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# Amplitude equation for SPDEs with quadratic nonlinearities*

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**Abstract**

In this paper we rigorously derive stochastic amplitude equations for a rather general class of SPDEs with quadratic nonlinearities forced by small additive noise. Near a change of stability we use the natural separation of time-scales to show that the solution of the original SPDE is approximated by the solution of an amplitude equation, which describes the evolution of dominant modes. Our results significantly improve older results.
We focus on equations with quadratic nonlinearities and give applications to the one-dimensional Burgers' equation and a model from surface growth.

**Key words:** Amplitude equations, quadratic nonlinearities, separation of time-scales, SPDE.

**AMS 2000 Subject Classification:** Primary 60H15; Secondary: 60H10, 35R60, 35Q99, 35K35.

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## 1 Introduction

Stochastic partial differentail equations (SPDEs) with quadratic nonlinearities arise in various applications in physics. One example is the stochastic Burgers' equation in the study of closure models for hydrodynamic turbulence [6]. Other examples are the growth of rough amorphous surfaces [23; 19], and the Kuramoto-Sivashinsky model, which originally models a fire front, but it is also used for surface erosion [7; 17]. All these models fit in the abstract framework of this paper.

Consider the following SPDE in Hilbert space $\mathscr{H}$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$:

$$du = \left[ \mathscr{A}u + \varepsilon^2 \mathscr{L}u + B(u,u) \right] dt + \varepsilon^2 dW. \tag{1}$$

We consider (1) near a change of stability, where the term $\varepsilon^2\mathscr{L}u$ represents the distance from bifurcation, which scales in the order of the noise strength $\varepsilon^2$. The operator $\mathscr{A}$ is assumed to be self-adjoint and non-positive, and we call the kernel of $\mathscr{A}$ the dominant modes. We allow for noise given by a fairly general $Q$-Wiener process.

Near the bifurcation the equation exhibits two widely separated characteristic time-scales and it is desirable to obtain a simplified equation which governs the evolution of the dominant modes. This is well known on a formal level in many examples in physics (see e.g. [8]). Moreover, for deterministic PDEs on unbounded domains, this method [16; 22; 24; 12] successfully overcomes the gap of a lacking center manifold theory. This is also useful for SPDEs on bounded domains [4], where no center manifold theory is available.

Moreover, there are numerous variants of this method. However, most of these results are non-rigorous approximations using this type of formal multi-scale analysis. A noteable example is [9].

Another interesting question, which can be tackled with similar methods, is the stabilization effect due to degenerate noise. Here noise is transported via nonlinear interaction to the dominant modes. Examples are [20; 5; 14; 15; 13; 21].

The purpose of this paper is to derive rigorously an amplitude equation for a quite general class of SPDEs (cf. (1)) with quadratic nonlinearities. This work is based on [5], where degenerate noise in a different scaling was considered, and it improves significantly previously know results of [1], where in a similar situation much more regular noise was considered. A related result can be found in [2], where a simple multiplicative noise was considered, but again with much weaker results.

In this paper we focus on quadratic nonlinearities only. The case of cubic equations is much simpler, as one can rely on nonlinear stability. This case was already considered in [3], for instance.

As an application of our main approximation result of Theorem 17, we discuss the stochastic Burgers' equation and surface growth model. Both models are scalar, but systems of PDEs are also covered. To illustrate our results consider the Burgers' equation

$$\partial_t u = \left(\partial_x^2 + 1\right) u + \varepsilon^2 \nu u + u\partial_x u + \varepsilon^2 \partial_t W, \tag{2}$$

on $[0,\pi]$ subject to Dirichlet boundary conditions.

We show in our main result that near a change of stability on a time-scale of order $\varepsilon^{-2}$ the solution of (2) is of the type

$$u(t,x) = \varepsilon b(\varepsilon^2 t)\sin(x) + \mathscr{O}(\varepsilon^2),$$

where $b$ is the solution of the amplitude equation on the slow time-scale

$$\partial_T b(T) = \nu b(T) - \frac{1}{12} b^3(T) + \partial_T \beta(T),$$

and $\beta$ is a Wiener process with a suitable variance.

For the proofs we rely on a cut-off technique, as in general we cannot control moments of solution and exclude the possibility of a blow up. Therefore all estimates are established only with high probability and not in moments. To be more precise, we use a stopping time, in order to look only at solutions that are not too large. Then we can use moments for time uniformly up to the stopping time. Later we use the amplitude equation itself to verify that the stopping is not small, at least with high probability.

As the general strategy we first show that all non-dominant modes are given by an Ornstein-Uhlenbeck process and a quadratic term in the dominant modes. Then we rely on Itô -Formula and some averaging argument, in order to transform the equation for the dominant modes to an amplitude equation with an additional small remainder.

The rest of this paper is organized as follows. In Section 2 we state the assumptions that we make. In Section 3 we give a formal derivation of the amplitude equation and state the main results. In Section 4 we give the main results. Finally, in Section 5 we apply our theory to the stochastic Burgers' equation and the surface growth model.

## 2 Main Assumptions and Definitions

This section summarizes all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (1) we assume the following:

**Assumption 1.** ***(Linear Operator $\mathcal{A}$)*** *Suppose $\mathcal{A}$ is a self-adjoint and non-positive operator on $\mathcal{H}$ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_k \leq ....$ and $\lambda_k \geq Ck^m$ for all large $k$. the corresponding complete orthonormal system of eigenvectors is $\{e_k\}_{k=1}^{\infty}$ with $\mathcal{A} e_k = -\lambda_k e_k$.*

We use the notation $\mathcal{N} := \ker \mathcal{A}$, $\mathcal{S} = \mathcal{N}^{\perp}$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$, and $P_c$ for the projection $P_c : \mathcal{H} \to \mathcal{N}$. Define, $P_s := I - P_c$, and suppose that $P_c$ and $P_s$ commute with $\mathcal{A}$. Suppose that $\mathcal{N}$ has finite dimension $n$ with basis $\left(e_1, ...., e_n\right)$ .

**Definition 2.** *For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^{\alpha}$ as*

$$\mathcal{H}^{\alpha} = \left\{ \sum_{k=1}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 \, k^{2\alpha} < \infty \right\} \quad \text{with norm} \quad \left\| \sum_{k=1}^{\infty} \gamma_k e_k \right\|_{\alpha}^2 = \sum_{k=1}^{\infty} \gamma_k^2 \, k^{2\alpha},$$

*where $\left(e_k\right)_{k\in\mathbb{N}}$ is the complete orthonormal basis in $\mathcal{H}$ defined by Assumption 1. We define the operator $D^{\alpha}$ by $D^{\alpha} e_k = k^{\alpha} e_k$, so that $\|u\|_{\alpha} = \|D^{\alpha} u\|$.*

**Remark 3.** *The operator $\mathcal{A}$ given by Assumption 1 generates an analytic semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$ defined by*

$$e^{\mathcal{A} t} \left( \sum_{k=1}^{\infty} \gamma_k e_k \right) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k \quad \forall \; t \geq 0.$$

The analytic semigroup has the following well known property:

**Lemma 4.** *Under Assumption 1 there are constants $M > 0$ and $\omega > 0$ such that for all $t > 0$, $\beta \le \alpha$, and all $u \in \mathscr{H}^\beta$*

$$\left\| e^{t\mathscr{A}} P_s u \right\|_\alpha \le M t^{-\frac{\alpha-\beta}{m}} e^{-\omega t} \left\| P_s u \right\|_\beta \ . \tag{3}$$

**Assumption 5.** ***(Operator $\mathscr{L}$)*** *Fix $\alpha \in \mathbb{R}$ and let $\mathscr{L} : \mathscr{H}^\alpha \to \mathscr{H}^{\alpha-\beta}$ for some $\beta \in [0, m)$ be a continuous linear mapping that in general does not commute with $P_c$ and $P_s$.*

**Assumption 6.** ***(Bilinear Operator** $B$**)*** *With $\alpha$, $\beta$ from Assumption 5 let $B$ be a bounded bilinear mapping from $\mathscr{H}^\alpha \times \mathscr{H}^\alpha$ to $\mathscr{H}^{\alpha-\beta}$. suppose without loss of generality that B is symmetric, i.e. $B(u, v) = B(v, u)$, and satisfies $P_c B(u, u) = 0$ for $u \in \mathscr{N}$.*

**Remark 7.** *If $B$ is not symmetric we can use*

$$\tilde{B}(u, v) := \frac{1}{2} B(u, v) + \frac{1}{2} B(v, u).$$

*Denote for shorthand notation $B_s = P_s B$ and $B_c = P_c B$.*

For the nonlinearity appearing later in the amplitude equation we define the following.

**Definition 8.** *Define $\mathscr{F} : \mathscr{N} \to \mathscr{N}$, for $u \in \mathscr{N}$, as*

$$\mathscr{F}(u, u, u) := B_c(u, \mathscr{A}_s^{-1} B_s(u, u)). \tag{4}$$

*Assume without loss of generality that $\mathscr{F}$ is given by a symmetric map $\mathscr{F} : \mathscr{N}^3 \to \mathscr{N}$, where we define $\mathscr{F}(u) = \mathscr{F}(u, u, u)$ for short.*

By Assumption 6 the operator $\mathscr{F}$ is already trilinear, continuous and therefore bounded. One standard example being a cubic like $u^3$.

**Remark 9.** *In order to obtain a symmetric map $\mathscr{F}$, we can always use*

$$\mathscr{F}(u, v, w) := \tfrac{1}{3} B_c(u, \mathscr{A}_s^{-1} B_s(v, w)) + \tfrac{1}{3} B_c(w, \mathscr{A}_s^{-1} B_s(u, v)) + \tfrac{1}{3} B_c(v, \mathscr{A}_s^{-1} B_s(w, u)).$$

Moreover, we assume the following:

**Assumption 10.** ***(Stability)*** *Assume that the nonlinearity $\mathscr{F}$ satisfies*

$$\langle \mathscr{F}(u, u, w), w \rangle > 0 \quad \forall\ u,\ w \in \mathscr{N} - \{0\}.$$

**Remark 11.** *Using the fact that $\mathscr{N}$ is finite dimensional and $\mathscr{F}$ is trilinear and symmetric, we easily derive the existence of some $\delta > 0$ such that*

$$\langle u, \mathscr{F}(u) \rangle \ge \delta \, \|u\|^4 \quad \forall\ u \in \mathscr{N}\ , \tag{5}$$

*and*

$$\langle \mathscr{F}(u, u, w), w \rangle \ge \delta \|u\|^2 \|w\|^2 \quad \forall\ u,\ w \in \mathscr{N}. \tag{6}$$

For the noise we suppose:

**Assumption 12.** ***(Wiener Process** $W$**)** Let $W$ be a cylindrical Wiener process on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a bounded covariance operator $Q : \mathcal{H} \to \mathcal{H}$ defined by $Qf_k = \alpha_k^2 f_k$ where $(\alpha_k)_k$ is a bounded sequence of real numbers and $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$. Using the orthonormal basis $e_k$ from Assumption 1, we assume*

$$\sum_{l=n+1}^{\infty} l^{2\alpha} \lambda_l^{2\gamma-1} \|Q^{\frac{1}{2}} e_l\|^2 < \infty \text{ for some } \gamma \in (0, \tfrac{1}{2}) . \tag{7}$$

We note that $W(t)$ and $\varepsilon W(\varepsilon^{-2}t)$ are in law the same process due to scaling properties of the Wiener process.

Let us discuss two different representations of $W$. One with the basis $e_k$ and the other one with $f_k$. For $t \geq 0$, we can write $W(t)$ (cf. Da Prato and Zabczyk [10]) as

$$W(t) := \sum_{k=1}^{\infty} \alpha_k \beta_k(t) f_k = \sum_{l=1}^{\infty} ß_l(t) e_l, \tag{8}$$

where $(\beta_k)_k$ are independent, standard Brownian motions in $\mathbb{R}$. Furthermore, the $ß_l := \sum_{k=1}^{\infty} \alpha_k \langle f_k, e_l \rangle \beta_k$ are real valued Brownian motions, which are in general not independent.

Moreover, it follows easily from the definition of $P_c$, $P_s$ and $W(t)$ that

$$P_c W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) P_c f_k = \sum_{l=1}^{n} ß_l(t) e_l, \tag{9}$$

and

$$P_s W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) P_s f_k = \sum_{l=n+1}^{\infty} ß_l(t) e_l, \tag{10}$$

**Definition 13.** *The stochastic convolution of $e^{\mathcal{A}t}$ and $W(t)$ is defined by*

$$W_{\mathcal{A}}(t) = \int_0^t e^{(t-s)\mathcal{A}} dW(s) = \sum_{l=1}^{\infty} \int_0^t e^{-(t-s)\lambda_l} dß_l(s) e_l . \tag{11}$$

For our result we rely on a cut off argument. We consider only solutions that are not too large. To be more precise we introduce a stopping time, at which the solution is larger than order one. Later we will show that this time is large with high probability.

**Definition 14.** ***(Stopping Time)** For the $\mathcal{N} \times \mathcal{S}$-valued stochastic process $(a, \psi)$ defined later in (14) we define, for some small $0 < \kappa < \frac{1}{7}$ and some time $T_0 > 0$, the stopping time*

$$\tau^* := T_0 \wedge \inf\left\{ T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \left\|\psi(T)\right\|_\alpha > \varepsilon^{-3\kappa} \right\} . \tag{12}$$

**Definition 15.** *For a real-valued family of processes $\left\{X_\varepsilon(t)\right\}_{t\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant $C_p$ such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \left|X_\varepsilon(t)\right|^p \leq C_p f_\varepsilon^p . \tag{13}$$

*We use also the analogous notation for time-independent random variables.*

Finally note, that we use the letter $C$ for all constants that depend only on other constants like $T_0$, $\kappa$, or $\alpha$ and the data of the equation given by $B$, $Q$, $\mathcal{L}$, and $\mathcal{A}$.

## 3 Formal Derivation and the Main Result

Let us first discuss a formal derivation of the Amplitude equation corresponding to Equation (1). We split the solution $u$ into

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2 \psi(\varepsilon^2 t)\,, \tag{14}$$

with $a \in \mathscr{N}$ and $\psi \in \mathscr{S}$, and rescale to the slow time scale $T = \varepsilon^2 t$, in order to obtain for the dominant modes

$$da = \left[\mathscr{L}_c a + \varepsilon \mathscr{L}_c \psi + 2B_c(a,\psi) + \varepsilon B_c(\psi,\psi)\right] dT + d\widetilde{W}_c. \tag{15}$$

For the fast modes we derive

$$\begin{aligned} d\psi = [&\varepsilon^{-2}\mathscr{A}_s \psi + \varepsilon^{-1}\mathscr{L}_s a + \mathscr{L}_s \psi + \varepsilon^{-2} B_s(a,a) + 2\varepsilon^{-1} B_s(a,\psi) \\ &+ B_s(\psi,\psi)]dT + \varepsilon^{-1} d\widetilde{W}_s\,, \end{aligned} \tag{16}$$

where $\widetilde{W}(T) := \varepsilon W(\varepsilon^{-2}T)$ is a rescaled version of the Wiener process. Now we use (16) in order to remove $\psi$ from Equation (15).

From (16) we obtain in lowest order of $\varepsilon$ that

$$\mathscr{A}_s \psi \approx -B_s(a,a).$$

As $\mathscr{A}_s$ is invertible on $\mathscr{S}$, we derive

$$\psi \approx -\mathscr{A}_s^{-1} B_s(a,a), \tag{17}$$

which we substitute into (15). Neglecting all small terms in $\varepsilon$ yields

$$da \approx \left[\mathscr{L}_c a - 2\mathscr{F}(a)\right] dT + d\widetilde{W}_c\,.$$

Thus we consider solutions $b : [0, T_0] \to \mathscr{N}$ of

$$db = \left[\mathscr{L}_c b - 2\mathscr{F}(b)\right] dT + d\widetilde{W}_c. \tag{18}$$

This approximating equation is the amplitude equation that approximates the dynamics of the original SPDE. The main aim of this paper to show that the solution of (1) is

$$u(t) = \varepsilon b(\varepsilon^2 t) + \mathscr{O}(\varepsilon^{2-})\,.$$

**Remark 16.** *In order to obtain higher order corrections in $\mathscr{N}$, we can add the higher order term $\varepsilon^2\eta(\varepsilon^2 t)$, with $\eta \in \mathscr{N}$, to the ansatz in (14).*

*Unfortunately, in order to deal with terms like $B_c(a,\eta)$ we then need a slightly stronger condition on $B$ like $P_c B(u,v) = 0$ for $u, v \in \mathscr{N}$. In this case we obtain in lowest order*

$$d_T \eta = \mathscr{L}_c \eta + 2B_c(\eta,\psi) + \mathscr{L}_c \psi + 2B_c(\psi,\psi). \tag{19}$$

*Thus using (17), we derive formally*

$$d_T \eta = \mathscr{L}_c \eta - 2B_c(\eta, \mathscr{A}_s^{-1} B_s(a,a)) + \Gamma(a)\,,$$

*where $\Gamma$ is a polynomial in $a$ given by*

$$\Gamma(a) = -\mathscr{L}_c \mathscr{A}_s^{-1} B_s(a,a) + 2B_c(\mathscr{A}_s^{-1} B_s(a,a), \mathscr{A}_s^{-1} B_s(a,a)) .$$

*In our cut-off technique, using the stopping time $\tau^*$, we only assume that $\psi = \mathcal{O}(\varepsilon^{-3\kappa})$. Thus, we cannot bound the term $B_c(\eta, \psi)$ in a usefull way.*

*We can easily add a cut off-condition on $\eta$ in the definition of $\tau^*$, but this at first glance will not improve the bound, and we will not be able to show directly that the modified $\tau^*$ is large.*

*The solution is to replace $\psi$ in the term $B_c(\eta, \psi)$ in Equation (19) by applying Itô-formula to $B_c(\eta, \mathscr{A}_s^{-1}\psi)$, which is quite similar to removing $\psi$ from the term $B_c(a, \psi)$ in equation (15), which is discussed below. Moreover, we need a similar argument for $B_c(\psi, \psi)$. This is an explicit averaging argument, but it will result in many and lengthy terms to estimate.*

*For simplicity and shortness of presentation we refrain from this analysis.*

In the following, let us be more precise. Applying Itô's formula to $B_c(a, \mathscr{A}_s^{-1}\psi)$ we obtain the amplitude equation with remainder

$$a(T) = a(0) + \int_0^T \mathscr{L}_c a(\tau) d\tau - 2\int_0^T \mathscr{F}(a(\tau)) d\tau + \widetilde{W}_c(T) + R(T), \tag{20}$$

where the remainder $R$ is given by

$$\begin{aligned} R(T) = {} & \varepsilon^2 B_c(a(T), \mathscr{A}_s^{-1}\psi(T)) - 2\varepsilon^2 \int_0^T B_c(B_c(a(\tau), \psi(\tau)), \mathscr{A}_s^{-1}\psi(\tau)) d\tau \\ & - \varepsilon^3 \int_0^T B_c(B_c(\psi(\tau), \psi(\tau)), \mathscr{A}_s^{-1}\psi(\tau)) d\tau - \varepsilon^2 \int_0^T B_c(\mathscr{L}_c a, \mathscr{A}_s^{-1}\psi) d\tau \\ & - 2\varepsilon \int_0^T B_c(a(\tau), \mathscr{A}_s^{-1} B_s(a(\tau), \psi(\tau))) d\tau - \varepsilon^3 \int_0^T B_c(\mathscr{L}_c \psi, \mathscr{A}_s^{-1}\psi) d\tau \\ & - \varepsilon \int_0^T B_c(a, \mathscr{A}_s^{-1}\mathscr{L}_s a) d\tau - \varepsilon^2 \int_0^T B_c(a, \mathscr{A}_s^{-1}\mathscr{L}_s \psi) d\tau + \varepsilon \int_0^T \mathscr{L}_c \psi(\tau) d\tau \\ & - \varepsilon^2 \int_0^T B_c(a(\tau), \mathscr{A}_s^{-1} B_s(\psi(\tau), \psi(\tau))) d\tau + \varepsilon \int_0^T B_c(\psi(\tau), \psi(\tau)) d\tau \\ & - \varepsilon^2 \int_0^T B_c(d\widetilde{W}_c(\tau), \mathscr{A}_s^{-1}\psi(\tau)) - \varepsilon \int_0^T B_c(a(\tau), \mathscr{A}_s^{-1} d\widetilde{W}_s(\tau)). \end{aligned} \tag{21}$$

For our main aim we need to show that the remainder $R$ is of order $\varepsilon$. This involves carefull analysis of all terms using moments of uniform bounds up to the stopping time like $\mathbb{E}\sup_{[0,\tau^*]} \|R\|_\alpha^p$. Later, we need an explicit error estimate to actually remove $R$ from the equation. Finally, we use the nonlinear stability of the amplitude equation to show that $\tau^* = T_0$ with high probability.

To be more precise, the main result is:

**Theorem 17.** ***(Approximation)*** *Under Assumptions 1, 5, 6 and 12, let $u$ be a solution of (1) defined in (14) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ where $a(0)$ and $\psi(0)$ are of order one. Suppose* *that $b$ is a solution of the amplitude equation (18). Then for all $p > 1$ and $T_0 > 0$ there exists $C > 0$ such that*

$$\mathbb{P}\Big( \sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-7\kappa}\Big) \le C\varepsilon^p \; . \tag{22}$$

**Remark 18.** *Let us finally remark without proof, that the scaling assumption on the initial conditions is not very restrictive. Using linear stability the following is easy to show: If $u(0) = \mathcal{O}(\varepsilon)$, then after some time $t_\varepsilon = \mathcal{O}(\ln(1/\varepsilon))$ the following attractively result holds true*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon + \varepsilon^2 \psi_\varepsilon \quad \textit{with } a_\varepsilon, \psi_\varepsilon = \mathcal{O}(1)\; .$$

# 4 Proof of the Main result

As a first step of the approximation result, we show that in (14) the modes $\psi \in \mathcal{S}$ are essentially an OU-process plus a quadratic term in the modes $a \in \mathcal{N}$. Later we will use this to replace the $\psi$ in (15). After this, we will proceed to show that $\psi$ is with high probability not too large.

**Lemma 19.** *Under Assumption 1, 5, 6 and 12 let $z(T)$, $T > 0$ be the $\mathcal{S}$-valued process solving the SDE*

$$dz = \varepsilon^{-2}\mathcal{A}_s z dT + \varepsilon^{-1} d\tilde{W}_s, \quad z(0) = \psi(0). \tag{23}$$

*Then for $\varepsilon \in (0,1)$ and $T \le \tau^*$*

$$\left\| \psi(T) - z(T) - \varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} B_s(a(\tau), a(\tau)) d\tau \right\|_\alpha \le C\varepsilon^{1-5\kappa}. \tag{24}$$

*Proof.* The mild formulation of (16) is

$$\psi(T) = z(T) + \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} \left[ \mathcal{L}_s\psi + \varepsilon^{-1}\mathcal{L}_s a + \varepsilon^{-2} B_s(a+\varepsilon\psi)\right] d\tau.$$

Thus we derive

$$\begin{aligned}
&\left\| \psi(T) - z(T) - \varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} B_s(a,a) d\tau \right\|_\alpha \\
\le\;& \left\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} \mathcal{L}_s\psi(\tau) d\tau \right\|_\alpha + \varepsilon^{-1}\left\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} \mathcal{L}_s a(\tau) d\tau \right\|_\alpha \\
& +2\varepsilon^{-1}\left\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} B_s(a(\tau),\psi(\tau)) d\tau \right\|_\alpha \\
& +\left\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} B_s(\psi(\tau),\psi(\tau)) d\tau \right\|_\alpha \\
=\;& : I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We now bound all four terms separately. Using Lemma 4 with $0 \le \beta < m$ we obtain for the first term for all $T \le \tau^*$

$$I_1 = \Big\| \int_0^T e^{\varepsilon^{-2}\mathscr{A}_s(T-\tau)} \mathscr{L}_s \psi(\tau) d\tau \Big\|_\alpha$$
$$\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|\psi(\tau)\right\|_\alpha d\tau$$
$$\le C\varepsilon^{2-3\kappa},$$

where we used the definition of $\tau^*$ and Assumption 5. Analogously, for the second term, we obtain for all $T \le \tau^*$

$$I_2 \le C\varepsilon^{\frac{2\beta}{m}-1} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|\mathscr{L}_s a(\tau)\right\|_{\alpha-\beta} d\tau \le C\varepsilon^{1-\kappa}.$$

For the third term, we obtain

$$I_3 \le C\varepsilon^{\frac{2\beta}{m}-1} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta} d\tau$$
$$\le C\varepsilon^{\frac{2\beta}{m}-1} \sup_{\tau\in[0,\tau^*]} \|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta} \cdot \int_0^T e^{-\varepsilon^{-2}\omega\tau} \tau^{-\frac{\beta}{m}} d\tau.$$

Using Assumption 6 yields for $T \le \tau^*$,

$$I_3 \le C\varepsilon \sup_{\tau\in[0,\tau^*]} \{\|a(\tau)\|_\alpha \|\psi(\tau)\|_\alpha\} \cdot \int_0^{\varepsilon^{-2}\omega T} e^{-\eta} \eta^{-\frac{\beta}{m}} d\eta \le C\varepsilon^{1-4\kappa}.$$

Analogously, we derive for the fourth term

$$I_4 \le \varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|B_s(\psi(\tau),\psi(\tau))\right\|_{\alpha-\beta} d\tau$$
$$\le C\varepsilon^{\frac{2\beta}{m}} \sup_{\tau\in[0,\tau^*]} \|B_s(\psi(\tau),\psi(\tau))\|_{\alpha-\beta} \cdot \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} d\tau$$
$$\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \|\psi(\tau)\|_\alpha^2 \cdot \int_0^{\varepsilon^{-2}\omega T} e^{-\eta} \eta^{-\frac{\beta}{m}} d\eta \le C\varepsilon^{2-6\kappa}.$$

Combining all four results yields (24). □

In the following we will show that $\psi \ll \mathscr{O}(\varepsilon^{-3\kappa})$. First, the next Lemma provides bounds for the stochastic convolution based on the well know factorization method. This also implies bounds for the process $z$ defined in (23).

**Lemma 20.** *Under Assumption 1 and 12, let $\|z(0)\|_\alpha = \mathscr{O}(1)$. Now for every $\kappa_0 > 0$, $p > 1$ and $T > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big( \sup_{t\in[0,T]} \|z(t)\|_\alpha^{2p} \Big) \le C\varepsilon^{-\kappa_0}. \tag{25}$$

*Proof.* The mild solution of equation (23) is given by

$$z(t) = e^{\varepsilon^{-2}\mathscr{A}_s t} z(0) + \varepsilon^{-1} \widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t). \tag{26}$$

The main part in the proof of a bound on $z(t)$ is the bound on $\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}$. For this, we use the celebrated factorization method introduced in [11]. Here, for $\gamma$ from Assumption 12

$$\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t) = C_\gamma \int_0^t e^{\varepsilon^{-2}\mathscr{A}_s(t-s)}(t-s)^{\gamma-1} y(s) ds, \tag{27}$$

with $y(s) := \int_0^s e^{\varepsilon^{-2}\mathscr{A}_s(s-\sigma)}(s-\sigma)^{-\gamma} d\widetilde{W}_s(\sigma)$. Hence, by Gaussianity

$$\mathbb{E} \left\| y(s) \right\|_\alpha^{2p} \leq C_p \left( \mathbb{E} \left\| y(s) \right\|_\alpha^2 \right)^p$$

Using the series expansion (cf. (10)) yields

$$y(s) = \sum_{l=n+1}^{\infty} \int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma} d\widetilde{ß}_l(\sigma) e_l.$$

From Itô-Isometry

$$\begin{aligned}\mathbb{E} \left\| y(s) \right\|_\alpha^{2p} &\leq C_p \left( \sum_{l=n+1}^{\infty} l^{2\alpha} \mathbb{E} \left( \int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma} d\widetilde{ß}_l(\sigma) \right)^2 \right)^p \\ &= C_p \varepsilon^{2p-4p\gamma} \left( \sum_{l=n+1}^{\infty} l^{2\alpha} \left(\lambda_l\right)^{2\gamma-1} \left\| Q^{\frac{1}{2}} e_l \right\|^2 \int_0^{\frac{\varepsilon^2 s}{2\lambda_l}} e^{-\tau} \tau^{-2\gamma} d\tau \right)^p,\end{aligned}$$

where we used

$$(d\widetilde{ß}_l(\sigma))^2 = \sum_{k=1}^{\infty} \alpha_k^2 \langle f_k, e_l \rangle^2 d\sigma = \|Q^{\frac{1}{2}} e_l\|^2 d\sigma. \tag{28}$$

Integrating from 0 to $T$ we obtain

$$\mathbb{E} \int_0^T \left\| y(s) \right\|_\alpha^{2p} ds \leq \text{Const} \cdot \varepsilon^{2p-4\gamma p}. \tag{29}$$

Taking the $\mathscr{H}^\alpha$ norm in (27) yields

$$\|\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t)\|_\alpha^{2p} \leq C \Big( \int_0^t e^{(-\varepsilon^{-2}\omega)(t-s)}(t-s)^{\gamma-1} \|y(s)\|_\alpha ds \Big)^{2p}.$$

Hölder inequality with $\frac{1}{2p} + \frac{1}{2q} = 1$ for sufficiently large $p$ implies

$$\|\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t)\|_\alpha^{2p} \leq \text{Const} \cdot \varepsilon^{4p\gamma-2} \int_0^t \|y(s)\|_\alpha^{2p} ds.$$

Hence, using (29) we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t)\|_\alpha^{2p}\le C\varepsilon^{4p\gamma-2}\int_0^T\mathbb{E}\|y(s)\|_\alpha^{2p}ds\le C\varepsilon^{2p-2}.$$

For the bound on $z$ take the norm in equation (26) to obtain for sufficiently large $p$

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|z(t)\|_\alpha^{2p}&\le C\left[\mathbb{E}\sup_{t\in[0,T]}\|e^{\varepsilon^{-2}\mathscr{A}_s}z(0)\|_\alpha^{2p}+\varepsilon^{-2p}\mathbb{E}\sup_{t\in[0,T]}\|\widetilde{W}_{\varepsilon^{-2}\mathscr{A}_s}(t)\|_\alpha^{2p}\right]\\&\le C\mathbb{E}\sup_{t\in[0,T]}e^{-2p\varepsilon^{-2}\omega t}\|z(0)\|_\alpha^{2p}+C\cdot\varepsilon^{-2p}\cdot\varepsilon^{2p-2}\\&\le C\varepsilon^{-2}.\end{aligned}$$

Using Hölder inequality we derive for all $p>1$ and sufficiently large $q>\frac{2}{\kappa_0}$

$$\mathbb{E}\sup_{t\in[0,T]}\|z(t)\|_\alpha^{2p}\le\mathbb{E}\Big(\sup_{t\in[0,T]}\|z(t)\|_\alpha^{2pq}\Big)^{\frac{1}{q}}\le C\varepsilon^{-\kappa_0},$$

where the constant $C$ depends among other things on $T$, $p$, and $\kappa_0$. □

We now need the following simple estimate.

**Lemma 21.** *Under Assumption 1 and 6, using $\tau^*$ defined in Definition 14,*

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]}\left\|\int_0^T e^{\varepsilon^{-2}\mathscr{A}_s(T-\tau)}B_s(a(\tau),a(\tau))d\tau\right\|_\alpha^{2p}\right)\le C\varepsilon^{4p-4p\kappa},\tag{30}$$

*for all $\varepsilon\in(0,1)$.*

*Proof.* Using Lemma 4 and Assumption 6 we obtain for $T<\tau^*$

$$\begin{aligned}\left\|\int_0^T e^{\varepsilon^{-2}\mathscr{A}_s(T-\tau)}B_s(a,a)d\tau\right\|_\alpha&\le C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\|B_s(a,a)\|_{\alpha-\beta}d\tau\\&\le C\varepsilon^2\sup_{\tau\in[0,\tau^*]}\|a(\tau)\|_\alpha^2\cdot\int_0^{\varepsilon^{-2}\omega T}e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta\\&\le C\varepsilon^{2-2\kappa}.\end{aligned}$$

□

Now we can proceed to bound $\psi$. The following lemma states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-3\kappa}$, as asserted by the Definition 14 for $T\le\tau^*$. Here a key fact is that in the Definition of $\tau^*$ that $a=\mathscr{O}(\varepsilon^{-\kappa})$, while $\psi=\mathscr{O}(\varepsilon^{-3\kappa})$, but we already proved that $\psi$ is essentially a quadratic term in $a$.

**Lemma 22.** *Let the assumptions of Lemmas 19, 20, and 21 be true. Then for all $p\ge1$ there is a constant $C>0$ such that*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^{2p}\le C\varepsilon^{-4p\kappa}.\tag{31}$$

*Proof.* From (24), by triangle inequality and Lemma 19, we obtain

$$\mathbb{E}\sup_{[0,\tau^*]}\|\psi\|_\alpha^{2p} \le C\varepsilon^{2p-10p\kappa} + C\mathbb{E}\sup_{[0,\tau^*]}\|z\|_\alpha^{2p} + C\varepsilon^{-4p}\mathbb{E}\sup_{[0,\tau^*]}\left\|\int_0^T e^{\varepsilon^{-2}\mathscr{A}_s(T-\tau)}B_s(a,a)d\tau\right\|_\alpha^{2p}.$$

Using Lemma 20 and 21 we finish the proof. □

**Corollary 23.** *Under the assumptions of Lemma 22, there is for every every $p > 1$ a constant $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha < \varepsilon^{-3\kappa}\Big) \ge 1 - C\varepsilon^{2p\kappa}. \tag{32}$$

*Proof.* From Chebychev inequality

$$\mathbb{P}\Big(\sup_{[0,\tau^*]}\|\psi\|_\alpha < \varepsilon^{-3\kappa}\Big) \ge 1 - \varepsilon^{6\kappa p}\cdot\mathbb{E}\sup_{[0,\tau^*]}\|\psi\|_\alpha^{2p}.$$

We finish the proof by using (31). □

Now the next step is to bound the remainder $R$ defined in (21), and use it in order to show the approximation result later.

**Lemma 24.** *We assume that Assumptions 1, 5, 6, and 12 hold. Then for all $p > 1$ there exists a constant $C > 0$ such that*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\|R(T)\|_\alpha^p \le C\varepsilon^{p-6p\kappa}. \tag{33}$$

*Proof.* For the bound on $R$ we bound all terms in (21) separately. The estimates rely on Assumption 6 and the inequality $\|\psi\|_\gamma \le C\|\psi\|_{\gamma+\delta}$ for all $\gamma\in\mathbb{R}$ and $\delta \ge 0$. Moreover, we use that $B_c(a(\tau),\mathscr{A}_s^{-1}\psi(\tau)) \in \mathscr{N}$ (finite dimensional) and $\mathscr{A}_s^{-1}$ being a bounded linear operator on $\mathscr{S}\subset\mathscr{H}^\alpha$ to obtain for all times up to the stopping time $\tau^*$ that

$$\begin{aligned}\left\|\varepsilon^2 B_c(a,\mathscr{A}_s^{-1}\psi)\right\|_\alpha &\le C\varepsilon^2\left\|B_c(a,\mathscr{A}_s^{-1}\psi)\right\|_{\alpha-\beta} \le C\varepsilon^2\|a\|_\alpha\left\|\mathscr{A}_s^{-1}\psi\right\|_\alpha\\ &\le C\varepsilon^2\|a\|_\alpha\left\|\psi\right\|_\alpha.\end{aligned}$$

Using the definition of $\tau^*$, we obtain

$$\mathbb{E}\sup_{[0,\tau^*]}\|\varepsilon^2 B_c(a,\mathscr{A}_s^{-1}\psi)\|_\alpha^p \le C\varepsilon^{2p-4p\kappa}. \tag{34}$$

For the second term in (21) with $T\le\tau^*\le T_0$

$$\begin{aligned}\left\|2\varepsilon^2\int_0^T B_c(B_c(a,\psi),\mathscr{A}_s^{-1}\psi)d\tau\right\|_\alpha &\le C\varepsilon^2\int_0^T \|B_c(B_c(a,\psi),\mathscr{A}_s^{-1}\psi)\|_{\alpha-\beta}d\tau\\ &\le C\varepsilon^2 T\cdot\sup_{[0,\tau^*]}\|B_c(a,\psi)\|_\alpha\|\mathscr{A}_s^{-1}\psi\|_\alpha\\ &\le C\varepsilon^2 T\cdot\sup_{[0,\tau^*]}\|a\|_\alpha\|\psi\|_\alpha^2\\ &\le C\varepsilon^{2-7\kappa}.\end{aligned} \tag{35}$$

Analogously, for the third term in (21)

$$\left\| \varepsilon^3 \int_0^T B_c(B_c(\psi,\psi), \mathscr{A}_s^{-1}\psi) d\tau \right\|_\alpha \le C\varepsilon^3 \int_0^T \|B_c(B_c(\psi,\psi), \mathscr{A}_s^{-1}\psi)\|_{\alpha-\beta} d\tau$$
$$\le C\varepsilon^3 T \cdot \sup_{[0,\tau^*]} \left\|\psi\right\|_\alpha^3 \le C\varepsilon^{3-9\kappa}. \tag{36}$$

The 4th term in (21) is bounded by

$$\begin{aligned}\left\| \varepsilon^2 \int_0^T B_c(\mathscr{L}_c a, \mathscr{A}_s^{-1}\psi) d\tau \right\|_\alpha &\le C\varepsilon^2 \int_0^T \|B_c(\mathscr{L}_c a, \mathscr{A}_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^2 \cdot \sup_{[0,\tau^*]} \|\mathscr{L}_c a\|_\alpha \|\mathscr{A}_s^{-1}\psi\|_\alpha \\ &\le C\varepsilon^2 \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\psi\|_\alpha \\ &\le C\varepsilon^{2-4\kappa},\end{aligned} \tag{37}$$

where we used $\|\mathscr{L}_c a\|_\alpha \le C\|\mathscr{L}_c a\|_{\alpha-\beta}$, as $\mathscr{N}$ is finite dimensional.

For the 5th term in (21)

$$\begin{aligned}\left\| 2\varepsilon \int_0^T B_c(a, \mathscr{A}_s^{-1}B_s(a,\psi)) d\tau \right\|_\alpha &\le C\varepsilon \int_0^T \|B_c(a, \mathscr{A}_s^{-1}B_s(a,\psi))\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\mathscr{A}_s^{-1}B_s(a,\psi)\|_\alpha \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha^2 \|\psi\|_\alpha \\ &\le C\varepsilon^{1-5\kappa}.\end{aligned} \tag{38}$$

The 6th term in (21) is bounded by

$$\begin{aligned}\left\| \varepsilon^3 \int_0^T B_c(\mathscr{L}_c\psi, \mathscr{A}_s^{-1}\psi) d\tau \right\|_\alpha &\le C\varepsilon^3 \int_0^T \|B_c(\mathscr{L}_c\psi(\tau), \mathscr{A}_s^{-1}\psi(\tau))\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^3 \cdot \sup_{[0,\tau^*]} \|\mathscr{L}_c\psi\|_\alpha \|\mathscr{A}_s^{-1}\psi)\|_\alpha \\ &\le C\varepsilon^3 \cdot \sup_{[0,\tau^*]} \|\psi\|_\alpha^2 \\ &\le C\varepsilon^{3-6\kappa}.\end{aligned} \tag{39}$$

The 7th term in (21) is bounded by

$$\begin{aligned}\left\| \varepsilon \int_0^T B_c(a, \mathscr{A}_s^{-1}\mathscr{L}_s a) d\tau \right\|_\alpha &\le C\varepsilon \int_0^T \|B_c(a, \mathscr{A}_s^{-1}\mathscr{L}_s a)\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\mathscr{A}_s^{-1}\mathscr{L}_s a\|_\alpha \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\mathscr{L}_s a\|_{\alpha-m} \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha^2 \\ &\le C\varepsilon^{1-2\kappa}.\end{aligned} \tag{40}$$

The 8th term in (21) is completely analogous. We have

$$\Big\|\varepsilon^2\int_0^T B_c(a,\mathscr{A}_s^{-1}\mathscr{L}_s\psi)d\tau\Big\|_\alpha \le C\varepsilon^{2-4\kappa}. \tag{41}$$

Moreover for the 9th term in (21):

$$\Big\|\varepsilon\int_0^T B_c(\psi,\psi)d\tau\Big\|_\alpha \le C\varepsilon\int_0^T \|B_c(\psi,\psi)\|_{\alpha-\beta}d\tau \le C\varepsilon^{1-6\kappa}\,. \tag{42}$$

For the 10th term in (21)

$$\begin{aligned}\Big\|\varepsilon\int_0^T \mathscr{L}_c\psi d\tau\Big\|_\alpha &\le C\varepsilon\int_0^T \|\mathscr{L}_c\psi\|_\alpha d\tau \le C\varepsilon\int_0^T \|\mathscr{L}_c\psi\|_{\alpha-\beta}d\tau\\ &\le C\varepsilon\cdot\sup_{[0,\tau^*]}\|\psi(\tau)\|_\alpha \le C\varepsilon^{1-3\kappa}.\end{aligned} \tag{43}$$

The 11th term in (21) is bounded by

$$\begin{aligned}\Big\|\varepsilon^2\int_0^T B_c(a,\mathscr{A}_s^{-1}B_s(\psi,\psi))d\tau\Big\|_\alpha &\le C\varepsilon^2\int_0^T \|B_c(a,\mathscr{A}_s^{-1}B_s(\psi,\psi))\|_{\alpha-\beta}d\tau\\ &\le C\varepsilon^2\cdot\sup_{[0,\tau^*]}\|a\|_\alpha\|\mathscr{A}_s^{-1}B_s(\psi,\psi)\|_\alpha\\ &\le C\varepsilon^2\sup_{[0,\tau^*]}\|a\|_\alpha\|\psi\|_\alpha^2\\ &\le C\varepsilon^{2-7\kappa}.\end{aligned} \tag{44}$$

For the stochastic integral $\varepsilon^2\int_0^T B_c(d\widetilde{W}_c,\mathscr{A}_s^{-1}\psi)$ in (21) note that the covariance operator of $W_c$ is $Q_c = P_cQP_c$. Define

$$\$(\tau)u := B_c(u(\tau),\mathscr{A}_s^{-1}\psi(\tau)),$$

to obtain

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T B_c(d\widetilde{W}_c(\tau),\mathscr{A}_s^{-1}\psi(\tau))\Big\|_\alpha^p = \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \$(\tau)d\widetilde{W}_c(\tau)\Big\|_\alpha^p.$$

By Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [18]) we derive

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \$d\widetilde{W}_c\Big\|_\alpha^p &= \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T D^\alpha\$d\widetilde{W}_c\Big\|^p\\ &\le C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\|D^\alpha\$Q_c^{\frac12}\|_{HS}^2d\tau\Big)^{\frac p2}\\ &= C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\|D^\alpha\$Q_c^{\frac12}g_k\|^2d\tau\Big)^{\frac p2},\end{aligned}$$

where $(g_k)_{k\in\mathbb{N}}$ is any orthonormal basis in $\mathscr{H}$ and $D^\alpha$ was defined in Definition 2. The space $HS$ is the space of Hilbert-Schmidt operators on $\mathscr{H}$, equipped with the norm $\|\Psi\|_{HS} = \mathrm{Trace}[\Psi\Psi^*]$. Hence,

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \$ d\widetilde{W}_c\Big\|_\alpha^p &\le C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty \|D^\alpha B_c(Q_c^{\frac{1}{2}}g_k,\mathscr{A}_s^{-1}\psi)\|^2 d\tau\Big)^{\frac{p}{2}} \\
&= C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty \|\underbrace{B_c(Q_c^{\frac{1}{2}}g_k,\mathscr{A}_s^{-1}\psi)}_{\in\mathscr{N}}\|_\alpha^2 d\tau\Big)^{\frac{p}{2}} \\
&\le C\mathbb{E}\Big(\sum_{k=1}^\infty \sup_{[0,\tau^*]}\|B_c(Q_c^{\frac{1}{2}}g_k,\mathscr{A}_s^{-1}\psi)\|_{\alpha-\beta}^2\Big)^{\frac{p}{2}} \\
&\le C\Big(\sum_{k=1}^\infty\|Q_c^{\frac{1}{2}}g_k\|_\alpha^2\Big)^{\frac{p}{2}}\cdot\mathbb{E}\sup_{[0,\tau^*]}\|\mathscr{A}_s^{-1}\psi(\tau)\|_\alpha^p \\
&\le C\varepsilon^{-3p\kappa}, \qquad (45)
\end{aligned}$$

where we used the fact that the norm in $HS$ is invariant under taking the adjoint, and independent of the choice of the basis, in order to obtain

$$\begin{aligned}
\sum_{k=1}^\infty \|Q_c^{\frac{1}{2}}g_k\|_\alpha^2 = \|D^\alpha Q_c^{\frac{1}{2}}\|_{HS}^2 = \|Q_c^{\frac{1}{2}}D^\alpha\|_{HS}^2 &= \sum_{k=1}^\infty \|Q_c^{\frac{1}{2}}D^\alpha e_k\|^2 \\
&= \sum_{k=1}^\infty \langle Q_c^{\frac{1}{2}}D^\alpha e_k, Q_c^{\frac{1}{2}}D^\alpha e_k\rangle = \sum_{k=1}^\infty k^{2\alpha}\langle P_cQP_ce_k,e_k\rangle \\
&= \sum_{k=1}^n k^{2\alpha}\langle Qe_k,e_k\rangle = \sum_{k=1}^n k^{2\alpha}\|Q^{\frac{1}{2}}e_k\|^2 \le C\,.
\end{aligned}$$

For $\varepsilon\int_0^T B_c(a,\mathscr{A}_s^{-1}d\widetilde{W}_s)$, the last stochastic integral in (21), note that the covariance operator of $\widetilde{W}_s$ is $Q_s = P_sQP_s$. Similar to the previous estimate we define

$$\$_1(\tau)u := B_c(a(\tau),\mathscr{A}_s^{-1}u).$$

Now by Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [18]) we obtain

$$
\begin{aligned}
\mathbb{E} \sup_{T\in[0,\tau^*]} \Big\| \varepsilon \int_0^T B_c(a, \mathscr{A}_s^{-1} d\widetilde{W}_s) \Big\|_\alpha^p &= \mathbb{E} \sup_{T\in[0,\tau^*]} \Big\| \varepsilon \int_0^T D^\alpha \$_1 d\widetilde{W}_s \Big\|^p \\
&= C \cdot \mathbb{E}\Big( \varepsilon^2 \int_0^{\tau^*} \|D^\alpha \$_1 Q_s^{\frac{1}{2}}\|_{HS}^2 d\tau \Big)^{\frac{p}{2}} \\
&= C \cdot \mathbb{E}\Big( \varepsilon^2 \int_0^{\tau^*} \sum_{k=1}^\infty \|D^\alpha \$_1 Q_s^{\frac{1}{2}} e_k\|^2 d\tau \Big)^{\frac{p}{2}} \\
&= C\varepsilon^p \cdot \mathbb{E}\Big( \int_0^{\tau^*} \sum_{k=1}^\infty \|D^\alpha B_c(a, \mathscr{A}_s^{-1} Q_s^{\frac{1}{2}} e_k)\|^2 d\tau \Big)^{\frac{p}{2}} \\
&\le C\varepsilon^p \cdot \mathbb{E}\Big( \sum_{k=1}^\infty \sup_{[0,\tau^*]} \|B_c(a, \mathscr{A}_s^{-1} Q_s^{\frac{1}{2}} e_k)\|_{\alpha-\beta}^2 \Big)^{\frac{p}{2}} \\
&\le C\varepsilon^{p-p\kappa} \Big( \sum_{k=1}^\infty \|\mathscr{A}_s^{-1} Q_s^{\frac{1}{2}} e_k\|_\alpha^2 \Big)^{\frac{p}{2}} \\
&\le C\varepsilon^{p-p\kappa},
\end{aligned} \tag{46}
$$

where we used

$$
\begin{aligned}
\sum_{k=1}^\infty \|\mathscr{A}_s^{-1} Q_s^{\frac{1}{2}} e_k\|_\alpha^2 = \|D^\alpha \mathscr{A}_s^{-1} Q_s^{\frac{1}{2}}\|_{HS}^2 = \|Q_s^{\frac{1}{2}} \mathscr{A}_s^{-1} D^\alpha\|_{HS}^2 &= \sum_{k=1}^\infty \|Q_s^{\frac{1}{2}} \mathscr{A}_s^{-1} D^\alpha e_k\|^2 \\
= \sum_{k=1}^\infty \frac{k^{2\alpha}}{\lambda_k^2} \|Q_s^{\frac{1}{2}} e_k\|^2 = \sum_{k=1}^\infty \frac{k^{2\alpha}}{\lambda_k^2} \langle P_s Q P_s e_k, e_k \rangle& \\
= \sum_{k=n+1}^\infty \frac{k^{2\alpha}}{\lambda_k^2} \|Q^{\frac{1}{2}} e_k\|^2 \le C\,.&
\end{aligned}
$$

The last step follows from Assumption 12, as $\lambda_k \to \infty$.
As we supposed $\kappa < \frac{1}{7}$ in the definition of $\tau^*$, we can collect all term in the equations from (34) until (46). This implies the result. □

In order to prove now the approximation result, we first need the following a-priori estimate for solutions of the amplitude equation.

**Lemma 25.** *Let Assumptions 1, 5, 10 and 12 hold. Define the stochastic process $b(T)$, with initial condition $\mathbb{E}\|b(0)\| \le C$, in $\mathscr{N}$ as the solution of*

$$
b(T) = b(0) + \int_0^T \mathscr{L}_c b(\tau) d\tau - 2\int_0^T \mathscr{F}(b(\tau)) d\tau + \widetilde{W}_c(T). \tag{47}
$$

*Then for $T_0 > 0$ there exists a constant $C > 0$ such that*

$$
\mathbb{E} \sup_{T\in[0,T_0]} \|b(T)\|_\alpha^p \le C\,. \tag{48}
$$

We note that all norms in a finite dimensional space are equivalent. Thus for simplicity of notation in the proof we use only the standard Eucledian norm and suppose that $b \in \mathbb{R}^n$.

*Proof.* The existence and uniqueness of solutions for equation (47) is standard. To verify the bound in (48) we define $X$ as

$$X(T) = b(T) - \tilde{W}_c(T) \, . \tag{49}$$

Substituting into (47), we obtain

$$\partial_T X = \mathscr{L}_c(X + \tilde{W}_c) - 2\mathscr{F}(X + \tilde{W}_c).$$

Taking the scalar product $\langle \cdot, X\rangle$ on both sides of (56) yields

$$\frac{1}{2}\partial_T \|X\|^2 = \langle \mathscr{L}_c(X + \tilde{W}_c), X\rangle - 2\langle \mathscr{F}(X + \tilde{W}_c), X\rangle.$$

Using Young and Cauchy-Schwarz inequalities and Assumption 10 yields

$$\partial_T \|X\|^2 \le C + C\left\|\tilde{W}_c\right\|^4 - \frac{\delta}{2}\|X\|^4.$$

Neglecting the fourth power, integrating from 0 to $T$, taking $\frac{p}{2}$-th power, and finally the expectation, we obtain

$$\mathbb{E} \sup_{[0,T_0]} \|X\|^p \le C T_0^{\frac{1}{2}p} + C T_0^{\frac{1}{2}p} \mathbb{E} \sup_{[0,T_0]} \left\|\tilde{W}_c\right\|^{2p} \le C \, .$$

Together with (49), this implies

$$\mathbb{E} \sup_{[0,T_0]} \|b\|^p \le C\mathbb{E} \sup_{[0,T_0]} \|X\|^p + C\mathbb{E} \sup_{[0,T_0]} \left\|\tilde{W}_c\right\|^p \le C.$$

□

**Definition 26.** *Define the set $\Omega^* \subset \Omega$ such that all these estimates*

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < C\varepsilon^{-3\kappa} \, , \tag{50}$$

$$\sup_{[0,\tau^*]} \|R\|_\alpha < C\varepsilon^{1-7\kappa} \, , \tag{51}$$

*and*

$$\sup_{[0,\tau^*]} \|b\|_\alpha < C\varepsilon^{-\frac{\kappa}{2}} \, , \tag{52}$$

*hold on $\Omega^*$.*

**Remark 27.** *The set $\Omega^*$ has approximately probability 1, as*

$$\begin{aligned}\mathbb{P}(\Omega^*) \ge 1 &- \mathbb{P}(\sup_{[0,\tau^*]} \|\psi\|_\alpha \ge C\varepsilon^{-3\kappa}) \\ &- \mathbb{P}(\sup_{[0,\tau^*]} \|R\|_\alpha \ge C\varepsilon^{1-7\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|b\|_\alpha \ge C\varepsilon^{-\frac{\kappa}{2}}).\end{aligned}$$

*Using Chebychev inequality and Lemmas 22, 24 and 25, we obtain for sufficient large $q$*

$$\mathbb{P}(\Omega^*) \ge 1 - C[\varepsilon^{q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa}] \ge 1 - C\varepsilon^p \, .$$

**Theorem 28.** *We assume that Assumption 1, 5, 6, 10 and 12 hold. Let $b$ be a solution of (47) and $a$ as defined in (20) with $\|a(0)\| \le C$ on $\Omega^*$. If the initial conditions satisfies $a(0) = b(0)$, then, for $\kappa < \frac{1}{7}$, we obtain*

$$\sup_{T\in[0,\tau^*]} \|a(T) - b(T)\|_\alpha \le C\varepsilon^{1-7\kappa}, \tag{53}$$

*and*

$$\sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha \le C\varepsilon^{-\frac{\kappa}{2}}, \tag{54}$$

*on $\Omega^*$.*

*Proof.* Define $\varphi(T)$ as

$$\varphi(T) := a(T) - R(T).$$

From (20) we obtain

$$\varphi(T) = a(0) + \int_0^T \mathscr{L}_c\left[\varphi(\tau) + R(\tau)\right] d\tau - 2\int_0^T \mathscr{F}(\varphi(\tau) + R(\tau))d\tau + \widetilde{W}_c(T). \tag{55}$$

Define now $h(T)$ by

$$h(T) := b(T) - \varphi(T).$$

Subtracting (55) from (47), we obtain

$$h(T) = \int_0^T \mathscr{L}_c h(\tau)d\tau - \int_0^T \mathscr{L}_c R(\tau)d\tau + 2\int_0^T [\mathscr{F}(b - h + R) - \mathscr{F}(b)](\tau)d\tau.$$

Thus

$$\partial_T h = \mathscr{L}_c h - \mathscr{L}_c R + 2[\mathscr{F}(b - h + R) - \mathscr{F}(b)]\,. \tag{56}$$

Taking the scalar product $\langle\cdot, h\rangle$ on both sides of (56) yields

$$\tfrac{1}{2}\partial_T \|h\|^2 = \langle\partial_T h, h\rangle = \langle\mathscr{L}_c h, h\rangle - \langle\mathscr{L}_c R, h\rangle + 2\langle\mathscr{F}(b - h + R) - \mathscr{F}(b), h\rangle\,.$$

Using Young and Cauchy-Schwarz inequalities and (6), we obtain the following linear ordinary differential inequality

$$\begin{aligned}\partial_T \|h\|^2 &\le C[\|h\|^2 + \|h\|^4] + C\|R\|^2\left[1 + \|R\|^2 + \|b\|^2 + \|b\|^4 + \|b\|^2\|R\|^2\right]\\ &\le C[\|h\|^2 + \|h\|^4] + C\|R\|^2\left[1 + \|R\|^4 + \|b\|^4\right]\,.\end{aligned}$$

Using (51) and (52), we obtain

$$\partial_T \|h\|^2 \le C[\|h\|^2 + \|h\|^4] + C\varepsilon^{2-14\kappa} \quad \text{on } \Omega^*.$$

Now we will show that $h$ stays small for a long time. As long as $\|h\| \le 1$, we obtain

$$\partial_T \|h\|^2 \le 2C\|h\|^2 + C\varepsilon^{2-14\kappa}.$$

Using Gronwall's Lemma, we obtain for sufficiently small $\varepsilon$ that

$$\|h\|^2 \le C\varepsilon^{2-14\kappa} < 1$$

and thus

$$\sup_{[0,\tau^*]} \|h\| \le C\varepsilon^{1-7\kappa} \quad \text{on} \;\; \Omega^*. \tag{57}$$

We finish the first part by using (51) and (57) together with

$$\sup_{[0,\tau^*]} \|a-b\| = \sup_{[0,\tau^*]} \|h-R\| \le \sup_{[0,\tau^*]} \|h\| + \sup_{[0,\tau^*]} \|R\|.$$

For the second part of the theorem consider

$$\sup_{[0,\tau^*]} \|a\| \le \sup_{[0,\tau^*]} \|a-b\| + \sup_{[0,\tau^*]} \|b\| \,.$$

Using the first part and (52), we obtain (54). □

Finally, we use the results previously obtained to prove the main result of Theorem 17 for the approximation of the solution of the SPDE (1).

*Proof of Theorem 17.* For the stopping time, we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{ \sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha < \varepsilon^{-\kappa}, \sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha < \varepsilon^{-3\kappa}\} \supseteq \Omega^*.$$

Now let us turn to the approximation result. Using (14) and triangle inequality, we obtain

$$\sup_{T\in[0,\tau^*]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha \le \varepsilon \sup_{[0,\tau^*]} \|a-b\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]} \|\psi\|_\alpha.$$

From (50) and (53), we obtain

$$\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha = \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha \le C\varepsilon^{2-7\kappa} \;\text{ on }\; \Omega^*.$$

□

# 5 Application

There are numerous examples in the physics literature of equations with quadratic nonlinearities where our theory does apply. For simplicity we focus on two scalar examples, although our result applies also to systems of PDEs.

Before we give examples, we suppose in our applications for simplicity that $W$ is a cylindrical Wiener process on $\mathscr{H}$ with a covariance operator $Q$ defined by $Qe_k = \alpha_k^2 e_k$ where $(\alpha_k)_k$ is a bounded sequence of real numbers and $e_k$ are the eigenfunctions of the dominant linear operator. Thus,

$$W(t) = \sum_k \alpha_k \beta_k(t) e_k$$

for a family of independent standard Brownian motions $\beta_k$.

### 5.1 Burgers' equation

One example is the Burgers' equation (cf. (2)) on the interval $[0,\pi]$, with Dirichlet boundary conditions. We take

$$\mathscr{H} = L^2([0,\pi]), \quad e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx) \quad \text{and} \quad \mathscr{N} = \text{span}\{\sin\}.$$

First note that Assumption 1 is true. All the eigenvalues of $-\mathscr{A} = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. If we fix $P_c$ to be the $\mathscr{H}$-orthogonal projection onto $\mathscr{N}$, then both $P_c$ and $P_s$ commute with $\mathscr{A}$.

Moreover, it is easy to check that all conditions of Assumption 6 are satisfied with

$$B(u,v) = \tfrac{1}{2}\partial_x(uv).$$

To be more precise:

$$P_c B(u,u) = P_c\left[\gamma^2\sin(x)\cos(x)\right] = 0 \ \text{ for } u = \gamma\sin \in \mathscr{N},$$

and for $\alpha = \frac{1}{4}$ and $\beta = \frac{5}{4} < m$, we obtain

$$\begin{aligned} 2\|B(u,v)\|_{\mathscr{H}^{-1}} &= \|\partial_x(uv)\|_{\mathscr{H}^{-1}} \le \|uv\|_{L^2} \\ &\le C\|u\|_{L^4}\|v\|_{L^4} \le C\|u\|_{\mathscr{H}^{\frac{1}{4}}}\|v\|_{\mathscr{H}^{\frac{1}{4}}}, \end{aligned}$$

where we used Sobolev embedding from $\mathscr{H}^{1/4}$ into $L^4$. After a straightforward calculation we derive

$$\mathscr{F}(\gamma_1\sin,\gamma_2\sin,\gamma_3\sin) = \frac{1}{24}\gamma_1\gamma_2\gamma_3\sin.$$

This function is trilinear, continuous and satisfies the conditions (5) and (6) as follows

$$\langle\gamma_1\sin, \mathscr{F}(\gamma_1\sin)\rangle = C\gamma_1^4 > 0, \text{ if } \gamma_1 \neq 0$$

and

$$\langle\mathscr{F}(\gamma_1\sin,\gamma_1\sin,\gamma_2\sin),\gamma_2\sin\rangle = \frac{\pi}{48}\gamma_1^2\gamma_2^2 > 0$$

for $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$.

Now our main theorem states that

$$u(t) = \varepsilon\gamma(\varepsilon^2 t)\sin + \mathscr{O}(\varepsilon^{2-}),$$

where

$$\gamma' = \nu\gamma - \tfrac{1}{12}\gamma^3 + \alpha_1\tilde{\beta}',$$

with a rescaled standard Brownian motion $\tilde{\beta}$ given by $\varepsilon\sqrt{\frac{2}{\pi}}P_c W(\varepsilon^{-2}T)$.

## 5.2 Surface growth model

The second example that falls into the scope of our work is the growth of rough amorphous surfaces. The equation is of the type

$$\partial_t h = -\triangle^2 h - \mu\triangle h - \triangle|\nabla h|^2 + \sigma\partial_t W(t). \tag{58}$$

Here $\triangle$ is the Laplacian with respect to periodic boundary conditions on $[0,2\pi]$. Usually one supposes initial condition $h(0)=0$ corresponding to an initially flat surface.

For this model we consider $\mu = 1+\varepsilon^2\nu$ and $\sigma=\varepsilon^2$, which reflects the fact that one is sufficiently close to the first bifurcation, where the flat surface gets unstable. The distance from bifurcation scales like the noise-strength.

For the abstract setting define

$$\mathscr{A} = -\triangle^2 - \triangle, \quad \mathscr{L} = -\nu\triangle \quad \text{and} \quad B(u,v) = -\triangle(\partial_x u\cdot\partial_x v).$$

We take

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k>0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k<0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k=0, \end{cases}$$

and

$$\mathscr{H} = \{u\in L^2([0,2\pi]) : \int_0^{2\pi} u\,dx = 0\} \quad \text{and} \quad \mathscr{N} = \text{span}\{\sin,\ \cos\}.$$

The eigenvalues of $-\mathscr{A} = \triangle^2 + \triangle$ are $\lambda_k = k^4 - k^2$ with $m=4$ and $\lim_{k\to\infty}\lambda_k = \infty$. Thus Assumption 1 is satisfied.

If we define $u(t) := h(t) - h_0(t)e_0$, then we obtain

$$\partial_t u = -\triangle^2 u - \mu\triangle u - \triangle|\nabla u|^2 + \sigma\sum_{k\neq 0}\alpha_k\partial_t\beta_k(t)e_k, \tag{59}$$

and

$$h_0(t) = \sigma\alpha_0\beta_0(t). \tag{60}$$

If $u = u_1\sin + u_{-1}\cos\in\mathscr{N}$, then

$$B(u,u) = 2\left[u_1^2 - u_{-1}^2\right]\cos(2x) - 4u_1u_{-1}\sin(2x),$$

and

$$P_cB(u,u) = 0,$$

and for $\alpha=\frac{5}{4}$ and $\beta = \frac{13}{4} < m$, we obtain

$$\begin{aligned} \|B(u,v)\|_{\mathscr{H}^{-2}} &= \|\triangle(\partial_x u\cdot\partial_x v)\|_{\mathscr{H}^{-2}} \le c\|\partial_x u\cdot\partial_x v\|_{L^2} \\ &\le c\|u\|_{\mathscr{H}^{\frac{5}{4}}}\|v\|_{\mathscr{H}^{\frac{5}{4}}}. \end{aligned}$$

Hence all conditions of Assumption 6 are satisfied. Moreover, it is easy to check that Assumption 10 also holds true. For $\mathscr{F}$ we derive

$$\mathscr{F}(u,u,u) = \frac{3}{18}(u_1^2 + u_{-1}^2)u,$$

and for the symmetric version of $\mathscr{F}$ we obtain

$$\begin{aligned}\mathscr{F}(u,u,w) &= \frac{2}{3}B_c(u,\mathscr{A}_s^{-1}B_s(u,w))+\frac{1}{3}B_c(w,\mathscr{A}_s^{-1}B_s(u,u))\\ &= \tfrac{1}{18}[(3u_1^2w_1+w_1u_{-1}^2+2u_1w_{-1}u_{-1})\sin\\ &\quad+(u_1^2w_{-1}+3w_{-1}u_{-1}^2+2u_1w_1u_{-1})\cos],\end{aligned}$$

where $u=u_1\sin+u_{-1}\cos,\ w=w_1\sin+w_{-1}\cos\in\mathscr{N},$. Now

$$\langle\mathscr{F}(u),u\rangle>0\quad\forall\ u\neq 0\ .$$

and if $u\neq 0$ and $v\neq 0$

$$\langle\mathscr{F}(u,u,w),w\rangle=\frac{\pi}{18}[3(u_1w_1+w_{-1}u_{-1})^2+(w_1u_{-1}-u_1w_{-1})^2]>0\ .$$

The amplitude equation for (59) is a system of two stochastic ordinary differential equations:

$$d\gamma_i=[\nu\gamma_i-\tfrac{1}{3}\gamma_i(\gamma_1^2+\gamma_{-1}^2)]dt+\tfrac{1}{\sqrt{\pi}}\alpha_i d\tilde\beta_i\ \text{ for } i=\pm1,$$

where

$$u(t)=\varepsilon\gamma(\varepsilon^2t)\cdot\begin{pmatrix}\sin\\ \cos\end{pmatrix}+\mathscr{O}(\varepsilon^{2-})\ .$$

and $\tilde\beta_i(T)=\varepsilon\beta_i(\varepsilon^{-2}T)$ rescaled Brownian motions.

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