_{S_{k+1}}) - (Y_{T_9} - X_{T_9}) \right) \right| + \left| \pi_{\Pi(X_{S_{k+1}})} \left((Y_{T_9} - X_{T_9}) - (Y_{T_1} - X_{T_1}) \right) \right| \\ &\le c_{41} \varepsilon_1^3 2^K + \left| \pi_{\Pi(X_{S_{k+1}})} \left(\int_{T_1}^{T_9} \mathbf{n}(X_s) dL_s \right) \right| + \left| \pi_{\Pi(X_{S_{k+1}})} \left(\int_{T_1}^{T_2} \mathbf{n}(Y_s) dL^y_s \right) \right| \le c_{52} \varepsilon_1^3 2^{(J \vee K)+K}. \end{aligned} \tag{3.99}$$

This implies that

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mathbf{1}_{A_1^c\cap A_2\cap F_2}\mid \mathscr{F}_{U_k}\right) \tag{3.100}$$
$$=\sum_{j=1}^{j_0}\sum_{k=1}^{j_0}\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mathbf{1}_{A_1^c\cap A_2\cap F_2}\mid J=j,K=k,\mathscr{F}_{U_k}\right)$$
$$\times\,\mathbf{P}\left(J=j,K=k\mid \mathscr{F}_{U_k}\right).$$

By (3.67) and an estimate similar to that in Lemma 3.5 (i),

$$\mathbf{P}\left(K=k\mid \mathscr{F}_{T_1}\right)\le c_{53}\varepsilon_1^2 2^J\varepsilon_1^{-1}2^{-k}=c_{53}\varepsilon_1 2^{J-k}.$$

This, (3.61) an the strong Markov property applied at T_1 yield,

$$\mathbf{P}\left(J=j,K=k\mid \mathscr{F}_{U_k}\right)\le c_{54}2^{-j}\varepsilon_1 2^{j-k}=c_{54}\varepsilon_1 2^{-k}. \tag{3.101}$$

For $K\ge J$ we have $2^{(J\vee K)+K}=2^{2K}$ so the the right hand side of (3.99) is bounded by $c_{55}\varepsilon_1^3 2^{2K}$. This and (3.101) imply that the corresponding contribution to the expectation in (3.100) is bounded by

$$\sum_{j=1}^{j_0}\sum_{k=j}^{j_0}c_{54}\varepsilon_1 2^{-k}c_{55}\varepsilon_1^3 2^{2k}\le c_{56}\varepsilon_1^3|\log\varepsilon_1|. \tag{3.102}$$

For $K<J$ we have $2^{(J\vee K)+K}=2^{J+K}$ so the corresponding contribution to the expectation in (3.100) is bounded by

$$\sum_{j=1}^{j_0}\sum_{k=1}^{j}c_{54}\varepsilon_1 2^{-k}c_{55}\varepsilon_1^3 2^{j+k}\le c_{57}\varepsilon_1^3|\log\varepsilon_1|.$$

Combining this with (3.102) yields

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mathbf{1}_{A_1^c\cap A_2\cap F_2}\mid \mathscr{F}_{U_k}\right)\le c_{58}\varepsilon_1^3|\log\varepsilon_1|. \tag{3.103}$$

The probability that A_2 does not occur, conditional on J and K, is bounded above by $c_{59}\varepsilon_1^2 2^K/\varepsilon_1 = c_{59}\varepsilon_1 2^K$. If $A_1^c\cap A_2^c$ holds, we use the following crude estimate,

$$\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\le c_5\varepsilon_1.$$

Therefore, using (3.101),

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mathbf{1}_{A_1^c\cap A_2^c}\mid \mathscr{F}_{U_k}\right) \tag{3.104}$$
$$\le\sum_{j=1}^{j_0}\sum_{k=1}^{j_0}c_{54}\varepsilon_1 2^{-k}c_{59}\varepsilon_1 2^k c_5\varepsilon_1\le c_{60}\varepsilon_1^3|\log\varepsilon_1|^2.$$

It remains to address the cases when F_1 or F_2 fail. The probability of $F_1^c \cap F_2^c$ is bounded by $c_{61}\varepsilon_1^2$. Hence,

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mathbf{1}_{F_1^c\cap F_2^c}\mid \mathscr{F}_{U_k}\right)\leq c_{61}\varepsilon_1^2 c_5\varepsilon_1 = c_{62}\varepsilon_1^3. \tag{3.105}$$

If F_1 fails but F_2 does not. we can repeat the analysis presented in Case (ii). Hence, (3.103) holds with $\mathbf{1}_{A_1^c\cap A_2\cap F_2}$ replaced with $\mathbf{1}_{F_1^c\cap A_2\cap F_2}$. The lemma follows from these remarks, (3.95), (3.103), (3.104) and (3.105). □

Lemma 3.16. *We have for some c_1,*

$$\mathbf{E}\left(\sum_{k=0}^{m'}|Y_{S_k}-X_{S_k}|\right)\leq c_1.$$

Proof. We will use modified versions of stopping times S_k and U_k by dropping σ_* from the definition (3.1). Let $S_0^* = U_0^* = \inf\{t\geq 0 : X_t\in\partial D\}$ and for $k\geq 1$ define

$$\begin{aligned}
S_k^* &= \inf\left\{t\geq U_{k-1}^* : \mathbf{d}(X_t,\partial D)\vee \mathbf{d}(Y_t,\partial D)\leq a_2|X_t-Y_t|^2\right\},\\
U_k^* &= \inf\left\{t\geq S_k^* : |X_t-X_{S_k^*}|\vee|Y_t-Y_{S_k^*}|\geq a_1|X_{S_k^*}-Y_{S_k^*}|\right\}.
\end{aligned}$$

Fix some k and let

$$\begin{aligned}
T_1 &= \inf\left\{t\geq S_k^* : \left\langle B_t-B_{S_k^*},\mathbf{n}(\Pi(X_{S_k^*}))\right\rangle\leq -(a_1/2)|X_{S_k^*}-Y_{S_k^*}|\right\},\\
T_2 &= \inf\left\{t\geq S_k^* : \left\langle B_t-B_{S_k^*},\mathbf{n}(\Pi(X_{S_k^*}))\right\rangle\geq (a_1/4)|X_{S_k^*}-Y_{S_k^*}|\right\},\\
T_3 &= \inf\left\{t\geq S_k^* : \left|\pi_{\Pi(X_{S_k^*})}\left(B_t-B_{S_k^*}\right)\right|\geq (a_1/10)|X_{S_k^*}-Y_{S_k^*}|\right\},\\
A &= \{T_1\leq T_2\leq T_3\},\\
\mathscr{F}_k^* &= \sigma\{B_t, t\leq S_k^*\}.
\end{aligned}$$

Let $\varepsilon = |X_0-Y_0|$ and recall that $|X_t-Y_t|<c_2\varepsilon$ for $t\leq\sigma_*$. By Brownian scaling and the strong Markov property, $\mathbf{P}(A\mid\mathscr{F}_k^*)\geq p_1$ on $\{S_k^*\leq\sigma^*\}$, for some $p_1>0$ that does not depend on ε or k. An argument similar to that in the proof of Lemma 3.8 (i) can be used to show that if ε, a_1 and a_2 are small and A holds then $T_1<U_k^*$ and $L_{T_1}-L_{S_k^*}>(a_1/4)|X_{S_k^*}-Y_{S_k^*}|$. Then $L_{U_k^*}-L_{S_k^*}>(a_1/4)|X_{S_k^*}-Y_{S_k^*}|$, so

$$\mathbf{E}(L_{U_k^*}-L_{S_k^*}\mid\mathscr{F}_k^*)>p_1(a_1/4)|X_{S_k^*}-Y_{S_k^*}|.$$

We use this estimate to see that

$$\begin{aligned}
\mathbf{E}\left(\sum_{k=0}^{m'}|Y_{S_k}-X_{S_k}|\right) &= \mathbf{E}\left(\sum_{k=0}^{m'}|Y_{S_k^*}-X_{S_k^*}|\right) \\
&= \mathbf{E}\left(\sum_{k=0}^{m'-1}|Y_{S_k^*}-X_{S_k^*}|\right)+|Y_{S_{m'}^*}-X_{S_{m'}^*}| \\
&\le \mathbf{E}\left(\sum_{k=0}^{m'-1}c_3\mathbf{E}\left(L_{U_k^*}-L_{S_k^*}\mid \mathscr{F}_k^*\right)\right)+|Y_{S_{m'}^*}-X_{S_{m'}^*}| \\
&\le c_3\mathbf{E}\left(\sum_{k=0}^{m'-1}\left(L_{U_k^*}-L_{S_k^*}\right)\right)+|Y_{S_{m'}^*}-X_{S_{m'}^*}| \\
&\le c_3\mathbf{E}\sigma_*+|Y_{S_{m'}^*}-X_{S_{m'}^*}|.
\end{aligned} \tag{3.106}$$

It is elementary to check that for all j,

$$\mathbf{P}(L_{j+1}-L_j>1\mid\sigma\{B_t,t\le j\})\ge p_2>0.$$

Hence, $\sigma_*\le\sigma_1$ is stochastically majorized by a geometric random variable with mean depending only on D, so

$$\mathbf{E}\sigma_*<c_4<\infty. \tag{3.107}$$

We have $|X_{S_{m'}^*}-Y_{S_{m'}^*}|<c_2\varepsilon$ because $S_{m'}^*\le\sigma_*$. We combine this, (3.106) and (3.107) to complete the proof. □

Lemma 3.17. *For some c_1 there exists $a_0>0$ such that if $a_1,a_2\in(0,a_0)$ and $|X_0-Y_0|=\varepsilon$ then,*

$$\mathbf{E}\left(\sum_{k=0}^{m'}|X_{S_k}-Y_{S_k}|\left((L_{U_k}^y-L_{S_k}^y)-(L_{U_k}-L_{S_k})\right)\right)\le c_1\varepsilon^2.$$

Proof. We have by Lemmas 3.13 and 3.16,

$$\mathbf{E}\left(\sum_{k=0}^{m'}|X_{S_k}-Y_{S_k}|\left((L_{U_k}^y-L_{S_k}^y)-(L_{U_k}-L_{S_k})\right)\right)\le c_2\varepsilon^2\mathbf{E}\left(\sum_{k=0}^{m'}|X_{S_k}-Y_{S_k}|\right)\le c_3\varepsilon^2.$$

□

Lemma 3.18. *For some c_1 there exists $a_0>0$ such that if $a_1,a_2\in(0,a_0)$ and $|X_0-Y_0|=\varepsilon$ then,*

$$\mathbf{E}\left(\sum_{k=0}^{m'}\left|\pi_{\Pi(X_{S_{k+1}})}\left(\mathbf{n}(\Pi(Y_{S_k}))\left((L_{U_k}^y-L_{S_k}^y)-(L_{U_k}-L_{S_k})\right)\right)\right|\right)\le c_1\varepsilon^2|\log\varepsilon|.$$

Proof. First, we will show that

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}(\mathbf{n}(\Pi(Y_{S_k}))\right|\mid\mathscr{F}_{U_k}\right)\le c_2|Y_{S_k}-X_{S_k}||\log|Y_{S_k}-X_{S_k}||. \tag{3.108}$$

Recall the notation from the proof of Lemma 3.15, in particular, $\varepsilon_1 = |Y_{U_k} - X_{U_k}|$, and note that by Lemma 3.4, $\varepsilon_1 \le c_3 |Y_{S_k} - X_{S_k}|$. If A_1 occurs then $S_{k+1} \le T_3 \le T_2$. This and definitions of S_k, U_k, T_2, T_3 and T_4 imply that

$$\begin{aligned}|Y_{S_k} - X_{S_{k+1}}| &\le |Y_{S_k} - X_{S_k}| + |X_{S_k} - X_{U_k}| + |X_{U_k} - X_{T_1}| + |X_{T_1} - X_{S_{k+1}}| \\ &\le c_4 |Y_{S_k} - X_{S_k}| 2^J.\end{aligned}$$

Therefore, (2.12) shows that $\left|\pi_{\Pi(X_{S_{k+1}})}(\mathbf{n}(\Pi(Y_{S_k}))\right| \le c_5 \varepsilon_1 2^J$. We calculate as in (3.95),

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}(\mathbf{n}(\Pi(Y_{S_k}))\right| \mathbf{1}_{A_1} \mid \mathscr{F}_{U_k}\right) \le \sum_{j=1}^{j_0} c_6 \varepsilon_1 2^j 2^{-j} \le c_7 \varepsilon_1 |\log \varepsilon_1|. \qquad (3.109)$$

We obtain from (3.96),

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}(\mathbf{n}(\Pi(Y_{S_k}))\right| \mathbf{1}_{A_1^c} \mid \mathscr{F}_{U_k}\right) \le \mathbf{E}\left(\mathbf{1}_{A_1^c} \mid \mathscr{F}_{U_k}\right) \le \sum_{j=1}^{j_0} c_8 \varepsilon_1 2^j 2^{-j} \le c_9 \varepsilon_1 |\log \varepsilon_1|.$$

This and (3.109) prove (3.108). By (3.108) and Lemma 3.13,

$$\mathbf{E}\left(\left|\mathbf{n}(\Pi(Y_{S_k}))\left((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\right)\right| \mid \mathscr{F}_{U_k}\right) \le c_{10} |Y_{S_k} - X_{S_k}|^3 |\log |Y_{S_k} - X_{S_k}||.$$

We use this estimate and Lemma 3.16 to conclude that

$$\begin{aligned}&\mathbf{E}\left(\sum_{k=0}^{m'}\left|\mathbf{n}(\Pi(Y_{S_k}))\left((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\right)\right|\right) \\ &\le \mathbf{E}\left(\sum_{k=0}^{m'} c_{11} \varepsilon^2 |\log \varepsilon| |Y_{S_k} - X_{S_k}|\right) \le c_{12} \varepsilon^2 |\log \varepsilon|.\end{aligned}$$

□

Lemma 3.19. *For some c_1 there exist $a_0, \varepsilon_0 > 0$ such that if $a_1, a_2 \in (0, a_0)$, $\varepsilon \le \varepsilon_0$ and $|X_0 - Y_0| = \varepsilon$ then,*

$$\mathbf{E}\left(\sum_{k=0}^{m'}\left|\pi_{\Pi(X_{S_{k+1}})}\left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})\right)\right|\right) \le c_1 \varepsilon^2 |\log \varepsilon|^2.$$

Proof. Lemmas 3.14 and 3.16 imply that

$$\begin{aligned}&\mathbf{E}\left(\sum_{k=0}^{m'}\left|\pi_{\Pi(X_{S_{k+1}})}\left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})\right)\right|\right) \\ &\le \mathbf{E}\left(\sum_{k=0}^{m'} \mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})\right)\right| \mid \mathscr{F}_{S_k}\right)\right) \\ &\le \mathbf{E}\left(\sum_{k=0}^{m'} c_2 \varepsilon |\log \varepsilon|^2 |Y_{S_k} - X_{S_k}|^2\right) \le \mathbf{E}\left(\sum_{k=0}^{m'} c_3 \varepsilon^2 |\log \varepsilon|^2 |Y_{S_k} - X_{S_k}|\right) \le c_4 \varepsilon^2 |\log \varepsilon|^2.\end{aligned}$$

□

Lemma 3.20. *For any $c_1, \varepsilon_0 > 0$ there exist $a_0 > 0$, a random variable Λ and c_2 such that if $\varepsilon \in (0, \varepsilon_0)$, $a_1, a_2 < a_0$ and $|X_0 - Y_0| = \varepsilon$ then $|\Lambda| \le c_1 \varepsilon$, a.s., and*

$$\mathbf{E}\left| \left|(Y_{\sigma_*} - X_{\sigma_*}) - \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0)\right| - \Lambda \right| \le c_2 \varepsilon^2 |\log \varepsilon|^2.$$

Proof. Note that $S_{m'+1} = \sigma_*$. We have

$$\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - (Y_{\sigma_*} - X_{\sigma_*}) \tag{3.110}$$
$$= \sum_{k=0}^{m'} \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \left(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{S_{k+1}} - X_{S_{k+1}}) \right)$$
$$= \sum_{k=0}^{m'} \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \left(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k}) \right) \tag{3.111}$$
$$+ \sum_{k=0}^{m'} \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \left((Y_{U_k} - X_{U_k}) - (Y_{S_{k+1}} - X_{S_{k+1}}) \right).$$

Recall Θ from Lemma 3.12. By (2.3), Lemma 3.4 and the triangle inequality, we have the following estimate for the first sum in (3.111),

$$\left| \sum_{k=0}^{m'} \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \left(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k}) \right) \right|$$
$$\le c_3 \sum_{k=0}^{m'} \left| \mathscr{G}_{k+1} \left(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k}) \right) \right|$$
$$\le c_3 \sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k})$$
$$+ \left(\mathbf{n}(\Pi(Y_{S_k})) + \Theta |X_{S_k} - Y_{S_k}| \right) \left((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k}) \right)$$
$$+ \pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k}) \Big) \Big|$$
$$+ c_3 \sum_{k=0}^{m'} \left| \mathscr{G}_{k+1} \left(\mathbf{n}(\Pi(Y_{S_k})) \left((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k}) \right) \right) \right|$$
$$+ c_3 \sum_{k=0}^{m'} \left| \mathscr{G}_{k+1} \left(\Theta |X_{S_k} - Y_{S_k}| \left((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k}) \right) \right) \right|$$
$$+ c_3 \sum_{k=0}^{m'} \left| \mathscr{G}_{k+1} \left(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k}) \right) \right|.$$

We combine this with (3.110) to obtain

$$\Big|\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - (Y_{\sigma_*} - X_{\sigma_*})\Big| \tag{3.112}$$

$$\le c_3 \sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\mathscr{G}_k(Y_{S_k} - X_{S_k}) - (Y_{U_k} - X_{U_k}) \tag{3.113}$$

$$+ \Big(\mathbf{n}(\Pi(Y_{S_k})) + \Theta|X_{S_k} - Y_{S_k}|\Big) \Big((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\Big) \tag{3.114}$$

$$+ \pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})\Big)\Big| \tag{3.115}$$

$$+ c_3 \sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\mathbf{n}(\Pi(Y_{S_k})) \Big((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\Big)\Big)\Big| \tag{3.116}$$

$$+ c_3 \sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\Theta|X_{S_k} - Y_{S_k}| \Big((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\Big)\Big)\Big| \tag{3.117}$$

$$+ c_3 \sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\pi_{\Pi(X_{S_k})}(Y_{S_k} - X_{S_k}) - (Y_{S_k} - X_{S_k})\Big)\Big| \tag{3.118}$$

$$+ \sum_{k=0}^{m'} \Big| \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \Big((Y_{U_k} - X_{U_k}) - (Y_{S_{k+1}} - X_{S_{k+1}})\Big)\Big|. \tag{3.119}$$

We need the following elementary fact about any non-negative real numbers b_1, b_2 and b_3. Suppose that $b_1 \le b_2 + b_3$. Let $\Lambda = \max(0, b_1 - b_2)$. Then $|\Lambda| \le b_3$. Moreover, $|b_1 - \Lambda| \le b_2$. To see this, suppose that $b_1 \ge b_2$. Then $\Lambda = b_1 - b_2$ and $|b_1 - \Lambda| = |b_1 - (b_1 - b_2)| = b_2$. If $b_1 < b_2$ then $\Lambda = 0$ and $|b_1 - \Lambda| = |b_1| < b_2$. We apply these observations to b_1 equal to (3.112), b_2 equal to the sum of the terms (3.116)-(3.119), and b_3 equal to (3.113)-(3.115). To finish the proof of the lemma, it will suffice to prove that

$$b_3 \le c_1 \varepsilon, \quad \text{a.s.}, \tag{3.120}$$

and

$$\mathbf{E} b_2 \le c_2 \varepsilon^2 |\log \varepsilon|^2. \tag{3.121}$$

Fix an arbitrarily small $c_1 > 0$. By Lemma 3.4, $|Y_{S_k} - X_{S_k}| \le c_4 \varepsilon$, for all k, a.s. By Lemma 3.12, if a_1 and a_2 are sufficiently small then with probability 1,

$$b_3 \le (c_1/c_4) \sum_{k=0}^{m'} |L_{U_k} - L_{S_k}| \cdot |Y_{S_k} - X_{S_k}| \le c_1 \varepsilon \sum_{k=0}^{m'} |L_{U_k} - L_{S_k}|.$$

We have $\sum_{k=0}^{m'} |L_{U_k} - L_{S_k}| \le 1$, so a.s., $b_3 \le c_1 \varepsilon$, that is, (3.120) holds true.
We estimate (3.116) using (2.3) and Lemma 3.18,

$$\mathbf{E} \left(\sum_{k=0}^{m'} \Big| \mathscr{G}_{k+1} \Big(\mathbf{n}(\Pi(Y_{S_k})) \Big((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\Big)\Big)\Big| \right) \tag{3.122}$$

$$\le c_5 \mathbf{E} \left(\sum_{k=0}^{m'} \Big| \pi_{\Pi(X_{S_{k+1}})} \Big(\mathbf{n}(\Pi(Y_{S_k})) \Big((L^y_{U_k} - L^y_{S_k}) - (L_{U_k} - L_{S_k})\Big)\Big)\Big| \right) \le c_6 \varepsilon^2 |\log \varepsilon|.$$

Similarly, (2.3) and Lemma 3.19 yield the following estimate for (3.118),

$$\begin{aligned}&\mathbf{E}\left(\sum_{k=0}^{m'}\left|\mathscr{G}_{k+1}\left(\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})-(Y_{S_k}-X_{S_k})\right)\right|\right) \\ &\leq c_5\mathbf{E}\left(\sum_{k=0}^{m'}\left|\pi_{\Pi(X_{S_{k+1}})}\left(\pi_{\Pi(X_{S_k})}(Y_{S_k}-X_{S_k})-(Y_{S_k}-X_{S_k})\right)\right|\right)\leq c_7\varepsilon^2|\log\varepsilon|^2.\end{aligned}\tag{3.123}$$

Recall from Lemma 3.12 that $|\Theta|\leq c_8$. By (2.3) and Lemmas 3.13 and 3.16,

$$\begin{aligned}&\mathbf{E}\left(\sum_{k=0}^{m'}\left|\mathscr{G}_{k+1}\left(\Theta|X_{S_k}-Y_{S_k}|\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)\right)\right|\right) \\ &\leq c_9\mathbf{E}\left(\sum_{k=0}^{m'}\left||X_{S_k}-Y_{S_k}|\left((L^y_{U_k}-L^y_{S_k})-(L_{U_k}-L_{S_k})\right)\right|\right) \\ &\leq c_{10}\mathbf{E}\left(\sum_{k=0}^{m'}|X_{S_k}-Y_{S_k}|^3\right)\leq c_{11}\varepsilon^2\mathbf{E}\left(\sum_{k=0}^{m'}|X_{S_k}-Y_{S_k}|\right)\leq c_{12}\varepsilon^2.\end{aligned}\tag{3.124}$$

By Lemma 3.15,

$$\mathbf{E}\left(\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\mid\mathscr{F}_{U_k}\right)\leq c_{13}|Y_{U_k}-X_{U_k}|^3|\log|Y_{U_k}-X_{U_k}||^2.$$

Hence, using (2.3) and Lemmas 3.4 and 3.16,

$$\begin{aligned}&\mathbf{E}\left(\left|\sum_{k=0}^{m'}\mathscr{G}_{m'}\circ\cdots\circ\mathscr{G}_{k+1}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\right) \\ &\leq c_{14}\mathbf{E}\left(\sum_{k=0}^{m'}\left|\pi_{\Pi(X_{S_{k+1}})}\left((Y_{U_k}-X_{U_k})-(Y_{S_{k+1}}-X_{S_{k+1}})\right)\right|\right) \\ &\leq c_{15}\mathbf{E}\left(\sum_{k=0}^{m'}|Y_{U_k}-X_{U_k}|^3|\log|Y_{U_k}-X_{U_k}||^2\right) \\ &\leq c_{16}\varepsilon^2|\log\varepsilon|^2\mathbf{E}\left(\sum_{k=0}^{m'}|Y_{U_k}-X_{U_k}|\right)\leq c_{17}\varepsilon^2|\log\varepsilon|^2.\end{aligned}\tag{3.125}$$

The inequality in (3.121) follows from (3.122)-(3.125). This completes the proof of the lemma. □

Recall operator $\mathscr{H}_k$ defined in (3.2).

Lemma 3.21. *For any $c_1,\varepsilon_0>0$ there exists $a_0>0$ such that if $a_1,a_2<a_0$ and $|X_0-Y_0|=\varepsilon$ then,*

$$\mathbf{E}|\mathscr{G}_{m'}\circ\cdots\circ\mathscr{G}_0(Y_0-X_0)-\mathscr{H}_{m'}\circ\cdots\circ\mathscr{H}_0(Y_0-X_0)|\leq c_1\varepsilon^2|\log\varepsilon|.$$

Proof. We have

$$\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) \tag{3.126}$$
$$= \sum_{k=0}^{m'} \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_{k+1} \left(\exp((L_{U_k} - L_{S_k})\mathscr{S}(\Pi(X_{S_k}))) - \exp((L_{S_{k+1}} - L_{S_k})\mathscr{S}(\Pi(X_{S_k}))) \right)$$
$$\circ \pi_{\Pi(X_{S_k})} \mathscr{H}_{k-1} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0).$$

By (2.6),

$$\| \exp((L_{U_k} - L_{S_k})\mathscr{S}(\Pi(X_{S_k}))) - \exp((L_{S_{k+1}} - L_{S_k})\mathscr{S}(\Pi(X_{S_k})))\| \leq c_2 |L_{U_k} - L_{S_{k+1}}|.$$

This, (2.3) and (3.126) imply that

$$|\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0)| \leq c_3 |(Y_0 - X_0)| \sum_{k=0}^{m'} |L_{U_k} - L_{S_{k+1}}|.$$

By Lemma 3.11, $\mathbf{E} \sum_{k=0}^{m'} |L_{U_k} - L_{S_{k+1}}| \leq c_4 \varepsilon |\log \varepsilon|$. Hence,

$$\mathbf{E} |\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0)| \leq c_4 \varepsilon^2 |\log \varepsilon|.$$

□

Recall notation from the beginning of this section.

Lemma 3.22. *We have for any $\beta_1 < 1$ and some c_0 and c_1, assuming that $|X_0 - Y_0| = \varepsilon$ and $\varepsilon_* \geq c_0 \varepsilon$,*

$$\mathbf{E} \left(\sum_{k=0}^{m'} \sum_{U_k \leq \xi_j \leq S_{k+1}} (L_{S_{k+1}} - L_{\xi_j}) |x_j^* - \Pi(X_{S_{k+1}})| \right) \leq c_1 \varepsilon^{1+\beta_1}.$$

Proof. By Lemma 3.8 (iv), for every k,

$$\mathbf{E} \left(\sum_{S_k \leq \xi_j \leq S_{k+1}} (L_{S_{k+1}} - L_{\xi_j}) |x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{S_k} \right) \leq c_2 |X_{S_k} - Y_{S_k}|^{2+\beta_1}.$$

This and Lemma 3.16 imply that

$$\mathbf{E} \left(\sum_{k=0}^{m'} \sum_{U_k \leq \xi_j \leq S_{k+1}} (L_{S_{k+1}} - L_{\xi_j}) |x_j^* - \Pi(X_{S_{k+1}})| \right)$$
$$\leq \mathbf{E} \left(\sum_{k=0}^{m'} \mathbf{E} \left(\sum_{U_k \leq \xi_j \leq S_{k+1}} (L_{S_{k+1}} - L_{\xi_j}) |x_j^* - \Pi(X_{S_{k+1}})| \mid \mathscr{F}_{S_k} \right) \right)$$
$$\leq \mathbf{E} \left(\sum_{k=0}^{m'} c_2 |X_{S_k} - Y_{S_k}|^{2+\beta_1} \right)$$
$$\leq \mathbf{E} \left(\sum_{k=0}^{m'} c_3 |X_{U_k} - Y_{U_k}| \varepsilon^{1+\beta_1} \right) \leq c_4 \varepsilon^{1+\beta_1}.$$

□

For the notation used in the following lemma and its proof, see the beginning of this section.

Lemma 3.23. *We have for any $\beta < 1$, some c_0 and c_1, assuming that $|X_0 - Y_0| = \varepsilon$ and $\varepsilon_* \geq c_0\varepsilon$,*

$$\mathbf{E}\left|\mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)\right| \leq c_1 \varepsilon^{1+\beta}.$$

Proof. We will follow closely the proof of Lemma 2.13 in [BL]. We will write $\mathscr{S}_i = \mathscr{S}(x_i'') = \mathscr{S}(x_i^*)$, $\pi_i = \pi_{x_i''} = \pi_{x_i^*}$. Recall that $m'' = m^*$. We have

$$\begin{aligned}
&\left|\mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(Y_0 - X_0)\right| \\
&= \left(e^{\Delta \ell^*_{m^*} \mathscr{S}_{m^*}} - e^{(\ell^*_{m^*+1} - \ell''_{m^*})\mathscr{S}_{m^*}}\right) \pi_{m^*} \circ \mathscr{J}_{m''-1} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) \\
&\quad + \sum_{i=1}^{m^*} e^{\Delta \ell^*_{m^*} \mathscr{S}_{m^*}} \pi_{m^*} \cdots e^{\Delta \ell^*_{i+1} \mathscr{S}_{i+1}} \pi_{i+1} \circ \\
&\qquad \left(e^{(\ell^*_{i+1} - \ell''_i)\mathscr{S}_i} \pi_i e^{\Delta \ell''_{i-1} \mathscr{S}_{i-1}} - e^{\Delta \ell^*_i \mathscr{S}_i} \pi_i e^{(\ell^*_i - \ell''_{i-1})\mathscr{S}_{i-1}}\right) \circ \\
&\qquad\quad \pi_{i-1} e^{\Delta \ell''_{i-2} \mathscr{S}_{i-2}} \cdots e^{\Delta \ell''_1 \mathscr{S}_1} \pi_1 e^{\Delta \ell''_0 \mathscr{S}_0} \pi_0 (Y_0 - X_0) \\
&\quad + \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_1 \left(e^{(\ell^*_1 - \ell''_0)\mathscr{S}_0} - e^{\Delta \ell''_0 \mathscr{S}_0}\right) \pi_0 (Y_0 - X_0).
\end{aligned} \tag{3.127}$$

By virtue of (2.3) and (2.4), the last term is bounded by a constant multiple of $|\ell_1^* - \ell_1''|\,|Y_0 - X_0|$. Since $\ell_1'' \geq \ell_1^*$, $\mathbf{E}|\ell_1^* - \ell_1''|\,|Y_0 - X_0| = \varepsilon \mathbf{E}(\ell_1'' - \ell_1^*)$. By the strong Markov property applied at ξ_1 and Lemma 3.8 (ii), $\mathbf{E}(\ell_1'' - \ell_1^*) \leq c_2 \varepsilon$. Hence

$$\mathbf{E}\left(\mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_1 \left(e^{(\ell^*_1 - \ell''_0)\mathscr{S}_0} - e^{\Delta \ell''_0 \mathscr{S}_0}\right) \pi_0 (Y_0 - X_0)\right) \leq c_3 \mathbf{E}|\ell_1^* - \ell_1''|\,|Y_0 - X_0| \leq c_4 \varepsilon^2. \tag{3.128}$$

We have $\ell''_{m^*+1} = \ell^*_{m^*+1} = 1$, so by (2.3) and (2.4), the first term on the right hand side of (3.127) is bounded by a constant multiple of $|\ell^*_{m^*} - \ell''_{m^*}|\,|Y_0 - X_0|$. We have $\ell''_{m^*} \geq \ell^*_{m^*}$ so $\mathbf{E}|\ell^*_{m^*} - \ell''_{m^*}|\,|Y_0 - X_0| \leq \varepsilon \mathbf{E}(1 - \ell^*_{m^*})$. The following estimate can be proved just like (3.10). We have for every $x \in \partial D$ and $b > 0$,

$$c_5/b \leq H^x\left(|e(0) - e(\zeta)| \geq b\right) \leq c_6/b. \tag{3.129}$$

This and the exit system formula (2.16) imply that $1 - \ell_1^*$ is stochastically majorized by an exponential random variable with mean $c_7\varepsilon$, so $\mathbf{E}(1 - \ell_1^*) \leq c_7 \varepsilon$. Hence

$$\begin{aligned}
&\mathbf{E}\left(\left(e^{\Delta \ell^*_{m^*} \mathscr{S}_{m^*}} - e^{(\ell^*_{m^*+1} - \ell''_{m^*})\mathscr{S}_{m^*}}\right) \pi_{m^*} \circ \mathscr{J}_{m''-1} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)\right) \\
&\leq c_8 \mathbf{E}|\ell^*_{m^*} - \ell''_{m^*}|\,|Y_0 - X_0| \leq c_9 \varepsilon^2.
\end{aligned} \tag{3.130}$$

The compositions before and after the parentheses in (3.127) in the summation are uniformly bounded in operator norm by (2.3), so we need only estimate the sum

$$\sum_{i=0}^{m^*} \left\| e^{(\ell^*_{i+1} - \ell''_i)\mathscr{S}_i} \pi_i e^{\Delta \ell''_{i-1} \mathscr{S}_{i-1}} - e^{\Delta \ell^*_i \mathscr{S}_i} \pi_i e^{(\ell^*_i - \ell''_{i-1})\mathscr{S}_{i-1}} \right\|.$$

Using the fact that π_i commutes with $\mathscr{S}_i$, we can rewrite the i-th term in this sum as

$$\begin{aligned}
&\left\| e^{\Delta \ell_i^* \mathscr{S}_i} \circ \pi_i \circ \left(e^{(\ell_i^* - \ell_i'')\mathscr{S}_i} - e^{(\ell_i^* - \ell_i'')\mathscr{S}_{i-1}} \right) e^{\Delta \ell_{i-1}'' \mathscr{S}_{i-1}} \right\| \\
&\qquad \le \left\| e^{\Delta \ell_i^* \mathscr{S}_i} \right\| \left\| e^{(\ell_i^* - \ell_i'')\mathscr{S}_i} - e^{(\ell_i^* - \ell_i'')\mathscr{S}_{i-1}} \right\| \left\| e^{\Delta \ell_{i-1}'' \mathscr{S}_{i-1}} \right\|.
\end{aligned}$$

From (2.3) and (2.5), this last expression is bounded by $c_{10} \left| \ell_i^* - \ell_i'' \right| \left| x_i'' - x_{i-1}'' \right|$. By Lemma 3.22, for any $\beta < 1$,

$$\mathbf{E} \sum_{i=1}^{m^*} \left| \ell_i^* - \ell_i'' \right| \left| x_i'' - x_{i-1}'' \right| \le c_{11} \varepsilon^{1+\beta}.$$

This combined with (3.128) and (3.130) yields the lemma. □

Once again, we ask the reader to consult the beginning of this section concerning notation used in the next lemma and its proof.

Lemma 3.24. *Suppose that $\varepsilon_* = c_0 \varepsilon$, where c_0 is as in Lemma 3.23. For some c_1, if we assume that $|X_0 - Y_0| = \varepsilon$ then,*

$$\mathbf{E} \left| \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) \right| \le c_1 \varepsilon^{4/3} |\log \varepsilon|.$$

Proof. Note that

$$\mathscr{H}_k = \exp(\Delta \ell_k') \mathscr{S}(x_k')) \pi_{x_k'}.$$

Let $\{(\ell_k, x_k)\}_{0 \le k \le m+1}$ be the sequence containing all the distinct elements of the union of $\{(\ell_k', x_k')\}_{0 \le k \le m'+1}$ and $\{(\ell_k'', x_k'')\}_{0 \le k \le m''+1}$. We will explain how the sequence $\{(\ell_k, x_k)\}_{0 \le k \le m+1}$ is ordered but first we note that ℓ_k''s need not be distinct, and neither do ℓ_k'''s, and, moreover, some ℓ_k''s may be equal to some ℓ_k'''s. We order the sequence $\{(\ell_k, x_k)\}_{0 \le k \le m+1}$ in such a way that

(i) $\ell_k \le \ell_{k+1}$ for all k.

(ii) If $\ell_{k_1} = \ell_{j_1}'$, $\ell_{k_2} = \ell_{j_2}'$, $\ell_{j_1}' = L_{S_{j_1}}$, $\ell_{j_2}' = L_{S_{j_2}}$, and $S_{j_1} < S_{j_2}$ then $k_1 < k_2$.

(iii) If $\ell_{k_1} = \ell_{j_1}''$, $\ell_{k_2} = \ell_{j_2}''$, $\ell_{j_1}'' = \lambda(\ell_{j_3}^*)$, $\ell_{j_2}'' = \lambda(\ell_{j_4}^*)$, and $\ell_{j_3}^* < \ell_{j_4}^*$ then $k_1 < k_2$.

(iv) If $(\ell_{k_1}, x_{k_1}) = (\ell_{j_1}', x_{j_1}')$, $(\ell_{k_2}, x_{k_2}) = (\ell_{j_2}'', x_{j_2}'')$ and $\ell_{j_1}' = \ell_{j_2}''$ then $k_1 < k_2$.

It is easy to check that the above conditions define one and only one ordering of $\{(\ell_k, x_k)\}_{0 \le k \le m+1}$.

We introduce the following shorthand notations, $\Delta_i = \ell_{i+1} - \ell_i$,

$$\begin{aligned}
\overline{x}_i &= \gamma'(\ell_i), & \widetilde{x}_i &= \gamma''(\ell_i), \\
\overline{\mathscr{S}}_i &= \overline{\mathscr{S}}(\overline{x}_i), & \widetilde{\mathscr{S}_i} &= \mathscr{S}(\widetilde{x}_i), \\
\overline{\pi}_i &= \pi_{\overline{x}_i}, & \widetilde{\pi}_i &= \pi_{\widetilde{x}_i}.
\end{aligned}$$

Observing that $\overline{\pi}_0 \widetilde{\pi}_0 = \overline{\pi}_0$ and $\widetilde{\pi}_{m+1} \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) = \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)$, we have,

$$\begin{aligned}
&\mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) \\
&= \sum_{i=0}^{m} e^{\Delta_m \overline{\mathscr{S}}_m} \overline{\pi}_m \cdots e^{\Delta_{i+1} \overline{\mathscr{S}}_{i+1}} \overline{\pi}_{i+1} \left(e^{\Delta_i \overline{\mathscr{S}}_i} \overline{\pi}_i - \widetilde{\pi}_{i+1} e^{\Delta_i \widetilde{\mathscr{S}_i}} \right) \widetilde{\pi}_i \cdots e^{\Delta_1 \widetilde{\mathscr{S}_1}} \widetilde{\pi}_1 e^{\Delta_0 \widetilde{\mathscr{S}_0}} \widetilde{\pi}_0 (Y_0 - X_0).
\end{aligned}$$

By (2.3), the compositions of operators before and after the parentheses in the summation above are uniformly bounded in operator norm by a constant. Therefore,

$$|\mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)| \tag{3.131}$$
$$\le c_2 \sum_{i=0}^{m} \left\| \overline{\pi}_{i+1} \circ \left(e^{\Delta_i \overline{\mathscr{S}}_i} \circ \overline{\pi}_i - \widetilde{\pi}_{i+1} \circ e^{\Delta_i \widetilde{\mathscr{S}_i}} \right) \circ \widetilde{\pi}_i \right\| |Y_0 - X_0|.$$

Using the fact that $\overline{\mathscr{S}}_i$ and $\overline{\pi}_i$ commute, as do $\widetilde{\mathscr{S}_i}$ and $\widetilde{\pi}_i$, we obtain,

$$\overline{\pi}_{i+1} \circ \left(e^{\Delta_i \overline{\mathscr{S}}_i} \circ \overline{\pi}_i - \widetilde{\pi}_{i+1} \circ e^{\Delta_i \widetilde{\mathscr{S}_i}} \right) \circ \widetilde{\pi}_i \tag{3.132}$$
$$= \overline{\pi}_{i+1} \circ \overline{\pi}_i \circ \left(e^{\Delta_i \overline{\mathscr{S}}_i} - e^{\Delta_i \widetilde{\mathscr{S}_i}} \right) \circ \widetilde{\pi}_i + \overline{\pi}_{i+1} \circ (\overline{\pi}_i - \widetilde{\pi}_{i+1}) \circ \widetilde{\pi}_i \circ e^{\Delta_i \widetilde{\mathscr{S}_i}}.$$

We will deal with each of these terms separately.

For the first term, we have by (2.5),

$$\left\| \overline{\pi}_{i+1} \circ \overline{\pi}_i \circ \left(e^{\Delta_i \overline{\mathscr{S}}_i} - e^{\Delta_i \widetilde{\mathscr{S}_i}} \right) \circ \widetilde{\pi}_i \right\| \le \left\| e^{\Delta_i \overline{\mathscr{S}}_i} - e^{\Delta_i \widetilde{\mathscr{S}_i}} \right\| \le c_3 \Delta_i |\overline{x}_i - \widetilde{x}_i|. \tag{3.133}$$

For the second term on the right hand side of (3.132), Lemma 2.2 and (2.3) allow us to conclude that

$$\left\| \overline{\pi}_{i+1} \circ (\overline{\pi}_i - \widetilde{\pi}_{i+1}) \circ \widetilde{\pi}_i \circ e^{\Delta_i \widetilde{\mathscr{S}_i}} \right\| \le c_4 \left(\left| \overline{x}_{i+1} - \overline{x}_i \right| \left| \overline{x}_i - \widetilde{x}_i \right| + \left| \overline{x}_{i+1} - \widetilde{x}_{i+1} \right| \left| \widetilde{x}_{i+1} - \widetilde{x}_i \right| \right) \left\| e^{\Delta_i \widetilde{\mathscr{S}_i}} \right\|$$
$$\le c_5 \left(\left| \overline{x}_{i+1} - \overline{x}_i \right| \left| \overline{x}_i - \widetilde{x}_i \right| + \left| \overline{x}_{i+1} - \widetilde{x}_{i+1} \right| \left| \widetilde{x}_{i+1} - \widetilde{x}_i \right| \right). \tag{3.134}$$

We will now analyze (3.133). Suppose that $\Delta_i > 0$ and $\overline{x}_i \neq \widetilde{x}_i$. Let j and k be defined by $\overline{x}_i = \gamma'(\ell'_j)$ and $\widetilde{x}_i = \gamma''(\ell''_k)$.

Suppose that $\ell_i = \ell'_j = \ell''_{k+1}$. Then, by our ordering of ℓ_r's, $\ell_{i+1} = \ell''_{k+1} = \ell_i$, so $\Delta_i = 0$. For the same reason, we have $\Delta_i = 0$ if any of the following conditions holds: $\ell''_k = \ell_i = \ell'_j$ or $\ell_i = \ell''_k = \ell'_{j+1}$. For this reason we consider only sharp versions of the corresponding inequalities in (3.135)-(3.138) below.

We have assumed that $\overline{x}_i \neq \widetilde{x}_i$ so one of the following four events holds,

$$F_i^1 = \{\ell''_k < \ell_i = \ell'_j < \ell''_{k+1},\ \xi_k < S_j \le t''_{k+1}\}, \tag{3.135}$$
$$F_i^2 = \{\ell''_k < \ell_i = \ell'_j < \ell''_{k+1},\ t''_{k+1} < S_j \le \xi_{k+1}\}, \tag{3.136}$$
$$F_i^3 = \{\ell'_j < \ell_i = \ell''_k < \ell'_{j+1},\ S_j < \xi_k \le U_j \le S_{j+1}\}, \tag{3.137}$$
$$F_i^4 = \{\ell'_j < \ell_i = \ell''_k < \ell'_{j+1},\ S_j < U_j \le \xi_k \le S_{j+1}\}. \tag{3.138}$$

If F_i^1 holds then,

$$\{\xi_k \le S_j \le t''_{k+1}\} \cap \{|\overline{x}_i - \widetilde{x}_i| > a\} \subset \bigcup_{1 \le r \le m} \left\{ \sup_{\xi_r < t < t''_{r+1}} |x''_r - X_t| > a \right\}. \tag{3.139}$$

This and Lemma 3.6 yield,

$$\mathbf{E}\left(\sum_{i=0}^{m}\Delta_i|\overline{x}_i-\widetilde{x}_i|\mathbf{1}_{F_i^1}\right)\leq\mathbf{E}\left(\left(\max_{0\leq k\leq m^*}\sup_{\xi_k<t<t^*_{k+1}}|x^*_k-X_t|\right)\sum_{i=0}^{m}\Delta_i\right) \tag{3.140}$$
$$=\mathbf{E}\left(\max_{0\leq k\leq m^*}\sup_{\xi_k<t<t^*_{k+1}}|x^*_k-X_t|\right)\leq c_6\varepsilon^{1/3}=c_7\varepsilon^{1/3}.$$

If F_i^2 holds then $\Delta_i=0$, because X does not hit ∂D in the interval (t''_{k+1},ξ_{k+1}), and, therefore, the local time L_t does not increase on this time interval. Hence,

$$\sum_{i=0}^{m}\Delta_i|\overline{x}_i-\widetilde{x}_i|\mathbf{1}_{F_i^2}=0. \tag{3.141}$$

If F_i^3 holds, the definition of U_j implies that $|\overline{x}_i-\widetilde{x}_i|\leq c_8\varepsilon$. Thus

$$\sum_{i=0}^{m}\Delta_i|\overline{x}_i-\widetilde{x}_i|\mathbf{1}_{F_i^3}\leq\sum_{i=0}^{m}c_8\Delta_i\varepsilon=c_8\varepsilon. \tag{3.142}$$

Suppose that F_i^4 occurred. It follows from the condition $U_j\leq\xi_k\leq S_{j+1}$ and the definition of ℓ''_k that $\ell''_k=\ell'_{j+1}$. We have already shown that in this case, $\Delta_i=0$. Hence,

$$\sum_{i=0}^{m}\Delta_i|\overline{x}_i-\widetilde{x}_i|\mathbf{1}_{F_i^4}=0. \tag{3.143}$$

Next we will consider the right hand side of (3.134). We start our discussion with the terms of the form $\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|$. Recall that we have defined j and k by $\overline{x}_i=\gamma'(\ell'_j)$ and $\widetilde{x}_i=\gamma''(\ell''_k)$. We will consider all possibilities listed in (3.135)-(3.138). If $\Delta_i=0$ then $\ell_i=\ell_{i+1}$ and $\overline{x}_i=\gamma'(\ell_i)=\gamma'(\ell_{i+1})=\overline{x}_{i+1}$. It follows that in this case, $\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|=0$. Hence, we can limit ourselves to (3.135)-(3.138), with sharp inequalities in the definitions.

Suppose that $F_i^1\cup F_i^2$ occurred. Then $\xi_k<S_j$, $\overline{x}_i=X_{S_j}$ and $\widetilde{x}_i=X_{\xi_k}$. By Lemma 3.8 (iii) and the strong Markov property applied at ξ_k,

$$\mathbf{E}\left(\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{\xi_k}\right)=\mathbf{E}\left(\left|X_{S_j}-X_{\xi_k}\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{\xi_k}\right) \tag{3.144}$$
$$\leq c_9|\log\mathbf{d}(Y_{\xi_k},D)|(\mathbf{d}(Y_{\xi_k},D)+\varepsilon^3)\leq c_{10}\varepsilon|\log\varepsilon|.$$

We have $\overline{x}_{i+1}=X_t$ for some $t\in(S_j,S_{j+1}]$. By Lemma 3.5 (ii), the strong Markov property applied at the stopping time $R_1=\inf\{t\geq S_j:X_t\in\partial D\}$ and Lemma 3.8 (iii),

$$\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{S_j}\right)\leq\mathbf{E}\left(\sup_{S_j\leq t\leq S_{j+1}}\left|X_t-X_{S_j}\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{S_j}\right) \tag{3.145}$$
$$\leq\mathbf{E}\left(\sup_{S_j\leq t\leq R_1}\left|X_t-X_{S_j}\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{S_j}\right)+\mathbf{E}\left(\sup_{R_1\leq t\leq S_{j+1}}\left|X_t-X_{R_1}\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid\mathscr{F}_{S_j}\right)$$
$$\leq c_{11}\varepsilon|\log\varepsilon|.$$

It follows from this and (3.144) that

$$\begin{aligned}&\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid \mathscr{F}_{\xi_k}\right) \\ &=\mathbf{E}\left(\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid \mathscr{F}_{S_j}\right)\mid \mathscr{F}_{\xi_k}\right)\le c_{12}\varepsilon^2|\log\varepsilon|^2.\end{aligned} \tag{3.146}$$

By (3.129) and the exit system formula (2.16), the expected value of m^* is bounded by c_{13}/ε. It follows from this estimate and (3.146) that

$$\begin{aligned}\mathbf{E}\left(\sum_{k=0}^{m}\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\right)&\le\mathbf{E}\left(\sum_{k=1}^{m^*}\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^1\cup F_i^2}\mid \mathscr{F}_{\xi_k}\right)\right)\\ &\le c_{14}\varepsilon|\log\varepsilon|^2.\end{aligned} \tag{3.147}$$

Next suppose that F_i^3 occurred. Then $\overline{x}_i=X_{S_j}$ and $\widetilde{x}_i=X_{\xi_k}$. Since $\xi_k\le U_j$, we have $\left|\overline{x}_i-\widetilde{x}_i\right|\le c_{15}\varepsilon$. As in the previous case, we have $\overline{x}_{i+1}=X_t$ for some $t\in(S_j,S_{j+1}]$, so we can use estimate (3.145). It follows that

$$\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^3}\mid \mathscr{F}_{\xi_k}\right)\le c_{16}\varepsilon^2|\log\varepsilon|.$$

The following estimate is analogous to (3.147),

$$\begin{aligned}\mathbf{E}\left(\sum_{k=0}^{m}\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^3}\right)&\le\mathbf{E}\left(\sum_{k=1}^{m^*}\mathbf{E}\left(\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^3}\mid \mathscr{F}_{\xi_k}\right)\right)\\ &\le c_{17}\varepsilon|\log\varepsilon|.\end{aligned} \tag{3.148}$$

We have already shown that if F_i^4 holds then $\Delta_i=0$ and, therefore, $\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|=0$. Hence

$$\mathbf{E}\left(\sum_{k=0}^{m}\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|\mathbf{1}_{F_i^4}\right)=0. \tag{3.149}$$

We continue our discussion of the right hand side of (3.134). We now consider the terms of the form $\left|\overline{x}_{i+1}-\widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1}-\widetilde{x}_i\right|$. The overall structure of our argument is similar to that used to analyze the terms of the form $\left|\overline{x}_{i+1}-\overline{x}_i\right|\left|\overline{x}_i-\widetilde{x}_i\right|$.

Suppose that $\overline{x}_{i+1}\neq\widetilde{x}_{i+1}$. Let j and k be defined by $\overline{x}_{i+1}=\gamma'(\ell'_j)$ and $\widetilde{x}_{i+1}=\gamma''(\ell''_k)$. We have assumed that $\overline{x}_{i+1}\neq\widetilde{x}_{i+1}$ so one of the following four events holds,

$$F_i^5=\{\ell''_k<\ell_{i+1}=\ell'_j<\ell''_{k+1},\ \xi_k<S_j\le t''_{k+1}\}, \tag{3.150}$$

$$F_i^6=\{\ell''_k<\ell_{i+1}=\ell'_j<\ell''_{k+1},\ t''_{k+1}<S_j\le\xi_{k+1}\}, \tag{3.151}$$

$$F_i^7=\{\ell'_j<\ell_{i+1}=\ell''_k<\ell'_{j+1},\ S_j<\xi_k\le U_j\le S_{j+1}\}, \tag{3.152}$$

$$F_i^8=\{\ell'_j<\ell_{i+1}=\ell''_k<\ell'_{j+1},\ S_j<U_j\le\xi_k\le S_{j+1}\}. \tag{3.153}$$

Suppose that $\ell_{i+1}=\ell'_j=\ell''_k$. Then because of the way we ordered (ℓ_i,x_i), we have $(\ell_i,x_i)=(\ell'_j,x'_j)$ and $(\ell_{i+1},x_{i+1})=(\ell''_k,x''_k)$. Therefore $\ell_i=\ell_{i+1}$. It follows that $\widetilde{x}_i=\gamma''(\ell_i)=\gamma''(\ell_{i+1})=\widetilde{x}_{i+1}$. In this case, $\left|\overline{x}_{i+1}-\widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1}-\widetilde{x}_i\right|=0$. We can reach the same conclusion in the same way in case we have $\ell''_{k+1} = \ell_{i+1} = \ell'_j$ or $\ell_{i+1} = \ell''_k = \ell'_{j+1}$. Hence, we can limit ourselves to (3.150)-(3.153), with sharp inequalities in the definitions.

Suppose that $F_i^5 \cup F_i^6$ occurred. Then $\overline{x}_{i+1} = X_{S_j}$ and $\widetilde{x}_{i+1} = X_{\xi_k}$. The following is a version of (3.144),

$$\mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right| \mathbf{1}_{F_i^5 \cup F_i^6} \mid \mathscr{F}_{\xi_k}\right) \le c_{18}\varepsilon|\log\varepsilon|. \tag{3.154}$$

We have $\widetilde{x}_i = X_t$ for some $t \in [\xi_{k-1}, \xi_k)$, so by Lemma 3.7 and the strong Markov property applied at ξ_{k-1},

$$\mathbf{E}\left(\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^5 \cup F_i^6} \mid \mathscr{F}_{\xi_{k-1}}\right) \le \mathbf{E}\left(\sup_{\xi_{k-1} \le t \le \xi_k} \left|X_t - X_{\xi_{k-1}}\right| \mid \mathscr{F}_{\xi_{k-1}}\right) \le c_{19}\varepsilon_*^{1/3} = c_{19}c_0^{1/3}\varepsilon^{1/3}. \tag{3.155}$$

It follows from this and (3.154) that

$$\begin{aligned} &\mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^5 \cup F_i^6} \mid \mathscr{F}_{\xi_{k-1}}\right) \\ &= \mathbf{E}\left(\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right| \mathbf{1}_{F_i^5 \cup F_i^6} \mid \mathscr{F}_{\xi_k}\right) \mid \mathscr{F}_{\xi_{k-1}}\right) \le c_{20}\varepsilon^{4/3}|\log\varepsilon|. \end{aligned} \tag{3.156}$$

Recall that the expected value of m^* is bounded by c_{13}/ε. It follows from this and (3.156) that

$$\begin{aligned} &\mathbf{E}\left(\sum_{k=0}^{m}\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^5 \cup F_i^6}\right) \\ &\le \mathbf{E}\left(\sum_{k=1}^{m^*}\mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^5 \cup F_i^6} \mid \mathscr{F}_{\xi_{k-1}}\right)\right) \le c_{21}\varepsilon^{1/3}|\log\varepsilon|. \end{aligned} \tag{3.157}$$

Next suppose that F_i^7 occurred. Then $\overline{x}_{i+1} = X_{S_j}$ and $\widetilde{x}_{i+1} = X_{\xi_k}$. Since $\xi_k \le U_j$, we have $\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right| \le c_{22}\varepsilon$. As in the previous case, we have $\widetilde{x}_i = X_t$ for some $t \in [\xi_{k-1}, \xi_k]$, so we can use estimate (3.155). It follows that

$$\mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^7} \mid \mathscr{F}_{\xi_{k-1}}\right) \le c_{23}\varepsilon^{4/3}.$$

The following estimate is analogous to (3.157)

$$\begin{aligned} &\mathbf{E}\left(\sum_{k=0}^{m}\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^7}\right) \\ &\le \mathbf{E}\left(\sum_{k=1}^{m^*}\mathbf{E}\left(\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^7} \mid \mathscr{F}_{\xi_{k-1}}\right)\right) \le c_{24}\varepsilon^{1/3}. \end{aligned} \tag{3.158}$$

Suppose that F_i^8 occurred. It follows from the condition $U_j \le \xi_k \le S_{j+1}$ and the definition of ℓ''_k that $\ell''_k = \ell'_{j+1}$. We have already argued that in this case, $\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| = 0$. Hence,

$$\sum_{k=0}^{m}\left|\overline{x}_{i+1} - \widetilde{x}_{i+1}\right|\left|\widetilde{x}_{i+1} - \widetilde{x}_i\right| \mathbf{1}_{F_i^8} = 0. \tag{3.159}$$

Recall that $|X_0 - Y_0| = \varepsilon$. The estimates in (3.140), (3.141), (3.142), (3.143), (3.147), (3.148), (3.149), (3.157), (3.158) and (3.159) are all less than or equal to $c_{25}\varepsilon^{1/3}|\log\varepsilon|$. We combine these remarks with (3.131)-(3.134) to conclude that,

$$\mathbf{E}|\mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)| \le c_{26}\varepsilon^{4/3}|\log\varepsilon|.$$

□

Proof of Theorem 3.1. Suppose that $|Y_0 - X_0| = \varepsilon$ and $\varepsilon_* = c_0\varepsilon$, where c_0 is as in Lemma 3.23. Consider an arbitrarily small $c_1 > 0$ let Λ be the random variable in the statement of Lemma 3.20. According to that lemma, for all sufficiently small $\varepsilon > 0$, we have a.s.,

$$|\Lambda| < c_1\varepsilon. \tag{3.160}$$

By the triangle inequality,

$$\begin{aligned} &|(Y_{\sigma_*} - X_{\sigma_*}) - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(Y_0 - X_0)| \\ &\le |\Lambda| + \Big||(Y_{\sigma_1} - X_{\sigma_1}) - \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0)| - \Lambda\Big| \\ &\quad + |\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0)| \\ &\quad + |\mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)| \\ &\quad + \Big|\mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(Y_0 - X_0)\Big| \\ &:= |\Lambda| + \Xi. \end{aligned} \tag{3.161}$$

By Lemma 3.20,

$$\mathbf{E}\Big|\Big|(Y_{\sigma_*} - X_{\sigma_*}) - \mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0)\Big| - \Lambda\Big| \le c_2\varepsilon^2|\log\varepsilon|^2. \tag{3.162}$$

By Lemma 3.21,

$$\mathbf{E}|\mathscr{G}_{m'} \circ \cdots \circ \mathscr{G}_0(Y_0 - X_0) - \mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0)| \le c_3\varepsilon^2|\log\varepsilon|. \tag{3.163}$$

Lemma 3.24 implies that

$$\mathbf{E}\Big|\mathscr{H}_{m'} \circ \cdots \circ \mathscr{H}_0(Y_0 - X_0) - \mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0)\Big| \le c_4\varepsilon^{4/3}|\log\varepsilon|. \tag{3.164}$$

Lemma 3.23 yields for any $\beta < 1$,

$$\mathbf{E}\Big|\mathscr{J}_{m''} \circ \cdots \circ \mathscr{J}_0(Y_0 - X_0) - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(Y_0 - X_0)\Big| \le c_5\varepsilon^{1+\beta}. \tag{3.165}$$

Combining (3.162)-(3.165), and using the definition of Ξ in (3.161), we see that

$$\mathbf{E}\Xi \le c_6\varepsilon^{4/3}|\log\varepsilon|. \tag{3.166}$$

Fix some $\beta_1 \in (1, 4/3)$ and $\beta_2 \in (0, 4/3 - \beta_1)$. By (3.166) and Chebyshev's inequality,

$$\mathbf{P}(\Xi > c_7\varepsilon^{\beta_1}) \le c_8\varepsilon^{\beta_2}. \tag{3.167}$$

Fix an arbitrary $b > 1$ and $\mathbf{v} \in \mathbf{R}^n$ with $|\mathbf{v}| = 1$. We apply the last estimate to a sequence of processes $Y = X^{z_0+\varepsilon\mathbf{v}}$ with $\varepsilon = b^{-k}$, $k \geq k_0$, for some fixed large k_0. We obtain

$$\mathbf{P}(\Xi > c_7 b^{-k\beta_1}) \leq c_8 b^{-k\beta_2}, \qquad k \geq k_0.$$

Since $\sum_{k \geq k_0} c_8 b^{-k\beta_2} < \infty$, the Borel-Cantelli Lemma shows that only a finite number of events $\{\Xi > c_7 b^{-k\beta_1}\}$ occur. This is the same as saying that only a finite number of events $\{\Xi/b^{-k} > c_7 b^{-k(\beta_1-1)}\}$ occur. We combine this fact with (3.160) and (3.161) to see that for any $c_1 > 0$, a.s.,

$$\limsup_{k\to\infty} \left| \frac{X^{z_0+b^{-k}\mathbf{v}}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{v}) \right| \leq c_1.$$

Since c_1 is arbitrarily small, we have in fact, a.s.,

$$\lim_{k\to\infty} \left| \frac{X^{z_0+b^{-k}\mathbf{v}}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{v}) \right| = 0. \tag{3.168}$$

It is easy to see that the last formula holds for all $\mathbf{v} \in \mathbf{R}^n$, not only those with $|\mathbf{v}| = 1$.

Consider an arbitrary compact set $K \subset \mathbf{R}^n$. Let c_9 be the same constant as c_1 in the statement of Lemma 3.4. It follows easily from (2.3) that $\|\mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0\| \leq c_{10}$, a.s. Fix any $c_{11} > 0$ and find $\mathbf{w}_1, \dots, \mathbf{w}_{j_1} \in \mathbf{R}^n$ such that for every $\mathbf{v} \in K$ there exists $j = j(\mathbf{v})$ such that $|\mathbf{v} - \mathbf{w}_j| < c_{11}/(2(c_9 + c_{10}))$. Note that $|(z_0 + b^{-k}\mathbf{v}) - (z_0 + b^{-k}\mathbf{w}_{j(\mathbf{v})})| < b^{-k} c_{11}/(2c_9)$ and, in view of (3.168),

$$\lim_{k\to\infty} \sup_{1\leq j\leq j_1} \left| \frac{X^{z_0+b^{-k}\mathbf{w}_j}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{w}_j) \right| = 0. \tag{3.169}$$

By Lemma 3.4, for $\mathbf{v} \in K$ and $j = j(\mathbf{v})$, a.s.,

$$\left| \frac{X^{z_0+b^{-k}\mathbf{w}_j}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \frac{X^{z_0+b^{-k}\mathbf{v}}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} \right| \leq c_9 |(z_0 + b^{-k}\mathbf{v}) - (z_0 + b^{-k}\mathbf{w}_j)|/b^{-k} \leq c_{11}/2. \tag{3.170}$$

Since $|\mathbf{v} - \mathbf{w}_j| < c_{11}/(2c_{10})$,

$$|\mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{w}_{j(\mathbf{v})}) - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{v})| \leq c_{11}/2. \tag{3.171}$$

Combining (3.169)-(3.171) yields a.s.,

$$\lim_{k\to\infty} \sup_{\mathbf{v}\in K} \left| \frac{X^{z_0+b^{-k}\mathbf{v}}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{v}) \right| \leq c_{11}.$$

Since $c_{11} > 0$ is arbitrarily small, we have a.s.,

$$\lim_{k\to\infty} \sup_{\mathbf{v}\in K} \left| \frac{X^{z_0+b^{-k}\mathbf{v}}_{\sigma_*} - X_{\sigma_*}}{b^{-k}} - \mathscr{I}_{m^*} \circ \cdots \circ \mathscr{I}_0(\mathbf{v}) \right| = 0. \tag{3.172}$$

Let $c_{12} = \sup\{|\mathbf{v}| \in K\}$. For $\varepsilon \in [b^{-k}, b^{-k+1})$, we have,

$$|(z_0 + b^{-k}\mathbf{v}) - (z_0 + \varepsilon\mathbf{v})|/\varepsilon \le c_{12}(1 - 1/b).$$

Hence, by Lemma 3.4, a.s.,

$$\begin{aligned}
&\left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{b^{-k}} - \frac{X_{\sigma_*}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_*}}{\varepsilon}\right| \\
&\le \left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{b^{-k}} - \frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{\varepsilon}\right| + \left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{\varepsilon} - \frac{X_{\sigma_*}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_*}}{\varepsilon}\right| \\
&\le (1 - 1/b)\left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{b^{-k}}\right| + c_9|(z_0 + \varepsilon\mathbf{v}) - (z_0 + b^{-k}\mathbf{v})|/\varepsilon \\
&\le (1 - 1/b)\left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{b^{-k}}\right| + c_9 c_{12}(1 - 1/b).
\end{aligned}$$

Let $\varepsilon_* = c_0 b^{-k}$, where k is defined by $\varepsilon \in [b^{-k}, b^{-k+1})$. The last formula and (3.172) yield,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|\frac{X_{\sigma_*}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_*}}{\varepsilon} - \mathscr{I}_{m^*}\circ\cdots\circ\mathscr{I}_0(\mathbf{v})\right| \\
&\le (1 - 1/b)\limsup_{k\to\infty}\sup_{\mathbf{v}\in K}\left|\frac{X_{\sigma_*}^{z_0+b^{-k}\mathbf{v}} - X_{\sigma_*}}{b^{-k}}\right| + c_9 c_{12}(1 - 1/b).
\end{aligned}$$

Let $\varepsilon^* = c_0\varepsilon$. We can take $b > 1$ arbitrarily close to 1, so, a.s.,

$$\limsup_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|\frac{X_{\sigma_*}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_*}}{\varepsilon} - \mathscr{I}_{m^*}\circ\cdots\circ\mathscr{I}_0(\mathbf{v})\right| = 0.$$

Recall the definition of σ_* from the beginning of this section. We let $k_* \to \infty$ to see that, a.s.,

$$\limsup_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|\frac{X_{\sigma_1}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_1}}{\varepsilon} - \mathscr{I}_{m^*}\circ\cdots\circ\mathscr{I}_0(\mathbf{v})\right| = 0.$$

We combine this with Theorem 2.5 to complete the proof of the theorem. □

Proof of Corollary 3.2. According to Theorem 3.1, for every $r > 0$ and compact set $K \subset \mathbf{R}^n$, we have $\lim_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|(X_{\sigma_r}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_r}^{z_0})/\varepsilon - \mathscr{A}_r\mathbf{v}\right| = 0$, a.s. By Fubini's Theorem, with probability 1, for almost all $r > 0$, we have

$$\limsup_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|(X_{\sigma_r}^{z_0+\varepsilon\mathbf{v}} - X_{\sigma_r}^{z_0})/\varepsilon - \mathscr{A}_r\mathbf{v}\right| = 0. \qquad (3.173)$$

Recall the definitions of $\mathscr{E}_r$, $\mathbf{v}_r$ and $\mathscr{A}_r$ from Section 2.3. Let $\mathscr{E}_{r_1,r_2} = \{e_s : \sigma_{r_1} \le s < \sigma_{r_2}\}$ and define $\mathbf{v}_{r_1,r_2}$ and $\mathscr{A}_{r_1,r_2}$ relative to $\mathscr{E}_{r_1,r_2}$ in the same way as $\mathscr{A}_r$ was defined relative to $\mathscr{E}_r$ and $\mathbf{v}_r$.

Fix any $t > 0$ and let $y_0 = X_t^{z_0}$ and $Y_s^x = X_{t+s}^x$ for $s \ge 0$. Let $\mathbf{w}$ be defined relative to $\mathbf{v}$ by $y_0 + \varepsilon\mathbf{w} = X_t^{z_0+\varepsilon\mathbf{v}}$. By the Markov property applied at time t, Theorem 3.1 and (3.173) hold for the flow $\{Y_s^{\cdot}, s \ge 0\}$ in place of the flow $\{X_s^{\cdot}, s \ge 0\}$. In other words, if $\mathscr{A}'_r = \mathscr{A}_{L_t,L_t+r}$ and $\sigma'_r = \sigma_{L_t+r} - t$ then with probability 1, for almost all $r > 0$, we have

$$\lim_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|(Y_{\sigma'_r}^{y_0+\varepsilon\mathbf{w}} - Y_{\sigma'_r}^{y_0})/\varepsilon - \mathscr{A}'_r\mathbf{w}\right| = 0. \tag{3.174}$$

Note that for any sequence $\mathbf{v}_n \in K$, $n \ge 1$, there exists $\mathbf{v}_* \in K$ such that a subsequence of $\mathbf{v}_n$ converges to $\mathbf{v}_*$, by compactness of K.

Suppose that it is not true that $\lim_{\varepsilon\to 0}\sup_{\mathbf{v}\in K}\left|(X_t^{z_0+\varepsilon\mathbf{v}} - X_t^{z_0})/\varepsilon - \mathscr{A}_{L_t}\mathbf{v}\right| = 0$ with probability 1. It will suffice to show that this assumption leads to a contradiction. The assumption implies that we can find $c_1, p_1 > 0$, $\varepsilon_n > 0$ for $n \ge 1$, $\mathbf{v}_* \in K$, and $\mathbf{v}_n \in K$ for $n \ge 1$, such that $\lim_{n\to\infty}\varepsilon_n = 0$, $\lim_{n\to\infty}\mathbf{v}_n = \mathbf{v}_*$, and with probability greater than p_1 we have $\liminf_n \left|(X_t^{z_0+\varepsilon_n\mathbf{v}_n} - X_t^{z_0})/\varepsilon_n - \mathscr{A}_{L_t}\mathbf{v}_n\right| > c_1$. Let A_1 be the event defined by the last formula.

Let $\pi_{\mathbb{H}}$ denote the orthogonal projection on an $(n-1)$-dimensional hyperplane $\mathbb{H}$. We can choose $\mathbb{H}$ so that for some $c_2 > 0$ and subsequence n_k, we have on the event A_1, $\left|\pi_{\mathbb{H}}\left((X_t^{z_0+\varepsilon_{n_k}\mathbf{v}_{n_k}} - X_t^{z_0})/\varepsilon_{n_k}\right) - \pi_{\mathbb{H}} \circ \mathscr{A}_{L_t}\mathbf{v}_{n_k}\right| > c_2$. Let $\mathbf{w}_k = (X_t^{z_0+\varepsilon_{n_k}\mathbf{v}_{n_k}} - X_t^{z_0})/\varepsilon_{n_k}$ so that the last formula can be written as $\left|\pi_{\mathbb{H}}(\mathbf{w}_k) - \pi_{\mathbb{H}} \circ \mathscr{A}_{L_t}\mathbf{v}_{n_k}\right| > c_2$. Since D is a bounded domain with C^2 boundary, there exists $x \in \partial D$ such that the tangent hyperplane at x is parallel to $\mathbb{H}$, so we can assume that $\pi_{\mathbb{H}} = \pi_x$. There exist $r_1 > 0$ and $c_3 \in (0, c_2)$ such that for $y \in M := \partial D \cap \mathscr{B}(x, r_1)$, we have on the event A_1, $\left|\pi_y(\mathbf{w}_k) - \pi_y \circ \mathscr{A}_{L_t}\mathbf{v}_{n_k}\right| > c_3$.

Let $T = \inf\{s > t : X_s^{z_0} \in \partial D\}$ and $A_2 = A_1 \cap \{X_T^{z_0} \in M\}$. By the support theorem for Brownian motion and the Markov property at time t, there exists $p_2 > 0$ such that $\mathbf{P}(A_2) > p_2$. If A_2 holds then

$$\left|\pi_{X_T^{z_0}}(\mathbf{w}_k) - \pi_{X_T^{z_0}} \circ \mathscr{A}_{L_t}\mathbf{v}_{n_k}\right| > c_3. \tag{3.175}$$

It follows from the definition of $\mathscr{A}_r$ and $\mathscr{A}'_r$ that $\lim_{r\downarrow 0}\|\mathscr{A}'_r - \pi_{X_T^{z_0}}\| = 0$ and $\lim_{r\downarrow 0}\|\mathscr{A}_{L_t+r} - \pi_{X_T^{z_0}} \circ \mathscr{A}_{L_t}\| = 0$. The rate of convergence to 0 may depend on the trajectory of the flow X. Let $r_2 > 0$ and $p_3 > 0$ be so small that with probability greater than p_3 the event A_2 holds and

$$\begin{aligned} |\mathscr{A}'_r\mathbf{w}_k - \pi_{X_T^{z_0}}(\mathbf{w}_k)| &\le c_3/4, \\ |\pi_{X_T^{z_0}} \circ \mathscr{A}_{L_t}\mathbf{v}_{n_k} - \mathscr{A}_{L_t+r}\mathbf{v}_{n_k}| &\le c_3/4, \end{aligned} \tag{3.176}$$

for $r \in (0, r_2)$ and $k \ge 1$. Let A_3 be the event that the last inequalities in (3.176) hold and A_2 holds. Combining (3.175) and (3.176), we see that on the event A_3 we have $|\mathscr{A}'_r\mathbf{w}_k - \mathscr{A}_{L_t+r}\mathbf{v}_{n_k}| \ge c_3/2$ for $r \in (0, r_2)$ and $k \ge 1$.

By (3.173) and (3.174), with probability 1, there exist some $r \in (0, r_2)$ such that,

$$\lim_{k\to\infty}\left|(X_{\sigma_{L_t+r}}^{z_0+\varepsilon_{n_k}\mathbf{v}_{n_k}} - X_{\sigma_{L_t+r}}^{z_0})/\varepsilon_{n_k} - \mathscr{A}_{L_t+r}\mathbf{v}_{n_k}\right| = 0$$

and

$$\lim_{k\to\infty}\left|(X^{z_0+\varepsilon_{n_k}\mathbf{v}_{n_k}}_{\sigma_{L_t+r}} - X^{z_0}_{\sigma_{L_t+r}})/\varepsilon_{n_k} - \mathscr{A}'_r\mathbf{w}_{n_k}\right| = \lim_{k\to\infty}\left|(Y^{y_0+\varepsilon_{n_k}\mathbf{w}_{n_k}}_{\sigma'_r} - Y^{y_0}_{\sigma'_r})/\varepsilon_{n_k} - \mathscr{A}'_r\mathbf{w}_{n_k}\right| = 0.$$

Since $|\mathscr{A}'_r\mathbf{w}_k - \mathscr{A}_{L_t+r}\mathbf{v}_{n_k}| \geq c_3/2$ on the event A_3 of positive probability, the last two formulas form a contradiction and this completes the proof. □

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