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Limiting spectral distribution of circulant type matrices with dependent inputs

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Abstract

Limiting spectral distribution (LSD) of scaled eigenvalues of circulant, symmetric circulant and a class of k-circulant matrices are known when the input sequence is independent and identically distributed with finite moments of suitable order. We derive the LSD of these matrices when the input sequence is a stationary, two sided moving average process of infinite order. The limits are suitable mixtures of normal, symmetric square root of the chisquare, and other mixture distributions, with the spectral density of the process involved in the mixtures.

Key words: Large dimensional random matrix, eigenvalues, circulant matrix, symmetric circulant matrix, reverse circulant matrix, k circulant matrix, empirical spectral distribution, limiting spectral distribution, moving average process, spectral density, normal approximation.

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1 Introduction

Suppose $\lambda_1, \lambda_2, ..., \lambda_k$ are all the eigenvalues of a square matrix A of order k. Then the *empirical spectral distribution (ESD)* of A is defined as

$$F_A(x,y) = k^{-1} \sum_{i=1}^{k} \mathbb{I}\{\mathscr{R}(\lambda_i) \le x,\ \mathscr{I}(\lambda_i) \le y\},$$

where for any $z \in \mathbb{C}$, $\mathscr{R}(z)$, $\mathscr{I}(z)$ denote the real and imaginary part of z respectively. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices (with growing dimension) with the corresponding ESD $\{F_{A_n}\}_{n=1}^{\infty}$. The *limiting spectral distribution* (or measure) (LSD) of the sequence is defined as the weak limit of the sequence $\{F_{A_n}\}_{n=1}^{\infty}$, if it exists.

Suppose elements of $\{A_n\}$ are defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that is $\{A_n\}$ are random. Then $\{F_{A_n}(\cdot)\}$ are random and are functions of $\omega \in \Omega$ but we suppress this dependence. Let F be a nonrandom distribution function. We say the ESD of A_n converges to F in L_2 if at all continuity points (x,y) of F,

$$\int_{\Omega} \left[F_{A_n}(x,y) - F(x,y)\right]^2 d\mathbb{P}(\omega) \to 0 \text{ as } n \to \infty. \tag{1.1}$$

If the eigenvalues are real then it is understood that $\{F_{A_n}\}$ and F are functions of one variable. It may be noted that (1.1) holds if $\mathbb{E}[F_{A_n}(t)] \to F(t)$ and $V[F_{A_n}(t)] \to 0$ at all continuity points t of F. We often write F_n for F_{A_n} when the matrix under consideration is clear from the context.

For detailed information on existence and identification of LSD in different contexts, see Bai [1999] and also Bose and Sen [2008]. There are many universality results available in this context. However, most of the existing work on LSD assumes the *input* sequence $\{x_i\}$ to be independent. With the current methods used to establish LSD, such as the moment method or the Stietljes transform method, it does not appear to be easy to extend the known results on LSD to general dependent situations for all types of matrices. There are very few works dealing with dependent inputs. For instance, Bose and Sen [2008] establishes LSD for some specific type of dependent entries for a few matrices and Bai and Zhou [2008] establishes LSD of large sample covariance matrices with AR(1) entries.

We focus on the LSD of circulant type matrices— the circulant, the symmetric circulant, the reverse circulant and the so called k-circulant for suitable values of k. We assume that $\{x_i\}$ is a stationary linear process. Stationary linear process is an important class of dependent sequence. For instance the widely used stationary time series models such as AR, MA, ARMA are all linear processes. Under very modest conditions on the process, we are able to establish LSD for these matrices. These LSD are functions of the spectral density of the process and hence are universal. Consider the following condition.

Assumption A $\{x_i\}$ are independent, $\mathbb{E}(x_i) = 0,\ V(x_i) = 1$ and $\sup_i \mathbb{E}|x_i|^3 < \infty$.

We first describe the different matrices that we deal with. These may be divided into four classes:

(i) The *circulant matrix* is given by

$$C_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-3} & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \dots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_0 \end{bmatrix}_{n\times n}.$$

Sen [2006] shows that if Assumption **A** is satisfied then the ESD of C_n converges in L_2 to the two-dimensional normal distribution given by $\mathbf{N}(0, D)$ where D is a 2×2 diagonal matrix with diagonal entries 1/2. Meckes [2009] shows similar type of result for independent complex entries. In particular, if $\mathbb{E}(x_j) = 0$, $\mathbb{E}|x_j|^2 = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}(|x_j|^2 1_{|x_j|>\epsilon\sqrt{n}}) = 0$$

for every $\epsilon > 0$, then the ESD converges in L_2 to the standard complex normal distribution.

(ii) The *symmetric circulant matrix* is defined as

$$SC_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \dots & x_3 & x_2 \\ x_2 & x_1 & x_0 & \dots & x_2 & x_3 \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \dots & x_1 & x_0 \end{bmatrix}_{n\times n}.$$

Bose and Mitra [2002] show that if $\{x_i\}$ satisfies Assumption **A**, then the ESD of SC_n converges weakly in L_2 to the standard normal distribution.

The *palindromic Toeplitz matrix* is the palindromic version of the usual symmetric Toeplitz matrix. It is defined as (see Massey et.al. [2007]),

$$PT_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_2 & x_1 & x_0 \\ x_1 & x_0 & x_1 & \dots & x_3 & x_2 & x_1 \\ x_2 & x_1 & x_0 & \dots & x_4 & x_3 & x_2 \\ & & & \vdots & & & \\ x_1 & x_2 & x_3 & \dots & x_1 & x_0 & x_1 \\ x_0 & x_1 & x_2 & \dots & x_2 & x_1 & x_0 \end{bmatrix}_{n\times n}.$$

Its behavior is closely related to the symmetric circulant matrix. It may be noted that the $n \times n$ principal minor of PT_{n+1} is SC_n. Massey et.al. [2007] establish the Gaussian limit for F_{PT_n}. Bose and Sen [2008] show that if the input sequence $\{x_i\}$ is independent with mean zero and variance 1 and are either (i) uniformly bounded or (ii) identically distributed, then the LSD of PT_n is standard Gaussian. They also observe that if the LSD of any one of PT_n and SC_n exist, then the other also exists and they are equal.

(iii) The *reverse circulant matrix* is given by

$$RC_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_0 \\ x_2 & x_3 & x_4 & \dots & x_0 & x_1 \\ & & & \vdots & & \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-3} & x_{n-2} \end{bmatrix}_{n\times n}.$$

Bose and Mitra [2002] show that if $\{x_i\}$ satisfies Assumption **A** then the ESD of RC_n converges weakly in L_2 to F, which is the symmetric square root of the chisquare with two degrees of freedom, having density

$$f(x) = |x|\exp(-x^2), \ -\infty < x < \infty. \tag{1.2}$$

(iv) For positive integers k and n, the $n \times n$ *k-circulant* matrix is defined as

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_{n-k+2} & \dots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_{n-2k+2} & \dots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \end{bmatrix}_{n\times n}.$$

We emphasize that all subscripts appearing above are calculated modulo n. For $1 \le j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, k mod n positions). We have dropped the $n^{-1/2}$ factor from the definition of the matrix so that subsequent formulae for eigenvalues remain simple. Establishing the LSD for general k-circulant matrices appears to be a difficult problem. Bose, Mitra and Sen [2008] show that if $\{x_i\}$ are i.i.d $N(0,1)$, $k = n^{o(1)}$ (≥ 2) and $gcd(k,n) = 1$ then the LSD of $F_{n^{-1/2}A_{k,n}}$ is degenerate at zero, in probability. They also derive the following LSD. Suppose $\{x_i\}$ are i.i.d. satisfying Assumption **C** given below. Let $\{E_i\}$ be i.i.d. $Exp(1)$, U_1 be uniformly distributed over $(2g)$-th roots of unity, U_2 be uniformly distributed over the unit circle where $\{U_i\}$, $\{E_i\}$ are mutually independent. Then as $n \to \infty$, $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to

(i) $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ if $k^g = -1 + sn, \ g \ge 1, \ s = o(n^{1/3})$,

(ii) $U_2(\prod_{i=1}^{g} E_i)^{1/2g}$ if $k^g = 1 + sn, \ g \ge 2$ and

$$s = \begin{cases} o(n) & \text{if } g \text{ is even} \\ o(n^{\frac{g+1}{g-1}}) & \text{if } g \text{ is odd.} \end{cases}$$

We investigate the existence of LSD of the above matrices under the following situation.

Assumption B $\{x_n; n \ge 0\}$ is a two sided moving average process

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}, \quad \text{where} \quad a_n \in \mathbb{R} \ \text{ and } \sum_{n\in\mathbb{Z}} |a_n| < \infty.$$

Assumption C $\{\epsilon_i\}, i \in \mathbb{Z}$ are i.i.d. random variables with mean zero, variance one and $\mathbb{E}|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$.

We show that the LSD of the above matrices continue to exist in this dependent situation under appropriate conditions on the spectral density of the process. The LSD turn out to be appropriate mixtures of, the normal distribution, the "symmetric" square root of the chisquare distribution, and, some other related distributions. Quite expectedly, the spectral density of the process is involved in these mixtures. Our results also reduce to the results quoted above for the i.i.d. situation.

In Section 2 we describe the nature of the eigenvalues of the above matrices, describe the spectral density and set up notation. In Section 3 we state the main results and report some simulation which demonstrate our theoretical results. The main proofs are given in Section 4 and the proofs of some auxiliary Lemma are given in the Appendix.

Some of the results reported in this article have been reported in the (not to be published) technical reports Bose and Saha [2008a], Bose and Saha [2008b], Bose and Saha [2009].

2 Preliminaries

2.1 Spectral density and related facts

Under Assumptions **B** and **C**, $\gamma_h = Cov\left(x_{t+h}, x_t\right)$ is finite and $\sum_{j\in\mathbb{Z}} |\gamma_j| < \infty$. The *spectral density function* f of $\{x_n\}$ exists, is continuous, and is given by

$$f(\omega) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\gamma_k \exp(ik\omega) = \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{k\geq 1}\gamma_k \cos(k\omega)\right] \text{ for } \omega\in[0,2\pi].$$

Let

$$I_n(\omega_k) = \frac{1}{n}\Big|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}\Big|^2, \quad k = 0, 1, \ldots, n-1, \tag{2.1}$$

denote the periodogram of $\{x_i\}$ where $\omega_k = 2\pi k/n$ are the Fourier frequencies. Let

$$C_0 = \{t\in[0,1] : f(2\pi t) = 0\} \text{ and } C_0' = \{t\in[0,1/2] : f(2\pi t) = 0\}. \tag{2.2}$$

Define

$$\psi(e^{i\omega}) = \sum_{j=-\infty}^{\infty} a_j e^{ij\omega}, \ \ \psi_1(e^{i\omega}) = \mathscr{R}[\psi(e^{i\omega})], \ \psi_2(e^{i\omega}) = \mathscr{I}[\psi(e^{i\omega})], \tag{2.3}$$

where a_i's are the moving average coefficients in the definition of x_n. It is easy to see that

$$|\psi(e^{i\omega})|^2 = [\psi_1(e^{i\omega})]^2 + [\psi_2(e^{i\omega})]^2 = 2\pi f(\omega).$$

Let

$$B(\omega) = \begin{pmatrix} \psi_1(e^{i\omega}) & -\psi_2(e^{i\omega}) \\ \psi_2(e^{i\omega}) & \psi_1(e^{i\omega}) \end{pmatrix} \text{ and for } g\geq 2,$$

$$B(\omega_1,\omega_2,..,\omega_g) = \begin{pmatrix} \psi_1(e^{i\omega_1}) & -\psi_2(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ \psi_2(e^{i\omega_1}) & \psi_1(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \psi_1(e^{i\omega_2}) & -\psi_2(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & \psi_2(e^{i\omega_2}) & \psi_1(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \psi_1(e^{i\omega_g}) & -\psi_2(e^{i\omega_g}) \\ 0 & 0 & 0 & \cdots & \psi_2(e^{i\omega_g}) & \psi_1(e^{i\omega_g}) \end{pmatrix}.$$

The above functions will play a crucial role in the statements and proofs of the main results later.

2.2 Description of eigenvalues

We now describe the eigenvalues of the four classes of matrices. Let $[x]$ be the largest integer less than or equal to x.

(i) **Circulant matrix**. Its eigenvalues $\{\lambda_i\}$ are (see for example Brockwell and Davis [2002]),

$$\lambda_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_k l} = b_k + ic_k \quad \forall\ k = 1, 2, \cdots, n,$$

where

$$\omega_k = \frac{2\pi k}{n}, \quad b_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \cos(\omega_k l), \quad c_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \sin(\omega_k l). \tag{2.4}$$

(ii) **Symmetric circulant matrix**. The eigenvalues $\{\lambda_i\}$ of SC_n are given by:

(a) for n odd:

$$\lambda_0 = \frac{1}{\sqrt{n}} \left[x_0 + 2 \sum_{j=1}^{[n/2]} x_j\right]$$

$$\lambda_k = \frac{1}{\sqrt{n}} \left[x_0 + 2 \sum_{j=1}^{[n/2]} x_j \cos \frac{2\pi k j}{n}\right], \quad 1 \le k \le [n/2]$$

(b) for n even:

$$\lambda_0 = \frac{1}{\sqrt{n}} \left[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2}\right]$$

$$\lambda_k = \frac{1}{\sqrt{n}} \left[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j \cos \frac{2\pi k j}{n} + (-1)^k x_{n/2}\right], \quad 1 \le k \le \frac{n}{2}$$

with $\lambda_{n-k} = \lambda_k$ for $1 \le k \le [n/2]$ in both the cases.

(iii) **Palindromic Toeplitz matrix.** As far as we know, there is no formula solution for the eigenvalues of the palindromic Toeplitz matrix. As pointed out already, since the $n \times n$ principal minor of PT_{n+1} is SC_n, by interlacing inequality PT_n and SC_n have identical LSD.

(iv) **Reverse circulant matrix**. The eigenvalues are given in Bose and Mitra [2002]:

$$\begin{cases} \lambda_0 & = n^{-1/2} \sum_{t=0}^{n-1} x_t \\ \lambda_{n/2} & = n^{-1/2} \sum_{t=0}^{n-1} (-1)^t x_t, \quad \text{if } n \text{ is even} \\ \lambda_k = -\lambda_{n-k} & = \sqrt{I_n(\omega_k)}, \ 1 \le k \le [\frac{n-1}{2}]. \end{cases}$$

(v) k**-circulant matrix**. The structure of its eigenvalues is available in Zhou (1996). A more detailed analysis and related properties of the eigenvalues, useful in the present context, have been developed in Section 2 of Bose, Mitra and Sen [2008]. Let

$$\nu = \nu_n := \cos(2\pi/n) + i\sin(2\pi/n) \text{ and } \lambda_k = \sum_{l=0}^{n-1} x_l \nu^{kl}, \quad 0 \le j < n. \tag{2.5}$$

For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here $\alpha_q,\ \beta_q \ge 1$ and $n',\ k',\ p_q$ are pairwise relatively prime. For any positive integer s, let $\mathbb{Z}_s = \{0, 1, 2, \ldots, s-1\}$. Define the following sets

$$S(x) = \{xk^b \bmod n' : b \ge 0\},\ 0 \le x < n'.$$

For any set A, let $|A|$ denote its cardinality. Let $g_x = |S(x)|$ and

$$\upsilon_{k,n'} = \left|\{x \in \mathbb{Z}_{n'} : g_x < g_1\}\right|. \tag{2.6}$$

We observe the following about the sets $S(x)$.

1. $S(x) = \{xk^b \bmod n' : 0 \le b < |S(x)|\}$.
2. For $x \ne u$, either $S(x) = S(u)$ or $S(x) \cap S(u) = \phi$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \le x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \ldots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_{l-1}\}, \text{ and } \quad n_i = |\mathscr{P}_i|,\ 0 \le i < l. \tag{2.7}$$

Define

$$y_j := \prod_{t \in \mathscr{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, l-1 \quad \text{where} \quad y = n/n'.$$

Then the characteristic polynomial of $A_{k,n}$ (whence its eigenvalues follow) is given by

$$\chi\left(A_{k,n}\right) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{n_j} - y_j\right). \tag{2.8}$$

3 Main results

For any Borel set B, Leb(B) will denote its Lebesgue measure in the appropriate dimension.

3.1 Circulant matrix

Define for $(x,y)\in\mathbb{R}^2$ and $\omega\in[0,2\pi]$,

$$H_C(\omega,x,y)=\begin{cases}\mathbb{P}\big(B(\omega)(N_1,N_2)'\le\sqrt{2}(x,y)'\big) & \text{if}\quad f(\omega)\neq 0,\\ \mathbb{I}(x\ge 0,y\ge 0) & \text{if}\quad f(\omega)=0,\end{cases}$$

where N_1 and N_2 are i.i.d. standard normal distributions.

Lemma 1. *(i) For fixed x,y, H_C is a bounded continuous function in ω.*
(ii) F_C defined as follows is a proper distribution function.

$$F_C(x,y)=\int_0^1 H_C(2\pi s,x,y)ds. \tag{3.1}$$

(iii) If $\mathrm{Leb}(C_0)=0$ then F_C is continuous everywhere and can be expressed as

$$F_C(x,y)=\iint \mathbb{I}_{\{(v_1,v_2)\le(x,y)\}}\Big[\int_0^1\frac{1}{2\pi^2 f(2\pi s)}e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}}ds\Big]dv_1dv_2. \tag{3.2}$$

Further, F_C is bivariate normal if and only if f is constant almost everywhere (Lebesgue).
(iv) If $\mathrm{Leb}(C_0)\neq 0$ then F_C is discontinuous only on $D_1=\{(x,y):xy=0\}$.

The proof of the Lemma is easy and we omit it. The normality claim in (iii) follows by applying Cauchy Schwartz inequality to compare fourth moment and square of the variance and using the fact that for the normal distribution their ratio equals 3. We omit the details.

Theorem 1. *Suppose Assumptions **B** and **C** hold. Then the ESD of C_n converges in L_2 to $F_C(\cdot)$ given in (3.1)–(3.2).*

Remark 1. *If $\{x_i\}$ are i.i.d with finite $(2+\delta)$ moment, then $f(\omega)\equiv 1/2\pi$, and F_C reduces to the bivariate normal distribution whose covariance matrix is diagonal with entries $1/2$ each. This agrees with Theorem 15, page 57 of Sen [2006] who proved the result under Assumption **A**.*

3.2 Symmetric Circulant Matrix

For $x\in\mathbb{R}$ and $\omega\in[0,\pi]$ define,

$$H_S(\omega,x)=\begin{cases}\mathbb{P}\big(\sqrt{2\pi f(\omega)}N(0,1)\le x\big) & \text{if}\quad f(\omega)\neq 0,\\ \mathbb{I}(x\ge 0) & \text{if}\quad f(\omega)=0.\end{cases} \tag{3.3}$$

As before, we now have the following Lemma. We omit the proof.

Lemma 2. *(i) For fixed x, H_S is a bounded continuous function in ω and $H_S(\omega,x)+H_S(\omega,-x)=1$.*
(ii) F_S defined below is a proper distribution function and $F_S(x)+F_S(-x)=1$.

$$F_S(x)=2\int_0^{1/2}H_S(2\pi s,x)ds. \tag{3.4}$$

(iii) If $\mathrm{Leb}(C_0') = 0$ *then* F_S *is continuous everywhere and may be expressed as*

$$F_S(x) = \int_{-\infty}^{x} \Big[\int_0^{1/2} \frac{1}{\pi\sqrt{f(2\pi s)}} e^{-\frac{t^2}{4\pi f(2\pi s)}} ds \Big] dt. \tag{3.5}$$

Further, F_S *is normal if and only if* f *is constant almost everywhere (Lebesgue).*

(iv) If $\mathrm{Leb}(C_0') \neq 0$ *then* F_S *is discontinuous only at* $x = 0$.

Theorem 2. *Suppose Assumptions **B** and **C** hold and*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{[np/2]} \big[f(\frac{2\pi k}{n}) \big]^{-3/2} \longrightarrow 0 \text{ for all } 0 < p < 1. \tag{3.6}$$

Then the ESD of SC_n *converges in* L_2 *to* F_S *given in (3.4)–(3.5). The same limit continues to hold for* PT_n.

Remark 2. *(i) (3.6) is satisfied if* $\inf_\omega f(\omega) > 0$.

(ii) It is easy to check that the variance, μ_2 *and the fourth moment* μ_4 *of* F_S *equal* $\int_0^{1/2} 4\pi f(2\pi s) ds$ *and* $\int_0^{1/2} 24\pi^2 f^2(2\pi s) ds$ *respectively. By Cauchy-Schwartz inequality it follows that* $\frac{\mu_4}{\mu_2^2} \geq 3$ *and equal to* 3 *iff* $f \equiv \frac{1}{2\pi}$. *In the latter case,* F_S *is standard normal. This agrees with Remark* 2 *of Bose and Mitra [2002] (under Assumption **A**).*

3.3 Reverse circulant matrix

Define $H_R(\omega, x)$ on $[0, 2\pi] \times \mathbb{R}$ as

$$H_R(\omega, x) = \begin{cases} G\Big(\frac{x^2}{2\pi f(\omega)}\Big) & \text{if } f(\omega) \neq 0 \\ 1 & \text{if } f(\omega) = 0, \end{cases}$$

where $G(x) = 1 - e^{-x}$ for $x > 0$, is the standard exponential distribution function.

Lemma 3. *(i) For fixed* x, $H_R(\omega, x)$ *is bounded continuous on* $[0, 2\pi]$.

(ii) F_R *defined below is a valid symmetric distribution function.*

$$F_R(x) = \begin{cases} \frac{1}{2} + \int_0^{1/2} H_R(2\pi t, x) dt & \text{if } x > 0 \\ \frac{1}{2} - \int_0^{1/2} H_R(2\pi t, x) dt & \text{if } x \leq 0. \end{cases} \tag{3.7}$$

(iii) If $\mathrm{Leb}(C_0') = 0$ *then* F_R *is continuous everywhere and can be expressed as*

$$F_R(x) = \begin{cases} 1 - \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x > 0 \\ \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x \leq 0. \end{cases} \tag{3.8}$$

Further, F_R *is the distribution of the symmetric version of the square root of chisquare variable with two degrees of freedom if and only if* f *is constant almost everywhere (Lebesgue).*

(iv) If $\mathrm{Leb}(C_0') \neq 0$ *then* F_R *is discontinuous only at* $x = 0$.

The proof of the above lemma is omitted.

Theorem 3. *Suppose Assumptions **B** and **C** hold. Then the ESD of RC_n converges in L_2 to F_R given in (3.7)–(3.8).*

Remark 3. *If $\{x_i\}$ are i.i.d, with finite $(2+\delta)$ moment, then $f(\omega)=1/2\pi$ for all $\omega \in [0,2\pi]$ and the LSD $F_R(\cdot)$ agrees with (1.2) given earlier.*

3.4 k-Circulant matrix

As mentioned before, it appears difficult to prove general results for all possible pairs $(k,\ n)$. We investigate two subclasses of the k-circulant.

3.4.1 $n = k^g + 1$ for some fixed $g \geq 2$

For any $d \geq 1$, let

$$G_d(x) = \mathbb{P}\Big(\prod_{i=1}^{d} E_i \leq x\Big),$$

where $\{E_i\}$ are i.i.d. $Exp(1)$. Note that G_d is continuous. For any integer $d \geq 1$, define $H_d(\omega_1,\ldots,\omega_d,x)$ on $[0,2\pi]^d \times \mathbb{R}_{\geq 0}$ as

$$H_d(\omega_1,\ldots,\omega_d,x) = \begin{cases} G_d\Big(\frac{x^{2d}}{(2\pi)^d \prod_{i=1}^{d} f(\omega_i)}\Big) & \text{if} \quad \prod_{i=1}^{d} f(\omega_i) \neq 0 \\ 1 & \text{if} \quad \prod_{i=1}^{d} f(\omega_i) = 0. \end{cases}$$

Lemma 4. *(i) For fixed x, $H_d(\omega_1,\ldots,\omega_d,x)$ is bounded continuous on $[0,2\pi]^d$.*

(ii) F_d defined below is a valid continuous distribution function.

$$F_d(x) \quad = \quad \int_0^1 \cdots \int_0^1 H_d(2\pi t_1,\ldots,2\pi t_d,x) \prod_{i=1}^{d} dt_i \ \ \textit{for} \ \ x \geq 0. \tag{3.9}$$

The proof of lemma is omitted.

Theorem 4. *Suppose Assumptions **B** and **C** hold. Suppose $n = k^g + 1$ for some fixed $g \geq 2$. Then as $n \to \infty$, $F_{n^{-1/2}A_{k,n}}$ converges in L_2 to the LSD $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. with distribution function F_g given in (3.9) and U_1 is uniformly distributed over the $(2g)$th roots of unity, independent of the $\{E_i\}$.*

Remark 4. *If $\{x_i\}$ are i.i.d, then $f(\omega) = 1/2\pi$ for all $\omega \in [0,2\pi]$ and the LSD is $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. $Exp(1)$, U_1 is as in Theorem 4 and independent of $\{E_i\}$. This limit agrees with Theorem* 3 *of Bose, Mitra and Sen [2008].*

Remark 5. *Using the expression (2.8) for the characteristic polynomial, it is then not difficult to manufacture $\{k = k(n)\}$ such that the LSD of $n^{-1/2}A_{k,n}$ has some positive mass at the origin. For example,* *suppose the sequences k and n satisfy $k^g = -1 + sn$ where $g \geq 1$ is fixed and $s = o(n^{1/3})$. Fix primes $p_1, p_2, \ldots, p_t$ and positive integers $\beta_1, \beta_2, \ldots, \beta_t$. Define*

$$\widetilde{n} = p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t} n.$$

Suppose $k = p_1 p_2 \ldots p_t m \to \infty$. Then the ESD of $\widetilde{n}^{-1/2} A_{k,\widetilde{n}}$ converges weakly in probability to the LSD which has $1 - \left(\Pi_{s=1}^{t} p_s^{\beta_s}\right)^{-1}$ mass at zero, and rest of the probability mass is distributed as $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where U_1 and $\{E_i\}$ are as in Theorem 4.

3.4.2 $n = k^g - 1$ for some $g \geq 2$

For $z_i, w_i \in \mathbb{R}, i = 1, 2, .., g$, and with $\{N_i\}$ i.i.d. $N(0,1)$, define

$$\mathscr{H}_g(\omega_i, z_i, w_i, i = 1, \ldots, g) = \mathbb{P}\big(B(\omega_1, \omega_2, .., \omega_g)(N_1, ..., N_{2g})' \leq (z_i, w_i, i = 1, 2, .., g)'\big).$$

Lemma 5. *(i) $\mathscr{H}_g$ is a bounded continuous in $(\omega_1, \ldots, \omega_g)$ for fixed $\{z_i, w_i, i = 1, \ldots, g\}$.*
(ii) $\mathscr{F}_g$ defined below is a proper distribution function.

$$\mathscr{F}_g(z_i, w_i, i = 1, \ldots, g) = \int_0^1 \cdots \int_0^1 \mathscr{H}_g(2\pi t_i, z_i, w_i, i = 1, \ldots, g) \prod dt_i. \qquad (3.10)$$

(iii) If Leb$(C_0) = 0$ *then $\mathscr{F}_g$ is continuous everywhere and may be expressed as*

$$\begin{aligned} &\mathscr{F}_g(z_i, w_i, i = 1, .., g) \\ = \;& \int \cdots \int \mathbb{I}_{\{\mathbf{t} \leq (z_k, w_k, k=1,..,g)\}} \Big[\int_0^1 \cdots \int_0^1 \frac{\mathbb{I}_{\{\prod f(2\pi u_i) \neq 0\}}}{(2\pi)^g \prod_{i=1}^{g} [\pi f(2\pi u_i)]} \prod_{i=1}^{g} e^{-\frac{1}{2} \frac{t_{2i-1}^2 + t_{2i}^2}{\pi f(2\pi u_i)}} \prod du_i \Big] d\mathbf{t}. \end{aligned}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_{2g-1}, t_{2g})$ and $d\mathbf{t} = \prod dt_i$. Further $\mathscr{F}_g$ is multivariate (with independent components) if and only if f is constant almost everywhere (Lebesgue).
(iv) If Leb$(C_0) \neq 0$ *then $\mathscr{F}_g$ is discontinuous only on $D_g = \{(z_i, w_i, i = 1, \ldots, g) : \prod_{i=1}^{g} z_i w_i = 0\}$.*

The proof of lemma is omitted.

Theorem 5. *Suppose Assumptions **B** and **C** hold. Suppose $n = k^g - 1$ for some $g \geq 2$. Then as $n \to \infty$, $F_{n^{-1/2} A_{k,n}}$ converges in L_2 to the LSD $(\prod_{i=1}^{g} G_i)^{1/g}$ where $(\mathscr{R}(G_i), \mathscr{I}(G_i);\ i = 1, 2, \ldots g)$ has the distribution $\mathscr{F}_g$ given in (3.10).*

Remark 6. *If $\{x_i\}$ are i.i.d, with finite $(2+\delta)$ moment, then $f(\omega) \equiv 1/2\pi$ and the LSD simplifies to $U_2(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. $Exp(1)$, U_2 is uniformly distributed over the unit circle independent of $\{E_i\}$. This agrees with Theorem* 4 *of Bose, Mitra and Sen [2008].*

3.5 Simulations

To demonstrate the limits we did some modest simulations with MA(1) and MA(2) processes. We performed numerical integration to obtain the LSD. In case of k-circulant ($n = k^2 + 1$), we have plotted the density of F_2 defined in (3.9).

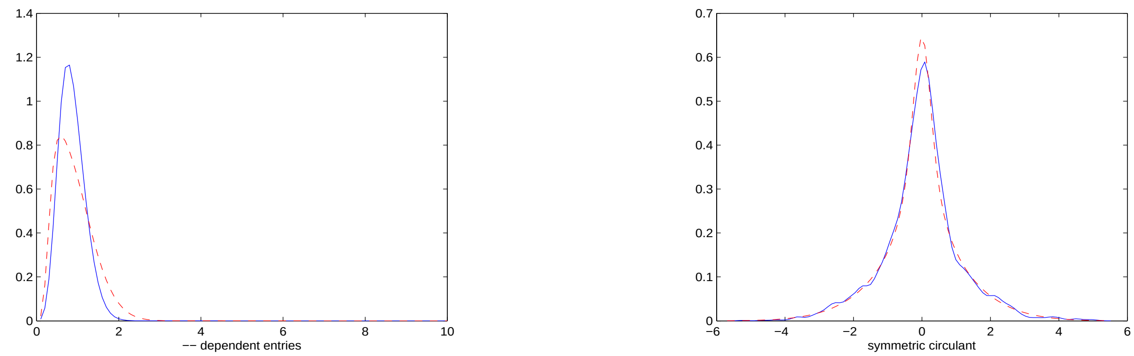

Figure 1: (i) (left) dashed line represents the density of F_2 when $f(\omega) = \frac{1}{2\pi}(1.25 + \cos x)$ and the continuous line represents the same with $f \equiv \frac{1}{2\pi}$. (ii) (right) dashed line represents the LSD of symmetric circulant matrix with entries $x_t = 0.3\epsilon_t + \epsilon_{t+1} + 0.5\epsilon_{t+2}$ where $\{\epsilon_i\}$ i.i.d. $N(0,1)$ and the continuous line represents the kernel density estimate of the ESD of the same matrix of order 5000×5000 and same $\{x_t\}$.

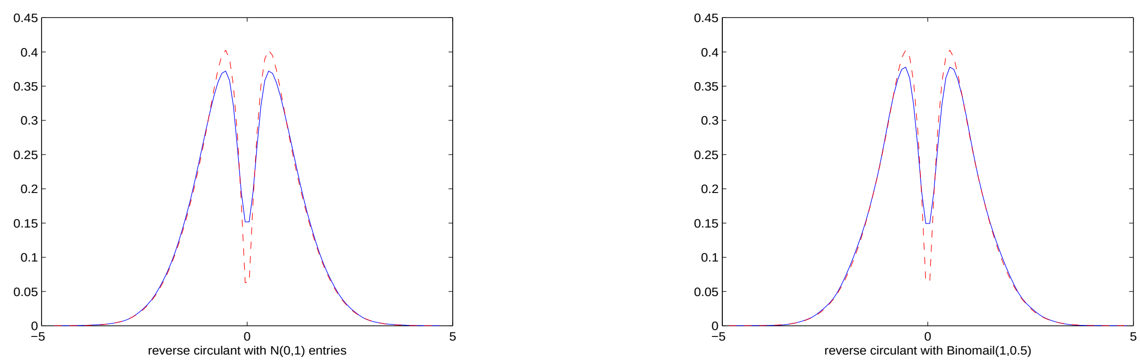

Figure 2: (i) (left) dashed line represents the LSD of the reverse circulant matrix with entries $x_t = 0.3\epsilon_t + \epsilon_{t+1} + 0.5\epsilon_{t+2}$ where $\{\epsilon_i\}$ i.i.d. $N(0,1)$. The continuous line represents the kernel density estimate of ESD of the same matrix of order 5000×5000 with same $\{x_t\}$. (ii) same graphs with centered and scaled Bernoulli$(1, 0.5)$.

4 Proofs of main results

Throughout c and C will denote generic constants depending only on d. We use the notation $a_n \sim b_n$ if $a_n - b_n \to 0$ and $a_n \approx b_n$ if $\frac{a_n}{b_n} \to 1$. As pointed out earlier, to prove that F_n converges to F (say) in L_2, it is enough to show that

$$\mathbb{E}[F_n(t)] \to F(t) \text{ and } V[F_n(t)] \to 0 \tag{4.1}$$

at all continuity points t of F. This is what we shall show in every case.
If the eigenvalues have the decomposition $\lambda_k = \eta_k + y_k$ for $1 \le k \le n$, where $y_k \to 0$ in probability then $\{\lambda_k\}$ and $\{\eta_k\}$ have similar behavior. We make this precise in the following lemma.

Lemma 6. *Suppose $\{\lambda_{n,k}\}_{1\le k\le n}$ is a triangular sequence of $\mathbb{R}^d$-valued random variables such that $\lambda_{n,k} = \eta_{n,k} + y_{n,k}$ for $1 \le k \le n$. Assume the following holds:*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\eta_{n,k} \le \tilde{x}) = F(\tilde{x})$*, for* $\tilde{x} \in \mathbb{R}^d$,

(ii) $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k,l=1}^{n} \mathbb{P}(\eta_{n,k} \le \tilde{x}, \eta_{n,l} \le \tilde{y}) = F(\tilde{x})F(\tilde{y})$*, for* $\tilde{x}, \tilde{y} \in \mathbb{R}^d$

(iii) For any $\epsilon > 0$, $\max_{1\le k\le n} \mathbb{P}(|y_{n,k}| > \epsilon) \to 0$ *as* $n \to \infty$.

Then,

1. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\lambda_{n,k} \le \tilde{x}) = F(\tilde{x})$.
2. $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k,l=1}^{n} \mathbb{P}(\lambda_{n,k} \le \tilde{x}, \lambda_{n,l} \le \tilde{y}) = F(\tilde{x})F(\tilde{y})$.

Proof. We define new random variables Λ_n with $\mathrm{P}(\Lambda_n = \lambda_{n,k}) = 1/n$ for $k = 1, \dots, n$. Then

$$\mathrm{P}(\Lambda_n \le \tilde{x}) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\lambda_{n,k} \le \tilde{x}).$$

Similarly define E_n with $\mathrm{P}(E_n = \eta_{n,k}) = 1/n$ for $1 \le k \le n$ and Y_n with $\mathrm{P}(Y_n = y_{n,k}) = 1/n$ for $1 \le k \le n$. Now observe that $\Lambda_n = E_n + Y_n$ and for any $\epsilon > 0$,

$$\mathrm{P}(|Y_n| > \epsilon) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(|y_{n,k}| > \epsilon) \to 0, \text{ as } n \to \infty$$

by assumption (iii). Therefore Λ_n and E_n have the same limiting distribution. Now as $n \to \infty$,

$$\mathrm{P}(E_n \le \tilde{x}) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\eta_{n,k} \le \tilde{x}) \to F(\tilde{x}). \text{ (by assumption } (i))$$

Therefore as $n \to \infty$,

$$\frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(\lambda_{n,k} \le \tilde{x}) = \mathrm{P}(\Lambda_n \le \tilde{x}) \to F(\tilde{x})$$

and this is conclusion (i). To prove (ii) we use similar type of argument. Here we define new random variables $\tilde{\Lambda}_n$ with $\mathrm{P}(\tilde{\Lambda}_n = (\lambda_{n,k}, \lambda_{n,l})) = 1/n^2$ for $1 \le k, l \le n$. Similarly define $\tilde{E}_n$ and $\tilde{Y}_n$. Again $\tilde{\Lambda}_n = \tilde{E}_n + \tilde{Y}_n$ and

$$\mathrm{P}(\|Y_n\| > \epsilon) = \frac{1}{n^2}\sum_{k,l=1}^{n} \mathbb{P}(\|(y_{n,k}, y_{n,l})\| > \epsilon) \to 0, \text{as } n \to \infty.$$

So $\tilde{\Lambda}_n$ and $\tilde{E}_n$ will have same limiting distribution and hence conclusion (ii) holds. □

We use normal approximation heavily in our proofs. Lemma 7 is a fairly standard consequence of normal approximation and follows easily from Bhattacharya and Ranga Rao [1976] (Corollary 18.1, page 181 and Corollary 18.3, page 184). We omit its proof. Part (i) will be used in Section 4.1– 4.4 and Part (ii) will be used in Section 4.4.

Lemma 7. *Let $X_1, \ldots, X_k$ be independent random vectors with values in $\mathbb{R}^d$, having zero means and an average positive-definite covariance matrix $V_k = k^{-1}\sum_{j=1}^{k} Cov(X_j)$. Let G_k denote the distribution of $k^{-1/2}T_k(X_1 + \ldots + X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \ge 1$. If for some $\delta > 0$, $\mathbb{E}\|X_j\|^{(2+\delta)} < \infty$, then there exists $C > 0$ (depending only on d), such that*

(i)

$$\sup_{B \in \mathscr{C}} |G_k(B) - \Phi_d(B)| \quad \le \quad Ck^{-\delta/2}[\lambda_{\min}(V_k)]^{-(2+\delta)}\rho_{2+\delta}$$

(ii) for any Borel set A,

$$|G_k(A) - \Phi_d(A)| \quad \le \quad Ck^{-\delta/2}[\lambda_{\min}(V_k)]^{-(2+\delta)}\rho_{2+\delta} + 2\sup_{y \in \mathbb{R}^d} \Phi_d((\partial A)^{\eta} - y)$$

where Φ_d is the standard d dimensional normal distribution function, $\mathscr{C}$ is the class of all Borel measurable convex subsets of $\mathbb{R}^d$, $\rho_{2+\delta} = k^{-1}\sum_{j=1}^{k}\mathbb{E}\|X_j\|^{(2+\delta)}$ and $\eta = C\rho_{2+\delta}n^{-\delta/2}$.

4.1 Proof of Theorem 1

The proof for circulant matrix mainly depends on Lemma 7 which helps to use normal approximation and, Lemma 8 given below which allows us to approximate the eigenvalues by appropriate partial sums of independent random variables. The latter follows easily from Fan and Yao [2003] (Theorem 2.14(ii), page 63). We have provided a proof in Appendix for completeness. For $k = 1, 2, \cdots, n$, define

$$\xi_{2k-1} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\epsilon_t \sin(\omega_k t).$$

Lemma 8. *Suppose Assumption **B** holds and $\{\epsilon_t\}$ are i.i.d random variables with mean 0, variance 1. For $k = 1, 2, \cdots, n$, write*

$$\lambda_{n,k} = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} x_l e^{i\omega_k l} = \psi(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k),$$

then we have $\max_{0 \le k < n}\mathbb{E}|Y_n(\omega_k)| \to 0$ as $n \to \infty$.

Proof of Theorem 1: We first assume Leb$(C_0) = 0$. Note that we may ignore the eigenvalue λ_n and also $\lambda_{n/2}$ whenever n is even since they contribute atmost $2/n$ to the ESD $F_n(x, y)$. So for $x, y \in \mathbb{R}$,

$$\mathbb{E}[F_n(x,y)] \quad \sim \quad n^{-1} \sum_{k=1, k\neq n/2}^{n-1} \mathbb{P}(b_k \leq x, c_k \leq y).$$

Define for $k = 1, 2, \cdots, n$,

$$\eta_k = (\xi_{2k-1}, \xi_{2k})', \quad Y_{1n}(\omega_k) = \mathscr{R}[Y_n(\omega_k)], \quad Y_{2n}(\omega_k) = \mathscr{I}[Y_n(\omega_k)],$$

where $Y_n(\omega_k)$ are same as defined in Lemma 8. Then $(b_k, c_k)' = B(\omega_k)\eta_k + (Y_{1n}(\omega_k), Y_{2n}(\omega_k))'$. Now in view of Lemma 6 and Lemma 8, to show $\mathbb{E}[F_n(x, y)] \to F_C(x, y)$ it is sufficient to show that

$$\frac{1}{n} \sum_{k=1, k\neq n/2}^{n-1} \mathbb{P}(B(\omega_k)\eta_k \leq (x,y)') \to F_C(x,y).$$

To show this, define for $1 \leq k \leq n-1$, (except for $k = n/2$) and $0 \leq l \leq n-1$,

$$X_{l,k} = \big(\sqrt{2}\epsilon_l \cos(\omega_k l), \quad \sqrt{2}\epsilon_l \sin(\omega_k l)\big)'.$$

Note that

$$\mathbb{E}(X_{l,k}) = 0 \tag{4.2}$$

$$n^{-1} \sum_{l=0}^{n-1} Cov(X_{l,k}) = I \tag{4.3}$$

$$\sup_n \sup_{1\leq k\leq n} [n^{-1} \sum_{l=0}^{n-1} \mathbb{E} \parallel X_{lk} \parallel^{(2+\delta)}] \leq C < \infty. \tag{4.4}$$

For $k \neq n/2$

$$\{B(\omega_k)\eta_k \leq (x,y)'\} = \{B(\omega_k)(n^{-1/2} \sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\}.$$

Since $\{(r,s) : B(\omega_k)(r,s)' \leq (\sqrt{2}x, \sqrt{2}y)'\}$ is a convex set in $\mathbb{R}^2$ and $\{X_{l,k},\ l = 0, 1, \ldots (n-1)\}$ satisfies (4.2)–(4.4), we can apply Part (i) of Lemma 7 for $k \neq n/2$ to get

$$\Big|\mathbb{P}\big(B(\omega_k)(n^{-1/2} \sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\big) - \mathbb{P}\big(B(\omega_k)(N_1,\ N_2)' \leq (\sqrt{2}x, \sqrt{2}y)'\big)\Big|$$
$$\leq Cn^{-\delta/2}[n^{-1} \sum_{l=0}^{n-1} \mathbb{E}\|X_{lk}\|^{(2+\delta)}] \leq Cn^{-\delta/2} \to 0, \text{ as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1, k\neq n/2}^{n-1} \mathbb{P}\big(B(\omega_k)\eta_k \leq (x,y)'\big) \quad = \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1, k\neq n/2}^{n-1} H_C(\frac{2\pi k}{n}, x, y) = \int_0^1 H_C(2\pi s, x, y) ds.$$

Hence

$$\mathbb{E}[F_n(x,y)] \sim n^{-1}\sum_{k=1,k\neq n/2}^{n-1}\mathbb{P}(b_k \le x, c_k \le y) \to \int_0^1 H_C(2\pi s, x, y)ds. \tag{4.5}$$

Now, to show $V[F_n(x,y)] \to 0$, it is enough to show that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} Cov(J_k, J_{k'}) = \frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n}\left[\mathbb{E}(J_k, J_{k'}) - \mathbb{E}(J_k)\mathbb{E}(J_{k'})\right] \to 0. \tag{4.6}$$

where for $1 \le k \le n$, J_k is the indicator that $\{b_k \le x, c_k \le y\}$. Now as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n}\mathbb{E}(J_k)\mathbb{E}(J_{k'}) = \big[\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(J_k)\big]^2 - \frac{1}{n^2}\sum_{k=1}^{n}\left[\mathbb{E}(J_k)\right]^2 \to \big[\int_0^1 H_C(2\pi s, x, y)ds\big]^2.$$

So to show (4.6), it is enough to show as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n}\mathbb{E}(J_k, J_{k'}) \to \big[\int_0^1 H_C(2\pi s, x, y)ds\big]^2.$$

Along the lines of the proof used to show (4.5) one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details. This completes the proof for the case $\text{Leb}(C_0) = 0$.
When $\text{Leb}(C_0) \neq 0$, we have to show (4.1) only on D_1^c (of Lemma 1). All the above steps in the proof will go through for all (x,y) in D_1^c. Hence if $\text{Leb}(C_0) \neq 0$, we have our required LSD. This completes the proof of Theorem 1. □

4.2 Proof of Theorem 2

For convenience, we prove the result for symmetric circulant matrix only for odd $n = 2m+1$. The even case follows by appropriate easy changes in the proof. The partial sum approximation now takes the following form. For the interested reader, we provide a proof in the Appendix.

Lemma 9. *Suppose Assumption **B** holds and $\{\epsilon_t\}$ are i.i.d random variables with mean* 0*, variance* 1*. For $n = 2m+1$ and $k = 1, 2, \cdots, m$, write*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{m} x_t \cos\frac{2\pi kt}{n} = \psi_1(e^{i\omega_k})\frac{1}{\sqrt{n}}\sum_{t=1}^{m}\epsilon_t \cos\frac{2\pi kt}{n} - \psi_2(e^{i\omega_k})\frac{1}{\sqrt{n}}\sum_{t=1}^{m}\sin\frac{2\pi kt}{n} + Y_{n,k},$$

where $\psi_1(e^{i\omega_k})$, $\psi_2(e^{i\omega_k})$ are same as defined in (2.3). Then we have $\max_{0\le k\le m}\mathbb{E}(Y_{n,k}) \to 0$ *as $n \to \infty$.*

Proof of Theorem 2: As before, we provide the detailed proof only when $\text{Leb}(C_0') = 0$. Note that we may ignore the eigenvalue λ_0 since it contributes $1/n$ to the ESD $F_n(\cdot)$. Further the term $\frac{x_0}{\sqrt{n}}$ can be ignored from the eigenvalue $\{\lambda_k\}$. So for $x \in \mathbb{R}$,

$$\mathbb{E}[F_n(x)] \sim \frac{2}{n}\sum_{k=1}^{m}\mathbb{P}(\frac{1}{\sqrt{n}}\lambda_k \le x) = \frac{2}{n}\sum_{k=1}^{m}\mathbb{P}(\frac{1}{\sqrt{n}}\sum_{t=1}^{m}2x_t\cos\frac{2\pi kt}{n} \le x).$$

Following the argument given in circulant case and using Lemma 6 and Lemma 9 it is sufficient to show that

$$\begin{aligned}
&\frac{2}{n}\sum_{k=1}^{m}\mathbb{P}\Big[\psi_1(e^{i\omega_k})\frac{2}{n}\sum_{t=1}^{m}\epsilon_t\cos\frac{2\pi kt}{n}-\psi_2(e^{i\omega_k})\frac{2}{\sqrt{n}}\sum_{t=1}^{m}\sin\frac{2\pi kt}{n}\le x\Big]\\
=\ &\frac{2}{n}\sum_{t=1}^{m}\mathbb{P}\Big\{m^{-1/2}\sum_{l=1}^{m}X_{l,k}\in C_k\Big\}\to F_S(x)
\end{aligned}$$

where

$$\begin{aligned}
X_{l,k} &= \big(2\sigma_n^{-1}\epsilon_l\cos\frac{2\pi kl}{n},\ 2\delta_n^{-1}\epsilon_l\sin\frac{2\pi kl}{n}\big),\ \sigma_n^2=2-1/m,\ \delta_n^2=2+1/m,\\
C_k &= \{(u,v):\sigma_n\psi_1(e^{i\omega_k})u+\delta_n\psi_2(e^{i\omega_k})v\le\sqrt{n/m}x\}.
\end{aligned}$$

Note that

$$\mathbb{E}(X_{l,k})=0,\ \frac{1}{m}\sum_{l=1}^{m}Cov(X_{l,k})=V_k\ \text{ and }\ \sup_m\sup_{1\le k\le m}m^{-1}\sum_{l=1}^{m}\mathbb{E}\|X_{l,k}\|^{2+\delta}\le C<\infty \tag{4.7}$$

where

$$V_k=\begin{pmatrix}1 & -\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1}\\ -\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1} & 1\end{pmatrix}.$$

Let α_k be the minimum eigenvalue of V_k. Then $\alpha_k\ge\alpha_m$ for $1\le k\le m$ and

$$\alpha_m=1-\frac{1}{\sqrt{4m^2-1}}\tan\frac{m\pi}{2m+1}\approx1-\frac{2m+1}{m\pi}\approx1-\frac{2}{\pi}=\alpha,\ \text{say.}$$

Since $\{X_{l,k}\}$ satisfies (4.7) and C_k is a convex set in $\mathbb{R}^2$, we can apply Part (i) of Lemma 7 for $k=1,2,\cdots,m$ to get

$$\Big|\frac{2}{n}\sum_{k=1}^{m}\Big[\mathbb{P}\Big\{m^{-1/2}\sum_{l=1}^{m}X_{l,k}\in C_k\Big\}-\Phi_{0,V_k}(C_k)\Big]\Big|\le Cm^{-\delta/2}\frac{2}{n}\sum_{k=1}^{m}\alpha_k^{-3/2}\le Cm^{-\delta/2}\alpha^{-3/2}\to0.$$

where Φ_{0,V_k} is a bivariate normal distribution with mean zero and covariance matrix V_k. Note that for large m, $\sigma_n^2\approx2$ and $\delta_n^2\approx2$. Hence $C_k'=\{(u,v):\psi_1(e^{i\omega_k})u+\psi_2(e^{i\omega_k})v\le\sqrt{x}\}$ serves as a good approximation to C_k and we get

$$\frac{2}{n}\sum_{k=1}^{m}\Phi_{0,V_k}(C_k)\sim\frac{2}{n}\sum_{k=1}^{m}\Phi_{0,V_k}(C_k')=\frac{2}{n}\sum_{k=1}^{m}\mathbb{P}(\mu_kN(0,1)\le x),$$

where $\mu_k^2=\psi_1(e^{i\omega_k})^2+\psi_2(e^{i\omega_k})^2+2\psi_1(e^{i\omega_k})\psi_2(e^{i\omega_k})\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1}$. Define $\nu_k^2=\psi_1(e^{i\omega_k})^2+\psi_2(e^{i\omega_k})^2$. Now we show that

$$\lim_{n\to\infty}\Big|\frac{2}{n}\sum_{k=1}^{m}\big[\mathbb{P}(\mu_kN(0,1)\le x)-\mathbb{P}(\nu_kN(0,1)\le x)\big]\Big|=0. \tag{4.8}$$

Let $0 < p < 1$. Now as $n \to \infty$, using Assumption (3.6),

$$\begin{aligned}\frac{2}{n}\Big|\sum_{k=1}^{[mp]}\big[\mathbb{P}(\mu_k N(0,1)\le x)-\mathbb{P}(\nu_k N(0,1)\le x)\big]\Big| &= \frac{2}{n}\sum_{k=1}^{[mp]}\Big|\int_{x/\nu_k}^{x/\mu_k}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\Big| \\ &\le \frac{2|x|}{n}\sum_{k=1}^{[mp]}\Big|\frac{\mu_k^2-\nu_k^2}{\mu_k\nu_k(\mu_k+\nu_k)}\Big| \\ &\le \frac{2|x|\tan\frac{p\pi}{2}}{m^2}\sum_{k=1}^{[mp]}\frac{1}{\nu_k^3\alpha(1+\alpha)}\longrightarrow 0\end{aligned}$$

On the other hand, for every n,

$$\frac{2}{n}\Big|\sum_{[mp]+1}^{m}\big[\mathbb{P}(\mu_k N(0,1)\le x)-\mathbb{P}(\nu_k N(0,1)\le x)\big]\Big|\le 4(1-p).$$

Therefore, by first letting $n\to\infty$ and then letting $p\to 1$, (4.8) holds. Hence

$$\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{m}\mathbb{P}\big(\nu_k N(0,1)\le x\big)=\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{m}\mathbb{P}\big(\sqrt{2\pi f(2\pi k/n)}N(0,1)\le x\big)\longrightarrow 2\int_0^{1/2}H_S(2\pi s,x)ds.$$

Rest of the argument in the proof for symmetric circulant is same as in the proof of Theorem 1.

To prove the result for PT_n, we use Cauchy's interlacing inequality (see Bhatia, 1997, page 59):

Interlacing inequality: Suppose A is an $n\times n$ symmetric real matrix with eigenvalues $\lambda_n\ge\lambda_{n-1}\ge\ldots\ge\lambda_1$. Let B be the $(n-1)\times(n-1)$ principal submatrix of A with eigenvalues $\mu_{n-1}\ge\ldots\ge\mu_1$. Then

$$\lambda_n\ge\mu_{n-1}\ge\lambda_{n-1}\ge\mu_{n-2}\ge\ldots\ge\lambda_2\ge\mu_1\ge\lambda_1.$$

As a consequence

$$\|F_A-F_B\|_\infty\le\frac{1}{n}$$

where F_A denote the ESD of the matrix A and $\|f\|_\infty=\sup_x|f(x)|$. We have already observed that the $n\times n$ principal minor of PT_{n+1} is SC_n. The result for PT_n follows immediately. □

4.3 Proof of Theorem 3

For reverse circulant we need the following Lemma. Its proof is given in Fan and Yao [2003] (Theorem 2.14(ii), page 63).

Lemma 10. *Suppose Assumption **B** holds and $\{\epsilon_t\}$ are i.i.d random variables with mean* 0*, variance* 1*. For $k=1,2,\cdots,[\frac{n-1}{2}]$, write*

$$I_n(\omega_k)=L_n(\omega_k)+R_n(\omega_k),\ \textit{where}\ L_n(\omega_k)=2\pi f(\omega_k)(\xi_{2k-1}^2+\xi_{2k}^2)$$

and then we have as $n\to\infty$, $\max_{1\le k\le[\frac{n-1}{2}]}\mathbb{E}|R_n(\omega_k)|\to 0$.

Proof of Theorem 3: As earlier, we give the proof only for the case $\mathrm{Leb}(C_0') = 0$. From the structure of the eigenvalues, the LSD, if it exists, is going to be that of a symmetric distribution. So, it is enough to concentrate on the case $x > 0$. As before we may ignore the two eigenvalues λ_0 and $\lambda_{n/2}$. Hence for $x > 0$,

$$\mathbb{E}[F_n(x)] \sim 1/2 + n^{-1} \sum_{k=1}^{[\frac{n-1}{2}]} \mathbb{P}(I_n(\omega_k) \leq x^2). \tag{4.9}$$

Along the same lines as in the proof of Theorem 1, using Lemma 6 and Lemma 10 it is sufficient to show

$$\frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} \mathbb{P}\big(L_n(\omega_k) \leq x^2\big) \to H_R(2\pi t, x),$$

where $L_n(\omega_k)$ is same as in Lemma 10. Define for $k = 1, 2, \cdots, [\frac{n-1}{2}]$ and $l = 0, 1, 2, \cdots, n-1$,

$$X_{l,k} = \big(\sqrt{2}\epsilon_l \cos(l\omega_k), \ \ \sqrt{2}\epsilon_l \sin(l\omega_k)\big)', \ \ A_{kn} = \big\{(r_1, r_2) : \pi f(\omega_k)(r_1^2 + r_2^2) \leq x^2\big\}.$$

Note that $\{X_{l,k}\}$ satisfies (4.2)– (4.4) and $\big\{L_n(\omega_k) \leq x^2\big\} = \big\{n^{-1/2} \sum_{l=0}^{n-1} X_{l,k} \in A_{kn}\big\}$. Since A_{kn} is a convex set in $\mathbb{R}^2$, we can apply Part (i) of Lemma 7 to get, as $n \to \infty$

$$\frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |\mathbb{P}(L_n(\omega_k) \leq x^2) - \Phi_{0,I}(A_{kn})| \leq C n^{-\delta/2} \to 0.$$

But

$$\frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} \Phi_{0,I}(A_{kn}) \ = \ \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} H_R(\frac{2\pi k}{n}, x) \to \int_0^{1/2} H_R(2\pi t, x) dt.$$

Hence for $x \geq 0$,

$$\mathbb{E}[F_n(x)] \to \frac{1}{2} + \int_0^{1/2} H_R(2\pi t, x) dt = F_R(x).$$

Now rest of the argument in the proof is same as in the proof of Theorem 1. □

4.4 Proofs for Theorems 4 and 5

The proofs of the two theorems for k circulant matrices draw substantially from the method of Bose and Mitra [2002] and uses the eigenvalue description given earlier.

4.4.1 $n = k^g + 1$ for some fixed $g \geq 2$

Proof of Theorem 4: For simplicity we first prove the result when $g = 2$. Note that $gcd(k, n) = 1$ and hence in this case $n' = n$ in (2.8). Thus the index of each eigenvalue belongs to *exactly one* of the sets $\mathscr{P}_l$ in the eigenvalue partition of $\{0, 1, 2, ..., n-1\}$. Recall that $v_{k,n}$ is the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathscr{P}_l$ and $|\mathscr{P}_l| < g_1$. In view of Lemma 7 of Bose, Mitra and Sen [2008], we have $v_{k,n}/n \le 2/n \to 0$ and hence these eigenvalues do not contribute to the LSD. Hence it remains to consider only the eigenvalues corresponding to the sets $\mathscr{P}_l$ which have size *exactly equal* to g_1.

Note that $S(1) = \{1, k, n-1, n-k\}$ and hence $g_1 = 4$. Recall the quantities $n_j = |\mathscr{P}_j|$, $y_j = \prod_{t \in \mathscr{P}_l} \lambda_t$, where $\lambda_j = \sum_{l=0}^{n-1} x_l \nu^{jl}$, $0 \le j < n$ given in (2.5). Also, for every integer $t \ge 0$, $tk^2 = -t \mod n$, so that, λ_t and λ_{n-t} belong to the same partition block $S(t) = S(n-t)$. Thus each y_t is real. Let us define

$$I_n = \{l : |\mathscr{P}_l| = 4\}.$$

It is clear that $n/|I_n| \to 4$. Without any loss, let $I_n = \{1, 2, ..., |I_n|\}$.

Let $1, \omega, \omega^2, \omega^3$ be all the fourth roots of unity. Note that for every j, the eigenvalues of $A_{k,n}$ corresponding to the set $\mathscr{P}_j$ are: $y_j^{1/4}, y_j^{1/4}\omega, y_j^{1/4}\omega^2, y_j^{1/4}\omega^3$. Hence it suffices to consider only the modulus of eigenvalues $y_j^{1/4}$ as j varies: if these have an LSD F, say, then the LSD of the whole sequence will be (r, θ) in polar coordinates where r is distributed according to F and θ is distributed uniformly across all the fourth roots of unity and r and θ are independent. With this in mind and remembering the scaling $\sqrt{n}$, we consider for $x > 0$,

$$F_n(x) = |I_n|^{-1} \sum_{i=1}^{|I_n|} \mathbb{I}\left(\left[\frac{y_j}{n^2} \right]^{\frac{1}{4}} \le x \right).$$

Since the set of λ values corresponding to any $\mathscr{P}_j$ is closed under conjugation, there exists a set $\mathscr{A}_i \subset \mathscr{P}_i$ of size 2 such that

$$\mathscr{P}_i = \{x : x \in \mathscr{A}_i \text{ or } n - x \in \mathscr{A}_i\}.$$

Combining each λ_j with its conjugate, we may write y_j in the form,

$$y_j = \prod_{t \in \mathscr{A}_j} (nb_t^2 + nc_t^2)$$

where $\{b_t\}$ and $\{c_t\}$ are given in (2.4). Note that for $x > 0$,

$$\mathbb{E}[F_n(x)] = |I_n|^{-1} \sum_{j=1}^{|I_n|} \mathbb{P}\left(\frac{y_j}{n^2} \le x^4\right).$$

Now our aim is to show

$$|I_n|^{-1} \sum_{j=1}^{|I_n|} \mathbb{P}\left(\frac{y_j}{n^2} \le x^4\right) \to F_2(x).$$

We can write $\frac{y_j}{n^2} = L_{n,j} + R_{n,j}$ for $1 \le j \le |I_n|$, where

$$\begin{aligned} L_{n,j} &= 4\pi^2 f_j \frac{\overline{y}_j}{n^2}, \quad \overline{y}_j = \prod_{t \in \mathscr{A}_j} (n\xi_{2t-1}^2 + n\xi_{2t}^2), \quad f_j = \prod_{t \in \mathscr{A}_j} f(\omega_t), \quad 1 \le j \le |I_n|, \\ R_{n,j} &= L_n(\omega_{j_1}) R_n(\omega_{j_2}) + L_n(\omega_{j_2}) R_n(\omega_{j_1}) + R_n(\omega_{j_1}) R_n(\omega_{j_2}), \\ L_n(\omega_{j_k}) &= 2\pi f(\omega_{j_k})(\xi_{2j_k-1}^2 + \xi_{2j_k}^2), \quad k = 1, 2. \end{aligned}$$

Now using Lemma 10 it is easy to see that for any $\epsilon > 0$, $\max_{1\le j\le |I_n|} \mathbb{E}(|R_{n,j}| > \epsilon) \to 0$ as $n \to \infty$. So in view of Lemma 6 it is enough to show

$$|I_n|^{-1} \sum_{j=1}^{|I_n|} \mathbb{P}\big(L_{n,j} \le x^4\big) \to F_2(x).$$

To do this we will use normal approximation. Define

$$X_{l,j} = 2^{1/2}\left(\epsilon_l \cos\left(\frac{2\pi t l}{n}\right),\ \ \epsilon_l \sin\left(\frac{2\pi t l}{n}\right),\ \ t \in \mathscr{A}_j\right) \quad 0 \le l < n, 1 \le j \le |I_n|,$$

$$A_{n,j} = \Big\{(a_1, b_1, a_2, b_2) : \prod_{i=1}^{2}[2^{-1}(a_i^2 + b_i^2)] \le \frac{x^4}{4\pi^2 f_j}\Big\},\ \ 1 \le j \le |I_n|.$$

Note that $\{X_{l,j}\}$ satisfies (4.2)–(4.4) and $\{L_{n,j} \le x^4\} = \{n^{-1/2}\sum_{l=1}^{n-1} X_{l,j} \in A_{n,j}\}$. Now using Part (ii) of Lemma 7 and Lemma 4 of Bose, Mitra and Sen [2008], arguing as in the previous proofs,

$$|I_n|^{-1} \sum_{l=1}^{|I_n|} \Big|\mathbb{P}\Big(L_{n,j} \le x^4\Big) - \Phi_4(A_{n,j})\Big| \to 0.$$

Therefore

$$\mathbb{E}[F_n(x)] = \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}\Big(\frac{y_j}{n^2} \le x^4\Big) \sim \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}\Big(L_{n,j} \le x^4\Big) \sim \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Phi_4(A_{n,j}).$$

To identify the limit, recall the structure of the sets $S(x), \mathscr{P}_j, \mathscr{A}_j$ and their properties. Since $|I_n|/n \to 1/4$, $v_{k,n} \le 2$ and either $S(x) = S(u)$ or $S(x) \cap S(u) = \phi$, we have

$$\lim_{n\to\infty} \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Phi_4(A_{n,j}) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1, |\mathscr{A}_j|=2}^{n}\Phi_4(A_{n,j}) \tag{4.10}$$

Also for $n = k^2 + 1$ we can write $\{1, 2, \ldots, n-1\}$ as $\{ak + b;\ 0 \le a \le k-1,\ 1 \le b \le k\}$ and using the construction of $S(x)$ we have (except for at most two values of j)

$$\mathscr{A}_j = \{ak + b, bk - a\} \text{ for } j = ak + b;\ 0 \le a \le k-1,\ 1 \le b \le k.$$

Recall that for fixed x, $H_2(\omega, \omega', x)$ is uniformly continuous on $[0, 2\pi] \times [0, 2\pi]$, . Therefore given any positive number ρ we can choose N large enough such that for all $n = k^2 + 1 > N$,

$$\sup_{0\le a\le k-1,\ 1\le b\le k}\Big|H_2\Big(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk-a)}{n}, x\Big) - H_2\Big(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, x\Big)\Big| < \rho. \tag{4.11}$$

Finally using (4.10), (4.11) we have

$$\lim_{n\to\infty}\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Phi_4(A_{n,j}) \quad = \quad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\Phi_4(A_{n,j})$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} G_2\big(\frac{x^4}{4\pi^2 f_j}\big) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{b=1}^{[\sqrt{n}]} \sum_{a=0}^{[\sqrt{n}]} H_2\Big(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk-a)}{n}, x\Big) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{b=1}^{[\sqrt{n}]} \sum_{a=0}^{[\sqrt{n}]} H_2\Big(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, x\Big) \\
&= \int_0^1 \int_0^1 H_2(2\pi s, 2\pi t, x) ds\ dt = F_2(x).
\end{aligned}
$$

To show that $V[F_n(x)] \to 0$, since the variables involved are all bounded, it is enough to show that

$$
n^{-2} \sum_{j \neq j'} Cov\left(\mathbb{I}(\frac{y_j}{n^2} \leq x^4),\ \mathbb{I}(\frac{y_{j'}}{n^2} \leq x^4)\right) \to 0.
$$

Along the lines of the proof used to show $\mathbb{E}[F_n(x)] \to F_2(x)$, one may now extend the vectors with 4 coordinates defined above to ones with 8 coordinates and proceed exactly as above to verify this. We omit the routine details. This proves the Theorem when $g = 2$. The above argument can be extended to cover the general ($g > 2$) case. We highlight only a few of the technicalities and omit the other details. We now need the following lemma.

Lemma 11. *Given any $\epsilon, \eta > 0$ there exist an $N \in \mathbb{N}$ such that*

$$
\mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) < \eta \text{ for all } n \geq N,
$$

where $L_n(\omega_j)$, $R_n(\omega_j)$ are as defined in Lemma 10.

Proof. Note

$$
\begin{aligned}
\mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) \quad \leq \quad & \mathbb{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) \\
& + \mathbb{P}\big(\big|\prod_{i=2}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M\big),
\end{aligned}
$$

and iterating this argument,

$$
\begin{aligned}
\mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) \quad \leq \quad & \mathbb{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) + \sum_{i=2}^{s} \mathbb{P}\big(\big|L_n(\omega_{j_i})\big| \geq M\big) \\
& + \mathbb{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big).
\end{aligned}
$$

Also note that

$$
\mathbb{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) \quad \leq \quad \mathbb{P}\big(\big|\prod_{i=s+2}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) + \mathbb{P}\big(\big|R_n(\omega_{j_{s+1}})\big| > 1\big)
$$

$$\leq \mathbb{P}\left(\left|R_n(\omega_{j_g})\right| > 1/M^s\right) + \sum_{i=1}^{g-1} \mathbb{P}\left(\left|R_n(\omega_{j_i})\right| > 1\right)$$
$$\leq \left(M^s + g - s - 1\right) \max_{1\leq k\leq n} \mathbb{E}|R_n(\omega_k)|.$$

Combining all the above we get

$$\begin{aligned}
\mathbb{P}\left(\left|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\right| > \epsilon\right) &\leq \mathbb{P}\left(\left|L_n(\omega_{j_1})\right| \geq M\epsilon\right) + \sum_{i=2}^{s} \mathbb{P}\left(\left|L_n(\omega_{j_i})\right| \geq M\right) \\
&\quad + \left(M^s + g - s - 1\right) \max_{1\leq k\leq n} \mathbb{E}|R_n(\omega_k)| \\
&\leq \frac{1}{M}(s - 1 + 1/\epsilon) 4\pi \max_{\omega\in[0,2\pi]} f(\omega) \\
&\quad + \left(M^s + g - s - 1\right) \max_{1\leq k\leq n} \mathbb{E}|R_n(\omega_k)|.
\end{aligned}$$

First term in the right side can be made smaller than $\eta/2$ by choosing M large enough and since $\max_{1\leq k\leq n} \mathbb{E}|R_n(\omega_k)| \to 0$ as $n \to \infty$, we can choose $N \in \mathbb{N}$ such that the second term is less than $\eta/2$ for all $n \geq N$, proving the lemma. □

For general $g \geq 2$, as before, $n' = n$, the index of each eigenvalue belongs to one of the sets $\mathscr{P}_l$ in the eigenvalue partition of $\{0, 1, 2, ..., n-1\}$ and $\nu_{k,n}/n \to 0$. Hence it remains to consider only the eigenvalues corresponding to the sets $\mathscr{P}_l$ which have size exactly equal to g_1 and it follows from the argument in the proof of Lemma 3(i) of Bose, Mitra and Sen [2008] that $g_1 = 2g$. We can now proceed as in $g = 2$ case. First we show

$$\frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \mathbb{P}\left(\frac{y_j}{n^g} \leq x^{2g}\right) \to F_g(x). \tag{4.12}$$

Now write $\frac{y_j}{n^g}$ as follows

$$\frac{y_j}{n^g} = L_{n,j} + R_{n,j} \quad \text{for } 1 \leq j \leq |I_n|, \text{ where } L_{n,j} = \prod_{t\in\mathscr{A}_j} L_n(\omega_t) = (2\pi)^g f_j \frac{\overline{y}_j}{n^g}.$$

Using Lemma 11 it is easy show that for any $\epsilon > 0$, $\max_{1\leq j\leq|I_n|} \mathbb{P}(|R_{n,j}| > \epsilon) \to 0$ as $n \to \infty$. So, by Lemma 6, to show (4.12) it is sufficient to show that

$$\frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \mathbb{P}\left(L_{n,j} \leq x^{2g}\right) \to F_g(x).$$

For this we use normal approximation as we did in $g = 2$ case and define

$$\bar{A}_{n,j} = \left\{(a_i, b_i, i = 1, 2, .., g) : \prod_{i=1}^{g} [2^{-1}(a_i^2 + b_i^2)] \leq \frac{x^{2g}}{(2\pi)^g f_j}\right\}.$$

Now using Part (ii) of Lemma 7 and Lemma 4 of Bose, Mitra and Sen [2008], we have

$$\left| |I_n|^{-1} \sum_{l=1}^{|I_n|} \mathbb{P}\left(L_{n,j} \leq x^4\right) - |I_n|^{-1} \sum_{l=1}^{|I_n|} \Phi_4(\bar{A}_{n,j}) \right| \to 0.$$

Now note that for $n = k^g + 1$ we can write $\{1, 2, \ldots, n-1\}$ as $\{b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_{g-1} k + b_g;\ 0 \le b_i \le k-1, \text{ for } 1 \le i \le k-1;\ 1 \le b_g \le k\}$. So we can write the sets $\mathscr{A}_j$ explicitly using this decomposition of $\{1, 2, \ldots, n-1\}$ as done in $g = 2$ case, that is, $n = k^2 + 1$ case. For example if $g = 3$, $\mathscr{A}_j = \{b_1 k^2 + b_2 k + b_3,\ b_2 k^2 + b_3 k - b_1,\ b_3 k^2 - b_1 k - b_2\}$ for $j = b_1 k^2 + b_2 k + b_3$ (except for finitely many j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we conclude that the LSD is now $F_g(\cdot)$, proving Theorem 4 completely. □

4.4.2 $n = k^g - 1$ for some fixed $g \ge 2$

Proof of Theorem 5. First we assume $\text{Leb}(C_0) = 0$. Note that $\gcd(k, n) = 1$. Since $k^g = 1 + n = 1 \bmod n$, we have $g_1 | g$. If $g_1 < g$, then $g_1 \le g/\alpha$ where $\alpha = 2$ if g is even and $\alpha = 3$ if g is odd. In either case, it is easy to check that

$$k^{g_1} \le k^{g/\alpha} \le (1+n)^{1/\alpha} = o(n).$$

Hence, $g = g_1$. By Lemma 3(ii) of Bose, Mitra and Sen [2008] the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathscr{A}_l$ and $|\mathscr{A}_l| < g$ is asymptotically negligible.

Unlike the previous theorem, here the partition sets $\mathscr{A}_l$ are not necessarily self-conjugate. However, the number of indices l such that $\mathscr{A}_l$ is self-conjugate is asymptotically negligible compared to n. To show this, we need to bound the cardinality of the following set for $1 \le l < g$:

$$D_l = \{t \in \{1, 2, \ldots, n\} : t k^l = -t \mod n\} = \{t \in \{1, 2, \ldots, n\} : n | t(k^l + 1)\}.$$

Note that $t_0 = n / \gcd(n, k^l + 1)$ is the minimum element of D_l and every other element is a multiple of t_0. Thus

$$|D_l| \le \frac{n}{t_0} \le \gcd(n, k^l + 1).$$

Let us now estimate $\gcd(n, k^l + 1)$. For $l > [g/2]$,

$$\gcd(n, k^l + 1) \le \gcd(k^g - 1, k^l + 1) = \gcd\left(k^{g-l}(k^l + 1) - (k^{g-l} - 1), k^l + 1\right) \le k^{g-l},$$

which implies $\gcd(n, k^l + 1) \le k^{[g/2]}$ for all $1 \le l < g$. Therefore,

$$\frac{\gcd(n, k^l + 1)}{n} = \frac{k^{[g/2]}}{(k^g - 1)} \le \frac{2}{k^{[(g+1)/2]}} \le \frac{2}{((n)^{1/g})^{[(g+1)/2]}} = o(1).$$

So, we can ignore the partition sets $\mathscr{P}_j$ which are self-conjugate. For other $\mathscr{P}_j$,

$$y_j = \prod_{t \in \mathscr{P}_j} (\sqrt{n} b_t + i \sqrt{n} c_t)$$

will be complex.

Now for simplicity we will provide the detailed argument assuming that $g = 2$. Then, $n = k^2 - 1$ and we can write $\{0, 1, 2, \ldots, n\}$ as $\{ak + b;\ 0 \le a \le k-1,\ 0 \le b \le k-1\}$ and using the construction of $S(x)$ we have $\mathscr{P}_j = \{ak + b, bk + a\}$ and $|\mathscr{P}_j| = 2$ for $j = ak + b;\ 0 \le a \le k-1,\ 0 \le b \le k-1$ (except for finitely many j and hence such indices do not contribute to the LSD). Let us define

$$I_n = \{j : |\mathscr{P}_j| = 2\}.$$

It is clear that $n/|I_n| \to 2$. Without any loss, let $I_n = \{1, 2, \ldots, |I_n|\}$. Suppose $\mathscr{P}_j = \{j_1, j_2\}$. We first find the limiting distribution of the empirical distribution of $\frac{1}{\sqrt{n}}(\sqrt{n}b_{j_1}, \sqrt{n}c_{j_1}, \sqrt{n}b_{j_2}, \sqrt{n}c_{j_2})$ for those j for which $|\mathscr{P}_j| = 2$ and show the convergence in L_2. Let $F_n(x, y, z, w)$ be the ESD of $\{(b_{j_1}, c_{j_1}, b_{j_2}, c_{j_2})\}$, that is

$$F_n(z_1, w_1, z_2, w_2) = \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{I}\big(b_{j_k} \le z_k, c_{j_k} \le w_k,\ k = 1, 2\big).$$

We show that for $z_1, w_1, z_2, w_2 \in \mathbb{R}$,

$$\mathbb{E}[F_n(z_1, w_1, z_2, w_2)] \to \mathscr{F}_2(z_1, w_1, z_2, w_2) \text{ and } V[F_n(z_1, w_1, z_2, w_2)] \to 0. \tag{4.13}$$

Define for $j = 1, 2, \cdots, n$,

$$\eta_j = (\xi_{2j_1-1}, \xi_{2j_1}, \xi_{2j_2-1}, \xi_{2j_2})',$$

and let $Y_{1n}(\omega_j) = \mathscr{R}(Y_n(\omega_j)),\ Y_{2n}(\omega_j) = \mathscr{I}(Y_n(\omega_j))$, where $Y_n(\omega_j)$ is same as defined in Lemma 8. Define

$$Y_{n,j} = \big(Y_{1n}(\omega_{j_1}), Y_{2n}(\omega_{j_1}), Y_{1n}(\omega_{j_2}), Y_{2n}(\omega_{j_2})\big).$$

Then $(b_{j_1}, c_{j_1}, b_{j_2}, c_{j_2}) = B(\omega_{j_1}, \omega_{j_2})\eta_j + Y'_{n,j}$. Note that by Lemma 8, for any $\epsilon > 0$, $\max_{1\le j\le n}\mathbb{P}(\|Y_{n,j}\| > \epsilon) \to 0$ as $n \to \infty$. So in view of Lemma 6 to show $\mathbb{E}[F_n(z_1, w_1, z_2, w_2)] \to \mathscr{F}_2(z_1, w_1, z_2, w_2)$ it is enough to show that

$$\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)') \to \mathscr{F}_2(z_1, w_1, z_2, w_2).$$

For this we use normal approximation and define

$$X_{l,j} = 2^{1/2}\left(\epsilon_l \cos\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \cos\left(\frac{2\pi j_2 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_2 l}{n}\right)\right)',$$

and $N = \big(N_1, N_2, N_3, N_4\big)'$, where $\{N_i\}$ are i.i.d. $N(0,1)$. Note

$$\big\{B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)'\big\} = \big\{B(\omega_{j_1}, \omega_{j_2})(n^{-1/2}\sum_{l=0}^{n-1}X_{l,j}) \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\big\}.$$

Since $\big\{(r_1, r_2, r_3, r_4) : B(\omega_{j_1}, \omega_{j_2})(r_1, r_2, r_3, r_4)' \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\big\}$ is a convex set in $\mathbb{R}^4$ and $\{X_{l,j};\ l = 0, 1, \ldots, (n-1)\}$ satisfies (4.2)–(4.4), we can show using Part (i) of Lemma 7 that

$$\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Big|\mathbb{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)') - \mathbb{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)')\Big| \to 0,$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty}\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)')$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \mathbb{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathscr{H}_2(\omega_{j_1}, \omega_{j_2}, z_1, w_1, z_2, w_2) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{[\sqrt{n}]} \sum_{b=0}^{[\sqrt{n}]} \mathscr{H}_2(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk+a)}{n}, z_1, w_1, z_2, w_2) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{[\sqrt{n}]} \sum_{b=0}^{[\sqrt{n}]} \mathscr{H}_2(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, z_1, w_1, z_2, w_2) \\
&= \int_0^1 \int_0^1 \mathscr{H}_2(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\, dt.
\end{aligned}
$$

Similarly we can show $V[F_n(x)] \to 0$ as $n \to \infty$.
Hence the empirical distribution of y_j for those j for which $|\mathscr{P}_j| = 2$ converges to the distribution of $\prod_{i=1}^{2} G_i$ such that $(\mathscr{R}(G_i), \mathscr{I}(G_i);\ i = 1, 2)$ has distribution $\mathscr{F}_2$. Hence the LSD of $n^{-1/2}A_{k,n}$ is $\big(\prod_{i=1}^{2} G_i\big)^{1/2}$, proving the result when $g = 2$ and $\text{Leb}(C_0) = 0$.
When $\text{Leb}(C_0) \neq 0$, we have to show (4.13) only on D_2^c (of Lemma 5). All the above steps in the proof will go through for all $(z_i, w_i; i = 1, 2)$ in D_2^c. Hence if $\text{Leb}(C_0) \neq 0$, we have our required LSD. This proves the Theorem when $g = 2$.

For general $g > 2$, note that we can write $\{0, 1, 2, \ldots, n\}$ as $\{b_1k^{g-1} + b_2k^{g-2} + \cdots + b_{g-1}k + b_g;\ 0 \le b_i \le k-1, \text{ for } 1 \le i \le k\}$. So we can write the sets $\mathscr{A}_j$ explicitly using this decomposition of $\{0, 1, 2, \ldots, n\}$ as done in $n = k^2 - 1$ case. For example if $g = 3$, $\mathscr{A}_j = \{b_1k^2 + b_2k + b_3,\ b_2k^2 + b_3k + b_1,\ b_3k^2 + b_1k + b_2\}$ for $j = b_1k^2 + b_2k + b_3$ (except for finitely many j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we will have the LSD as $\big(\prod_{i=1}^{g} G_i\big)^{1/g}$ such that $(\mathscr{R}(G_i), \mathscr{I}(G_i);\ i = 1, 2, \ldots g)$ has distribution $\mathscr{F}_g$. □

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5 Appendix

Proof of Lemma 8.

$$
\begin{aligned}
\lambda_k &= \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t e^{i\omega_k t} \\
&= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \sum_{t=0}^{n-1} \epsilon_{t-j} e^{i\omega_k (t-j)}
\end{aligned}
$$

$$= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \left(\sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t} + U_{nj} \right)$$
$$= \psi(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k),$$

where

$$U_{nj} = \sum_{t=-j}^{n-1-j} \epsilon_t e^{i\omega_k t} - \sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t}, \quad Y_n(\omega_k) = n^{-1/2} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} U_{nj}.$$

Note that if $|j| < n$, U_{nj} is a sum of $2|j|$ independent random variables, whereas if $|j| \geq n$, U_{nj} is a sum of $2n$ independent random variables. Thus $\mathbb{E}|U_{nj}|^2 \leq 2\min(|j|, n)$. Therefore, for any fixed positive integer l and $n > l$,

$$\begin{aligned}
\mathbb{E}|Y_n(\omega_k)| &\leq \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} |a_j| (\mathbb{E}U_{nj}^2)^{1/2} \quad (\because \sum_{-\infty}^{\infty} |a_j| < \infty) \\
&\leq \sqrt{\frac{2}{n}} \sum_{j=-\infty}^{\infty} |a_j| \{\min(|j|, n)\}^{1/2} \\
&\leq \sqrt{2} \left(\frac{1}{\sqrt{n}} \sum_{|j| \leq l} |a_j| |j|^{1/2} + \sum_{|j| > l} |a_j| \right).
\end{aligned}$$

The right side of the above expression is independent of k and as $n \to \infty$, it can be made smaller than any given positive constant by choosing l large enough. Hence, $\max_{1 \leq k \leq n} \mathbb{E}|Y_n(\omega_k)| \to 0$. □

Proof of Lemma 9.

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{t=1}^{m} x_t \cos \frac{2\pi kt}{n} \\
= & \frac{1}{\sqrt{n}} \sum_{t=1}^{m} \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j} \Big[\cos \frac{2\pi k(t-j)}{n} \cos \frac{2\pi kj}{n} - \sin \frac{2\pi k(t-j)}{n} \sin \frac{2\pi kj}{n} \Big] \\
= & \frac{\psi_1(e^{i\omega_k})}{\sqrt{n}} \sum_{t=1}^{m} \epsilon_t \cos \frac{2\pi kt}{n} - \frac{\psi_2(e^{i\omega_k})}{\sqrt{n}} \sum_{t=1}^{m} \sin \frac{2\pi kt}{n} + Y_{n,k},
\end{aligned}$$

where

$$Y_{n,k} = \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j \Big[\cos \frac{2\pi kj}{n} U_{k,j} - \sin \frac{2\pi kj}{n} V_{k,j} \Big],$$

$$U_{k,j} = \sum_{t=1}^{m} \Big[\epsilon_{t-j} \cos \frac{2\pi k(t-j)}{n} - \epsilon_t \cos \frac{2\pi kt}{n} \Big], \; V_{k,j} = \sum_{t=1}^{m} \Big[\epsilon_{t-j} \sin \frac{2\pi k(t-j)}{n} - \epsilon_t \sin \frac{2\pi kt}{n} \Big].$$

Note that if $|j| < m$, $U_{k,j}, U'_{k,j}$ are sums of $2|j|$ independent random variables, whereas if $|j| \geq m$, $U_{k,j}, U'_{k,j}$ are sums of $2m$ independent random variables. Thus $\mathbb{E}|U_{k,j}|^2 \leq 2\min(|j|, m)$. Therefore, for any fixed positive integer l and $m > l$,

$$\mathbb{E}|Y_{n,k}| \leq \frac{1}{\sqrt{n}} \Big[\sum_{j=-\infty}^{\infty} |a_j| \mathbb{E}(U_{k,j}^2)^{1/2} + \sum_{j=-\infty}^{\infty} |a_j| (\mathbb{E}V_{k,j}^2)^{1/2} \Big] \quad (\because \sum_{-\infty}^{\infty} |a_j| < \infty)$$

$$\leq \frac{2\sqrt{2}}{\sqrt{n}} \sum_{j=-\infty}^{\infty} |a_j| \{\min(|j|, m)\}^{1/2}$$

$$\leq 2\sqrt{2}\Big(\frac{1}{\sqrt{n}} \sum_{|j|\leq l} |a_j||j|^{1/2} + \sum_{|j|>l} |a_j|\Big).$$

The right side of the above expression is independent of k and as $n \to \infty$, it can be made smaller than any given positive constant by choosing l large enough. Hence, $\max_{1\leq k\leq m} \mathbb{E}(Y_{n,k}) \to 0$. □

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