Electronic Journal of Probability

Vol. 14 (2009), Paper no. 78, pages 2287–2309.

Journal URL
http://www.math.washington.edu/~ejpecp/

# Density formula and concentration inequalities with Malliavin calculus

Ivan Nourdin*
Université Paris 6

Frederi G. Viens†
Purdue University

**Abstract**

We show how to use the Malliavin calculus to obtain a new exact formula for the density $\rho$ of the law of any random variable $Z$ which is measurable and differentiable with respect to a given isonormal Gaussian process. The main advantage of this formula is that it does not refer to the divergence operator $\delta$ (dual of the Malliavin derivative $D$). The formula is based on an auxilliary random variable $G := \langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}$, where $L$ is the generator of the so-called Ornstein-Uhlenbeck semigroup. The use of $G$ was first discovered by Nourdin and Peccati (Probab. Theory Relat. Fields **145**, 2009) in the context of rates of convergence in law. Here, thanks to $G$, density lower bounds can be obtained in some instances. Among several examples, we provide an application to the (centered) maximum of a general Gaussian process. We also explain how to derive concentration inequalities for $Z$ in our framework.

**Key words:** Malliavin calculus; density; concentration inequality; suprema of Gaussian processes; fractional Brownian motion.

**AMS 2000 Subject Classification:** Primary 60G15; 60H07.

Submitted to EJP on May 22, 2009, final version accepted September 29, 2009.

*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France, ivan.nourdin@upmc.fr

†Dept. Statistics and Dept. Mathematics, Purdue University, 150 N. University St., West Lafayette, IN 47907-2067, USA, viens@purdue.edu

# 1 Introduction

Let $X = \{X(h) : h \in \mathfrak{H}\}$ be a centered isonormal Gaussian process defined on a real separable Hilbert space $\mathfrak{H}$. This just means that $X$ is a collection of centered and jointly Gaussian random variables indexed by the elements of $\mathfrak{H}$, defined on some probability space $(\Omega, \mathscr{F}, P)$, such that the covariance of $X$ is given by the inner product in $\mathfrak{H}$: for every $h, g \in \mathfrak{H}$,

$$E\big(X(h)X(g)\big) = \langle h, g\rangle_{\mathfrak{H}}.$$

The process $X$ has the interpretation of a Wiener (stochastic) integral. As usual in Malliavin calculus, we use the following notation (see Section 2 for precise definitions):

- $L^2(\Omega, \mathscr{F}, P)$ is the space of square-integrable functionals of $X$; this means in particular that $\mathscr{F}$ is the $\sigma$-field generated by $X$;
- $\mathbb{D}^{1,2}$ is the domain of the Malliavin derivative operator $D$ with respect to $X$; this implies that the Malliavin derivative $DZ$ of $Z \in \mathbb{D}^{1,2}$ is a random element with values in $\mathfrak{H}$, and that $E\left[\|DZ\|_{\mathfrak{H}}^2\right] < \infty$.
- $\operatorname{Dom}\delta$ is the domain of the divergence operator $\delta$. This operator will only play a marginal role in our study; it is simply used in order to simplify some proof arguments, and for comparison purposes.

From now on, $Z$ will always denote a random variable in $\mathbb{D}^{1,2}$ with *zero mean*.

The following result on the density of a random variable is a well-known fact of the Malliavin calculus: if $DZ/\|DZ\|_{\mathfrak{H}}^2$ belongs to $\operatorname{Dom}\delta$, then the law of $Z$ has a continuous and bounded density $\rho$ given, for all $z \in \mathbb{R}$, by

$$\rho(z) = E\left[\mathbf{1}_{(z,+\infty]}(Z)\,\delta\left(\frac{DZ}{\|DZ\|_{\mathfrak{H}}^2}\right)\right]. \tag{1.1}$$

From this expression, it is sometimes possible to deduce upper bounds for $\rho$. Several examples are detailed in Section 2.1.1 of Nualart's book [16].

In the first main part of our paper (Section 3), we prove a new general formula for $\rho$, which does not refer to $\delta$. For $Z$ a mean-zero r.v. in $\mathbb{D}^{1,2}$, define the function $g_Z : \mathbb{R} \to \mathbb{R}$ almost everywhere by

$$g_Z(z) = E\left(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}\Big| Z = z\right). \tag{1.2}$$

The $L$ appearing here is the so-called generator of the Ornstein-Uhlenbeck semigroup; it is defined, as well as its pseudo-inverse $L^{-1}$, in the next section. By [15, Proposition 3.9], we know that $g_Z$ is non-negative on the support of the law of $Z$. Under some general conditions on $Z$ (see Theorem 3.1 for a precise statement), the density $\rho$ of the law of $Z$ (provided it exists) is given by the following new formula, valid for almost all $z$ in the support of $\rho$:

$$P(Z \in dz) = \rho(z)dz = \frac{E|Z|}{2g_Z(z)}\exp\left(-\int_0^z \frac{x\,dx}{g_Z(x)}\right)dz. \tag{1.3}$$

We also show that one simple condition under which $\rho$ exists and is strictly positive on $\mathbb{R}$ is that $g(z) \geqslant \sigma_{\min}^2$ hold almost everywhere for some constant $\sigma_{\min}^2 > 0$ (see Corollary 3.3 for a precise statement). In this case, formula (1.3) immediately implies that $\rho(z) \geqslant E|Z| / \left(2g_Z(z)\right) \exp\left(-z^2 / \left(2\sigma_{\min}^2\right)\right)$, so that if some a-priori upper bound is known on $g$, then $\rho$ is, up to a constant, bounded below by a Gaussian density.

Another main point in our approach, also discussed in Section 3, is that it is often possible to express $g_Z$ relatively explicitly, via the following formula (see Proposition 3.7):

$$g_Z(z) = \int_0^\infty e^{-u}\,\mathbf{E}\big(\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_{\mathfrak{H}} | Z = z\big)\,du. \tag{1.4}$$

This is a consequence of the so-called Mehler formula of Malliavin calculus; here $X'$, which stands for an independent copy of $X$, is such that $X$ and $X'$ are defined on the product probability space $(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', P \times P')$; $\mathbf{E}$ denotes the mathematical expectation with respect to $P \times P'$; and the mapping $\Phi_Z : \mathbb{R}^{\mathfrak{H}} \to \mathfrak{H}$ is defined $P \circ X^{-1}$-a.s. through the identity $DZ = \Phi_Z(X)$ (note that $e^{-u}X + \sqrt{1-e^{-2u}}X' \stackrel{\mathscr{L}}{=} X$ for all $u \geqslant 0$, so that $\Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')$ is well-defined for all $u \geqslant 0$).

As an important motivational example of our density formula (1.3) combined with the explicit expression (1.4) for $g_Z$, let $X = (X_t,\ t \in [0,T])$ be a centered Gaussian process with continuous paths, such that $E(X_t - X_s)^2 \neq 0$ for all $s \neq t$. Consider $Z = \sup_{[0,T]} X - E\big(\sup_{[0,T]} X\big)$. Understanding the distribution of $Z$ is a topic of great historical and current interest. For detailed accounts, one may consult textbooks by Robert Adler et al., from 1990, 2007, and in preparation: [1], [2], [3]. Expressing the density of $Z$, even implicitly, is a subject of study in its own right; in the case of differentiable random fields, geometric methods have been used by Azaïs and Wschebor, based on the so-called Rice-Kac formula, to express the density of $Z$ in a way which allows them to derive sharp bounds on the tail of $Z$: see [4] and references therein.

Herein we will apply our density formula to $Z$, resulting in an expression which is not restricted to differentiable fields, and is not related to the Azaïs-Wschebor formula. To achieve this, we will use specific facts about $Z$ (see explanations and references in Section 3.2.4). It is known that $Z \in \mathbb{D}^{1,2}$, that, almost surely, the supremum of $X$ on $[0,T]$ is attained at a single point in $I_0 \in [0,T]$, and that the law of $Z$ has a density $\rho$. The underlying Wiener space can then be parametrized to imply $DZ = \mathbf{1}_{[0,I_0]}$. We note by $R$ the covariance function of $X$, defined by $R(s,t) = E(X_sX_t)$. Let $I_u := \mathrm{argmax}_{[0,T]}\big(e^{-u}X + \sqrt{1-e^{-2u}}X'\big)$ where $X'$ stands for an independent copy of $X$ (as defined above). Using the Mehler-type formula (1.4), we show

$$g_Z(z) = \int_0^\infty e^{-u}\,\mathbf{E}\big(R(I_0, I_u) | Z = z\big)\,du.$$

Therefore by (1.3), for almost all $z$ in the support of $\rho$, we have

$$\rho(z) = \frac{E|Z|}{2\int_0^\infty e^{-u}\,\mathbf{E}\big(R(I_0,I_u)|Z=z\big)\,du}\ \exp\left(-\int_0^z \frac{x\,dx}{\int_0^\infty e^{-u}\,\mathbf{E}\big(R(I_0,I_u)|Z=x\big)\,du}\right). \tag{1.5}$$

In particular, if $R$ is bounded above and below on $[0,T]$, we immediately get some Gaussian lower and upper bounds for $\rho$ over all of $\mathbb{R}$. Moreover, now that we have a formula for $\rho$, it is not difficult to derive a formula for the variance of $Z$. We get

$$\mathrm{Var}(Z) = \int_0^\infty e^{-u} \mathbf{E}\big(R(I_0, I_u)\big)\,du. \tag{1.6}$$

Our general density formula (1.3) has found additional applications reported in other publications. In the context of Brownian directed polymers in Gaussian and non-Gaussian environments, this paper's second author obtained fully diffusive fluctuations for some polymer partition functions in [20] using slightly different tools than we have here, but the techniques in this paper which lead to (1.5) would yield the results in [20, Section 5] as well. In a direct application of (1.3) in the same style as (1.5), Nualart and Quer-Sardanyons proved in the preprint [18] that the stochastic heat and wave equations have solutions whose densities are bounded above and below by Gaussian densities in many cases.

In the second main part of the paper (Section 4), we explain what can be done when one knows that $g_Z$ is sub-affine. More precisely, if the law of $Z$ has a density and if $g_Z$ verifies $g_Z(Z) \leqslant \alpha Z + \beta$ $P$-a.s. for some $\alpha \geqslant 0$ and $\beta > 0$, we prove the following concentration inequalities (Theorem 4.1): for all $z > 0$,

$$P(Z \geqslant z) \leqslant \exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right) \quad \text{and} \quad P(Z \leqslant -z) \leqslant \exp\left(-\frac{z^2}{2\beta}\right). \tag{1.7}$$

As an application of (1.7), we prove the following result. Let $B = (B_t,\, t \in [0,1])$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. Let $Q : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^1$ function such that the Lebesgue measure of the set $\{u \in \mathbb{R} : Q'(u) = 0\}$ is zero, and $|Q'(u)| \leqslant C|u|$ and $Q(u) \geqslant cu^2$ for some positive constants $c, C$ and all $u \in \mathbb{R}$. The square function satisfies this assumption, but we may also allow many perturbations of the square. Let $Z = \int_0^1 Q\left(B_s\right) ds - \mu$ where $\mu = E\left[\int_0^1 Q\left(B_s\right) ds\right]$. Then inequality (1.7) holds with

$$\alpha = \frac{C^2}{(2H+1)c}\left(\frac{1}{2} + \frac{1}{2H+1}\sqrt{\frac{\pi}{8}}\right) \text{ and } \beta = \frac{C^2}{(2H+1)c}\left(\frac{\mu}{2} + \frac{\mu}{2H+1}\sqrt{\frac{\pi}{8}} + \frac{c}{4}\sqrt{\frac{\pi}{8}}\right). \tag{1.8}$$

The interest of this result lies in the fact that the exact distribution of $\int_0^1 Q\left(B_u\right) du$ is unknown; even when $Q(x) = x^2$, it is still an open problem for $H \neq 1/2$. Note also that classical results by Borell [5] can only be applied when $Q(x) = x^2$ (because then, $Z$ is a second-chaos random variable) and would give a bound like $A\exp(-Cz)$. The behavior for large $z$ is always of exponential type. The proof of (1.7) with $\alpha$ and $\beta$ as above for this class of examples is given at the end of Section 4.1.

A related application of relation (1.7) from our Theorem 4.1 is reported in [6], by this paper's first author, together with J.C. Breton and G. Peccati: they describe an application in statistics where they build exact confidence intervals for the Hurst parameter associated with a one-dimensional fractional Brownian motion.

Section 4 also contains a general lower bound result, Theorem 4.3, again based on the quantity $\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}$ via the function $g_Z$ defined in (1.2). This quantity was introduced recently in [15] for the purpose of using Stein's method in order to show that the standard deviation of

$\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}$ provides an error bound of the normal approximation of $Z$, see also Remark 3.2 below for a precise statement. Here, in Theorem 4.3 and in Theorem 4.1 as a special case ($\alpha = 0$ therein), $g_Z(Z) = E(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}|Z)$ can be instead assumed to be bounded either above *or* below *almost surely* by a constant; the role of this constant is to be a measure of the dispersion of $Z$, and more specifically to ensure that the tail of $Z$ is bounded either above or below by a normal tail with that constant as its variance. Our Section 4 can thus be thought as a way to extend the phenomena described in [15] when comparison with the normal distribution can only be expected to go one way. Theorem 4.3 shows that we may have no control over how heavy the tail of $Z$ may be (beyond the existence of a second moment), but the condition $g_Z(Z) \geqslant \sigma^2 > 0$ $P$-a.s. essentially guarantees that it has to be no less heavy than a Gaussian tail with variance $\sigma^2$.

The rest of the paper is organized as follows. In Section 2, we recall the notions of Malliavin calculus that we need in order to perform our proofs. In Section 3, we state and discuss our density estimates. Section 4 deals with concentration inequalities, i.e. tail estimates.

## 2 Some elements of Malliavin calculus

Details of the exposition in this section are in Nualart's book [16, Chapter 1]. As stated in the introduction, we let $X$ be a centered isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$. For any $m \geqslant 1$, let $\mathfrak{H}^{\otimes m}$ be the $m$th tensor product of $\mathfrak{H}$ and $\mathfrak{H}^{\odot m}$ be the $m$th symmetric tensor product. Let $\mathscr{F}$ be the $\sigma$-field generated by $X$. It is well-known that any random variable $Z$ belonging to $L^2(\Omega, \mathscr{F}, P)$ admits the following chaos expansion:

$$Z = \sum_{m=0}^{\infty} I_m(f_m), \tag{2.9}$$

where $I_0(f_0) = E(Z)$, the series converges in $L^2(\Omega)$ and the kernels $f_m \in \mathfrak{H}^{\odot m}$, $m \geqslant 1$, are uniquely determined by $Z$. In the particular case where $\mathfrak{H}$ is equal to a separable space $L^2(A, \mathscr{A}, \mu)$, for $(A, \mathscr{A})$ a measurable space and $\mu$ a $\sigma$-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot m} = L^2_s(A^m, \mathscr{A}^{\otimes m}, \mu^{\otimes m})$ is the space of symmetric and square integrable functions on $A^m$ and, for every $f \in \mathfrak{H}^{\odot m}$, $I_m(f)$ coincides with the multiple Wiener-Itô integral of order $m$ of $f$ with respect to $X$. For every $m \geqslant 0$, we write $J_m$ to indicate the orthogonal projection operator on the $m$th Wiener chaos associated with $X$. That is, if $Z \in L^2(\Omega, \mathscr{F}, P)$ is as in (2.9), then $J_m F = I_m(f_m)$ for every $m \geqslant 0$.

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables of the form

$$Z = g\big(X(\phi_1), \ldots, X(\phi_n)\big)$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathscr{C}_b^\infty$ (the set of bounded and infinitely differentiable functions $g$ with bounded partial derivatives), and $\phi_i \in \mathfrak{H}$, $i = 1, \ldots, n$. The Malliavin derivative of $Z$ with respect to $X$ is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DZ = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(X(\phi_1), \ldots, X(\phi_n)\big)\phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the $m$th derivative $D^m Z$ (which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$) for every $m \geqslant 2$. For $m \geqslant 1$, $\mathbb{D}^{m,2}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{m,2}$, defined by the relation

$$\|Z\|_{m,2}^2 = E(Z^2) + \sum_{i=1}^{m} E\big(\|D^i Z\|_{\mathfrak{H}^{\otimes i}}^2\big).$$

Note that a random variable $Z$ as in (2.9) is in $\mathbb{D}^{1,2}$ if and only if

$$\sum_{m=1}^{\infty} m\, m!\, \|f_m\|_{\mathfrak{H}^{\otimes m}}^2 < \infty,$$

and, in this case, $E\big(\|DZ\|_{\mathfrak{H}}^2\big) = \sum_{m\geqslant 1} m\, m!\, \|f_m\|_{\mathfrak{H}^{\otimes m}}^2$. If $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$ (with $\mu$ non-atomic), then the derivative of a random variable $Z$ as in (2.9) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a Z = \sum_{m=1}^{\infty} m I_{m-1}\big(f_m(\cdot, a)\big), \quad a \in A.$$

The Malliavin derivative $D$ satisfies the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathscr{C}^1$ with bounded derivatives, and if $\{Z_i\}_{i=1,\dots,n}$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(Z_1, \dots, Z_n) \in \mathbb{D}^{1,2}$ and

$$D\,\varphi(Z_1, \dots, Z_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(Z_1, \dots, Z_n) DZ_i. \tag{2.10}$$

Formula (2.10) still holds when $\varphi$ is only Lipschitz but the law of $(Z_1, \dots, Z_n)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^n$ (see e.g. Proposition 1.2.3 in [16]).

We denote by $\delta$ the adjoint of the operator $D$, also called the divergence operator. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of $\delta$, denoted by $\operatorname{Dom}\delta$, if and only if it satisfies

$$\Big|E\langle DZ, u\rangle_{\mathfrak{H}}\Big| \leqslant c_u\, E(Z^2)^{1/2} \quad \text{for any } Z \in \mathscr{S},$$

where $c_u$ is a constant depending only on $u$. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is uniquely defined by the duality relationship

$$E(Z\delta(u)) = E\langle DZ, u\rangle_{\mathfrak{H}}, \tag{2.11}$$

which holds for every $Z \in \mathbb{D}^{1,2}$. Notice that all chaos variables $I_m(f_m)$ are in $\operatorname{Dom}\delta$.

The operator $L$ is defined through the projection operators as $L = \sum_{m=0}^{\infty} -mJ_m$, and is called the generator of the Ornstein-Uhlenbeck semigroup. It satisfies the following crucial property. A random variable $Z$ is an element of $\operatorname{Dom} L$ ($= \mathbb{D}^{2,2}$) if and only if $Z \in \operatorname{Dom}\delta D$ (i.e. $Z \in \mathbb{D}^{1,2}$ and $DZ \in \operatorname{Dom}\delta$), and in this case:

$$\delta DZ = -LZ. \tag{2.12}$$

We also define the operator $L^{-1}$, which is the pseudo-inverse of $L$, as follows. For every $Z \in L^2(\Omega, \mathscr{F}, P)$, we set $L^{-1}Z = \sum_{m\geqslant 1} -\frac{1}{m} J_m(Z)$. Note that $L^{-1}$ is an operator with values in $\mathbb{D}^{2,2}$, and that $LL^{-1}Z = Z - E(Z)$ for any $Z \in L^2(\Omega, \mathscr{F}, P)$, so that $L^{-1}$ does act as $L$'s inverse for centered r.v.'s.

The family $(T_u, u \geqslant 0)$ of operators is defined as $T_u = \sum_{m=0}^{\infty} e^{-mu} J_m$, and is called the Orstein-Uhlenbeck semigroup. Assume that the process $X'$, which stands for an independent copy of $X$, is such that $X$ and $X'$ are defined on the product probability space $(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', P \times P')$. Given a random variable $Z \in \mathbb{D}^{1,2}$, we can write $DZ = \Phi_Z(X)$, where $\Phi_Z$ is a measurable mapping from $\mathbb{R}^{\mathfrak{H}}$ to $\mathfrak{H}$, determined $P \circ X^{-1}$-almost surely. Then, for any $u \geqslant 0$, we have the so-called Mehler formula:

$$T_u(DZ) = E'\big(\Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\big), \tag{2.13}$$

where $E'$ denotes the mathematical expectation with respect to the probability $P'$.

# 3 Formula for the density

As said in the introduction, we consider a random variable $Z \in \mathbb{D}^{1,2}$ with zero mean. Recall the function $g_Z$ introduced in (1.2):

$$g_Z(z) = E(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} | Z = z).$$

It is useful to keep in mind throughout this paper that, by [15, Proposition 3.9], $g_Z(z) \geqslant 0$ for almost all $z$ in the support of the law of $Z$.

## 3.1 General formulae

We begin with the following theorem.

**Theorem 3.1.** *The law of $Z$ has a density $\rho$ if and only if the random variable $g_Z(Z)$ is strictly positive almost surely. In this case, the support of $\rho$, denoted by* $\operatorname{supp}\rho$*, is a closed interval of $\mathbb{R}$ containing zero and we have, for almost all $z \in \operatorname{supp}\rho$:*

$$\rho(z) = \frac{E|Z|}{2g_Z(z)} \exp\left(-\int_0^z \frac{x\,dx}{g_Z(x)}\right). \tag{3.14}$$

*Proof.* Let us first prove a useful identity. For any $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$ with bounded derivative, we have

$$\begin{aligned}
E\big(Zf(Z)\big) &= E\big(L(L^{-1}Z) \times f(Z)\big) = E\big(\delta(D(-L^{-1}Z)) \times f(Z)\big) \quad \text{by (2.12)}\\
&= E\big(\langle Df(Z), -DL^{-1}Z\rangle_{\mathfrak{H}}\big) \quad \text{by (2.11)}\\
&= E\big(f'(Z)\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}\big) \quad \text{by (2.10)}\\
&= E\big(f'(Z)g_Z(Z)\big).
\end{aligned} \tag{3.15}$$

Now, assume that the random variable $g_Z(Z)$ is strictly positive almost surely. Combining (3.15) with an approximation argument, we get, for any Borel set $B \in \mathscr{B}(\mathbb{R})$, that

$$E\big(Z\int_{-\infty}^{Z} \mathbf{1}_B(y)dy\big) = E\big(\mathbf{1}_B(Z)g_Z(Z)\big). \tag{3.16}$$

Suppose that the Lebesgue measure of $B \in \mathscr{B}(\mathbb{R})$ is zero. Then $E\big(\mathbf{1}_B(Z)g_Z(Z)\big) = 0$ by (3.16). Consequently, since $g_Z(Z) > 0$ a.s. by assumption, we have $P(Z \in B) = 0$. Therefore, the Radon-Nikodym criterion implies that the law of $Z$ has a density.

Conversely, assume that the law of $Z$ has a density, say $\rho$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support, and let $F$ denote any antiderivative of $f$. Note that $F$ is necessarily bounded. Following Stein himself (see [19, Lemma 3, p. 61]), we can write:

$$
\begin{aligned}
E\big(f(Z)g_Z(Z)\big) &= E\big(F(Z)Z\big) \quad \text{by (3.15)} \\
&= \int_{\mathbb{R}} F(z)\, z\, \rho(z)dz \underset{(*)}{=} \int_{\mathbb{R}} f(z) \left( \int_z^\infty y\rho(y)dy \right) dz \\
&= E\left( f(Z) \frac{\int_Z^\infty y\rho(y)dy}{\rho(Z)} \right).
\end{aligned}
$$

Equality (*) was obtained by integrating by parts, after observing that

$$\int_z^\infty y\rho(y)dy \longrightarrow 0 \text{ as } |z| \to \infty$$

(for $z \to +\infty$, this is because $Z \in L^1(\Omega)$; for $z \to -\infty$, this is because $Z$ has mean zero). Therefore, we have shown

$$g_Z(Z) = \frac{\int_Z^\infty y\rho(y)dy}{\rho(Z)}, \quad P\text{-a.s.}. \tag{3.17}$$

Since $Z \in \mathbb{D}^{1,2}$, it is known (see e.g. [16, Proposition 2.1.7]) that $\operatorname{supp}\rho = [\alpha, \beta]$ with $-\infty \leqslant \alpha < \beta \leqslant +\infty$. Since $Z$ has zero mean, note that $\alpha < 0$ and $\beta > 0$ necessarily. For every $z \in (\alpha, \beta)$, define

$$\varphi(z) = \int_z^\infty y\rho(y)\,dy. \tag{3.18}$$

The function $\varphi$ is differentiable almost everywhere on $(\alpha, \beta)$, and its derivative is $-z\rho(z)$. In particular, since $\varphi(\alpha) = \varphi(\beta) = 0$ and $\varphi$ is strictly increasing before 0 and strictly decreasing afterwards, we have $\varphi(z) > 0$ for all $z \in (\alpha, \beta)$. Hence, (3.17) implies that $g_Z(Z)$ is strictly positive almost surely.

Finally, let us prove (3.14). Let $\varphi$ still be defined by (3.18). On the one hand, we have $\varphi'(z) = -z\rho(z)$ for almost all $z \in \operatorname{supp}\rho$. On the other hand, by (3.17), we have, for almost all $z \in \operatorname{supp}\rho$,

$$\varphi(z) = \rho(z)g_Z(z). \tag{3.19}$$

By putting these two facts together, we get the following ordinary differential equation satisfied by $\varphi$:

$$\frac{\varphi'(z)}{\varphi(z)} = -\frac{z}{g_Z(z)} \quad \text{for almost all } z \in \operatorname{supp}\rho.$$

Integrating this relation over the interval $[0, z]$ yields

$$\log\varphi(z) = \log\varphi(0) - \int_0^z \frac{x\,dx}{g_Z(x)}.$$

Taking the exponential and using $0 = E(Z) = E(Z_+) - E(Z_-)$ so that $E|Z| = E(Z_+) + E(Z_-) = 2E(Z_+) = 2\varphi(0)$, we get

$$\varphi(z) = \frac{1}{2} E|Z| \exp\left(-\int_0^z \frac{x\,dx}{g_Z(x)}\right).$$

Finally, the desired conclusion comes from (3.19).

□

**Remark 3.2.** The 'integration by parts formula' (3.15) was proved and used for the first time by Nourdin and Peccati in [15], in order to perform error bounds in the normal approximation of $Z$. Specifically, [15] shows, by combining Stein's method with (3.15), that, if $\mathrm{Var}(Z) > 0$, then

$$\sup_{z\in\mathbb{R}} \left|P(Z \leqslant z) - P(N \leqslant z)\right| \leqslant \frac{\sqrt{\mathrm{Var}\big(g_Z(Z)\big)}}{\mathrm{Var}(Z)}, \tag{3.20}$$

where $N \sim \mathscr{N}(0, \mathrm{Var}Z)$. In reality, the inequality stated in [15] is with $\mathrm{Var}\big(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}\big)$ instead of $\mathrm{Var}\big(g_Z(Z)\big)$ on the right-hand side; but the same proof allows to write this slight improvement; it was not stated or used in [15] because it did not improve the applications therein.

As a corollary of Theorem 3.1, we can state the following.

**Corollary 3.3.** *Assume that there exists $\sigma^2_{\min} > 0$ such that*

$$g_Z(Z) \geqslant \sigma^2_{\min}, \quad P\text{-a.s.} \tag{3.21}$$

*Then the law of $Z$, which has a density $\rho$ by Theorem 3.1, has $\mathbb{R}$ for support and (3.14) holds a.e. in $\mathbb{R}$.*

*Proof.* It is an immediate consequence of Theorem 3.1, except the fact that $\operatorname{supp}\rho = \mathbb{R}$. For the moment, we just know that $\operatorname{supp}\rho = [\alpha, \beta]$ with $-\infty \leqslant \alpha < 0 < \beta \leqslant +\infty$. Identity (3.17) yields

$$\int_z^\infty y\rho\left(y\right)dy \geqslant \sigma^2_{\min}\,\rho\left(z\right) \quad \text{for almost all } z \in (\alpha, \beta). \tag{3.22}$$

Let $\varphi$ be defined by (3.18), and recall that $\varphi(z) > 0$ for all $z \in (\alpha, \beta)$. When multiplied by $z \in [0, \beta)$, the inequality (3.22) gives $\frac{\varphi'(z)}{\varphi(z)} \geqslant -\frac{z}{\sigma^2_{\min}}$. Integrating this relation over the interval $[0, z]$ yields $\log\varphi\left(z\right) - \log\varphi\left(0\right) \geqslant -\frac{z^2}{2\sigma^2_{\min}}$, i.e., since $\varphi(0) = \frac{1}{2}E|Z|$,

$$\varphi\left(z\right) = \int_z^\infty y\rho\left(y\right)dy \geqslant \frac{1}{2}E|Z|e^{-\frac{z^2}{2\sigma^2_{\min}}}. \tag{3.23}$$

Similarly, when multiplied by $z \in (\alpha, 0]$, inequality (3.22) gives $\frac{\varphi'(z)}{\varphi(z)} \leqslant -\frac{z}{\sigma^2_{\min}}$. Integrating this relation over the interval $[z, 0]$ yields $\log\varphi\left(0\right) - \log\varphi\left(z\right) \leqslant \frac{z^2}{2\sigma^2_{\min}}$, i.e. (3.23) still holds for $z \in (\alpha, 0]$. Now, let us prove that $\beta = +\infty$. If this were not the case, by definition, we would have $\varphi\left(\beta\right) = 0$; on the other hand, by letting $z$ tend to $\beta$ in the above inequality, because $\varphi$ is continuous, we would have $\varphi\left(\beta\right) \geqslant \frac{1}{2}E|Z|e^{-\frac{\beta^2}{2\sigma^2_{\min}}} > 0$, which contradicts $\beta < +\infty$. The proof of $\alpha = -\infty$ is similar. In conclusion, we have shown that $\operatorname{supp}\rho = \mathbb{R}$.

□

Using Corollary 3.3, we can deduce the following interesting criterion for normality, which one will compare with (3.20).

**Corollary 3.4.** *Assume that $Z$ is not identically zero. Then $Z$ is Gaussian if and only if* $\mathrm{Var}(g_Z(Z)) = 0$.

*Proof*: By (3.15) (choose $f(z) = z$), we have

$$E(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}) = E(Z^2) = \mathrm{Var}Z.$$

Therefore, the condition $\mathrm{Var}(g_Z(Z)) = 0$ is equivalent to

$$g_Z(Z) = \mathrm{Var}Z, \quad P\text{-a.s.}$$

Let $Z \sim \mathcal{N}(0, \sigma^2)$ with $\sigma > 0$. Using (3.17), we immediately check that $g_Z(Z) = \sigma^2$, $P$-a.s. Conversely, if $g_Z(Z) = \sigma^2 > 0$ $P$-a.s., then Corollary 3.3 implies that the law of $Z$ has a density $\rho$, given by $\rho(z) = \frac{E|Z|}{2\sigma^2} e^{-\frac{z^2}{2\sigma^2}}$ for almost all $z \in \mathbb{R}$, from which we immediately deduce that $Z \sim \mathcal{N}(0, \sigma^2)$. □

Observe that if $Z \sim \mathcal{N}(0, \sigma^2)$ with $\sigma > 0$, then $E|Z| = \sqrt{2/\pi}\,\sigma$, so that the formula (3.14) for $\rho$ agrees, of course, with the usual one in this case.

When $g_Z$ can be bounded above and away from zero, we get the following density estimates:

**Corollary 3.5.** *If there exists* $\sigma_{\min}, \sigma_{\max} > 0$ *such that*

$$\sigma_{\min}^2 \leqslant g_Z(Z) \leqslant \sigma_{\max}^2 \quad P\text{-a.s.},$$

*then the law of $Z$ has a density $\rho$ satisfying, for almost all $z \in \mathbb{R}$,*

$$\frac{E|Z|}{2\,\sigma_{\max}^2} \exp\left(-\frac{z^2}{2\sigma_{\min}^2}\right) \leqslant \rho(z) \leqslant \frac{E|Z|}{2\,\sigma_{\min}^2} \exp\left(-\frac{z^2}{2\sigma_{\max}^2}\right).$$

*Proof*: One only needs to apply Corollary 3.3.

□

**Remark 3.6.** General lower bound results on densities are few and far between. The case of uniformly elliptic diffusions was treated in a series of papers by Kusuoka and Stroock: see [14]. This was generalized by Kohatsu-Higa [13] in Wiener space via the concept of uniformly elliptic random variables; these random variables proved to be well-adapted to studying diffusion equations. E. Nualart [17] showed that fractional exponential moments for a divergence-integral quantity known to be useful for bounding densities from above (see formula (1.1) above), can also be useful for deriving a scale of exponential lower bounds on densities; the scale includes Gaussian lower bounds. However, in all these works, the applications are largely restricted to diffusions.

### 3.2 Computations and examples

We now show how to 'compute' $g_Z(Z) = E(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}|Z)$ in practice. We then provide several examples using this computation.

**Proposition 3.7.** *Write $DZ = \Phi_Z(X)$ with a measurable function $\Phi_Z : \mathbb{R}^{\mathfrak{H}} \to \mathfrak{H}$. We have*

$$\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} = \int_0^\infty e^{-u}\,\langle \Phi_Z(X), E'\big(\Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\big)\rangle_{\mathfrak{H}} du,$$

*so that*

$$g_Z(Z) = \int_0^\infty e^{-u}\,\mathbf{E}\big(\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_{\mathfrak{H}}|Z\big) du,$$

*where $X'$ stands for an independent copy of $X$, and is such that $X$ and $X'$ are defined on the product probability space $(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', P \times P')$. Here $\mathbf{E}$ denotes the mathematical expectation with respect to $P \times P'$, while $E'$ is the mathematical expectation with respect to $P'$.*

*Proof*: We follow the arguments contained in Nourdin and Peccati [15, Remark 3.6]. Without loss of generality, we can assume that $\mathfrak{H}$ is equal to $L^2(A, \mathscr{A}, \mu)$, where $(A, \mathscr{A})$ is a measurable space and $\mu$ is a $\sigma$-finite measure without atoms. Let us consider the chaos expansion of $Z$, given by $Z = \sum_{m=1}^\infty I_m(f_m)$, with $f_m \in \mathfrak{H}^{\odot m}$. Therefore $-L^{-1}Z = \sum_{m=1}^\infty \frac{1}{m} I_m(f_m)$ and

$$-D_a L^{-1}Z = \sum_{m=1}^\infty I_{m-1}(f_m(\cdot, a)), \quad a \in A.$$

On the other hand, we have $D_a Z = \sum_{m=1}^\infty m I_{m-1}(f_m(\cdot, a))$. Thus

$$\begin{aligned}\int_0^\infty e^{-u} T_u(D_a Z) du &= \int_0^\infty e^{-u} \left( \sum_{m=1}^\infty m e^{-(m-1)u} I_{m-1}(f_m(\cdot, a)) \right) du \\ &= \sum_{m=1}^\infty I_{m-1}(f_m(\cdot, a)).\end{aligned}$$

Consequently,

$$-DL^{-1}Z = \int_0^\infty e^{-u} T_u(DZ) du.$$

By Mehler's formula (2.13), and since $DZ = \Phi_Z(X)$ by assumption, we deduce that

$$-DL^{-1}Z = \int_0^\infty e^{-u} E'\big(\Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\big) du,$$

so that the formula for $\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}$ follows. Using $E(E'(\ldots)|Z) = \mathbf{E}(\ldots|Z)$, the formula for $g_Z(Z)$ holds.

□

By combining Theorem 3.1 with Proposition 3.7, we get the following formula:

**Corollary 3.8.** *Let the assumptions of Theorem 3.1 prevail. Let $\Phi_Z : \mathbb{R}^{\mathfrak{H}} \to \mathfrak{H}$ be measurable and such that $DZ = \Phi_Z(X)$. Then, for almost all $z$ in $\operatorname{supp}\rho$, the density $\rho$ of the law of $Z$ is given by*

$$\rho(z) = \frac{E|Z|}{2\int_0^\infty e^{-u}\,\mathbf{E}\big(\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_{\mathfrak{H}} | Z = z\big)du}$$
$$\times \exp\left( -\int_0^z \frac{x\,dx}{\int_0^\infty e^{-v}\,\mathbf{E}\big(\langle \Phi_Z(X), \Phi_Z(e^{-v}X + \sqrt{1-e^{-2v}}X')\rangle_{\mathfrak{H}} | Z = x\big)dv} \right).$$

Now, we give several examples of application of this corollary.

### 3.2.1 First example: monotone Gaussian functional, finite case.

Let $N \sim \mathscr{N}_n(0,K)$ with $K$ positive definite, and $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^1$ function having bounded derivatives. Consider an isonormal Gaussian process $X$ over the Euclidean space $\mathfrak{H} = \mathbb{R}^n$, endowed with the inner product $\langle h_i, h_j\rangle_{\mathfrak{H}} = E(N_i N_j) = K_{ij}$. Here, $\{h_i\}_{1\leqslant i\leqslant n}$ stands for the canonical basis of $\mathfrak{H} = \mathbb{R}^n$. Without loss of generality, we can identify $N_i$ with $X(h_i)$ for any $i = 1,\dots,n$. Set $Z = f(N) - E(f(N))$. The chain rule (2.10) implies that $Z \in \mathbb{D}^{1,2}$ and that $DZ = \Phi_Z(N) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(N) h_i$. Therefore

$$\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_{\mathfrak{H}} = \sum_{i,j=1}^n K_{ij}\frac{\partial f}{\partial x_i}(N)\frac{\partial f}{\partial x_j}(e^{-u}N + \sqrt{1-e^{-2u}}N'),$$

for $N_i' = X'(h_i)$, $i = 1,\dots,n$. In particular, Corollary 3.5 combined with Proposition 3.7 yields the following.

**Proposition 3.9.** *Let $N \sim \mathscr{N}_n(0,K)$ with $K$ positive definite, and $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^1$ function with bounded derivatives. If there exist $\alpha_i, \beta_i \geqslant 0$ such that $\alpha_i \leqslant \frac{\partial f}{\partial x_i}(x) \leqslant \beta_i$ for any $i \in \{1,\dots,n\}$ and $x \in \mathbb{R}^n$, if $K_{ij} \geqslant 0$ for any $i,j \in \{1,\dots,n\}$ and if $\sum_{i,j=1}^n \alpha_i\alpha_j K_{ij} > 0$, then the law of $Z = f(N) - E(f(N))$ admits a density $\rho$ which satisfies, for almost all $z \in \mathbb{R}$,*

$$\frac{E|Z|}{2\sum_{i,j=1}^n \beta_i\beta_j K_{ij}} \exp\left(-\frac{z^2}{2\sum_{i,j=1}^n \alpha_i\alpha_j K_{ij}}\right)$$
$$\leqslant \rho(z) \leqslant \frac{E|Z|}{2\sum_{i,j=1}^n \alpha_i\alpha_j K_{ij}} \exp\left(-\frac{z^2}{2\sum_{i,j=1}^n \beta_i\beta_j K_{ij}}\right).$$

### 3.2.2 Second example: monotone Gaussian functional, continuous case.

Assume that $X = (X_t,\, t \in [0,T])$ is a centered Gaussian process with continuous paths, and that $f : \mathbb{R} \to \mathbb{R}$ is $\mathscr{C}^1$ with a bounded derivative. The Gaussian space generated by $X$ can be identified with an isonormal Gaussian process of the type $X = \{X(h) : h \in \mathfrak{H}\}$, where the real and separable

Hilbert space $\mathfrak{H}$ is defined as follows: (i) denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions on $[0,T]$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}} = E(X_s X_t).$$

In particular, with such a notation, we identify $X_t$ with $X(\mathbf{1}_{[0,t]})$. Now, let $Z = \int_0^T f(X_v)dv - E\left(\int_0^T f(X_v)dv\right)$. Then $Z \in \mathbb{D}^{1,2}$ and we have $DZ = \Phi_Z(X) = \int_0^T f'(X_v)\mathbf{1}_{[0,v]}dv$. Therefore

$$\begin{aligned}
&\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_{\mathfrak{H}} \\
&= \iint_{[0,T]^2} f'(X_v) f'(e^{-u}X_w + \sqrt{1-e^{-2u}}X'_w)E(X_v X_w)dvdw.
\end{aligned}$$

Using Corollary 3.5 combined with Proposition 3.7, we get the following.

**Proposition 3.10.** *Assume that $X = (X_t,\, t \in [0,T])$ is a centered Gaussian process with continuous paths, and that $f : \mathbb{R} \to \mathbb{R}$ is $\mathscr{C}^1$. If there exists $\alpha, \beta, \sigma_{\min}, \sigma_{\max} > 0$ such that $\alpha \leqslant f'(x) \leqslant \beta$ for all $x \in \mathbb{R}$ and $\sigma^2_{\min} \leqslant E(X_v X_w) \leqslant \sigma^2_{\max}$ for all $v, w \in [0,T]$, then the law of $Z = \int_0^T f(X_v)dv - E\left(\int_0^T f(X_v)dv\right)$ has a density $\rho$ satisfying, for almost all $z \in \mathbb{R}$,*

$$\frac{E|Z|}{2\beta^2\sigma^2_{\max}T^2}\, e^{-\frac{z^2}{2\alpha^2\sigma^2_{\min}T^2}} \leqslant \rho(z) \leqslant \frac{E|Z|}{2\alpha^2\sigma^2_{\min}T^2}\, e^{-\frac{z^2}{2\beta^2\sigma^2_{\max}T^2}}.$$

### 3.2.3 Third example: maximum of a Gaussian vector.

Let $N \sim \mathscr{N}_n(0,K)$ with $K$ positive definite. Once again, we assume that $N$ can be written $N_i = X(h_i)$, for $X$ and $h_i$, $i = 1,\dots,n$, defined as in the section 3.2.1. Since $K$ is positive definite, note that the members $h_1,\dots,h_n$ are necessarily different in pairs. Let $Z = \max N - E(\max N)$, and set

$$I_u = \operatorname{argmax}_{1\leqslant i\leqslant n}(e^{-u}X(h_i) + \sqrt{1-e^{-2u}}X'(h_i)) \quad \text{for } u \geqslant 0.$$

**Lemma 3.11.** *For any $u \geqslant 0$, $I_u$ is a well-defined random element of $\{1,\dots,n\}$. Moreover, $Z \in \mathbb{D}^{1,2}$ and we have $DZ = \Phi_Z(N) = h_{I_0}$.*

*Proof*: Fix $u \geqslant 0$. Since, for any $i \neq j$, we have

$$\begin{aligned}
&P\big(e^{-u}X(h_i) + \sqrt{1-e^{-2u}}X'(h_i) = e^{-u}X(h_j) + \sqrt{1-e^{-2u}}X'(h_j)\big) \\
&= P\big(X(h_i) = X(h_j)\big) = 0,
\end{aligned}$$

the random variable $I_u$ is a well-defined element of $\{1,\dots,n\}$. Now, if $\Delta_i$ denotes the set $\{x \in \mathbb{R}^n : x_j \leqslant x_i \text{ for all } j\}$, observe that $\frac{\partial}{\partial x_i}\max(x_1,\dots,x_n) = \mathbf{1}_{\Delta_i}(x_1,\dots,x_n)$ almost everywhere. The desired conclusion follows from the Lipschitz version of the chain rule (2.10), and the following Lipschitz property of the max function, which is easily proved by induction (on $n \geqslant 1$):

$$\big|\max(y_1,\dots,y_n) - \max(x_1,\dots,x_n)\big| \leqslant \sum_{i=1}^n |y_i - x_i| \quad \text{for any } x, y \in \mathbb{R}^n. \tag{3.24}$$

□

In particular, we deduce from Lemma 3.11 that

$$\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X') \rangle_{\mathfrak{H}} = K_{I_0,I_u}. \tag{3.25}$$

so that, by Corollary 3.8, the density $\rho$ of the law of $Z$ is given, for almost all $z$ in $\operatorname{supp}\rho$, by:

$$\rho(z) = \frac{E|Z|}{2\int_0^\infty e^{-u}\mathbf{E}\big(K_{I_0,I_u}|Z=z\big)du} \exp\left(-\int_0^z \frac{x\,dx}{\int_0^\infty e^{-v}\mathbf{E}\big(K_{I_0,I_v}|Z=x\big)dv}\right).$$

As a by-product (see also Corollary 3.5), we get the density estimates in the next proposition, and a variance formula.

**Proposition 3.12.** *Let $N \sim \mathscr{N}_n(0,K)$ with $K$ positive definite.*

- *If there exists $\sigma_{\min}, \sigma_{\max} > 0$ such that $\sigma_{\min}^2 \leqslant K_{ij} \leqslant \sigma_{\max}^2$ for any $i,j \in \{1,\dots,n\}$, then the law of $Z = \max N - E(\max N)$ has a density $\rho$ satisfying*

$$\frac{E|Z|}{2\sigma_{\max}^2}\exp\left(-\frac{z^2}{2\sigma_{\min}^2}\right) \leqslant \rho(z) \leqslant \frac{E|Z|}{2\sigma_{\min}^2}\exp\left(-\frac{z^2}{2\sigma_{\max}^2}\right)$$

  *for almost all $z \in \mathbb{R}$.*

- *With $N'$ an independent copy of $N$ and $I_u := \operatorname{argmax}(e^{-u}N + \sqrt{1-e^{-2u}}N')$, we have*

$$\operatorname{Var}(\max N) = \int_0^\infty e^{-u}\mathbf{E}\big(K_{I_0,I_u}\big)du.$$

The variance formula above is a discrete analogue of formula (1.6): the reader can check that it is established identically to the proof of (1.6) found in the next section (Proposition 3.13), by using formula (3.25) instead of formula (3.26) therein. It appears that this discrete-case variance formula was established recently using non-Malliavin-calculus tools in the preprint [8, Lemma 3.1].

### 3.2.4 Fourth example: supremum of a Gaussian process.

Assume that $X = (X_t, t \in [0,T])$ is a centered Gaussian process with continuous paths. Fernique's theorem [12] implies that $E(\sup_{[0,T]} X^2) < \infty$. Assume $E(X_t - X_s)^2 \neq 0$ for all $s \neq t$. As in the section above, we can see $X$ as an isonormal Gaussian process (over $\mathfrak{H}$). Set $Z = \sup_{[0,T]} X - E(\sup_{[0,T]} X)$, and let $I_u$ be the (unique) random point where $e^{-u}X + \sqrt{1-e^{-2u}}X'$ attains its maximum on $[0,T]$. Note that $I_u$ is well-defined, see e.g. Lemma 2.6 in [11]. Moreover, we have that $Z \in \mathbb{D}^{1,2}$ and the law of $Z$ has a density, see Proposition 2.1.11 in [16], and $DZ = \Phi_Z(X) = \mathbf{1}_{[0,I_0]}$, see Lemma 3.1 in [9]. Therefore

$$\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X') \rangle_{\mathfrak{H}} = R(I_0, I_u) \tag{3.26}$$

where $R(s,t) = E(X_sX_t)$ is the covariance function of $X$. Hence, (1.5) is a direct application of Corollary 3.8. The first statement in the next proposition now follows straight from Corollary 3.5. The proposition's second statement is the variance formula (1.6), and its proof is given below.

**Proposition 3.13.** *Let $X = (X_t,\ t \in [0,T])$ be a centered Gaussian process with continuous paths, and $E(X_t - X_s)^2 \neq 0$ for all $s \neq t$.*

- *Assume that, for some real $\sigma_{\min}, \sigma_{\max} > 0$, we have $\sigma_{\min}^2 \leqslant E(X_s X_t) \leqslant \sigma_{\max}^2$ for any $s,t \in [0,T]$. Then, $Z = \sup_{[0,T]} X - E(\sup_{[0,T]} X)$ has a density $\rho$ satisfying, for almost all $z \in \mathbb{R}$,*

$$\frac{E|Z|}{2\sigma_{\max}^2} e^{-\frac{z^2}{2\sigma_{\min}^2}} \leqslant \rho(z) \leqslant \frac{E|Z|}{2\sigma_{\min}^2} e^{-\frac{z^2}{2\sigma_{\max}^2}}. \tag{3.27}$$

- *Let $R(s,t) = E(X_s X_t)$, let $X'$ be an independent copy of $X$, and let*

$$I_u = \operatorname{argmax}_{[0,T]}(e^{-u}X + \sqrt{1-e^{-2u}}X'), \quad u \geqslant 0.$$

  *Then* $\operatorname{Var}(\sup X) = \int_0^\infty e^{-u} \mathbf{E}\big(R\left(I_0, I_u\right)\big) du$.

*Proof*: The first bullet comes immediately from Corollary 3.5. For the variance formula of the second bullet, with $Z = \sup_{[0,T]} X - E(\sup_{[0,T]} X)$, using (3.15) with $f(z) = z$, we get $E(Z^2) = E(\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}})$, so that the desired conclusion is obtained immediately by combining (3.26) with Proposition 3.7.

□

When applied to the case of fractional Brownian motion, we get the following.

**Corollary 3.14.** *Let $b > a > 0$, and $B = (B_t,\ t \geqslant 0)$ be a fractional Brownian motion with Hurst index $H \in [1/2, 1)$. Then the random variable $Z = \sup_{[a,b]} B - E\big(\sup_{[a,b]} B\big)$ has a density $\rho$ satisfying (3.27) with $\sigma_{\min} = a^H$ and $\sigma_{\max} = b^H$.*

*Proof*: The desired conclusion is a direct application of Proposition 3.13 since, for all $a \leqslant s < t \leqslant b$,

$$E(B_s B_t) \leqslant \sqrt{E(B_s^2)}\sqrt{E(B_t^2)} = (st)^H \leqslant b^{2H}$$

and

$$\begin{aligned} E(B_s B_t) &= \frac{1}{2}\big(t^{2H} + s^{2H} - (t-s)^{2H}\big) = H(2H-1)\iint_{[0,s]\times[0,t]} |v-u|^{2H-2} du dv \\ &\geqslant H(2H-1)\iint_{[0,a]\times[0,a]} |v-u|^{2H-2} du dv = E(B_a^2) = a^{2H}. \end{aligned}$$

□

## 4 Concentration inequalities

In this whole section, we continue to assume that $Z \in \mathbb{D}^{1,2}$ has zero mean, and to work with $g_Z$ defined by (1.2).

Now, we investigate what can be said when $g_Z(Z)$ just admits a lower (resp. upper) bound. Results under such hypotheses are more difficult to obtain than in the previous section, since there we could use bounds on $g_Z(Z)$ in both directions to good effect; this is apparent, for instance, in the appearance of both the lower and upper bounding values $\sigma_{\min}$ and $\sigma_{\max}$ in each of the two bounds in (3.27), or more generally in Corollary 3.5. However, given our previous work, tails bounds can be readily obtained: most of the analysis of the role of $g_Z(Z)$ in tail estimates is already contained in the proof of Corollary 3.3.

Before stating our own results, let us cite a work which is closely related to ours, insofar as some of the preoccupations and techniques are similar. In [10], Houdré and Privault prove concentration inequalities for functionals of Wiener and Poisson spaces: they have discovered almost-sure conditions on expressions involving Malliavin derivatives which guarantee upper bounds on the tails of their functionals. This is similar to the upper bound portion of our work (Section 4.1), and closer yet to the first-chaos portion of the work in [21]; they do not, however, address lower bound issues.

### 4.1 Upper bounds

The next result allows comparisons both to the Gaussian and exponential tails.

**Theorem 4.1.** *Fix $\alpha \geqslant 0$ and $\beta > 0$. Assume that*

*(i) $g_Z(Z) \leqslant \alpha Z + \beta$, P-a.s.;*

*(ii) the law of $Z$ has a density $\rho$.*

*Then, for all $z > 0$, we have*

$$P(Z \geqslant z) \leqslant \exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right) \quad \textit{and} \quad P(Z \leqslant -z) \leqslant \exp\left(-\frac{z^2}{2\beta}\right).$$

*Proof*: We follow the same line of reasoning as in [7, Theorem 1.5]. For any $A > 0$, define $m_A : [0, +\infty) \to \mathbb{R}$ by $m_A(\theta) = E\left(e^{\theta Z}\mathbf{1}_{\{Z \leqslant A\}}\right)$. By Lebesgue differentiation theorem, we have

$$m_A'(\theta) = E(Z e^{\theta Z}\mathbf{1}_{\{Z \leqslant A\}}) \quad \text{for all } \theta \geqslant 0.$$

Therefore, we can write

$$
\begin{aligned}
m_A'(\theta) &= \int_{-\infty}^{A} z\, e^{\theta z}\, \rho(z)dz \\
&= -e^{\theta A} \int_A^{\infty} y\rho(y)dy + \theta \int_{-\infty}^{A} e^{\theta z} \left( \int_z^{\infty} y\rho(y)dy \right) dz \quad \text{by integration by parts} \\
&\leqslant \theta \int_{-\infty}^{A} e^{\theta z} \left( \int_z^{\infty} y\rho(y)dy \right) dz \quad \text{since } \int_A^{\infty} y\rho(y)dy \geqslant 0 \\
&= \theta E\big(g_Z(Z)\, e^{\theta Z}\, \mathbf{1}_{\{Z \leqslant A\}}\big),
\end{aligned}
$$

where the last line follows from identity (3.17). Due to the assumption $(i)$, we get

$$
m_A'(\theta) \leqslant \theta\, \alpha\, m_A'(\theta) + \theta\, \beta\, m_A(\theta),
$$

that is, for any $\theta \in (0, 1/\alpha)$:

$$
m_A'(\theta) \leqslant \frac{\theta\beta}{1-\theta\alpha}\, m_A(\theta). \tag{4.28}
$$

By integration and since $m_A(0) = P(Z \leqslant A) \leqslant 1$, this gives, for any $\theta \in (0, 1/\alpha)$:

$$
m_A(\theta) \leqslant \exp\left( \int_0^{\theta} \frac{\beta u}{1-\alpha u} du \right) \leqslant \exp\left( \frac{\beta\theta^2}{2(1-\theta\alpha)} \right).
$$

Using Fatou's lemma (as $A \to \infty$) in the previous relation implies

$$
E\big(e^{\theta Z}\big) \leqslant \exp\left( \frac{\beta\theta^2}{2(1-\theta\alpha)} \right)
$$

for all $\theta \in (0, 1/\alpha)$. Therefore, for all $\theta \in (0, 1/\alpha)$, we have

$$
P(Z \geqslant z) = P(e^{\theta Z} \geqslant e^{\theta z}) \leqslant e^{-\theta z} E\big(e^{\theta Z}\big) \leqslant \exp\left( \frac{\beta\theta^2}{2(1-\theta\alpha)} - \theta z \right).
$$

Choosing $\theta = \frac{z}{\alpha z + \beta} \in (0, 1/\alpha)$ gives the desired bound for $P(Z \geqslant z)$.

Now, let us focus on the lower tail. Set $Y = -Z$. Observe that assumptions $(i)$ and $(ii)$ imply that $Y$ has a density and satisfies $g_Y(Y) \leqslant -\alpha Y + \beta$, $P$-a.s. For $A > 0$, define $\widetilde{m}_A : [0, +\infty) \to \mathbb{R}$ by $\widetilde{m}_A(\theta) = E\big(e^{\theta Y} \mathbf{1}_{\{Y \leqslant A\}}\big)$. Here, instead of (4.28), we get similarly that $\widetilde{m}_A'(\theta) \leqslant \frac{\theta\beta}{1+\theta\alpha} \widetilde{m}_A(\theta) \leqslant \theta\beta\, \widetilde{m}_A(\theta)$ for all $\theta \geqslant 0$. Therefore, we can use the same arguments as above in order to obtain, this time, firstly that $E\big(e^{\theta Y}\big) \leqslant e^{\frac{\beta\theta^2}{2}}$ for all $\theta \geqslant 0$ and secondly that $P(Y \geqslant z) \leqslant \exp\left(-\frac{z^2}{2\beta}\right)$ (choosing $\theta = z/\beta$), which is the desired bound for $P(Z \leqslant -z)$.

□

**Remark 4.2.** In Theorem 4.1, when $\alpha > 0$, by the non-negativity of $g_Z(Z)$, (i) automatically implies that $Z$ is bounded below by the non-random constant $-\beta/\alpha$, and therefore the left hand tail $P(Z \leqslant -z)$ is zero for $z \leqslant -\beta/\alpha$. Therefore the upper bound on this tail in the above theorem is only asymptotically of interest in the "sub-Gaussian" case where $\alpha = 0$.

We will now give an example of application of Theorem 4.1. Assume that $B = (B_t,\, t \in [0,T])$ is a fractional Brownian motion with Hurst index $H \in (0,1)$. For any choice of the parameter $H$, as already mentioned in section 3.2.2, the Gaussian space generated by $B$ can be identified with an isonormal Gaussian process of the type $X = \{X(h) : h \in \mathfrak{H}\}$, where the real and separable Hilbert space $\mathfrak{H}$ is defined as follows: (i) denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions on $[0,T]$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = E(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

In particular, with such a notation one has that $B_t = X(\mathbf{1}_{[0,t]})$.

Now, let $Q$ be a $\mathscr{C}^1$ function such that the Lebesgue measure of the set $\{u \in \mathbb{R} : Q'(u) = 0\}$ is zero, and $\left|Q'(u)\right| \leqslant C\,|u|$ and $Q(u) \geqslant cu^2$ for some positive constants $c, C$ and all $u \in \mathbb{R}$. Let

$$Z = \int_0^1 Q\left(B_s\right) ds - E\left[\int_0^1 Q\left(B_s\right) ds\right].$$

Observe that $Z \in \mathbb{D}^{1,2}$, with $DZ = \int_0^1 Q'(B_s)\mathbf{1}_{[0,s]} ds$. Denoting $B^{(u)} = e^{-u}B + \sqrt{1-e^{-2u}}B'$, and $Z^{(u)} = \int_0^1 Q(B_s^{(u)})ds - E\left[\int_0^1 Q(B_s^{(u)})ds\right]$, we first note that the Malliavin derivative of $Z$ easily accommodates the transformation from $Z$ to $Z^{(u)}$. In the notation of formula (1.4), we simply have

$$\Phi_Z(Z^{(u)}) = \int_0^1 Q'(B_s^{(u)})\mathbf{1}_{[0,s]} ds.$$

Thus, by Proposition 3.7, we calculate

$$\begin{aligned}
&\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} \\
&= \int_0^\infty du e^{-u} \left\langle \int_0^1 Q'(B_s)\mathbf{1}_{[0,s]} ds; E'\left(\int_0^1 Q'(B_t^{(u)})\mathbf{1}_{[0,t]} dt\right)\right\rangle_{\mathfrak{H}} \\
&= \int_0^\infty du e^{-u} \int_{[0,1]^2} ds dt Q'(B_s) E'\left(Q'(B_t^{(u)})\right) \left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathfrak{H}} \\
&= \int_0^\infty du e^{-u} \int_{[0,1]^2} ds dt Q'(B_s) E'\left(Q'(B_t^{(u)})\right) E(B_s B_t).
\end{aligned}$$

We now estimate this expression from above using the fact that $\left|E\left(B_s B_t\right)\right| \leqslant s^H t^H$ and the upper bound on $\left|Q'\right|$:

$$\begin{aligned}
&\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} \\
&\leqslant C^2 \int_0^\infty due^{-u} \int_{[0,1]^2} dsdts^H t^H |B_s| E'\left(|B_t^{(u)}|\right) \\
&= C^2 \int_0^\infty due^{-u} \int_{[0,1]^2} dsdts^H t^H |B_s| E'\left(|e^{-u}B_t + \sqrt{1-e^{-2u}}B'_t|\right) \\
&\leqslant C^2 \int_0^\infty due^{-u} \int_{[0,1]^2} dsdts^H t^H |B_s| \left(e^{-u}|B_t| + \sqrt{\frac{2(1-e^{-2u})}{\pi}}\, t^H\right) \\
&= C^2 \int_0^\infty due^{-u} \left(e^{-u}\left(\int_0^1 |B_s|s^H ds\right)^2 + \frac{1}{2H+1}\sqrt{\frac{2(1-e^{-2u})}{\pi}} \int_0^1 |B_s|s^H ds\right) \\
&= \frac{C^2}{2}\left(\int_0^1 |B_s|s^H ds\right)^2 + \frac{C^2}{2H+1}\sqrt{\frac{\pi}{8}} \int_0^1 |B_s|s^H ds.
\end{aligned}$$

Now we wish to make $Z$ appear inside the right-hand side above. Note first that, using Cauchy-Schwarz's inequality and thanks to the lower bound on $Q$,

$$\left(\int_0^1 |B_s|s^H ds\right)^2 \leqslant \frac{1}{2H+1}\int_0^1 B_s^2 ds \leqslant \frac{1}{(2H+1)c}\int_0^1 Q(B_s)ds = \frac{Z+\mu}{(2H+1)c},$$

where $\mu = E\left[\int_0^1 Q(B_s)ds\right]$. By using $|x| \leqslant x^2 + \frac{1}{4}$ in order to bound $\int_0^1 |B_s|s^H ds$, we finally get that $\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}$ is less than

$$\frac{C^2}{(2H+1)c}\left[\left(\frac{1}{2} + \frac{1}{2H+1}\sqrt{\frac{\pi}{8}}\right)(Z+\mu) + \frac{c}{4}\sqrt{\frac{\pi}{8}}\right],$$

so that $g_Z(Z) \leqslant \alpha Z + \beta$, with $\alpha, \beta$ defined as in (1.8). Therefore, due to Theorem 4.1, the desired conclusion (1.7) is proved, once we show that the law of $Z$ has a density.

For that purpose, recall the so-called Bouleau-Hirsch criterion from [16, Theorem 2.1.3]: if $Z \in \mathbb{D}^{1,2}$ is such that $\|DZ\|_{\mathfrak{H}} > 0$ $P$-a.s., then the law of $Z$ has a density. Here, we have

$$\|DZ\|_{\mathfrak{H}}^2 = \int_{[0,1]^2} Q'(B_s)Q'(B_t)E(B_sB_t)dsdt,$$

from the computations performed above. We can express it as

$$\|DZ\|_{\mathfrak{H}}^2 = \int_{[0,1]^2} Q'(B_s)Q'(B_t)\hat{E}(\hat{B}_s\hat{B}_t)dsdt, = \hat{E}\left(\int_0^1 Q'(B_s)\hat{B}_s ds\right)^2,$$

where $\hat{B}$ is a fractional Brownian motion of Hurst index $H$, independent of $B$, and $\hat{E}$ denotes the expectation with respect to $\hat{B}$. Then, $\|DZ\|_{\mathfrak{H}}^2 = 0$ implies that $\int_0^1 Q'(B_s)\hat{B}_s ds = 0$ for almost all the trajectories $\hat{B}$, which implies that $Q'(B_s) = 0$ for all $s \in [0,1]$, so that $Q'(u) = 0$ for all $u$ in the interval $[\min_{s\in[0,1]} B_s, \max_{s\in[0,1]} B_s]$, which is a contradiction with the fact that the Lebesgue measure of the set $\{u \in \mathbb{R} : Q'(u) = 0\}$ is zero. Therefore, $\|DZ\|_{\mathfrak{H}} > 0$ $P$-a.s., and the law of $Z$ has a density according to the Bouleau-Hirsch criterion. The proof of (1.7) is concluded.

## 4.2 Lower bounds

We now investigate a lower bound analogue of Theorem 4.1. Recall we still use the function $g_Z$ defined by (1.2), for $Z \in \mathbb{D}^{1,2}$ with zero mean.

**Theorem 4.3.** *Fix $\sigma_{\min}, \alpha > 0$ and $\beta > 1$. Assume that*

*(i)* $g_Z(Z) \geqslant \sigma^2_{\min}$, *P-a.s.*

*The existence of the density $\rho$ of the law of $Z$ is thus ensured by Corollary 3.3. Also assume that*

*(ii) the function $h(x) = x^{1+\beta}\rho(x)$ is decreasing on $[\alpha, +\infty)$.*

*Then, for all $z \geqslant \alpha$, we have*

$$P(Z \geqslant z) \geqslant \frac{1}{2}\left(1 - \frac{1}{\beta}\right) E|Z| \, \frac{1}{z} \exp\left(-\frac{z^2}{2\sigma^2_{\min}}\right).$$

*Alternately, instead of $(ii)$, assume that there exists $0 < \alpha < 2$ such that*

*(ii)'* $\limsup_{z\to\infty} z^{-\alpha} \log g_Z(z) < \infty$.

*Then, for any $0 < \varepsilon < 2$, there exist $K, z_0 > 0$ such that, for all $z > z_0$,*

$$P(Z \geqslant z) \geqslant K \exp\left(-\frac{z^2}{(2-\varepsilon)\sigma^2_{\min}}\right).$$

*Proof*: First, let us relate the function $\varphi(z) = \int_z^\infty y\rho(y)dy$ to the tail of $Z$. By integration by parts, we get

$$\varphi(z) = z\,P(Z \geqslant z) + \int_z^\infty P(Z \geqslant y)dy. \tag{4.29}$$

If we assume $(ii)$, since $h$ is decreasing, for any $y > z \geqslant \alpha$ we have $\frac{y\rho(y)}{z\rho(z)} \leqslant \left(\frac{z}{y}\right)^\beta$. Then we have, for any $z \geqslant \alpha$ (observe that $\rho(z) > 0$):

$$P(Z \geqslant z) = z\rho(z)\int_z^\infty \frac{1}{y}\frac{y\rho(y)}{z\rho(z)}dy \leqslant z\rho(z) z^\beta \int_z^\infty \frac{dy}{y^{1+\beta}} = \frac{z\rho(z)}{\beta}.$$

By putting that inequality into (4.29), we get

$$\varphi(z) \leqslant z\,P(Z \geqslant z) + \frac{1}{\beta}\int_z^\infty y\rho(y)dy = z\,P(Z \geqslant z) + \frac{1}{\beta}\varphi(z)$$

so that $P(Z \geqslant z) \geqslant \left(1 - \frac{1}{\beta}\right)\frac{\varphi(z)}{z}$. Combined with (3.23), this gives the desired conclusion.

Now assume $(ii)'$ instead. Here the proof needs to be modified. From the result of Corollary 3.3 and condition $(i)$, we have

$$\rho(z) \geqslant \frac{E|Z|}{2\, g_Z(z)} \exp\left(-\frac{z^2}{2\,\sigma_{\min}^2}\right).$$

Let $\Psi(z)$ denote the unnormalized Gaussian tail $\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) dy$. We can write, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\Psi^2(z) &= \left(\int_z^\infty \exp\left(-\frac{y^2}{2\,\sigma_{\min}^2}\right) \sqrt{g_Z(y)}\, \frac{1}{\sqrt{g_Z(y)}}\, dy\right)^2 \\
&\leqslant \int_z^\infty \exp\left(-\frac{y^2}{2\,\sigma_{\min}^2}\right) g_Z(y)\, dy \times \int_z^\infty \exp\left(-\frac{y^2}{2\,\sigma_{\min}^2}\right) \frac{1}{g_Z(y)}\, dy
\end{aligned}$$

so that

$$\begin{aligned}
P(Z \geqslant z) &= \int_z^\infty \rho\,(y)\, dy \\
&\geqslant \frac{E\,|Z|}{2} \int_z^\infty e^{-y^2/(2\sigma_{\min}^2)}\, \frac{1}{g_Z(y)}\, dy \\
&\geqslant \frac{E\,|Z|}{2} \frac{\Psi^2\,(z)}{\int_z^\infty e^{-y^2/(2\sigma_{\min}^2)} g_Z\,(y)\, dy}.
\end{aligned}$$

Using the classical inequality $\int_z^\infty e^{-y^2/2} dy \geqslant \frac{z}{1+z^2} e^{-z^2/2}$, we get

$$P(Z \geqslant z) \geqslant \frac{E|Z|}{2}\, \frac{\sigma_{\min}^4 z^2}{(\sigma_{\min}^2 + z^2)^2}\, \frac{\exp\left(-\frac{z^2}{\sigma_{\min}^2}\right)}{\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) g_Z(y) dy}. \tag{4.30}$$

Under condition $(ii)'$, we have that there exists $c > 0$ such that, for $y$ large enough, $g_Z(y) \leqslant e^{c y^\alpha}$ with $0 < \alpha < 2$. We leave it to the reader to check that the conclusion now follows by an elementary calculation from (4.30).

□

**Remark 4.4.** *1.* Inequality (4.30) itself may be of independent interest, when the growth of $g_Z$ can be controlled, but not as efficiently as in $(ii)'$.

2. Condition $(ii)$ implies that $Z$ has a moment of order greater than $\beta$. Therefore it can be considered as a technical regularity and integrability condition. Condition $(ii)'$ may be easier to satisfy in cases where a good handle on $g_Z$ exists. Yet the use of the Cauchy-Schwarz inequality in the above proof means that conditions $(ii)'$ is presumably stronger than it needs to be.

3. In general, one can see that deriving lower bounds on tails of random variables with little upper bound control is a difficult task, deserving of further study.

**Acknowledgment**: We are grateful to Jean-Christophe Breton, Christian Houdré, Paul Malliavin, David Nualart, Giovanni Peccati and Nicolas Privault, for helpful comments. We warmly thank an anonymous Referee for suggesting and proving that $g_Z(Z)$ is strictly positive almost surely if and only if the law of $Z$ has a density (see Theorem 3.1), as well as a simplification of the argument allowing to show (1.7).

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