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Confinement of the two dimensional discrete Gaussian free field between two hard walls

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Abstract

We consider the two dimensional discrete Gaussian free field confined between two hard walls. We show that the field becomes massive and identify the precise asymptotic behavior of the mass and the variance of the field as the height of the wall goes to infinity. By large fluctuation of the field, asymptotic behaviors of these quantities in the two dimensional case differ greatly from those of the higher dimensional case studied by [14].

Key words: Gaussian field, hard wall, random interface, mass, random walk representation.

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1 Introduction and main results

The discrete Gaussian free field is represented as a Gibbs random field with massless interaction potentials and is interpreted as a probabilistic model of phase separating random interfaces. Because of its long range correlations the field exhibits many interesting behaviors, especially under the effect of various external forces (wall, pinning, etc.) and its study has been quite active in recent years (cf. [16] and references therein). Also, the two dimensional discrete Gaussian free field is related to the scaling limit of a number of discrete random surface models (cf. [15, Section 1.3] and references therein). In this paper, we study the behavior of the two dimensional discrete Gaussian free field confined between two hard walls.

At first, we introduce the model. Let $d \geq 2$, $\Lambda_N = [-N,N]^d \cap \mathbb{Z}^d$. For a configuration $\phi = \{\phi_x\}_{x\in\Lambda_N} \in \mathbb{R}^{\Lambda_N}$, consider the following massless Hamiltonian with quadratic interaction potential:

$$H_N(\phi) = \frac{1}{8d} \sum_{\substack{\{x,y\}\cap\Lambda_N \neq \phi \\ |x-y|=1}} (\phi_x - \phi_y)^2.$$

We define the corresponding Gibbs measure with 0-boundary conditions by

$$P_N^0(d\phi) = \frac{1}{Z_N^0} \exp\{-H_N(\phi)\} \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N} \delta_0(d\phi_x),$$

where $d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$ and Z_N^0 is a normalization factor. By summation by parts, we have an identity:

$$H_N(\phi)\Big|_{\phi\equiv 0 \text{ on } \Lambda_N^c} = \frac{1}{2} \langle \phi, (-\Delta_N)\phi \rangle_{\Lambda_N},$$

where Δ_N is a discrete Laplacian on $\mathbb{Z}^d$ with Dirichlet boundary condition outside Λ_N and $\langle \cdot , \cdot \rangle_{\Lambda_N}$ denotes $l^2(\Lambda_N)$-scalar product. Hence the measure P_N^0 coincides with the law of a centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with the covariance matrix $(-\Delta_N)^{-1}$. This model is called discrete Gaussian free field or harmonic crystal. The configuration ϕ is interpreted as an effective modelization of (discretized) phase separating random interface embedded in the $d+1$-dimensional space and the spin ϕ_x denotes the height of the interface at the position $x \in \Lambda_N$.

Confinement of the field between two hard walls is one of the problems related to the study of random interface. This was originally investigated by Bricmont et al. [4]. They showed that under the condition:

$$\mathscr{W}_N(L) = \{\phi : |\phi_x| \leq L \text{ for every } x \in \Lambda_N\},$$

the field turns to be massive and the following large L asymptotics holds.

- Free energy :

$$-\frac{1}{N^d} \log P_N^0(\mathscr{W}_N(L)) = \begin{cases} e^{-O(L)} & \text{if } d = 2, \\ e^{-O(L^2)} & \text{if } d \geq 3, \end{cases}$$

  for every N large enough (N depends on L).

- Mass :

$$\lim_{|x|\to\infty} -\frac{1}{|x|}\log E^{P^L_\infty}[\phi_0\phi_x] = \begin{cases} e^{-O(L)} & \text{if } d=2, \\ e^{-O(L^2)} & \text{if } d\ge 3. \end{cases}$$

- Variance :

$$\mathrm{Var}_{P^L_\infty}(\phi_0) = O(L) \quad \text{if } d=2,$$

$$0 \le \mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P^L_\infty}(\phi_0) \le e^{-O(L^2)} \quad \text{if } d\ge 3.$$

Here, P^L_∞ is the thermodynamic limit of the conditioned measure $P^0_N(\,\cdot\,|\mathscr{W}_N(L))$. Note that the family of conditioned measures automatically satisfies tightness and the uniqueness of the infinite limit is well known (e.g. [7] and references therein). P_∞ denotes the thermodynamic limit of P^0_N, namely the centered Gaussian field on $\mathbb{R}^{\mathbb{Z}^d}$ with the covariance matrix $(-\Delta)^{-1}$, Δ is a discrete Laplacian of $\mathbb{Z}^d$. Refinement of these results in the higher dimensional case was studied in [13] and [14]. When $d\ge 3$, the free energy behaves as $e^{-\frac{1}{2g_d}L^2(1+o(1))}$, the mass behaves as $e^{-\frac{1}{4g_d}L^2(1+o(1))}$ and the difference of the variance $\mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P^L_\infty}(\phi_0)$ is $e^{-\frac{1}{2g_d}L^2(1+o(1))}$ as $L\to\infty$, where $g_d = (-\Delta)^{-1}(0,0)$ for $d\ge 3$.

The main aim of this paper is to make refinement of the above results when $d=2$. The difficulty of the two dimensional case arises from the fact that the field is *rough*. Namely, we have the following property of the field:

$$\mathrm{Var}_{P^0_N}(\phi_0) = (-\Delta_N)^{-1}(0,0) \sim \begin{cases} g_2\log N & \text{if } d=2, \\ g_d & \text{if } d\ge 3, \end{cases} \tag{1.1}$$

as $N\to\infty$, where $g_2 = \frac{2}{\pi}$. By the uniform bound of variance the field is said to be *smooth* in the higher dimensional case and this guarantees the existence of the infinite volume limit P_∞ in $d\ge 3$, while the infinite volume limit does not exist in $d=2$. For our problem, large fluctuation of the field makes difficult to handle the effect of confinement in the two dimensional case. For example, the precise upper bound of the free energy in the higher dimensional case simply follows from a combination of Griffiths' inequality:

$$P^0_N(\mathscr{W}_N(L)) \ge \prod_{x\in\Lambda_N} P^0_N(|\phi_x|\le L),$$

Gaussian tail estimate:

$$P^0_N(|\phi_x|\ge L) \le \exp\Big\{-\frac{L^2}{2\mathrm{Var}_{P^0_N}(\phi_x)}\Big\},$$

and the variance estimate (1.1). Actually the asymptotics is the same as the i.i.d. Gaussian random variables with the same one site marginal distribution to P_∞. On the other hand, in the two dimensional case this argument does not work well because the variance diverges as $N\to\infty$. Large L asymptotics of the confined field in the two dimensional case is not clear at all since we first consider the confinement of the rough field and take the limit $N\to\infty$. We also remark that there are several numerical studies about this problem in the two dimensional case (e.g. [11] and references therein).

Now we are in the position to state the results of this paper. We first give the precise asymptotic behavior of the free energy.

Theorem 1.1. *Let $d=2$ and $g=\frac{2}{\pi}$. For every $\varepsilon>0$, there exists $L_0=L_0(\varepsilon)>0$ large enough such that the following holds for every $L\geq L_0$: there exists $N_0=N_0(L)$ and it holds that*

$$e^{-(\frac{1}{\sqrt{g}}+\varepsilon)L}\leq -\frac{1}{N^2}\log P_N^0(\mathscr{W}_N(L))\leq e^{-(\frac{1}{\sqrt{g}}-\varepsilon)L}, \tag{1.2}$$

for every $N\geq N_0$.

For the mass and variance of the confined field, we treat the following slightly modified model: let T_N be a d-dimensional lattice torus with size $2N$ (we identify N and $-N$ in Λ_N) and consider the following Hamiltonian with quadratic interaction potential and self-potential:

$$H_{N,m}(\phi)=\frac{1}{8d}\sum_{\substack{\{x,y\}\subset T_N\\|x-y|=1}}(\phi_x-\phi_y)^2+\frac{1}{2}m^2\sum_{x\in T_N}\phi_x^2. \tag{1.3}$$

$P_{N,m}^{\mathrm{per}}$ is the corresponding Gibbs measure on $\mathbb{R}^{T_N}$ with periodic boundary conditions and $P_{\infty,m}^L$ denotes the thermodynamic limit of the conditioned measure $P_{N,m}^{\mathrm{per}}(\,\cdot\,|\mathscr{W}_{T_N}(L))$ where $\mathscr{W}_A(L)=\{\phi : |\phi_x|\leq L \text{ for every } x\in A\}$ for $A\subset\mathbb{Z}^d$. Similarly to the higher dimensional case [14], the reason of this modification of the model is for the use of *reflection positivity/chessboard estimate* in the proof. We stress that the periodic boundary conditions are crucial for reflection positivity, but the mass term in the Hamiltonian (1.3) serves only to replace the 0-boundary conditions and to make the model well-defined. See [14, Section 1] for detail. Then we have the following precise asymptotic behavior of the mass and variance.

Theorem 1.2. *Let $d=2$ and $g=\frac{2}{\pi}$. For every $\varepsilon>0$, there exists $L_0=L_0(\varepsilon)>0$ large enough such that the following holds for every $L\geq L_0$:*

1.

$$\liminf_{m\to 0}\liminf_{l\to\infty}\Big\{-\frac{1}{l}\log E^{P_{\infty,m}^L}\big[\phi_0\phi_{[lz]}\big]\Big\}\geq e^{-(\frac{1}{2\sqrt{g}}+\varepsilon)L}, \tag{1.4}$$

$$\limsup_{m\to 0}\limsup_{l\to\infty}\Big\{-\frac{1}{l}\log E^{P_{\infty,m}^L}\big[\phi_0\phi_{[lz]}\big]\Big\}\leq e^{-(\frac{1}{2\sqrt{g}}-\varepsilon)L}, \tag{1.5}$$

for every $z\in\mathbb{S}^{d-1}=\{z\in\mathbb{R}^d;|z|=1\}$.

2.

$$\begin{aligned}(\frac{\sqrt{g}}{2}-\varepsilon)L&\leq\liminf_{m\to 0}\mathrm{Var}_{P_{\infty,m}^L}(\phi_0)\\&\leq\limsup_{m\to 0}\mathrm{Var}_{P_{\infty,m}^L}(\phi_0)\leq(\frac{\sqrt{g}}{2}+\varepsilon)L.\end{aligned} \tag{1.6}$$

Remark 1.1. *Reflection positivity is needed only for the proof of the lower bound of mass (1.4) and the upper bound of variance (1.6). The upper bound of mass (1.5) and the lower bound of variance (1.6) hold for the original massless Gaussian field (with 0-boundary conditions). But for consistency, we treat the case with periodic boundary conditions.*

The rest of this paper is organized as follows. In Section 2, we give the proof of the free energy estimate. The strategy of the proof of the upper bound is that to suppress the large fluctuation of the two dimensional Gaussian field, we insert a mass term (quadratic self-potential) to the Hamiltonian. This enables us to use Griffiths' inequality and Gaussian tail estimate similarly to the higher dimensional case. In the argument, certainly we need a cost to insert the mass term. But we have precise asymptotic behaviors of several quantities as the inserted mass vanishes and the optimal choice of the mass give the correct upper bound. For the proof of the lower bound, we use the result of entropic repulsion [1]. The mass estimate under confinement is given in Sections 3 and 4. The proof of the lower bound (1.4) is based on a combination of a random walk representation of the correlation of the field by Brydges-Fröhlich-Spencer [5] and reflection positivity. The precise asymptotic behavior of the free energy plays an important role in the argument. The proof of the upper bound (1.5) is also based on the random walk representation. The key ingredient is a precise estimate on the number of multiple points of ballistically pinned random walk which represents the fact that ballistically pinned two dimensional simple random walk turns to be transient. The proof of the variance estimate (1.6) is given in Section 5.

Finally we remark that throughout this paper below, C represents a positive constant which does not depend on the size of the system N, height of the wall L and mass m but may depend on other parameters. Also, this C in estimates may change from place to place in the paper.

2 Free energy estimates

For the proof of the results, we prepare some notations. Let $P^0_{N,m}$ be a Gibbs measure which corresponds to the Hamiltonian $H_N(\phi) + \frac{1}{2}m^2 \sum_{x\in\Lambda_N} \phi_x^2$ with 0-boundary conditions and $Z^0_{N,m}$ be its partition function. Actually, $P^0_{N,m}$ coincides with the law of the centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with the covariance matrix $(m^2-\Delta_N)^{-1}$. In the limit $N\to\infty$, $P^0_{N,m}$ weakly converges to $P_{\infty,m}$, the law of the centered Gaussian field on $\mathbb{R}^{\mathbb{Z}^d}$ with the covariance matrix $(m^2-\Delta)^{-1}$ for every $d\geq 1$. $\{S_n\}_{n\geq 0}$ is a simple random walk on $\mathbb{Z}^d$, $\mathbb{P}_x$ denotes its law starting at $x\in\mathbb{Z}^d$ and $\mathbb{E}_x$ denotes the corresponding expectation.

Proof of the upper bound of (1.2). At first, we have

$$\begin{aligned} P^0_N(\mathscr{W}_N(L)) &\geq \frac{Z^0_{N,m}}{Z^0_N} P^0_{N,m}(\mathscr{W}_N(L)) \\ &\geq \frac{Z^0_{N,m}}{Z^0_N} \prod_{x\in\Lambda_N} P^0_{N,m}(|\phi_x|\leq L) \\ &\geq \frac{Z^0_{N,m}}{Z^0_N} \prod_{x\in\Lambda_N} \Big\{1-\exp\{-\frac{L^2}{2\mathrm{Var}_{P^0_{N,m}}(\phi_x)}\}\Big\}, \end{aligned} \tag{2.1}$$

for every $m>0$, where the second inequality follows from Griffiths' inequality which holds for the massless field with symmetric self-potentials (cf. [6, Appendix A] and [10]) and the last inequality follows from the Gaussian tail estimate. The crucial point is that we have the asymptotics

$$\begin{aligned}\mathrm{Var}_{P^0_{N,m}}(\phi_x) &\le \mathrm{Var}_{P_{\infty,m}}(\phi_0) \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{m^2+1}\Big)^n \mathbb{P}_0(S_n=0) = g|\log m|(1+o(1)),\end{aligned} \tag{2.2}$$

as $m \downarrow 0$, where we used the local limit theorem

$$\mathbb{P}_0(S_n=0) = \frac{1}{\pi n} + O(n^{-\frac{3}{2}}), \tag{2.3}$$

as $n \to \infty$ and the expansion $-\log(1-x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}, 0 \le x < 1$ for the last equality. Therefore, if $\frac{L^2}{|\log m|}$ is large enough then

$$\begin{aligned}&\prod_{x\in\Lambda_N}\Big\{1-\exp\{-\frac{L^2}{2\mathrm{Var}_{P^0_{N,m}}(\phi_x)}\}\Big\} \\ &\qquad \ge \exp\Big\{-C|\Lambda_N|\exp\{-\frac{L^2}{2g|\log m|(1+o(1))}\}\Big\}.\end{aligned}$$

Next, by a random walk representation (cf. [2, section 4.1]),

$$\begin{aligned}\log Z^0_{N,m} &= \log\big((2\pi)^{\frac{|\Lambda_N|}{2}}(\det(m^2-\Delta_N))^{-\frac{1}{2}}\big) \\ &= \frac{1}{2}|\Lambda_N|\log(2\pi) - \frac{1}{2}|\Lambda_N|\log(m^2+1) \\ &\qquad + \frac{1}{2}\sum_{x\in\Lambda_N}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{1}{m^2+1}\Big)^n \mathbb{P}_x(S_n=x, n<\tau_{\Lambda_N}),\end{aligned}$$

for every $m \ge 0$, where $\tau_A = \inf\{n \ge 1; S_n \notin A\}$ is the first exit time from $A \subset \mathbb{Z}^d$. Then we have

$$\begin{aligned}&\frac{1}{|\Lambda_N|}\log\frac{Z^0_{N,m}}{Z^0_N} \\ &= -\frac{1}{2}\log(m^2+1) - \frac{1}{2|\Lambda_N|}\sum_{x\in\Lambda_N}\sum_{n=1}^{\infty}\frac{1}{n}\Big\{1-\Big(\frac{1}{m^2+1}\Big)^n\Big\}\mathbb{P}_x(S_n=x, n<\tau_{\Lambda_N}) \\ &\ge -\frac{1}{2}m^2 - C\sum_{n=1}^{\infty}\frac{1}{n^2}\Big\{1-\Big(\frac{1}{m^2+1}\Big)^n\Big\},\end{aligned}$$

for some constant $C>0$ where we used the estimate $\mathbb{P}_0(S_n=0) \le \frac{C}{n}$ for the last inequality. Now, let X be a random variable whose distribution is given by $P(X=n) = \frac{m^2}{m^2+1}(\frac{1}{m^2+1})^{n-1}, n \in \mathbb{N}$, namely geometric distribution with parameter $\frac{m^2}{m^2+1}$. Then

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\Big\{1-\Big(\frac{1}{m^2+1}\Big)^n\Big\} = \sum_{n=1}^{\infty}\frac{1}{n^2}P(X \le n)$$

$$= E\Big[\sum_{n=X}^{\infty}\frac{1}{n^2}\Big]$$
$$\le CE\Big[\frac{1}{X}\Big] = Cm^2|\log m|(1+o(1)),$$

as $m \downarrow 0$.

By collecting all the estimates, we obtain

$$-\frac{1}{|\Lambda_N|}\log P_N^0(\mathscr{W}_N(L))$$
$$\le \frac{1}{2}m^2 + Cm^2|\log m|(1+o(1)) + \exp\Big\{-\frac{L^2}{2g|\log m|(1+o(1))}\Big\}.$$

Finally, $m = e^{-\frac{L}{2\sqrt{g}}}$ optimizes the right hand side and we get the lower bound of (1.2). □

Remark 2.1. *Actually, the above optimal choice of m indicates the precise asymptotic behavior of the mass under confinement. This is also used in the proof of (1.5).*

Proof of the lower bound of (1.2). The idea of the proof of the upper bound is the same as the higher dimensional case. We divide Λ_N into small boxes by 0-boundary conditions and apply the result about entropic repulsion for the massless field.

By Griffiths' inequality and Markov property of the field, we can divide Λ_N into boxes with side-length $2M+1$ by 0-boundary conditions and we have

$$P_N^0\big(\mathscr{W}_N(L)\big) \le P_M^0\big(\mathscr{W}_M(L)\big)^{C(\frac{N}{M})^2}$$
$$\le P_M^0\big(\phi_x \ge -L \text{ for every } x \in \Lambda_M\big)^{C(\frac{N}{M})^2},$$

for every $1 \le M \ll N$. Now, take M as $M = e^{\frac{1-\varepsilon}{2\sqrt{g}}L}$ for $\varepsilon > 0$. Then, by the result of [1] there exists a constant $C \in (0,1)$ such that

$$P_M^0\big(\phi_x \ge -L \text{ for every } x \in \Lambda_M\big) \le C,$$

for every L large enough and we obtain

$$P_N^0\big(\mathscr{W}_N(L)\big) \le \exp\{-C(\frac{N}{M})^2\} \le \exp\{-CN^2 e^{-\frac{1-\varepsilon}{\sqrt{g}}L}\}.$$

This gives the lower bound of (1.2). □

3 Lower bound of mass

In this section, we prove (1.4). Once we have the precise estimate of the free energy (1.2), lower bound of mass can be proved by a similar argument to the higher dimensional case. We explain mainly the difference to the higher dimensional case.

At first we recall Brydges-Fröhlich-Spencer's random walk representation (cf. [5, Theorem 2.2]) which is applied to our setting.

Lemma 3.1. *It holds that*

$$E^{P^{\mathrm{per}}_{N,m}}\left[\phi_0\phi_x \,\middle|\, \mathscr{W}_{T_N}(L)\right] = {\sum_{\omega:0\to x}}' \Big(\frac{1}{2d(m^2+1)}\Big)^{|\omega|} \int \frac{\Xi^L_{N,m}(\frac{2}{m^2+1}\psi)}{\Xi^L_{N,m}} \mu_\omega(d\psi),$$

where the primed sum represents a summation with respect to paths of simple random walk on T_N connecting 0 and x. For a path ω, $|\omega|$ denotes its length. $\mu_\omega(d\psi) = \prod_{z\in T_N} \mu_{n(z,\omega)}(d\psi_z)$ is a product measure, μ_n is a measure on $[0,\infty)$ defined by

$$\mu_n(dt) = \begin{cases} \delta_0(dt) & \text{if } n = 0, \\ e^{-t}\frac{t^{n-1}}{(n-1)!} I(t\geq 0)dt & \text{if } n \in \mathbb{N}. \end{cases}$$

$n(z,\omega)$ denotes the total number of visits of site $z \in T_N$ in the path ω. Also,

$$\begin{aligned} &\Xi^L_{N,m}\Big(\frac{2}{m^2+1}\psi\Big) = P^{\mathrm{per}}_{N,m}\Big(\phi_z^2 + \frac{2}{m^2+1}\psi_z \leq L^2 \text{ for every } z \in T_N\Big), \\ &\Xi^L_{N,m} = P^{\mathrm{per}}_{N,m}(\mathscr{W}_{T_N}(L)). \end{aligned}$$

Then, by the argument in [14, Section 2] using reflection positivity, we know that

$$E^{P^{\mathrm{per}}_{N,m}}\left[\phi_0\phi_x \,\middle|\, \mathscr{W}_{T_N}(L)\right] \leq \sum_{k=0}^{\infty} \Big(\frac{1}{2d(m^2+1)}\Big)^k {\sum_{\substack{\omega:0\to x\\ |\omega|=k}}}' (q^L_{N,m})^{|R(\omega)|}, \tag{3.1}$$

where

$$q^L_{N,m} = \int P^{\mathrm{per}}_{N,m}\Big(\phi_x^2 + \frac{2}{m^2+1}t \leq L^2 \text{ for every } x \in T_N \,\Big|\, \mathscr{W}_{T_N}(L)\Big)^{\frac{1}{|T_N|}} \mu_1(dt),$$

and $R(\omega) = \{z \in T_N ; n(z,\omega) \geq 1\}$ is the range of the random walk path ω.
We have the following estimate for $q^L_{N,m}$. The proof is given later.

Lemma 3.2. *For every $\varepsilon > 0$, there exists $L_0 = L_0(\varepsilon) > 0$ such that*

$$\limsup_{N\to\infty} q^L_{N,m} \leq 1 - e^{-\frac{1+\varepsilon}{\sqrt{g}}L},$$

for every $L \geq L_0$ and every $m > 0$ small enough.

By this lemma and (3.1), after taking the limit $N \to \infty$ (if necessary along subsequence) we obtain

$$\begin{aligned} E^{P^L_{\infty,m}}[\phi_0\phi_x] &\leq \sum_{k=0}^{\infty} \mathbb{E}_0\left[I(S_k = x)(1-p^L)^{|S_{[0,k]}|}\right] \\ &= \sum_{k=0}^{\infty} \nu_{p^L} \otimes \mathbb{P}_0\left(S_k = x, S_{[0,k]} \subset \mathscr{A}^c\right), \end{aligned} \tag{3.2}$$

for every $m > 0$, where $p^L = e^{-\frac{1+\varepsilon}{\sqrt{g}}L}$ and $S_{[0,k]}$ denotes the set of points visited by the random walk up to time k. ν_q denotes a Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ with parameter $q \in (0,1)$. For i.i.d. $\{0,1\}$-valued random variables $\sigma = \{\sigma_z\}_{z\in\mathbb{Z}^d}$ with $\nu_q(\sigma_z = 1) = q = 1 - \nu_q(\sigma_z = 0)$, $\mathscr{A}$ denotes the set $\{z \in \mathbb{Z}^d; \sigma_z = 1\}$.

Asymptotic behavior of the right hand side of (3.2) as $p^L \to 0$ has been precisely studied in [3]. By (3.2) and the proof of [3, Lemma 5.1.2], we have

$$\begin{aligned}\liminf_{l\to\infty}&\Big\{-\frac{1}{l}\log E^{P^L_{\infty,m}}\big[\phi_0\phi_{[lz]}\big]\Big\}\\ &\geq \liminf_{l\to\infty}\Big\{-\frac{1}{l}\log \nu_{p^L}\otimes\mathbb{P}_0\big(T_{\{[lz]\}} < T_{\mathscr{A}}\big)\Big\},\end{aligned} \tag{3.3}$$

for every $z \in \mathbb{S}^{d-1}$, where $T_A = \inf\{k \geq 0; S_k \in A\}$ is the first hitting time to the set $A \subset \mathbb{Z}^d$. Also, by the proof of the lower bound of [3, Theorem 2.3] we know that

$$\begin{aligned}\nu_{p^L}\otimes\mathbb{P}_0\big(T_{\{x\}} < T_{\mathscr{A}}\big) &= \mathbb{E}_0\big[(1-p^L)^{|S_{[0,T_{\{x\}}]}|}\big]\\ &\leq e^{-C(p^L)^{\frac{1}{2}}|\log p^L|^{-\frac{1}{2}}|x|},\end{aligned} \tag{3.4}$$

for every $x \in \mathbb{Z}^d$ and L large enough. Since $p^L = e^{-\frac{1+\varepsilon}{\sqrt{g}}L}$ and $\varepsilon > 0$ is arbitrary, we obtain (1.4) by (3.3) and (3.4). □

The rest is to prove Lemma 3.2. For this purpose we prepare the following lemma.

Lemma 3.3. *For every $\varepsilon > 0$ and $0 < \lambda_0 < 1$, there exists $L_0 = L_0(\varepsilon, \lambda_0) > 0$ large enough such that the following holds: for every $L \geq L_0$, there exist $N_0 = N_0(L) > 0$ large enough and $m_0 = m_0(L) > 0$ small enough and it holds that*

$$P^{\mathrm{per}}_{N,m}\big(\mathscr{W}_{T_N}(\lambda L)\,\big|\,\mathscr{W}_{T_N}(L)\big) \leq e^{-N^2 e^{-\frac{\lambda}{\sqrt{g}}(1+\varepsilon)L}},$$

for every $N \geq N_0$, $0 < m < m_0$ and $\lambda_0 \leq \lambda < 1$.

Proof. We first note that the conditioned measure $P^{\mathrm{per}}_{N,m}(\,\cdot\,|\mathscr{W}_{T_N}(L))$ can be obtained as a weak limit of the measure:

$$dQ^{\mathrm{per}}_{N,\beta} \propto \exp\{-\beta\sum_{x\in T_N} I(|\phi_x| > L)\}dP^{\mathrm{per}}_{N,m},$$

as $\beta \to \infty$. This also holds for the 0-boundary conditions case. By Griffiths' inequality, we have

$$Q^{\mathrm{per}}_{N+1,\beta}\big(\mathscr{W}_{T_{N+1}}(\lambda L)\big) \leq Q^0_{N,\beta}\big(\mathscr{W}_{\Lambda_N}(\lambda L)\big).$$

Then, by taking the limits $\beta \to \infty$, we obtain

$$P^{\mathrm{per}}_{N+1,m}\big(\mathscr{W}_{T_{N+1}}(\lambda L)\,\big|\,\mathscr{W}_{T_{N+1}}(L)\big) \leq P^0_{N,m}\big(\mathscr{W}_{\Lambda_N}(\lambda L)\,\big|\,\mathscr{W}_{\Lambda_N}(L)\big).$$

Next, we have

$$P^0_{N,m}\big(\mathscr{W}_{\Lambda_N}(\lambda L)\,\big|\,\mathscr{W}_{\Lambda_N}(L)\big) = \frac{P^0_{N,m}\big(\mathscr{W}_{\Lambda_N}(\lambda L)\big)}{P^0_{N,m}\big(\mathscr{W}_{\Lambda_N}(L)\big)} \leq \frac{Z^0_N}{Z^0_{N,m}}\frac{P^0_N\big(\mathscr{W}_{\Lambda_N}(\lambda L)\big)}{P^0_N\big(\mathscr{W}_{\Lambda_N}(L)\big)},$$

where we used (2.1) for the numerator and Griffiths' inequality for the estimate on the denominator. Also, by the proof of the lower bound of (1.2), we know that $\frac{Z_N^0}{Z_{N,m}^0} \le e^{C|\Lambda_N|m^2|\log m|}$. By these estimates with the free energy estimate (1.2), we obtain the lemma.

□

Proof of Lemma 3.2. Let $\varepsilon > 0$ and $0 < \delta < 1$. We compute that

$$
\begin{aligned}
q_{N,m}^L &= \int P_{N,m}^{\mathrm{per}}\big(\phi_x^2 + \frac{2}{m^2+1}t \le L^2 \text{ for every } x \in T_N \,\big|\, \mathscr{W}_{T_N}(L)\big)^{\frac{1}{|T_N|}} \mu_1(dt) \\
&= \int_0^1 \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t} \\
&\qquad\qquad \times P_{N,m}^{\mathrm{per}}\big(\phi_x^2 \le L^2(1-t) \text{ for every } x \in T_N \,\big|\, \mathscr{W}_{T_N}(L)\big)^{\frac{1}{|T_N|}} dt \\
&\le 1 - \int_0^{1-\delta} \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t} \\
&\qquad\qquad \times \big\{1 - P_{N,m}^{\mathrm{per}}\big(\mathscr{W}_{T_N}(\sqrt{1-t}L) \,\big|\, \mathscr{W}_{T_N}(L)\big)^{\frac{1}{|T_N|}}\big\} dt \\
&=: 1 - I_1.
\end{aligned}
$$

By Lemma 3.3 and Fatou's lemma, there exists $L_1 = L_1(\varepsilon, \delta) > 0$ such that the following holds: for every $L \ge L_1$, there exists $m_1 = m_1(L) > 0$ small enough and for every $0 < m < m_1$ it holds that

$$
\begin{aligned}
\liminf_{N\to\infty} I_1 &\ge \int_0^{1-\delta} \frac{1}{2}(m^2+1)L^2 e^{-\frac{1}{2}(m^2+1)L^2 t}\big(1 - e^{-e^{-\frac{\sqrt{1-t}}{\sqrt{g}}(1+\varepsilon)L}}\big) dt \\
&\ge CL^2 \int_0^{1-\delta} e^{-\frac{1}{2}(m^2+1)L^2 t} e^{-\frac{\sqrt{1-t}}{\sqrt{g}}(1+\varepsilon)L} dt \\
&\ge e^{-\frac{1+2\varepsilon}{\sqrt{g}}L},
\end{aligned}
$$

where for the second inequality, we used an estimate: if $0 < C < 1$ then $1 - e^{-x} \ge Cx$ for every $x > 0$ small enough. The last estimate follows from changing the variable $\sqrt{1-t}$ as $1-s$ and an elementary computation. By these estimates we complete the proof of Lemma 3.2. □

4 Upper bound of mass

In this section, we prove the upper bound of mass (1.5). Similarly to the proof of the lower bound of (1.2), we first insert a mass term. By Lemma 3.1,

$$
\begin{aligned}
&E^{P_{N,m}^{\mathrm{per}}}\big[\phi_0\phi_x \,\big|\, \mathscr{W}_{T_N}(L)\big] \\
&\quad\ge E^{P_{N,m'}^{\mathrm{per}}}\big[\phi_0\phi_x \,\big|\, \mathscr{W}_{T_N}(L)\big] \\
&\quad= \sum_{k=0}^{\infty}\Big(\frac{1}{2d((m')^2+1)}\Big)^k \sideset{}{'}\sum_{\substack{\omega:0\to x\\ |\omega|=k}} \int \frac{\Xi_{N,m'}^L\big(\frac{2}{(m')^2+1}\psi\big)}{\Xi_{N,m'}^L} \mu_\omega(d\psi),
\end{aligned}
$$

for every $0 < m < m'$. Note that $E^{P^{\mathrm{per}}_{N,m}}\big[\phi_0\phi_x \,\big|\, \mathscr{W}_{T_N}(L)\big]$ decreases as $m > 0$ increases by Griffiths' inequality. Then, by the argument of [14, Section 3], we have

$$\begin{aligned}
&\log E^{P^L_{\infty,m}}\big[\phi_0\phi_x\big] \\
&\quad \geq k\log\frac{1}{(m')^2+1} + \sum_{j\geq 1}\mathbb{E}_0\big[R_k(j)\,\big|\, S_k = x\big]\log q^L_{m'}(j) + \log\mathbb{P}_0(S_k = x) \qquad (4.1)\\
&\quad =: I_1 + I_2 + I_3,
\end{aligned}$$

for every $0 < m < m'$, where $k = [\alpha|x|]$ with $\alpha > 0$ is specified later on, $R_k(j) = \sharp\{z \in \mathbb{Z}^d; n(z, S_{[0,k]}) = j\}$ denotes the total number of sites that the simple random walk visits exactly j times in the first k steps and $q^L_{m'}(j) = P_{\infty,m'}\otimes\mu_j\big(\phi_0^2 + 2\psi_0 \leq L^2\big)$.

Now, the proof of the upper bound of (1.2) indicates that $m' = e^{-\frac{L}{2\sqrt{g}}}$ gives the correct asymptotic behavior of the mass under confinement. So we estimate the right hand side of (4.1) for this choice of m'. Trivially, $I_1 \geq -k(m')^2$. For I_2, we first estimate $q^L_{m'}(j)$ from below.

$$\begin{aligned}
q^L_{m'}(j) &= P_{\infty,m'}\otimes\mu_j\big(\phi_0^2 + 2\psi_0 \leq L^2\big)\\
&= E^{P_{\infty,m'}}\Big[\mu_j\big(\psi_0 \leq \frac{1}{2}(L^2 - \phi_0^2)\big)I(|\phi_0| \leq L)\Big].
\end{aligned}$$

Since $E^{\mu_j}[e^{\theta\psi_0}] = (\frac{1}{1-\theta})^j$ for every $0 \leq \theta < 1$, for given ϕ_0 with $|\phi_0| \leq L$, we have

$$\mu_j\big(\psi_0 \leq \frac{1}{2}(L^2 - \phi_0^2)\big) \geq 1 - e^{-\frac{1}{2}(L^2-\phi_0^2)\theta}\big(\frac{1}{1-\theta}\big)^j,$$

and this yields

$$q^L_{m'}(j) \geq P_{\infty,m'}(|\phi_0| \leq L) - e^{-\frac{1}{2}\theta L^2}\big(\frac{1}{1-\theta}\big)^j E^{P_{\infty,m'}}\big[e^{\frac{1}{2}\theta\phi_0^2}I(|\phi_0| \leq L)\big].$$

We choose θ as $\theta = \theta_{m'} := (\mathrm{Var}_{P_{\infty,m'}}(\phi_0))^{-1}$. Then it holds that

$$E^{P_{\infty,m'}}\big[e^{\frac{1}{2}\theta_{m'}\phi_0^2}I(|\phi_0| \leq L)\big] = \sqrt{\frac{2\theta_{m'}}{\pi}}L,$$

and by combining these estimates with Gaussian tail estimate, we get

$$q^L_{m'}(j) \geq 1 - e^{-\frac{1}{2}\theta_{m'}L^2}\Big\{\big(\frac{1}{1-\theta_{m'}}\big)^j\sqrt{\frac{2\theta_{m'}}{\pi}}L + 1\Big\}.$$

Now, let $\varepsilon > 0$ be fixed. Since $m' = e^{-\frac{L}{2\sqrt{g}}}$, by (2.2) we have $\theta_{m'} = \frac{2}{\sqrt{g}L}(1 + o(1))$ as $L \to \infty$ and

$$q^L_{m'}(j) \geq 1 - e^{-\frac{1-\varepsilon}{\sqrt{g}}L}\big(\frac{1}{1-\theta_{m'}}\big)^j,$$

for every $j \geq 1$ and $L > 0$ large enough. Especially, if j is less than $b_L := \big[\frac{1}{-\log(1-\theta_{m'})}\frac{1-2\varepsilon}{\sqrt{g}}L\big]$ then

$$e^{-\frac{1-\varepsilon}{\sqrt{g}}L}\big(\frac{1}{1-\theta_{m'}}\big)^j \leq e^{-\frac{\varepsilon}{\sqrt{g}}L},$$

and we obtain

$$q^L_{m'}(j) \geq \exp\Big\{-Ce^{-\frac{1-\varepsilon}{\sqrt{g}}L}\big(\frac{1}{1-\theta_{m'}}\big)^j\Big\}, \tag{4.2}$$

for L large enough in this case. We remark that $b_L = O(L^2)$ as $L \to \infty$. On the other hand, by [14, (3.3)] we know that

$$\begin{aligned} q^L_{m'}(j) &\geq P_{\infty,m'}\big(\phi_0^2 \leq \frac{1}{2}L^2\big)\mu_j\big(2\psi_0 \leq \frac{1}{2}L^2\big) \\ &\geq e^{-CL^2}e^{-Cj\log j}, \end{aligned} \tag{4.3}$$

for for every $j \geq 1$ and L large enough.

By using estimate (4.2) for $j \leq b_L$ and (4.3) for $j \geq b_L + 1$, we obtain

$$\begin{aligned} I_2 &\geq \sum_{j\leq b_L} \mathbb{E}_0\big[R_k(j)\,\big|\,S_k = x\big]\Big\{-Ce^{-\frac{1-\varepsilon}{\sqrt{g}}L}\big(\frac{1}{1-\theta_{m'}}\big)^j\Big\} \\ &\quad + \sum_{j\geq b_L+1} \mathbb{E}_0\big[R_k(j)\,\big|\,\eta_k = x\big]\{-CL^2 - Cj\log j\} \\ &=: J_1 + J_2. \end{aligned}$$

The ingredient of our proof is the following estimate on the number of multiple points of the ballistically pinned random walk. The proof is given in the end of this section.

Proposition 4.1. *Let $\{S_n\}_{n\geq 0}$ be a simple random walk on $\mathbb{Z}^d$. If $d = 2$, then there exists a constant $C > 0$ such that for every $\varepsilon > 0$ there exists $\alpha_0 = \alpha_0(\varepsilon) > 0$ and for every $\alpha \geq \alpha_0$ it holds that*

$$\frac{1}{k}\mathbb{E}_0\big[R_k(j)|S_k = x\big] \leq C\Big(1 - \frac{1}{g(1+\varepsilon)\log\alpha}\Big)^{j-1},$$

for every x with $|x|$ large enough and $j \geq 1$, where we set $k = [\alpha|x|]$.

Remark 4.1. *A similar estimate in the higher dimensional case is proved in [14]. For a simple random walk on $\mathbb{Z}^d$ without pinned condition, it is well known that $\frac{\log n}{n}R_n(j) \to \pi$ as $n \to \infty$ a.s. if $d = 2$ (cf. [8]) and $\frac{1}{n}R_n(j) \to \gamma^2(1-\gamma)^{j-1}$ as $n \to \infty$ a.s. if $d \geq 3$, where $\gamma = \mathbb{P}_0(S_n \neq 0$ for every $n \geq 1)$ (cf. [12]).*

Now, for $k = [\alpha|x|]$, choose α as $\alpha = \alpha_L := \exp\{\frac{1}{g(1+\varepsilon)\theta_{m'}}\}$ where $m' = e^{-\frac{L}{2\sqrt{g}}}$. Then, by Proposition 4.1,

$$\begin{aligned} J_1 &\geq \sum_{j\leq b_L} C(1-\theta_{m'})^{j-1}\alpha_L|x|\Big\{-Ce^{-\frac{1-\varepsilon}{\sqrt{g}}L}\big(\frac{1}{1-\theta_{m'}}\big)^j\Big\} \\ &\geq -\alpha_L|x|e^{-\frac{1-2\varepsilon}{\sqrt{g}}L}, \end{aligned}$$

for every L large enough. For J_2, we have

$$J_2 \geq \sum_{j\geq b_L+1} C(1-\theta_{m'})^{j-1}\alpha_L|x|\{-CL^2 - Cj\log j\}$$

$$\geq -\alpha_L |x| e^{-\frac{1-3\varepsilon}{\sqrt{g}}L},$$

for every L large enough. Note that $(1-\theta_{m'})^{b_L+1} \leq e^{-\frac{1-2\varepsilon}{\sqrt{g}}L}$ by the definition of b_L.

For I_3 in (4.1), by a version of the local limit theorem [3, Proposition B.2] we obtain that

$$\begin{aligned}\log \mathbb{P}_0(S_k = x) &\geq \log\Big\{\frac{1}{Ck^{\frac{d}{2}}}\exp\{-C\frac{|x|^2}{k}\}\Big\}\\ &\geq -C\frac{|x|}{\alpha_L}(1+o_{|x|}(1)),\end{aligned}$$

for every L large enough and x with $|x|$ large enough, where $k = [\alpha|x|]$ with $\alpha = \alpha_L$ and $o_{|x|}(1)$ represents a term which goes to 0 as $|x| \to \infty$.

By collecting all the estimates, we have

$$\begin{aligned}&\limsup_{m\to 0}\limsup_{l\to\infty}\Big\{-\frac{1}{l}\log E^{P^L_{\infty,m}}\big[\phi_0\phi_{[lz]}\big]\Big\}\\ &\leq \alpha_L(m')^2 + \alpha_L e^{-\frac{1-2\varepsilon}{\sqrt{g}}L} + \alpha_L e^{-\frac{1-3\varepsilon}{\sqrt{g}}L} + \frac{C}{\alpha_L} \qquad (4.4)\\ &\leq e^{-\frac{1-3\varepsilon}{2\sqrt{g}}L},\end{aligned}$$

for every L large enough and every $z \in \mathbb{S}^{d-1}$. Recall that $m' = e^{-\frac{L}{2\sqrt{g}}}$, $\alpha_L = \exp\{\frac{1}{g(1+\varepsilon)\theta_{m'}}\}$ and $\theta_{m'} = \frac{2}{\sqrt{g}L}(1+o(1))$. Since $\varepsilon > 0$ is arbitrary we obtain (1.5). □

Remark 4.2. *Our choice of α_L optimizes (4.4). Furthermore, if we proceed the same argument without identifying m' in the beginning, then we can see that the choice of $m' = e^{-\frac{L}{2\sqrt{g}}}$ optimizes (4.4).*

Proof of Proposition 4.1. Let $\{p(x)\}_{x\in\mathbb{Z}^d}$ be a transition kernel of the simple random walk, namely $p(x) = \frac{1}{2d}$ if $|x| = 1$ and $p(x) = 0$ otherwise. For $\lambda \in \mathbb{R}^d$, we define a tilted measure $p^\lambda(x) = \frac{1}{Z(\lambda)}e^{\langle\lambda,x\rangle}p(x), x \in \mathbb{Z}^d$ where $Z(\lambda) = \sum_{x\in\mathbb{Z}^d} e^{\langle\lambda,x\rangle}p(x)$. We first remark that for a function $f(S) = f(S_0, S_1, \cdots, S_n)$, we have

$$\begin{aligned}\mathbb{E}_0[f(S)I(S_n = x)] &= \sum_{\substack{\omega:0\to x\\ |\omega|=n}} f(\omega)\prod_{i=1}^n p(\omega_i - \omega_{i-1})\\ &= \sum_{\substack{\omega:0\to x\\ |\omega|=n}} f(\omega)\prod_{i=1}^n Z(\lambda)e^{-\langle\lambda,\omega_i-\omega_{i-1}\rangle}p^\lambda(\omega_i - \omega_{i-1})\\ &= Z(\lambda)^n e^{-\langle\lambda,x\rangle}\mathbb{E}^\lambda_0[f(S)I(S_n = x)],\end{aligned}$$

where we denote the law of a random walk on $\mathbb{Z}^d$ with the transition kernel $\{p^\lambda(x)\}_{x\in\mathbb{Z}^d}$ and starting at $x \in \mathbb{Z}^d$ by $\mathbb{P}^\lambda_x$. $\mathbb{E}^\lambda_x$ denotes the corresponding expectation. Especially, we have

$$\mathbb{P}_0(S_n = x) = Z(\lambda)^n e^{-\langle\lambda,x\rangle}\mathbb{P}^\lambda_0(S_n = x), \qquad (4.5)$$

and

$$\mathbb{E}_0[R_n(j)I(S_n = x)] = Z(\lambda)^n e^{-\langle \lambda, x\rangle} \mathbb{E}_0^{\lambda}[R_n(j)I(S_n = x)], \tag{4.6}$$

for every $\lambda \in \mathbb{R}^d$.

Next, let $T_x^{(0)} = 0$ and $T_x^{(j)} = \inf\{n > T_x^{(j-1)}; S_n = x\}$, $j \geq 1$ be the j-th hitting time to site $x \in \mathbb{Z}^d$. By Markov property,

$$\begin{aligned}
&\mathbb{E}_0^{\lambda}[R_n(j)I(S_n = x)] \\
&\quad = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_0^{\lambda}\big(T_y^{(j)} \leq n, T_y^{(j+1)} > n, S_n = x\big) \\
&\quad = \sum_{y \in \mathbb{Z}^d} \sum_{0 \leq s+t \leq n} \mathbb{P}_0^{\lambda}\big(T_y^{(1)} = s\big)\mathbb{P}_0^{\lambda}\big(T_0^{(j-1)} = t\big) q_{n-(s+t)}^{\lambda}(x-y) \\
&\quad = \sum_{0 \leq t \leq n} \mathbb{P}_0^{\lambda}\big(T_0^{(j-1)} = t\big) \sum_{y \in \mathbb{Z}^d} \sum_{0 \leq s \leq n-t} \mathbb{P}_0^{\lambda}\big(T_y^{(1)} = s\big) q_{n-(s+t)}^{\lambda}(x-y),
\end{aligned}$$

where $q_l^{\lambda}(x) = \mathbb{P}_0^{\lambda}\big(S_n \neq 0 \text{ for every } 1 \leq n \leq l-1 \text{ and } S_l = x\big)$. Also,

$$\begin{aligned}
\sum_{y \in \mathbb{Z}^d} \sum_{0 \leq s \leq n} \mathbb{P}_0^{\lambda}\big(T_y^{(1)} = s\big) q_{n-s}^{\lambda}(x-y) &= \sum_{y \in \mathbb{Z}^d} \mathbb{P}_0^{\lambda}\big(T_y^{(1)} \leq n, T_y^{(2)} > n, S_n = x\big) \\
&= \mathbb{E}_0^{\lambda}[R_n(1)I(S_n = x)] \\
&\leq n\mathbb{P}_0^{\lambda}\big(S_n = x\big).
\end{aligned}$$

Therefore, we have

$$\mathbb{E}_0^{\lambda}[R_n(j)I(S_n = x)] \leq \sum_{0 \leq t \leq n} (n-t)\mathbb{P}_0^{\lambda}\big(T_0^{(j-1)} = t\big)\mathbb{P}_0^{\lambda}\big(\eta_{n-t} = x\big), \tag{4.7}$$

for every $n \geq 1$ and $x \in \mathbb{Z}^d$.

Now, set $k = [\alpha|x|]$. We show that there exists a constant $C > 0$ such that for every $\alpha > 0$ large enough there exists $r_0 = r_0(\alpha) > 0$ and it holds that for every x with $|x| \geq r_0$, if we choose $\lambda = \lambda(\alpha, x)$ appropriately then we have

$$\mathbb{P}_0^{\lambda}\big(\eta_l = x\big) \leq C\mathbb{P}_0^{\lambda}\big(\eta_k = x\big), \tag{4.8}$$

for every $l \leq k$. Since $\nabla \log Z(0) = 0$ and $\nabla^2 \log Z(0) = \operatorname{diag}\{\frac{1}{d}, \frac{1}{d}, \cdots, \frac{1}{d}\}$, the mapping $\lambda \to \nabla \log Z(\lambda)$ is an analytic diffeomorphism from a neighborhood of $0 \in \mathbb{R}^d$ to a neighborhood of $0 \in \mathbb{R}^d$. Hence, by inverse function theorem, for any ξ in a neighborhood of $0 \in \mathbb{R}^d$ there exists a unique $\widetilde{\lambda}(\xi) \in \mathbb{R}^d$ with $\nabla \log Z(\widetilde{\lambda}(\xi)) = \xi$. Now, by choosing λ as $\lambda_0 := \widetilde{\lambda}(\frac{x}{[\alpha|x|]})$, we have $\mathbb{E}_0^{\lambda_0}[S_k] = x$ and (4.8) easily follows from the local central limit theorem (see also the proof of [14, Lemma 3.1] and [3, Proposition B.2] for this type argument.)

By (4.7) and (4.8), for the above choice of λ we obtain

$$\begin{aligned}
\mathbb{E}_0^{\lambda_0}[R_k(j)I(S_k = x)] &\leq C \sum_{0 \leq t \leq k} k\mathbb{P}_0^{\lambda_0}\big(T_0^{(j-1)} = t\big)\mathbb{P}_0^{\lambda_0}\big(S_k = x\big) \\
&\leq Ck\mathbb{P}_0^{\lambda_0}\big(T_0^{(j-1)} < \infty\big)\mathbb{P}_0^{\lambda_0}\big(S_k = x\big)
\end{aligned}$$

$$= Ck(1-\gamma_{\lambda_0})^{j-1}\mathbb{P}_0^{\lambda_0}\big(S_k = x\big),$$

where $\gamma_\lambda = \mathbb{P}_0^\lambda\big(S_n \neq 0 \text{ for every } n \geq 1\big)$, $\lambda \in \mathbb{R}^d$. By combining this estimate with (4.5) and (4.6), we get

$$\mathbb{E}_0\big[R_k(j)\,\big|\,S_k = x\big] \leq Ck(1-\gamma_{\lambda_0})^{j-1}.$$

Next, we compute the asymptotics of γ_{λ_0} as $\alpha \to \infty$ when $d = 2$. For this purpose we first compute the asymptotics of $Z(\lambda_0)$. By definition, we have

$$Z(\lambda) = \frac{1}{4}(e^{\lambda^{(1)}} + e^{-\lambda^{(1)}} + e^{\lambda^{(2)}} + e^{-\lambda^{(2)}}),$$

for every $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathbb{R}^2$. Hence, by Taylor's expansion

$$Z(\lambda) = 1 + \frac{1}{4}((\lambda^{(1)})^2 + (\lambda^{(2)})^2)(1 + o(1)), \tag{4.9}$$

as $|\lambda| \to 0$. Also, by the choice of λ_0,

$$\mathbb{E}_0^{\lambda_0}[S_1^{(i)}] = \frac{1}{4Z(\lambda_0)}(e^{\lambda_0^{(i)}} - e^{-\lambda_0^{(i)}}) = \frac{x^{(i)}}{\alpha|x|}, \tag{4.10}$$

for $i = 1, 2$. Especially, since $Z(\lambda) \geq 1$ for every $\lambda \in \mathbb{R}^2$, this yields $\lambda_0^{(i)} = 0$ if $x^{(i)} = 0$ and $|\lambda_0| \to 0$ as $\alpha \to \infty$ uniformly in x with $|x|$ large enough. (4.10) also yields

$$\frac{1}{16(Z(\lambda_0))^2}\{(e^{\lambda_0^{(1)}} - e^{-\lambda_0^{(1)}})^2 + (e^{\lambda_0^{(2)}} - e^{-\lambda_0^{(2)}})^2\} = \frac{1}{\alpha^2}.$$

Then, by the fact that $\lim_{|\lambda|\to 0} Z(\lambda) = 1$ and Taylor's expansion we have

$$(\lambda_0^{(1)})^2 + (\lambda_0^{(2)})^2 = \frac{4}{\alpha^2}(1 + o(1)), \tag{4.11}$$

as $\alpha \to \infty$. By (4.9) and (4.11) we get

$$Z(\lambda_0) = 1 + \frac{1}{\alpha^2}(1 + o(1)), \tag{4.12}$$

as $\alpha \to \infty$ uniformly in x with $|x|$ large enough.

Now, let $\tau = \inf\{n \geq 1; S_n = 0\}$. Then

$$\begin{aligned} G^\lambda := \sum_{n=0}^{\infty}\mathbb{P}_0^\lambda(S_n = 0) &= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\mathbb{P}_0^\lambda(\tau = k)\mathbb{P}_0^\lambda(S_{n-k} = 0) \\ &= 1 + \sum_{k=1}^{\infty}\Big(\sum_{n=k}^{\infty}\mathbb{P}_0^\lambda(S_{n-k} = 0)\Big)\mathbb{P}_0^\lambda(\tau = k) \\ &= 1 + G^\lambda\mathbb{P}_0^\lambda(\tau < \infty), \end{aligned}$$

and we obtain

$$\gamma_\lambda = 1 - \mathbb{P}_0^\lambda(\tau < \infty) = \frac{1}{G^\lambda}.$$

Finally, by using (4.5), (4.12), local limit theorem (2.3) and an expansion $-\log(1-x) = \sum_{n=1} \frac{x^n}{n}, 0 \le x < 1$, we have

$$G^{\lambda_0} = \sum_{n=0}^{\infty} \frac{1}{(Z(\lambda_0))^n} \mathbb{P}_0(S_n = 0) = \frac{2}{\pi} \log \alpha(1 + o(1)),$$

as $\alpha \to \infty$. Hence we get $\gamma_{\lambda_0} = \frac{1}{g \log \alpha (1+o(1))}$ as $\alpha \to \infty$ uniformly in x with $|x|$ large enough and this completes the proof. □

5 Variance estimates

Proof of the upper bound of (1.6). By (3.2), we have

$$\begin{aligned} E^{P^L_{\infty,m}}[(\phi_0)^2] &\le \sum_{k=0}^{\infty} \mathbb{E}_0\left[I(S_k = 0)(1 - p^L)^{|S_{[0,k]}|}\right] \\ &\le \sum_{k=0}^{N_0} \mathbb{P}_0(S_k = 0) + \sum_{k=N_0}^{\infty} \mathbb{E}_0\left[(1 - p^L)^{|S_{[0,k]}|}\right] \\ &=: I_1 + I_2, \end{aligned}$$

where $p^L = e^{-\frac{1+\varepsilon}{\sqrt{g}} L}, \varepsilon > 0$. Now, by [3, Section 4.2] we know that $I_1 = \frac{1}{\pi} \log N_0 (1 + o(1))$ and $I_2 \le \frac{2}{p^L \kappa} e^{-\frac{1}{4} p^L \kappa N_0} + o(1)$ as $N_0 \to \infty$ for $\kappa > 0$ small enough. Therefore, if we choose N_0 as $N_0 = \frac{\beta L}{p^L}$ with $\beta = \beta(\kappa) > 0$ large enough, then $I_2 = o(1)$ as $L \to \infty$ and I_1 gives the upper bound of (1.6). □

Proof of the lower bound of (1.6). The idea of the proof of the lower bound owes to Y.Velenik (cf.[17]). For every $N \ge M$ and $m > 0$, by Griffiths' inequality,

$$\begin{aligned} &E^{P^{\mathrm{per}}_{N,m}}\left[(\phi_0)^2 | \mathscr{W}_{T_N}(L)\right] \\ &\quad \ge E^{P^0_{N,m}}\left[(\phi_0)^2 | \mathscr{W}_{\Lambda_N}(L)\right] \\ &\quad \ge E^{P^0_{M,m}}\left[(\phi_0)^2 | \mathscr{W}_{\Lambda_M}(L)\right] \\ &\quad \ge E^{P^0_{M,m}}\left[(\phi_0)^2 I(\mathscr{W}_{\Lambda_M}(L))\right] \\ &\quad = E^{P^0_{M,m}}\left[(\phi_0)^2\right] - E^{P^0_{M,m}}\left[(\phi_0)^2 I(|\phi_x| > L \text{ for some } x \in \Lambda_M)\right] \\ &\quad \ge E^{P^0_{M,m}}\left[(\phi_0)^2\right] - E^{P^0_{M,m}}\left[(\phi_0)^4\right]^{\frac{1}{2}} P^0_{M,m}(|\phi_x| > L \text{ for some } x \in \Lambda_M)^{\frac{1}{2}}. \end{aligned}$$

After taking the limit $N \to \infty$ and $m \downarrow 0$, we have

$$\begin{aligned} &\liminf_{m \to 0} E^{P^L_{\infty,m}}\left[(\phi_0)^2\right] \\ &\quad \ge E^{P^0_M}\left[(\phi_0)^2\right] - E^{P^0_M}\left[(\phi_0)^4\right]^{\frac{1}{2}} P^0_M(|\phi_x| > L \text{ for some } x \in \Lambda_M)^{\frac{1}{2}}. \end{aligned}$$

Also, by Gaussian computation,

$$E^{P_M^0}\left[(\phi_0)^2\right] = g\log M(1+o(1)),$$

$$E^{P_M^0}\left[(\phi_0)^4\right] \le C(\log M)^2,$$

and

$$P_M^0(|\phi_x| > L \text{ for some } x\in\Lambda_M) \le CM^2 e^{-\frac{L^2}{2g\log M}}.$$

Now, take M as $M = e^{\frac{1-\varepsilon}{2\sqrt{g}}L}$ for $\varepsilon > 0$. Then,

$$E^{P_M^0}\left[(\phi_0)^2\right] \ge g\cdot\frac{1-\varepsilon}{2\sqrt{g}}L(1+o(1)) = \frac{\sqrt{g}(1-\varepsilon)}{2}L(1+o(1)),$$

and

$$E^{P_{M,m}}\left[(\phi_0)^4\right]P_{M,m}(|\phi_x| > L \text{ for some } x\in\Lambda_M) \le C(\log M)^2 M^2 e^{-\frac{L^2}{2g\log M}} = o(1),$$

as $L\to\infty$. By these estimates we obtain the lower bound of (1.6). □

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References

[1] E. Bolthausen, J.–D. Deuschel and G. Giacomin, *Entropic repulsion and the maximum of two dimensional harmonic crystal*, Ann. Prob. **29** (2001) 1670-1692. MR1880237

[2] E. Bolthausen and D. Ioffe, *Harmonic crystal on the wall: a microscopic approach*, Comm. Math. Phys. **187** (1997) 523–566. MR1468313

[3] E. Bolthausen and Y. Velenik, *Critical behavior of the massless free field at the depinning transition*, Comm. Math. Phys. **223** (2001) 161–203. MR1860764

[4] J. Bricmont, A. El Mellouki and J. Fröhlich, *Random surfaces in Statistical Mechanics: roughening, rounding, wetting*, J. Stat. Phys. **42** (5/6) (1986) 743–798. MR0833220

[5] D. Brydges, J. Fröhlich and T. Spencer, *The random walk representation of classical spin systems and correlation inequalities*, Comm. Math. Phys. **83** (1982), 123–150. MR0648362

[6] F. Dunlop, J. Magnen, V. Rivasseau and P. Roche, *Pinning of an interface by a weak potential*, J. Stat. Phys. **66** (1992) 71–98. MR1149482

[7] P. Ferrari and S. Grynberg, *No phase transition for Gaussian fields with bounded spins*, J. Stat. Phys. **130** (2008) 195–202. MR2375961

[8] L. Flatto, *The multiple range of two-dimensional recurrent walk*, Ann. Probab. **4** (1976) 229–248. MR0431388

[9] T. Funaki, *Stochastic interface models*, In: Lectures on Probability Theory and Statistics, Ecole d'Eté de Probabilités de Saint -Flour XXXIII-2003 (ed. J. Picard), 103–274, Lect. Notes Math., **1869** (2005), Springer. MR2228384

[10] J. Ginibre, *General formulation of Griffith's inequalities*, Comm. Math. Phys. **16** (1970) 310–328. MR0269252

[11] D.-S. Lee, *Distribution of extremes in the fluctuations of two-dimensional equilibrium interfaces*, Phys. Rev. Lett. **95** (2005), 150601.

[12] J. Pitt, *Multiple points of transient random walks*, Proc. Amer. Math. Soc. **43** (1974), 195–199. MR0386021

[13] H. Sakagawa, *Entropic repulsion for the high dimensional Gaussian lattice field between two walls*, J. Stat. Phys. **124** (2006), 1255–1274. MR2265851

[14] H. Sakagawa, *Bounds on the mass for the high dimensional Gaussian lattice field between two hard walls,* J. Stat. Phys. **129** (2007), 537–553. MR2351416

[15] O. Schramm and S. Sheffield, *Contour lines of the two-dimensional discrete Gaussian free field*, Acta Math. **202** (2009) 21–137. MR2486487

[16] Y. Velenik, *Localization and delocalization of random interfaces*, Probab. Surv., **3** (2006) 112–169. MR2216964

[17] Y. Velenik, *private communication*.