Electronic Journal of Probability

Vol. 14 (2009), Paper no. 94, pages 2691–2719.

Journal URL
http://www.math.washington.edu/~ejpecp/

Series Representations of Fractional Gaussian Processes by Trigonometric and Haar Systems

Antoine Ayache* Werner Linde†

Abstract

The aim of the present paper is to investigate series representations of the Riemann–Liouville process R^α, $\alpha > 1/2$, generated by classical orthonormal bases in $L_2[0,1]$. Those bases are, for example, the trigonometric or the Haar system. We prove that the representation of R^α via the trigonometric system possesses the optimal convergence rate if and only if $1/2 < \alpha \le 2$. For the Haar system we have an optimal approximation rate if $1/2 < \alpha < 3/2$ while for $\alpha > 3/2$ a representation via the Haar system is not optimal. Estimates for the rate of convergence of the Haar series are given in the cases $\alpha > 3/2$ and $\alpha = 3/2$. However, in this latter case the question whether or not the series representation is optimal remains open. [1] .

Key words: Approximation of operators and processes, Rie-mann–Liouville operator, Riemann–Liouville process, Haar system, trigonometric system.

AMS 2000 Subject Classification: Primary 60G15; Secondary: 26A33, 47B06, 41A30.

Submitted to EJP on December 07, 2008, final version accepted November 13, 2009.

*UMR CNRS 8524, Laboratoire Paul Painlevé, Bat. M2, Université Lille 1, 59655 Villeneuve d'Ascq, France, Antoine.Ayache@math.univ-lille1.fr

†FSU Jena, Institut für Stochastik, Ernst–Abbe–Platz 2, 07743 Jena, Germany, lindew@minet.uni-jena.de

[1] Recently M. A. Lifshits answered this question (cf. [13]). Using a different approach he could show that in the case $\alpha = 3/2$ a representation of the Riemann–Liouville process via the Haar system is also not optimal.

1 Introduction

Let $X = (X(t))_{t\in T}$ be a centered Gaussian process over a compact metric space (T,d) possessing a.s. continuous paths. Then it admits a representation

$$X(t) = \sum_{k=1}^{\infty} \epsilon_k \, \psi_k(t), \quad t \in T, \tag{1.1}$$

with $(\epsilon_k)_{k\geq 1}$ i.i.d. standard (real) normal random variables and with continuous real–valued functions ψ_k on T. Moreover, the right hand sum converges a.s. uniformly on T. Since representation (1.1) is not unique, one may ask for optimal ones, i.e. those for which the error

$$\mathbb{E} \sup_{t\in T} \left| \sum_{k=n}^{\infty} \epsilon_k \, \psi_k(t) \right| \tag{1.2}$$

tends to zero, as $n \to \infty$, of the best possible order. During past years several optimal representations were found, e.g. for the fractional Brownian motion, the Lévy fractional motion or for the Riemann–Liouville process (cf. [7], [2], [5], [8], [1], [18] and [20]). Thereby the representing functions ψ_k were either constructed by suitable wavelets or by Bessel functions.

In spite of this progress a, to our opinion, natural question remained unanswered. Are the "classical" representations also optimal ? To make this more precise, suppose that the process X has a.s. continuous paths and admits an integral representation

$$X(t) = \int_I K(t,x) \, \mathrm{d}W(x), \quad t \in T,$$

for some interval $I \subseteq \mathbb{R}$ and with white noise W on I. Given any ONB $\Phi = (\varphi_k)_{k\geq 1}$ in $L_2(I)$ we set

$$\psi_k(t) := \int_I K(t,x)\varphi_k(x) \, \mathrm{d}x \, .$$

By the Itô–Nisio–Theorem (cf. [10], Theorem 2.1.1) the sum in (1.1) converges a.s. uniformly on T, thus leading to a series representation of X. For example, if $I = [0,1]$, then one may choose for Φ natural bases as e.g. the ONB of the trigonometric functions $\mathbf{T} = \{\mathbf{1}\} \cup \left\{\sqrt{2}\cos(k\pi\,\cdot) : k \geq 1\right\}$ or that of the Haar functions $\mathbf{H}$ (see (5.2)). There is no evidence that in some interesting cases these "classical" bases do not lead to optimal expansions as well.

The aim of the present paper is to investigate those questions for the Riemann–Liouville process R^α defined by

$$R^\alpha(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \, \mathrm{d}W(x), \quad 0 \leq t \leq 1 \, .$$

This process is known to have a.s. continuous paths whenever $\alpha > 1/2$ (cf. [15] for further properties of this process). Thus, for example, representation (1.1) of R^α by the basis $\mathbf{T}$ leads to

$$R^\alpha(t) = \epsilon_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\sqrt{2}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \epsilon_k \int_0^t (t-x)^{\alpha-1} \cos(k\pi x) \, \mathrm{d}x \tag{1.3}$$

and in similar way it may be represented by the Haar system $\mathbf{H}$ as

$$R^{\alpha}(t)=\epsilon_{-1}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\sum_{k=0}^{2^{j}-1}\epsilon_{j,k}\int_{0}^{t}(t-x)^{\alpha-1}h_{j,k}(x)\,\mathrm{d}x \tag{1.4}$$

where the $h_{j,k}$ are the usual Haar functions. The basic question we investigate is whether or not representations (1.3) and (1.4) are optimal. The answer is quite surprising.

Theorem 1.1. *If* $1/2<\alpha\le 2$*, then representation* (1.3) *is optimal while for* $\alpha>2$ *it is rearrangement non–optimal, i.e., it is not optimal with respect to any order chosen in* $\mathbf{T}$*. If* $1/2<\alpha<3/2$*, then representation* (1.4) *is optimal and rearrangement non–optimal for* $\alpha>3/2$*.*

For the proof we refer to Theorems 4.1, 5.4, 5.5 and 5.11. As recently shown by M. A. Lifshits (oral communication) representation (1.4) is also not optimal for $\alpha=3/2$. Let us recall that the assertions for $\alpha>2$ or $\alpha\ge 3/2$, respectively, say that these bases are not only non–optimal in their natural order but also after any rearrangement of the bases.

Even in the cases where these representations are not optimal it might be of interest how fast (or slow) the error in (1.2) tends to zero as $n\to\infty$. Here we have lower and upper estimates which differ by $\sqrt{\log n}$.

Another process, tightly related to R^{α}, is the Weyl process I^{α} which is stationary and 1–periodic. It may be defined, for example, by

$$I^{\alpha}(t)=\sqrt{2}\sum_{k=1}^{\infty}\epsilon_{k}\,\frac{\cos(2k\pi t-\alpha\pi/2)}{(2\pi k)^{\alpha}}+\sqrt{2}\sum_{l=1}^{\infty}\epsilon_{l}'\,\frac{\sin(2l\pi t-\alpha\pi/2)}{(2\pi l)^{\alpha}}\,. \tag{1.5}$$

Here (ϵ_k) and (ϵ_l') are two independent sequences of i.i.d. standard (real) normal random variables. We refer to [3] or [14] for more information about this process.

In fact, (1.5) is already a series representation and we shall prove in Theorem 4.8 that it is optimal for all $\alpha>1/2$. In comparison with Theorem 1.1 this is quite unexpected. Note that the processes R^{α} and I^{α} differ by a very smooth process (cf. [3]). Moreover, if $\alpha>1/2$ is an integer, then their difference is even a process of finite rank.

2 Approximation by a Fixed Basis

Let $\mathscr{H}$ be a separable Hilbert space. Then $\mathscr{G}(\mathscr{H},E)$ denotes the set of those (bounded) operators u from $\mathscr{H}$ into a Banach space E for which the sum

$$\sum_{k=1}^{\infty}\epsilon_k u(\varphi_k)$$

converges a.s. in E for one (then for each) ONB $\Phi=(\varphi_k)_{k\ge1}$ in $\mathscr{H}$. As above $(\epsilon_k)_{k\ge1}$ denotes an i.i.d. sequence of standard (real) normal random variables. If $u\in\mathscr{G}(\mathscr{H},E)$, we set

$$l(u):=\mathbb{E}\left\|\sum_{k=1}^{\infty}\epsilon_k u(\varphi_k)\right\|_E$$

which is independent of the special choice of the basis.

For $u \in \mathscr{G}(\mathscr{H}, E)$ the sequence of l–approximation numbers is then defined as follows:

$$l_n(u) := \inf\left\{ l(u - uP) : P \text{ orthogonal projection in } \mathscr{H} \,,\ \operatorname{rk}(P) < n \right\} \ .$$

Note that, of course, $l(u) = l_1(u) \geq l_2(u) \geq \cdots \geq 0$ and $l_n(u) \to 0$ as $n \to \infty$ and that

$$l_n(u) = \inf\left\{ \mathbb{E} \left\| \sum_{k=n}^{\infty} \epsilon_k u(\varphi_k) \right\| : \Phi = (\varphi_k)_{k\geq 1} \text{ ONB in } \mathscr{H} \right\} \ .$$

We refer to [1], [9] or [19] for more information about these numbers.

For our purposes we need to specify the definition of the l–approximation numbers as follows. Let $\Phi = (\varphi_k)_{k\geq 1}$ be a **fixed** ONB in the Hilbert space $\mathscr{H}$. Then we define the l–approximation numbers of u with respect to Φ by

$$l_n^{\Phi}(u) := \inf\left\{ \mathbb{E} \left\| \sum_{k\notin N} \epsilon_k u(\varphi_k) \right\| \ :\ N \subseteq \mathbb{N} \,,\ \#N < n \right\} .$$

Let us state some properties of $l_n^{\Phi}(u)$ for later use.

Proposition 2.1. *Let $u \in \mathscr{G}(\mathscr{H}, E)$ and let $\Phi = (\varphi_k)_{k\geq 1}$ be some ONB in $\mathscr{H}$. Then the following are valid.*

(1) *We have*

$$l(u) = l_1^{\Phi}(u) \geq l_2^{\Phi}(u) \geq \cdots \to 0 \,.$$

(2) *If $u_1, u_2 \in \mathscr{G}(\mathscr{H}, E)$, then it follows that*

$$l_{n_1+n_2-1}^{\Phi}(u_1 + u_2) \leq l_{n_1}^{\Phi}(u_1) + l_{n_2}^{\Phi}(u_2) \,.$$

(3) *If v is a (bounded) operator from E into another Banach space F, then it holds*

$$l_n^{\Phi}(v \circ u) \leq \|v\|\ l_n^{\Phi}(u) \,.$$

(4) *We have*

$$l_n(u) = \inf\left\{ l_n^{\Phi}(u) : \Phi \text{ ONB in } \mathscr{H} \right\} \ .$$

(5) *If $E = \mathscr{H}$, then it holds*

$$l_n^{\Phi}(u) \approx \inf\left\{ \left(\sum_{k\notin N} \left\| u(\varphi_k) \right\|_{\mathscr{H}}^2 \right)^{1/2} \ :\ N \subseteq \mathbb{N} \,,\ \#N < n \right\} .$$

Proof. Properties (1), (2) and (4) follow directly from the definition of $l_n^\Phi(u)$. Thus we omit their proofs. In order to verify (3) let us choose a subset $N \subset \mathbb{N}$ with $\#N < n$ such that

$$\mathbb{E}\left\|\sum_{k\notin N}\epsilon_k u(\varphi_k)\right\| \le l_n^\Phi(u)+\varepsilon$$

for some given $\varepsilon > 0$. Then we get

$$\begin{aligned} l_n^\Phi(v\circ u) &\le \mathbb{E}\left\|\sum_{k\notin N}\epsilon_k (v\circ u)(\varphi_k)\right\| = \mathbb{E}\left\| v\left(\sum_{k\notin N}\epsilon_k u(\varphi_k)\right)\right\| \\ &\le \|v\|\,\mathbb{E}\left\|\sum_{k\notin N}\epsilon_k u(\varphi_k)\right\| \le \|v\|\left[l_n^\Phi(u)+\varepsilon\right] \end{aligned}$$

which completes the proof of (3) by letting $\varepsilon \to 0$.

Property (5) is an easy consequence of

$$\mathbb{E}\left\|\sum_k \epsilon_k x_k\right\|_{\mathscr{H}}^2 = \sum_k \left\|x_k\right\|_{\mathscr{H}}^2$$

for any elements x_k in a Hilbert space $\mathscr{H}$. Note, moreover, that all moments of a Gaussian vector are equivalent by Fernique's Theorem (cf. [6]). □

Remark 2.1. It is worthwhile to mention that in general $\mathrm{rk}(u) < n$ does not imply $l_n^\Phi(u) = 0$. This is in contrast to the properties of the usual l–approximation numbers.

In order to define optimality of a given representation in its natural order we have to introduce a quantity tightly related to $l_n^\Phi(u)$. For $u \in \mathscr{G}(\mathscr{H}, E)$ and an ONB $\Phi = (\varphi_k)_{k\ge 1}$ in $\mathscr{H}$ we set

$$l_n^{o,\Phi}(u) := \mathbb{E}\left\|\sum_{k=n}^{\infty}\epsilon_k u(\varphi_k)\right\| .$$

The "o" in the notation indicates that $l_n^{o,\Phi}(u)$ depends on the order of the elements in Φ while, of course, $l_n^\Phi(u)$ does not depend on it.

Clearly $l_n^\Phi(u) \le l_n^{o,\Phi}(u)$ and, moreover, it is not difficult to show (cf. Prop. 2.1 in [9]) that $l_n^\Phi(u) \le c_1\, n^{-\alpha}(\log n)^\beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$ implies $l_n^{o,\Phi'}(u) \le c_2\, n^{-\alpha}(\log n)^\beta$ where Φ' coincides with Φ after a suitable rearrangement of its elements.

We may now introduce the notion of optimality for a given basis (cf. also [9] and [1]).

Definition 2.1. *An ONB Φ is said to be **optimal** for u (in the given order of Φ) provided there is some $c > 0$ such that*

$$l_n^{o,\Phi}(u) \le c\, l_n(u)\,, \quad n = 1, 2, \ldots$$

*It is **rearrangement non–optimal** if*

$$\limsup_{n\to\infty}\frac{l_n^\Phi(u)}{l_n(u)} = \infty .$$

In particular, the approximation error $l_n^{o,\Phi}(u)$ tends to zero slower than the optimal rate.

For later purposes we state the following result. Since the proof is quite standard we omit it (cf. also Prop. 2.1 in [9]).

Proposition 2.2. *Suppose that there is a constant $c_1 > 0$ such that for $\alpha > 0$, $\beta \in \mathbb{R}$ and all $n \in \mathbb{N}$*

$$\mathbb{E}\left\|\sum_{k=2^n}^{2^{n+1}-1} \epsilon_k u(\varphi_k)\right\| \leq c_1\, n^\beta\, 2^{-n\alpha}\,.$$

Then this implies

$$l_n^{o,\Phi}(u) \leq c_2\, n^{-\alpha}\, (\log n)^\beta$$

for a suitable $c_2 > 0$.

The following lemma is elementary, but very helpful to prove lower estimates for $l_n^\Phi(u)$.

Lemma 2.3. *Let u and Φ be as before. Suppose that there exists a subset $M \subseteq \mathbb{N}$ of cardinality m possessing the following property: For some $\delta > 0$ and some $n \leq m$ each subset $L \subset M$ with $\#L > m-n$ satisfies*

$$\mathbb{E}\left\|\sum_{k\in L} \epsilon_k u(\varphi_k)\right\| \geq \delta\,.$$

Then this implies $l_n^\Phi(u) \geq \delta$.

Proof. Let $N \subset \mathbb{N}$ be an arbitrary subset of cardinality strictly less than n. Set $L := M \cap N^c$. Clearly, $\#L > m - n$, hence the assumption leads to

$$\mathbb{E}\left\|\sum_{k\notin N} \epsilon_k u(\varphi_k)\right\| \geq \mathbb{E}\left\|\sum_{k\in L} \epsilon_k u(\varphi_k)\right\| \geq \delta\,. \tag{2.1}$$

Taking on the left hand side of (2.1) the infimum over all subsets N with $\#N < n$ proves the assertion. □

Let us shortly indicate how the preceding statements are related to the problem of finding optimal series expansions of Gaussian processes. If the process $X = (X(t))_{t\in T}$ is represented by a sequence $\Psi = (\psi_k)_{k\geq 1}$ as in (1.1), then we choose an arbitrary separable Hilbert space $\mathscr{H}$ and an ONB $\Phi = (\varphi_k)_{k\geq 1}$ in it. By

$$u(\varphi_k) := \psi_k\,, \quad k \geq 1\,, \tag{2.2}$$

a unique operator $u \in \mathscr{G}(\mathscr{H}, C(T))$ is defined. Of course, this construction implies $l_n^\Phi(u) = l_n^\Psi(X)$ where the latter expression is defined by

$$l_n^\Psi(X) := \inf\left\{\mathbb{E}\left\|\sum_{k\notin N} \epsilon_k \psi_k\right\|_\infty \,:\, N \subseteq \mathbb{N}\,,\ \#N < n\right\}\,.$$

Similarly, $l_n^{o,\Phi}(u) = l_n^{o,\Psi}(X)$ with

$$l_n^{o,\Psi}(X) := \mathbb{E}\left\|\sum_{k=n}^{\infty} \epsilon_k \psi_k\right\|_\infty\,.$$

As shown in [1] we have $l_n(u) = l_n(X)$ where

$$l_n(X) := \inf\left\{ \mathbb{E}\left\| \sum_{k=n}^{\infty} \epsilon_k \rho_k \right\|_\infty : X = \sum_{k=1}^{\infty} \epsilon_k \rho_k \right\} .$$

Hence representation (1.1) is optimal for X, i.e. there is some $c > 0$ such that for all $n \geq 1$ we have $l_n^{o,\Psi}(X) \leq c\, l_n(X)$, if and only if this is so for Φ and u related to $\Psi = (\psi_k)_{k\geq 1}$ via (2.2). In the same way Φ is rearrangement non–optimal for u if and only if this is so for the representing functions ψ_k of X. Consequently, all our results about series expansions may be formulated either in the language of ONB and operators $u \in \mathscr{G}(\mathscr{H}, C(T))$ or in that of series expansions of centered Gaussian processes $X = (X(t))_{t\in T}$ with a.s. continuous paths.

3 A General Approach

We start with a quite general result which is in fact the abstract version of Theorem 5.7 in [9]. Before let us recall the definition of the covering numbers of a compact metric space (T, d). If $\varepsilon > 0$, then we set

$$N(T, d, \varepsilon) := \min\left\{ n \geq 1 : \exists\, t_1, \ldots, t_n \in T \text{ s.t. } \min_{1\leq j\leq n} d(t, t_j) < \varepsilon\,,\ t \in T \right\}.$$

With these notation the following is valid.

Proposition 3.1. *Let (T, d) be a compact metric space such that*

$$N(T, d, \varepsilon) \leq c\, \varepsilon^{-\gamma} \tag{3.1}$$

for some $\gamma > 0$. Let $u \in \mathscr{G}(\mathscr{H}, C(T))$ and let $\Phi = (\varphi_k)_{k\geq 1}$ be some fixed ONB in $\mathscr{H}$. Suppose there are $\beta > 0$ and $\alpha > 1/2$ such that for all $k \geq 1$ and all $t, s \in T$ we have

$$\left|(u\varphi_k)(t) - (u\varphi_k)(s)\right| \leq c\, d(t, s)^\beta \tag{3.2}$$

and

$$\left\|u(\varphi_k)\right\|_\infty \leq c\, k^{-\alpha} . \tag{3.3}$$

Then for each $n \geq 1$ it follows that

$$\mathbb{E}\left\| \sum_{k=2^n}^{2^{n+1}-1} \epsilon_k u(\varphi_k) \right\|_\infty \leq c\, n^{1/2}\, 2^{-n(\alpha-1/2)} . \tag{3.4}$$

Proof. Let ε_n be a decreasing sequence of positive numbers which will be specified later on. Set $N_n := N(T, d, \varepsilon_n)$ and note that (3.1) implies

$$N_n \leq c\, \varepsilon_n^{-\gamma} . \tag{3.5}$$

Then there are $t_1, \ldots, t_{N_n} \in T$ such that $T = \bigcup_{j=1}^{N_n} B_j$ where $B_j := B(t_j, \varepsilon_n)$ are the open d–balls in T with radius ε_n and center t_j. To simplify the notation set $J_n := \left\{2^n, \ldots, 2^{n+1} - 1\right\}$ and write

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k\in J_n} \epsilon_k u(\varphi_k)\right\|_\infty &\leq \mathbb{E} \sup_{1\leq j\leq N_n} \sup_{t\in B_j} \left|\sum_{k\in J_n} \epsilon_k \left[(u\varphi_k)(t) - (u\varphi_k)(t_j)\right]\right| \\
&+ \mathbb{E} \sup_{1\leq j\leq N_n} \left|\sum_{k\in J_n} \epsilon_k (u\varphi_k)(t_j)\right| . \qquad (3.6)
\end{aligned}$$

We estimate both terms in (3.6) separately. Since for $t \in B_j$ we have $d(t, t_j) < \varepsilon_n$ condition (3.2) leads to

$$\left|(u\varphi_k)(t) - (u\varphi_k)(t_j)\right| \leq c\, \varepsilon_n^\beta$$

for those $t \in T$. Hence the first term in (3.6) can be estimated by

$$\begin{aligned}
&\mathbb{E} \sup_{1\leq j\leq N_n} \sup_{t\in B_j} \left|\sum_{k\in J_n} \epsilon_k \left[(u\varphi_k)(t) - (u\varphi_k)(t_j)\right]\right| \\
&\leq c\, \varepsilon_n^\beta\, \mathbb{E} \sum_{k\in J_n} \left|\epsilon_k\right| = c' \,(\#J_n)\, \varepsilon_n^\beta \leq c\, 2^n\, \varepsilon_n^\beta . \qquad (3.7)
\end{aligned}$$

In order to estimate the second term in (3.6) we need the following result (cf. Lemma 4.14 in [19]).

Lemma 3.2. *There is a constant $c > 0$ such that for any $N \geq 1$ and any centered Gaussian sequence $Z_1, \ldots, Z_N$ one has,*

$$\mathbb{E}\left\{\sup_{1\leq k\leq N} |Z_k|\right\} \leq c\,(1 + \log N)^{1/2} \sup_{1\leq k\leq N} \left(E|Z_k|^2\right)^{1/2} . \qquad (3.8)$$

We apply (3.8) to $Z_j = \sum_{k\in J_n} \epsilon_k (u\varphi_k)(t_j)$ and use (3.3). Then similar arguments as in the proof of Theorem 5.7 in [9], p. 686, lead to

$$\mathbb{E} \sup_{1\leq j\leq N_n} \left|\sum_{k\in J_n} \epsilon_k (u\varphi_k)(t_j)\right| \leq c\,(1 + \log N_n)^{1/2}\, 2^{-n(\alpha-1/2)} . \qquad (3.9)$$

Summing up, (3.6), (3.7) and (3.9) yield

$$\mathbb{E}\left\|\sum_{k\in J_n} \epsilon_k u(\varphi_k)\right\|_\infty \leq c_1\, 2^n\, \varepsilon_n^\beta + c_2\,(1 + \log N_n)^{1/2}\, 2^{-n(\alpha-1/2)} . \qquad (3.10)$$

Now we choose $\varepsilon_n := 2^{-\delta n}$ with $\delta > (\alpha + 1/2)/\beta$. By (3.5) we get $\log N_n \leq c\, n$ and by the choice of δ the first term in (3.10) is of lower order than the second one. This implies

$$\mathbb{E}\left\|\sum_{k\in J_n} \epsilon_k u(\varphi_k)\right\|_\infty \leq c\, n^{1/2}\, 2^{-n(\alpha-1/2)}$$

as asserted and completes the proof. $\square$

Combining Propositions 2.2 and 3.1 gives the following.

Corollary 3.3. *Suppose that* (3.1)*,* (3.2) *and* (3.3) *hold for some* $\gamma, \beta > 0$ *and* $\alpha > 1/2$*. Then this implies*

$$l_n^{o,\Phi}(u) \le c\, n^{-\alpha+1/2} \sqrt{\log n}$$

for all $n > 1$.

Remark 3.1. Results more or less similar to Proposition 3.1 and Corollary 3.3 were obtained in [16; 17] (see Proposition 4 in [17] and also the proof of Theorem 1 in [16]).

Let us formulate the preceding result in probabilistic language. We shall do so in a quite general way. Let $X = (X(t))_{t\in T}$ be an a.s. bounded centered Gaussian process on an arbitrary index set T (we do not suppose that there is a metric on T). Define the Dudley metric d_X on T by

$$d_X(t,s) := \left(\mathbb{E}\,|X(t) - X(s)|^2\right)^{1/2}, \quad t,s \in T\,.$$

Since X is a.s. bounded (T, d_X) is known to be compact (cf. [6]). We assume that (T, d_X) satisfies a certain degree of compactness, i.e., we assume that

$$N(T, d_X, \varepsilon) \le c\, \varepsilon^{-\gamma} \tag{3.11}$$

for some $\gamma > 0$. Suppose now that

$$X(t) = \sum_{k=1}^{\infty} \epsilon_k \psi_k(t), \quad t \in T\,, \tag{3.12}$$

where the right hand sum converges a.s. uniformly on T. Note that the ψ_k are necessarily continuous with respect to d_X. This easily follows from

$$d_X(t,s) = \left(\sum_{k=1}^{\infty} \left|\psi_k(t) - \psi_k(s)\right|^2\right)^{1/2}\,. \tag{3.13}$$

Then the following holds:

Proposition 3.4. *Suppose* (3.11) *and* (3.12) *and that there is some* $\alpha > 1/2$ *such that for all* $k \ge 1$

$$\sup_{t\in T} \left|\psi_k(t)\right| \le c\, k^{-\alpha}\,. \tag{3.14}$$

Then this implies

$$\mathbb{E} \sup_{t\in T} \left|\sum_{k=n}^{\infty} \epsilon_k \psi_k(t)\right| \le c\, n^{-\alpha+1/2} \sqrt{\log n}\,. \tag{3.15}$$

Proof. We will prove the proposition by using Corollary 3.3. Choose $\mathscr{H}$ and $\Phi = (\varphi_k)_{k\ge 1}$ as above and construct u as in (2.2). Then (3.1) and (3.3) hold by assumption and it remains to show that

(3.2) is satisfied. Yet this follows by the special choice of the metric on T. Indeed, for $h \in \mathscr{H}$ and $t, s \in T$ by (3.13) we have

$$
\begin{aligned}
|(uh)(t) - (uh)(s)| &= \left| \sum_{k=1}^{\infty} \langle h, \varphi_k \rangle \left[\psi_k(t) - \psi_k(s) \right] \right| \\
&\leq \left(\sum_{k=1}^{\infty} \left| \langle h, \varphi_k \rangle \right|^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{\infty} \left| \psi_k(t) - \psi_k(s) \right|^2 \right)^{1/2} = \|h\|_{\mathscr{H}} \, d_X(t,s) \,.
\end{aligned}
$$

Consequently, (3.2) holds with $\beta = 1$ and the assertion follows by (2.2) and by Corollary 3.3. □

Remark 3.2. Let us demonstrate on a well–known example how Proposition 3.4 applies. If $B = (B(t))_{0 \le t \le 1}$ denotes the Brownian Motion on $[0,1]$, then it admits the representation

$$
B(t) = \epsilon_0 \, t + \sqrt{2} \sum_{k=1}^{\infty} \epsilon_k \frac{\sin(k\pi t)}{k\pi} \,. \tag{3.16}
$$

Clearly, condition (3.11) holds with $\gamma = 2$ and the representing functions ψ_k satisfy $\left\|\psi_k\right\|_\infty \le c\, k^{-1}$. Consequently, Proposition 3.4 leads to the classical estimate

$$
\mathbb{E} \sup_{t \in [0,1]} \left| \sum_{k=n}^{\infty} \epsilon_k \frac{\sin(k\pi t)}{k\pi} \right| \le c\, n^{-1/2} \sqrt{\log n}
$$

and representation (3.16) of B is optimal. On the other hand, there exist optimal representations of B with functions ψ_k where $\left\|\psi_k\right\|_\infty \approx k^{-1/2}$ (take the representation by the Faber–Schauder system). Thus, in general, neither (3.14) nor (3.3) are necessary for (3.15) or (3.4), respectively.

4 The Trigonometric System

The aim of this section is to investigate whether or not the ONB

$$
\mathbf{T} := \{\mathbf{1}\} \cup \left\{ \sqrt{2}\, \cos(k\pi\, \cdot\,) : k \ge 1 \right\}
$$

is optimal for the Riemann–Liouville operator $R_\alpha : L_2[0,1] \mapsto C[0,1]$ defined by

$$
(R_\alpha h)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} h(x) \,\mathrm{d}x \,, \quad 0 \le t \le 1 \,. \tag{4.1}
$$

Equivalently, we may ask whether or not the representation of the Riemann–Liouville process $R^\alpha = (R^\alpha(t))_{0 \le t \le 1}$ given by

$$
R^\alpha(t) = \epsilon_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\sqrt{2}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \epsilon_k \int_0^t (t-x)^{\alpha-1} \cos(k\pi x) \,\mathrm{d}x
$$

is an optimal one.

Here we shall prove the following:

Theorem 4.1. *The trigonometric system* $\mathbf{T}$ *is optimal for* R_α *from* $L_2[0,1]$ *to* $C[0,1]$ *in the case* $1/2 < \alpha \le 2$. *If* $\alpha > 2$, *then* $\mathbf{T}$ *is rearrangement non–optimal.*

Proof. In a first step we prove that $\mathbf{T}$ is optimal for R_α provided that $1/2 < \alpha \le 2$. We want to apply Proposition 3.1 with $T = [0,1]$ (here the metric on T is the Euclidean metric), $\mathscr{H} = L_2[0,1]$, $\varphi_k(t) = \sqrt{2}\cos(k\pi t)$ and with $u = R_\alpha$. Of course, $T = [0,1]$ satisfies condition (3.1) with $\gamma = 1$, hence it remains to prove that (3.2) and (3.3) are valid as well.

We claim that (3.2) holds for all $\alpha > 1/2$ with $\beta = \alpha - 1/2$ if $\alpha \le 3/2$ and with $\beta = 1$ for $\alpha > 3/2$. Indeed, if $1/2 < \alpha \le 3/2$, then the operator $R_\alpha : L_2[0,1] \mapsto C[0,1]$ is known to be $(\alpha - 1/2)$–Hölder (cf. [21], vol. II, p. 138), i.e. for all $h \in \mathscr{H}$ and $t, s \in [0,1]$ we have

$$\left|(R_\alpha h)(t) - (R_\alpha h)(s)\right| \le c\ \|h\|_2\ |t-s|^{\alpha-1/2}\ .$$

Clearly, this implies (3.2) with $\beta = \alpha - 1/2$ provided that $1/2 < \alpha \le 3/2$.

If $\alpha > 3/2$, then for all $h \in L_2[0,1]$ we have $(R_\alpha h)'(t) = (R_{\alpha-1}h)(t)$, hence

$$\left|(R_\alpha h)(t) - (R_\alpha h)(s)\right| = |t-s|\ \left|(R_{\alpha-1}h)(x)\right| \tag{4.2}$$

for a certain $x \in (t,s)$. Since $\left|(R_{\alpha-1}h)(x)\right| \le \|h\|_2\ \left\|R_{\alpha-1}\right\|$, where the last expression denotes the operator norm of $R_{\alpha-1} : L_2[0,1] \mapsto C[0,1]$, by (4.2) we conclude that (3.2) holds with $\beta = 1$ in the remaining case.

In order to verify (3.3) we first mention the following general result:

If $-1 < \alpha < 0$, then it follows that

$$\sup_{x>0}\left|\int_0^x s^\alpha \sin(x-s)\,\mathrm{d}s\right| < \infty \text{ and } \sup_{x>0}\left|\int_0^x s^\alpha \cos(x-s)\,\mathrm{d}s\right| < \infty\,. \tag{4.3}$$

This is a direct consequence of the well–known fact that $\int_0^\infty s^\alpha \cos(s)\,\mathrm{d}s$ as well as $\int_0^\infty s^\alpha \sin(s)\,\mathrm{d}s$ exist for those α.

The following lemma, which is more or less similar to Lemma 4 in [16], shows that (3.3) holds for $\alpha \le 2$.

Lemma 4.2. *Suppose* $1/2 < \alpha \le 2$ *and let as before* $\varphi_k(t) = \sqrt{2}\cos(k\pi t)$. *Then it follows that*

$$\left\|R_\alpha \varphi_k\right\|_\infty \le c\,k^{-\alpha}\,. \tag{4.4}$$

Proof. For $1/2 < \alpha \le 1$ this was proved in [9], Lemma 5.6. The case $\alpha = 2$ follows by direct calculations. Thus it remains to treat the case $1 < \alpha < 2$. If $\alpha > 1$, then integrating by parts gives

$$\begin{aligned}(R_\alpha\varphi_k)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\cos(k\pi x)\,\mathrm{d}x \\ &= k^{-1}\,\frac{1}{\pi\Gamma(\alpha-1)}\int_0^t (t-x)^{\alpha-2}\sin(k\pi x)\,\mathrm{d}x \\ &= k^{-\alpha}\,\frac{\pi^{-\alpha}}{\Gamma(\alpha-1)}\int_0^{k\pi t} s^{\alpha-2}\sin(k\pi t - s)\,\mathrm{d}s\,. \end{aligned} \tag{4.5}$$

Moreover, if $1 < \alpha < 2$, then (4.3) and (4.5) imply

$$\left\|R_\alpha \varphi_k\right\|_\infty = \sup_{0\le t\le 1} \left|(R_\alpha \varphi_k)(t)\right| \le c\, k^{-\alpha}$$

as asserted. Observe that in that case $-1 < \alpha - 2 < 0$. □

Summing up, R_α and $\mathbf{T}$ satisfy (3.2) and (3.3) of Proposition 3.1 provided that $1/2 < \alpha \le 2$. Thus, we get

$$\mathbb{E}\left\|\sum_{k=2^n}^{2^{n+1}-1} \epsilon_k R_\alpha(\varphi_k)\right\|_\infty \le c\, n^{1/2}\, 2^{-n(\alpha-1/2)}\,,$$

for all $n \ge 1$ and Proposition 2.2 or Corollary 3.3 imply

$$l_n^{o,\mathbf{T}}(R_\alpha) \le c\, n^{-\alpha+1/2}\sqrt{\log n}\,.$$

Recall (cf. [9]) that $l_n(R_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n}$, thus the trigonometric system $\mathbf{T}$ is optimal for R_α in the case $1/2 < \alpha \le 2$.

To complete the proof of Theorem 4.1 we have to show that the basis $\mathbf{T}$ is rearrangement non–optimal for R_α whenever $\alpha > 2$. To verify this we need the following lemma.

Lemma 4.3. *If $\alpha > 2$, then there is a $c > 0$ such that for $k \ge 1$*

$$\left\|R_\alpha \varphi_k\right\|_2 \ge c\, k^{-2}\,. \tag{4.6}$$

Proof. We start with $2 < \alpha < 3$. Using (4.5) we get

$$(R_\alpha \varphi_k)(t) = c_\alpha\, k^{-\alpha} \int_0^{k\pi t} s^{\alpha-2} \sin(k\pi t - s)\,\mathrm{d}s\,.$$

Another integration by parts gives

$$(R_\alpha \varphi_k)(t) = c_\alpha\, k^{-\alpha} \left\{(tk)^{\alpha-2} + g_k(t)\right\} \tag{4.7}$$

where

$$g_k(t) := -(\alpha-2)\int_0^{k\pi t} s^{\alpha-3} \cos(k\pi t - s)\,\mathrm{d}s\,.$$

Since $-1 < \alpha - 3 < 0$, by (4.3) it follows that

$$\sup_{k\ge 1}\left\|g_k\right\|_2 \le \sup_{k\ge 1}\left\|g_k\right\|_\infty < \infty\,. \tag{4.8}$$

Consequently, (4.7) and (4.8) lead to

$$\left\|R_\alpha \varphi_k\right\|_2 \ge c_1\, k^{-2} - c_\alpha\, k^{-\alpha}\, \left\|g_k\right\|_2 \ge c_1\, k^{-2} - c_2\, k^{-\alpha}$$

which proves our assertion in the case $2 < \alpha \le 3$ where $\alpha = 3$ follows by direct calculations.

Suppose now $3 < \alpha < 4$. Another integration by parts in the integral defining g_k gives $g_k = c'_\alpha\, \tilde{g}_k$ with $\sup_{k\ge 1}\left\|\tilde{g}_k\right\|_\infty < \infty$. Then the above arguments lead to (4.6) in this case as well.

We may proceed in that way (in the next step a term of order k^{-4} appears) for all $\alpha > 2$. This completes the proof of the lemma. □

Now we are in position to complete the proof of Theorem 4.1. This is done by using Lemma 2.3. In the notation of this lemma we set $M := \{1, \ldots, 2n\}$ for some given $n \geq 1$, hence we have $m = 2n$. Let L be an arbitrary subset in M with $\#L > m - n$. Using that all moments of Gaussian sums are equivalent it follows that

$$\begin{aligned}\mathbb{E}\Big\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\Big\|_\infty &\geq c\left(\mathbb{E}\Big\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\Big\|_\infty^2\right)^{1/2}\\ &\geq c\left(\mathbb{E}\Big\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\Big\|_2^2\right)^{1/2} = c\left(\sum_{k\in L}\big\|R_\alpha\varphi_k\big\|_2^2\right)^{1/2} \geq c\left(\sum_{k\in L}k^{-4}\right)^{1/2}\end{aligned}$$

where the last estimate follows by Lemma 4.3. Because of $\#L > n$ and $L \subseteq M$ we have

$$\left(\sum_{k\in L}k^{-4}\right)^{1/2} \geq \left(\sum_{k=n}^{2n}k^{-4}\right)^{1/2} \geq c\,n^{-3/2}\,,$$

hence

$$\mathbb{E}\Big\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\Big\|_\infty \geq c\,n^{-3/2}\,.$$

Since $L \subseteq M$ was arbitrary with $\#L > m - n$, Lemma 2.3 leads to

$$l_n^{\mathbf{T}}(R_\alpha) \geq c\,n^{-3/2}\,. \tag{4.9}$$

Yet $l_n(R_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n}$, thus (4.9) shows that $\mathbf{T}$ is rearrangement non–optimal for $\alpha > 2$. $\square$

Remark 4.1. The optimality of the trigonometric system for $1/2 < \alpha \leq 2$ is implicitly known. Indeed, as shown in [17], Proposition 4, estimate (4.4) implies optimality in the Riemann–Liouville setting.

Corollary 4.4. *If $\alpha > 2$, then are constants $c_1, c_2 > 0$ only depending on α such that*

$$c_1\,n^{-3/2} \leq l_n^{\mathbf{T}}(R_\alpha) \leq c_2\,n^{-3/2}\sqrt{\log n}\,. \tag{4.10}$$

Proof. The left hand estimate in (4.10) was proved in (4.9). In order to verify the right hand one we use property (3) of Proposition 2.1. If $\alpha > 2$ this implies

$$l_n^{\mathbf{T}}(R_\alpha) \leq \big\|R_{\alpha-2} : C[0,1] \mapsto C[0,1]\big\|\; l_n^{\mathbf{T}}(R_2) \leq c\,n^{-3/2}\sqrt{\log n}\,.$$

This completes the proof. $\square$

Remark 4.2. We conjecture that for $\alpha > 2$ the right hand side of (4.10) is the correct asymptotic of $l_n^{\mathbf{T}}(R_\alpha)$.

Our next objective is to investigate the ONB

$$\tilde{\mathbf{T}} := \{\mathbf{1}\} \cup \left\{\sqrt{2}\cos(2k\pi\,\cdot), \sqrt{2}\sin(2l\pi\,\cdot) : k, l \geq 1\right\}$$

in $L_2[0,1]$. Let us fix the order of the elements in $\tilde{\mathbf{T}}$ by setting $\varphi_0(t) \equiv 1$, $\varphi_1(t) = \sqrt{2}\cos(2\pi t)$, $\varphi_2(t) = \sqrt{2}\sin(2\pi t)$ and so on.

Theorem 4.5. *Let $\tilde{\mathbf{T}}$ be the ONB defined above. Then $\tilde{\mathbf{T}}$ is optimal for R_α provided that $1/2 < \alpha \le 1$. If $\alpha > 1$ it is rearrangement non–optimal.*

Proof. We start with the case $1/2 < \alpha \le 1$. By using the same method as in the proof of Lemma 5.6 in [9] we have

$$\left\|R_\alpha(\varphi_k)\right\|_\infty \le c\, k^{-\alpha}$$

where as before $\varphi_k \in \tilde{\mathbf{T}}$ are ordered in the natural way. Condition (3.2) holds by the same arguments as in the proof of Theorem 4.1. Thus Corollary 3.3 applies and proves that $\tilde{\mathbf{T}}$ is optimal.

To treat the case $\alpha > 1$ we need the following lemma.

Lemma 4.6. *If $\alpha > 1$, then it follows that*

$$\left\|R_\alpha(\sin(2k\pi\,\cdot))\right\|_\infty \approx \left\|R_\alpha(\sin(2k\pi\,\cdot))\right\|_2 \approx k^{-1} \tag{4.11}$$

Proof. Suppose first $1 < \alpha < 2$ and write

$$\begin{aligned}(R_\alpha \sin(2k\pi\,\cdot))(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\sin(2k\pi x)\,\mathrm{d}x \\ &= \frac{t^{\alpha-1}}{2k\pi\Gamma(\alpha)} - \frac{(2\pi k)^{-\alpha}}{\Gamma(\alpha-1)}\int_0^{2\pi t k} s^{\alpha-2}\cos(2k\pi t - s)\,\mathrm{d}s\,. \end{aligned} \tag{4.12}$$

Since $-1 < \alpha - 2 < 0$, by (4.3) this implies

$$(R_\alpha \sin(2k\pi\,\cdot))(t) = c_1\, k^{-1}\, t^{\alpha-1} + c_2 k^{-\alpha} g_k(t)$$

with $\sup_k \left\|g_k\right\|_\infty < \infty$. Clearly, from this we derive (recall $\alpha > 1$) that

$$\left\|R_\alpha(\sin(2k\pi\,\cdot))\right\|_\infty \le c\, k^{-1}\,.$$

On the other hand,

$$\begin{aligned}\left\|R_\alpha(\sin(2k\pi\,\cdot))\right\|_2 &\ge c_3\, k^{-1} - c_4 k^{-\alpha}\left\|g_k\right\|_2 \\ &\ge c_3\, k^{-1} - c_4 k^{-\alpha}\left\|g_k\right\|_\infty \ge c_3 k^{-1} - c_5\, k^{-\alpha}\end{aligned}$$

completing the proof of the lemma in that case.

If $\alpha = 2$ the assertion follows by direct calculations and for $\alpha > 2$ as in the proof of Theorem 4.1 we integrate by parts as long as we get in (4.12) an exponent of s which is in $(-1, 0)$. □

Now we may finish the proof of Theorem 4.5. To this end fix $n \in \mathbb{N}$ and define $M \subset \mathbb{N}$ as $M := \{2, 4, \ldots, 4n\}$. Recall that the φ_k with even index correspond to the sin–terms. Take now an arbitrary subset $L \subset M$ satisfying $\#L > \#M - n = n$. By the choice of M it follows that

$$\begin{aligned}\mathbb{E}\left\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\right\|_\infty &\ge c\left(\mathbb{E}\left\|\sum_{k\in L}\epsilon_k R_\alpha\varphi_k\right\|_2^2\right)^{1/2} \\ = \;\; c\left(\sum_{k\in L}\left\|R_\alpha\varphi_k\right\|_2^2\right)^{1/2} &\ge c\left(\sum_{l=n}^{2n} l^{-2}\right)^{1/2} \ge c' n^{-1/2}\end{aligned}$$

where we used (4.11) in the last step. The set $L \subset M$ was arbitrary with $\#L > n$, hence, by Lemma 2.3 we obtain

$$l_n^{\tilde{\mathbf{T}}}(R_\alpha) \geq c'\, n^{-1/2} .$$

In view of $l_n(R_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n}$ this implies that $\tilde{\mathbf{T}}$ is rearrangement non–optimal whenever $\alpha > 1$ completing the proof. □

Finally we investigate series representations of the Weyl process. Recall the definition of the Weyl operator of fractional integration. For $\alpha > 1/2$, the Weyl operator I_α is given on exponential functions for all $t \in [0,1]$ by

$$I_\alpha(\mathrm{e}^{2\pi \mathrm{i}k\cdot})(t) := \frac{\mathrm{e}^{2\pi i k t}}{(2\pi \mathrm{i}k)^\alpha}, \quad k \in \mathbb{Z} \setminus \{0\},$$

where for $\alpha \notin \mathbb{N}$, the denominator has to be understood as

$$(2\pi \mathrm{i}k)^{-\alpha} := |2\pi k|^{-\alpha} \cdot \exp(-\frac{\pi\alpha}{2}\,\mathrm{i}\,\mathrm{sgn}(k)) .$$

By linearity and continuity, the definition of I_α can be extended to the complex Hilbert space

$$L_2^0[0,1] := \left\{ f \in L_2[0,1] \, : \, \int_0^1 f(x)\,dx = 0 \right\},$$

thus, I_α is a well–defined operator from $L_2^0[0,1]$ into $C[0,1]$. Note, that it maps real valued functions onto real ones.

Proposition 4.7. *For all $\alpha > 1/2$ it holds*

$$l_n(I_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n} .$$

Proof. Let $e_n(u)$ denote the n–th (dyadic) entropy number of an operator u from $\mathscr{H}$ into a Banach space E (cf. [4] for more information about these numbers). As proved in [11], Proposition 2.1, whenever an operator $u \in \mathscr{G}(\mathscr{H}, E)$ satisfies

$$e_n(u) \leq c_1\, n^{-a}\,(\log n)^\beta$$

for some $a > 1/2$ and $\beta \in \mathbb{R}$, then this implies

$$l_n(u) \leq c_2\, n^{-a+1/2}(\log n)^{\beta+1} .$$

Moreover, as shown in [3], for any $\alpha > 1/2$ it follows that

$$e_n(R_\alpha - I_\alpha) \leq c_1\, \mathrm{e}^{-c_2\, n^{1/3}} .$$

In particular, we have $e_n(R_\alpha - I_\alpha) \leq c_\gamma\, n^{-\gamma}$ for any $\gamma > 0$. Thus, by the above implication, for any $\gamma > 0$ holds $l_n(R_\alpha - I_\alpha) \leq c'_\gamma\, n^{-\gamma}$ as well. Of course, since $l_{2n-1}(I_\alpha) \leq l_n(I_\alpha - R_\alpha) + l_n(R_\alpha)$ by $l_n(R_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n}$ we get $l_n(I_\alpha) \leq c\, n^{-\alpha+1/2}\sqrt{\log n}$. The reverse estimate is proved by exactly the same methods. This completes the proof. □

Remark 4.3. The upper estimate $l_n(I_\alpha) \le c\, n^{-\alpha+1/2}\sqrt{\log n}$ may also be derived from Theorem 4.8 below.

Before proceeding further, let us choose a suitable ONB in the real $L_2^0[0,1]$. We take

$$\mathbf{T}' := \tilde{\mathbf{T}} \setminus \{\mathbf{1}\} = \left\{\sqrt{2}\cos(2k\pi\,\cdot\,),\ \sqrt{2}\sin(2l\pi\,\cdot\,) : k,l \ge 1\right\}$$

and order it in the natural way. Then it holds

Theorem 4.8. *For any $\alpha > 1/2$ the basis $\mathbf{T}'$ is optimal for I_α.*

Proof. Direct calculations give

$$I_\alpha(\cos(2k\pi\,\cdot\,))(t) = \frac{\cos(2k\pi t - \alpha\pi/2)}{(2\pi k)^\alpha}$$

as well as

$$I_\alpha(\sin(2l\pi\,\cdot\,))(t) = \frac{\sin(2l\pi t - \alpha\pi/2)}{(2\pi l)^\alpha}\ .$$

Consequently, condition (3.3) in Proposition 3.4 holds for I_α and $\mathbf{T}'$.

We claim now that for $1/2 < \alpha < 3/2$ it follows that

$$\left|(I_\alpha h)(t) - (I_\alpha h)(s)\right| \le c\ \|h\|_2\ |t-s|^{\alpha-1/2} \tag{4.13}$$

for all $h \in L_2^0[0,1]$ and $t,s \in [0,1]$. This is probably well–known, yet since it is easy to prove we shortly verify (4.13). It is a direct consequence of

$$\begin{aligned}\sum_{k=1}^\infty \frac{\left|1-\mathrm{e}^{2\pi i k\varepsilon}\right|^2}{k^{2\alpha}} &= \sum_{k\le 1/\varepsilon} \frac{\left|1-\mathrm{e}^{2\pi i k\varepsilon}\right|^2}{k^{2\alpha}} + \sum_{k>1/\varepsilon} \frac{\left|1-\mathrm{e}^{2\pi i k\varepsilon}\right|^2}{k^{2\alpha}}\\ &\le c_1\,\varepsilon^2 \sum_{k\le 1/\varepsilon} k^{-2\alpha+2} + 4\sum_{k>1/\varepsilon} k^{-2\alpha} \le c\,\varepsilon^{-2\alpha+1}\end{aligned}$$

for any $k \ge 1$ and $\varepsilon > 0$ small enough. Clearly, (4.13) shows that (3.2) holds with $\beta = \alpha - 1/2$ as long as $1/2 < \alpha < 3/2$. The identity $I_{\alpha_1} \circ I_{\alpha_2} = I_{\alpha_1+\alpha_2}$ implies the following: Suppose that

$$\left|(I_\alpha h)(t) - (I_\alpha h)(s)\right| \le c\ \|h\|_2\ |t-s|^\beta \tag{4.14}$$

for some $\alpha > 1/2$ and $\beta \in (0,1]$. Then for any $\alpha' \ge \alpha$ estimate (4.14) is also valid (with the same β). This, for example, follows from the fact that (4.14) is equivalent to

$$\left\|I_\alpha^*\delta_t - I_\alpha^*\delta_s\right\|_2 \le c\ |t-s|^\beta$$

together with the semi–group property of I_α, hence also of I_α^*. Here I_α^* denotes the dual operator of I_α, mapping $C[0,1]^*$ into $L_2^0[0,1]$, and δ_t denotes the Dirac point measure concentrated at $t \in [0,1]$.

Summing up, we see that I_α and $\mathbf{T}'$ satisfy (3.2) and (3.3). Clearly, (3.1) holds for $T = [0,1]$. Thus Corollary 3.3 applies and completes the proof. Recall that by Proposition 4.7 we have $l_n(I_\alpha) \approx n^{-\alpha+1/2}\sqrt{\log n}$. □

Remark 4.4. It may be a little bit surprising that for all $\alpha > 1/2$ the basis $\mathbf{T}'$ is optimal for I_α while $\tilde{\mathbf{T}}$ is not for R_α in the case $\alpha > 1$. Recall that $l_n(I_\alpha - R_\alpha)$ tends to zero exponentially and, if $\alpha \in \mathbb{N}$, then I_α and R_α differ only by a finite rank operator, i.e., we even have $l_n(I_\alpha - R_\alpha) = 0$ for large n. The deeper reason for this phenomenon is that $l_n^{\mathbf{T}'}(I_\alpha - R_\alpha)$ tends to zero slower than $l_n^{\mathbf{T}'}(I_\alpha)$.

5 Haar Basis

5.1 Some useful notations and some preliminary results

Recall (cf. (1.4)) that for any parameter $\alpha > 1/2$, the Riemann-Liouville process can be written as

$$R^{\alpha}(t) = \epsilon_{-1}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\epsilon_{j,k}(R_{\alpha}h_{j,k})(t), \tag{5.1}$$

where R_{α} is the Riemann-Liouville operator and the $h_{j,k}$'s are the usual Haar functions, i.e.

$$h_{j,k}(x) = 2^{j/2}\Big\{\mathbf{1}_{[\frac{2k}{2^{j+1}},\frac{2k+1}{2^{j+1}})}(x) - \mathbf{1}_{[\frac{2k+1}{2^{j+1}},\frac{2k+2}{2^{j+1}})}(x)\Big\} \tag{5.2}$$

and where the series converges almost surely uniformly in t (i.e. in the sense of the norm $\|\cdot\|_{\infty}$). For any $t \in [0,1]$ and $J \in \mathbb{N}$ we set,

$$R_J^{\alpha}(t) := \epsilon_{-1}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\epsilon_{j,k}(R_{\alpha}h_{j,k})(t), \tag{5.3}$$

$$\widetilde{\sigma}_J^2(t) := \mathbb{E}\left|R^{\alpha}(t) - R_J^{\alpha}(t)\right|^2 = \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}\left|(R_{\alpha}h_{j,k})(t)\right|^2 \tag{5.4}$$

and

$$\widetilde{\sigma}_J^2 := \sup_{t\in[0,1]}\widetilde{\sigma}_J^2(t). \tag{5.5}$$

In order to conveniently express the coefficients $(R_{\alpha}h_{j,k})(t)$, for any reals ν and x we set

$$(x)_+^{\nu} = \begin{cases} x^{\nu} & \text{when } x > 0, \\ 0 & \text{else .}\end{cases} \tag{5.6}$$

Then it follows from (4.1), (5.2) and (5.6) that one has for every integers $j \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $0 \le k \le 2^j - 1$ and real $t \in [0,1]$,

$$(R_{\alpha}h_{j,k})(t) = \frac{2^{j/2}}{\Gamma(\alpha+1)}\left\{\Big(t - \frac{2k+2}{2^{j+1}}\Big)_+^{\alpha} - 2\Big(t - \frac{2k+1}{2^{j+1}}\Big)_+^{\alpha} + \Big(t - \frac{2k}{2^{j+1}}\Big)_+^{\alpha}\right\}. \tag{5.7}$$

Let us now give some useful lemmas. The following lemma can be proved similarly to Lemma 1 in [2], this is why we omit its proof.

Lemma 5.1. *Let $\{\epsilon_{j,k} : j \in \mathbb{N}_0, 0 \le k \le 2^j - 1\}$ be a sequence of standard Gaussian variables. Then there exists a random variable $C_1 > 0$ of finite moment of any order such that one has almost surely, for every $j \in \mathbb{N}_0$ and $0 \le k < 2^j - 1$,*

$$|\epsilon_{j,k}| \le C_1\sqrt{1+j}\,.$$

Let us now give a lemma that allows to control the increments of the Riemann-Liouville process. This result is probably known however we will give its proof for the sake of completeness.

Lemma 5.2.

(i) For any $\alpha \in (1/2, 3/2)$, there is a random variable $C_2 > 0$ of finite moment of any order such that almost surely for every $t_1, t_2 \in [0,1]$,

$$|R^\alpha(t_1) - R^\alpha(t_2)| \le C_2 |t_1 - t_2|^{\alpha - 1/2} \sqrt{\log(2 + |t_1 - t_2|^{-1})}. \tag{5.8}$$

(ii) For any $\alpha > 3/2$, there is a random variable $C_3 > 0$ of finite moment of any order such that one has almost surely for every $t_1, t_2 \in [0,1]$,

$$|R^\alpha(t_1) - R^\alpha(t_2)| \le C_3 |t_1 - t_2|.$$

Proof. (of Lemma 5.2) Part (ii) is a straightforward consequence of the fact that the trajectories of R^α are continuously differentiable functions when $\alpha > 3/2$. Let us now prove part (i). First observe (see for instance relation (7.6) in [12]) that inequality (5.8) is satisfied when the Riemann-Liouville process is replaced by the fractional Brownian motion (fBm) $(B^H(t))_{0 \le t \le 1}$ with Hurst index $H := \alpha - 1/2$. Recall that for some $c_\alpha > 0$ (again H and α are related via $H = \alpha - 1/2$)

$$B^H(t) = Q^\alpha(t) + c_\alpha R^\alpha(t), \tag{5.9}$$

where the process $(Q^\alpha(t))_{t \in [0,1]}$ is called the low-frequency part of fBm and up to a positive constant it is defined as

$$Q^\alpha(t) = \int_{-\infty}^{0} \left\{ (t - x)^{\alpha - 1} - (-x)^{\alpha - 1} \right\} \mathrm{d}W(x).$$

Finally, it is well-known that the trajectories of the process $(Q^\alpha(t))_{t \in [0,1]}$ are C^∞-functions. Therefore, it follows from (5.9) that $(R^\alpha(t))_{0 \le t \le 1}$ satisfies (5.8) as well. □

Remark 5.1. It is very likely that (5.8) also holds for $\alpha = 3/2$. But in this case our approach does not apply. Observe that $\alpha = 3/2$ corresponds to $H = 1$ and (5.9) is no longer valid.

For any reals $\gamma > 0$ and $t \in [0,1]$ and for any integers $j \in \mathbb{N}_0$ and $0 \le k \le 2^j - 1$ we set

$$A_{\gamma,j,k}(t) := \Big(t - \frac{2k+2}{2^{j+1}} \Big)_+^\gamma - 2\Big(t - \frac{2k+1}{2^{j+1}} \Big)_+^\gamma + \Big(t - \frac{2k}{2^{j+1}} \Big)_+^\gamma. \tag{5.10}$$

Observe that (5.10) and (5.7) imply that

$$(R_\alpha h_{j,k})(t) = \frac{2^{j/2}}{\Gamma(\alpha+1)} A_{\alpha,j,k}(t). \tag{5.11}$$

Furthermore, we denote by $\widetilde{k}_j(t)$ the unique integer satisfying the following property:

$$\frac{\widetilde{k}_j(t)}{2^j} \le t < \frac{\widetilde{k}_j(t) + 1}{2^j}, \tag{5.12}$$

with the convention that $\widetilde{k}_j(1) = 2^j - 1$.

The following lemma allows us to estimate $(R_\alpha h_{j,k})(t)$ suitably.

Lemma 5.3.

(i) *For each $k \geq \widetilde{k}_j(t)+1$, one has $A_{\gamma,j,k}(t)=0$.*

(ii) *There is a constant $c_4>0$, only depending on γ, such that the inequality*

$$|A_{\gamma,j,k}(t)| \leq c_4 2^{-j\gamma}\Big(1+\widetilde{k}_j(t)-k\Big)^{\gamma-2} \tag{5.13}$$

holds when $0 \leq k \leq \widetilde{k}_j(t)$.

(iii) *If $\gamma \neq 1$, then there is a constant $c_5>0$, only depending on γ, such that the inequality*

$$|A_{\gamma,j,k}(t)| \geq c_5 2^{-j\gamma}\Big(1+\widetilde{k}_j(t)-k\Big)^{\gamma-2} \tag{5.14}$$

holds when $0 \leq k \leq \widetilde{k}_j(t)-2$.

Proof. (of Lemma 5.3) Part (i) is a straightforward consequence of (5.6), (5.10) and (5.12), so we will focus on parts (ii) and (iii). Inequality (5.13) clearly holds when $\gamma=1$, this is why we will assume in all the sequel that $\gamma \neq 1$. Let us first show that (5.13) is satisfied when

$$\widetilde{k}_j(t)-1 \leq k \leq \widetilde{k}_j(t). \tag{5.15}$$

Putting together (5.12) and (5.15) one has for any $l \in \{0,1,2\}$,

$$\Big(t-\frac{2k+l}{2^{j+1}}\Big)_+^{\gamma} \leq \Big|t-\frac{\widetilde{k}_j(t)-1}{2^j}\Big|^{\gamma} \leq 2^{-(j-1)\gamma}.$$

Therefore, it follows from (5.10) and (5.15) that

$$|A_{\gamma,j,k}(t)| \leq 2^{2-(j-1)\gamma} \leq c_6 2^{-j\gamma}\Big(1+\widetilde{k}_j(t)-k\Big)^{\gamma-2},$$

where the constant $c_6=2^{2+\gamma}\max\{1,2^{2-\gamma}\}$. Let us now show that the inequalities (5.13) and (5.14) are verified when

$$0 \leq k \leq \widetilde{k}_j(t)-2. \tag{5.16}$$

We denote by $f_{\gamma,j,t}$ the function defined for every real $x \leq k/2^j$ as

$$f_{\gamma,j,t}(x)=\Big(t-x-2^{-j}\Big)^{\gamma}-\Big(t-x-2^{-j-1}\Big)^{\gamma}.$$

By applying the Mean Value Theorem to $f_{\gamma,j,t}$ on the interval $[\frac{2k-1}{2^{j+1}},\frac{k}{2^j}]$, it follows that there exists $a_1 \in (\frac{2k-1}{2^{j+1}},\frac{k}{2^j})$ such that

$$\begin{aligned} A_{\gamma,j,k}(t) &= f_{\gamma,j,t}\Big(\frac{k}{2^j}\Big)-f_{\gamma,j,t}\Big(\frac{2k-1}{2^{j+1}}\Big)=2^{-j-1}f'_{\gamma,j,t}(a_1)\\ &= -\gamma\, 2^{-j-1}\Big\{\Big(t-a_1-2^{-j}\Big)^{\gamma-1}-\Big(t-a_1-2^{-j-1}\Big)^{\gamma-1}\Big\}. \end{aligned} \tag{5.17}$$

Next, by applying the Mean Value Theorem to the function

$$y \mapsto \Big(t - a_1 - y\Big)^{\gamma-1}$$

on the interval $[2^{-j-1}, 2^{-j}]$, it follows that there exists $a_2 \in (2^{-j-1}, 2^{-j})$ such that

$$\begin{aligned}&\Big(t-a_1-2^{-j}\Big)^{\gamma-1} - \Big(t-a_1-2^{-j-1}\Big)^{\gamma-1}\\ &\quad = -(\gamma-1)2^{-j-1}\Big(t-a_1-a_2\Big)^{\gamma-2}.\end{aligned} \tag{5.18}$$

Observe that the inequalities (5.12), $\frac{k}{2^j} < a_1 + a_2 < \frac{k+1}{2^j}$ and (5.16) imply that

$$\begin{aligned}\frac{1}{2^j} \le \frac{\widetilde{k}_j(t)-k-1}{2^j} < t-a_1-a_2 &< \frac{\widetilde{k}_j(t)-k-1+2}{2^j}\\ &\le \frac{3(\widetilde{k}_j(t)-k-1)}{2^j}.\end{aligned} \tag{5.19}$$

Next setting $c_4 = c_6 + \gamma|\gamma-1|\max\{1, 3^{\gamma-2}\}$ and $c_5 = \gamma|\gamma-1|\min\{1, 3^{\gamma-2}\}$ and combining (5.17) with (5.18) and (5.19), it follows that the inequalities (5.13) and (5.14) are verified when (5.16) holds. □

5.2 Optimality when $1/2 < \alpha < 1$

The goal of this section is to prove the following theorem.

Theorem 5.4. *Suppose* $1/2 < \alpha < 1$. *Then there is a random variable* $C_7 > 0$ *of finite moments of any order such that one has almost surely, for every* $J \in \mathbb{N}$,

$$\|R^\alpha - R^\alpha_J\|_\infty \le C_7 2^{-(\alpha-1/2)J}\sqrt{1+J}.$$

In particular, this implies that in this case representation (5.1) *possesses the optimal approximation rate.*

Proof. (of Theorem 5.4) Putting together (5.1), (5.3), Lemma 5.1, (5.11) and (5.13), one obtains that almost surely, for every $t \in [0,1]$, and every integer $J \in \mathbb{N}$,

$$\begin{aligned}|R^\alpha(t) - R^\alpha_J(t)| &\le \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} |\epsilon_{j,k}||(R_\alpha h_{j,k})(t)|\\ &\le C_1\Gamma(\alpha+1)^{-1}c_4\sum_{j=J}^{\infty}2^{-j(\alpha-1/2)}\sqrt{j+1}\sum_{k=0}^{\widetilde{k}_j(t)}\Big(1+\widetilde{k}_j(t)-k\Big)^{\alpha-2}\\ &\le C_7 2^{-J(\alpha-1/2)}\sqrt{J+1}.\end{aligned}$$

□

Observe that the condition $1/2 \le \alpha < 1$ plays a crucial role in the proof of Theorem 5.4. Indeed, one has $\sum_{k=0}^{\widetilde{k}_j(t)}\Big(1+\widetilde{k}_j(t)-k\Big)^{\alpha-2} \le \sum_{l=1}^{\infty} l^{\alpha-2} < \infty$ only when it is satisfied.

5.3 Optimality when $1 < \alpha < 3/2$

The goal of this subsection is to show that the following theorem holds.

Theorem 5.5. *Suppose $1 < \alpha < 3/2$. Then there is a constant $c_8 > 0$ such that for every $J \in \mathbb{N}$ one has*

$$\mathbb{E}\,\|R^{\alpha} - R_J^{\alpha}\|_{\infty} \le c_8 2^{-J(\alpha-1/2)}\sqrt{J+1}.$$

In particular, this implies that also in this case representation (5.1) *possesses the optimal approximation rate.*

First we need to prove some preliminary results.

Proposition 5.6. *If $1/2 < \alpha < 3/2$, there exists a constant $c_9 > 0$ such that one has for any $J \in \mathbb{N}$,*

$$\widetilde{\sigma}_J^2 \le c_9^2\, 2^{-J(2\alpha-1)}.$$

Proof. (of Proposition 5.6) It follows from (5.4), (5.11) and parts (i) and (ii) of Lemma 5.3, that

$$\widetilde{\sigma}_J^2(t) \le c_4^2\Gamma(\alpha+1)^{-2}\sum_{j=J}^{\infty} 2^{-j(2\alpha-1)}\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{-2(2-\alpha)} \le c_9^2 2^{-J(2\alpha-1)},$$

where the constant $c_9^2 = c_4^2\Gamma(\alpha+1)^{-2}(1-2^{-(2\alpha-1)})\sum_{l=1}^{\infty} l^{-2(2-\alpha)} < \infty$. □

Lemma 5.7. *For any $\alpha \in (1,3/2)$, there exists a random variable $C_{10} > 0$ of finite moment of any order such that one has almost surely for any real $t \in [0,1]$ and any integer $J \in \mathbb{N}$,*

$$\left|R_J^{\alpha}(t) - R_J^{\alpha}\left(\widetilde{k}_J(t)2^{-J}\right)\right| \le C_{10}2^{-J(\alpha-1/2)}\sqrt{J+1}.$$

We refer to (5.12) *for the definition of the integer $\widetilde{k}_J(t)$.*

In order to be able to prove Lemma 5.7 we need the following lemma.

Lemma 5.8. *For any real $\alpha > 1$, there exists a constant $c_{11} > 0$ such that for all $t \in [0,1]$, $J \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ satisfying*

$$0 \le j \le J \text{ and } 0 \le k \le \widetilde{k}_j(t),$$

one has

$$\left|(R_{\alpha}h_{j,k})(t) - (R_{\alpha}h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right| \le c_{11}2^{(3/2-\alpha)j-J}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3}. \tag{5.20}$$

Proof. (Proof of Lemma 5.8) It is clear that (5.20) holds when $t = \widetilde{k}_J(t)2^{-J}$, so we will assume that $t \neq \widetilde{k}_J(t)2^{-J}$. By applying the Mean Value Theorem, it follows that there exists $a \in (\widetilde{k}_J(t)2^{-J}, t)$ such that

$$\begin{aligned}&\left|(R_{\alpha}h_{j,k})(t) - (R_{\alpha}h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right| \\ &= \frac{\alpha 2^{j/2-J}}{\Gamma(\alpha)}\left|\left(a-\frac{2k+2}{2^{j+1}}\right)_+^{\alpha-1} - 2\left(a-\frac{2k+1}{2^{j+1}}\right)_+^{\alpha-1} + \left(a-\frac{2k}{2^{j+1}}\right)_+^{\alpha-1}\right|.\end{aligned} \tag{5.21}$$

Observe that one has $\widetilde{k}_j(a) = \widetilde{k}_j(t)$ since $a \in \left(\frac{\widetilde{k}_J(t)}{2^J}, t\right) \subset \left(\frac{\widetilde{k}_j(t)}{2^j}, \frac{\widetilde{k}_j(t)+1}{2^j}\right)$. Thus, putting together, (5.21), (5.10) and (5.13) in which we replace t by a and γ by $\alpha-1$, we obtain the lemma. □

We are now in position to prove Lemma 5.7.

Proof. (of Lemma 5.7) By using Lemma 5.1 and the fact that

$$0 \le t^{\alpha} - \left(\widetilde{k}_J(t)2^{-J}\right)^{\alpha} \le \alpha 2^{-J},$$

one gets that

$$\begin{aligned}
\left|R_J^{\alpha}(t) - R_J^{\alpha}\left(2^{-J}\widetilde{k}_J(t)\right)\right| &\le |\epsilon_{-1}|\frac{\alpha 2^{-J}}{\Gamma(\alpha+1)} \\
&+ \quad C_1\sqrt{J+1}\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left|(R_{\alpha}h_{j,k})(t) - (R_{\alpha}h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right|. \qquad (5.22)
\end{aligned}$$

On the other hand, it follows from Lemma 5.8 that

$$\begin{aligned}
&\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left|(R_{\alpha}h_{j,k})(t) - (R_{\alpha}h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right| \\
&\le \quad c_{11}\sum_{j=0}^{J-1}2^{(3/2-\alpha)j-J}\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3} \le c_{12}2^{-(\alpha-1/2)J} \qquad (5.23)
\end{aligned}$$

where $c_{12} = c_{11}\left(2^{3/2-\alpha}-1\right)^{-1}\sum_{l=1}^{\infty} l^{\alpha-3} < \infty$. Finally combining (5.22) with (5.23) one obtains the lemma. □

Lemma 5.9. *There is a random variable $C_{13} > 0$ of finite moments of any order such that one has almost surely for every $J \in \mathbb{N}$*

$$\begin{aligned}
&\sup_{t\in[0,1]}\left|R^{\alpha}(t) - R_J^{\alpha}(t)\right| \\
&\le \sup_{0\le K<2^J,\, K\in\mathbb{N}_0}\left|R^{\alpha}(K2^{-J}) - R_J^{\alpha}(K2^{-J})\right| + C_{13}\,2^{-J(\alpha-1/2)}\sqrt{J+1}.
\end{aligned}$$

Proof. (of Lemma 5.9) Let us fix ω. As the function $t \mapsto R^{\alpha}(t,\omega) - R_J^{\alpha}(t,\omega)$ is continuous over the compact interval $[0,1]$, there exist a $t_0 \in [0,1]$ such that

$$\sup_{t\in[0,1]}\left|R^{\alpha}(t,\omega) - R_J^{\alpha}(t,\omega)\right| = \left|R^{\alpha}(t_0,\omega) - R_J^{\alpha}(t_0,\omega)\right|.$$

Using the triangular inequality and Lemmas 5.2 (i) and 5.7, it follows that

$$\begin{aligned}
\left|R^{\alpha}(t_0,\omega) - R_J^{\alpha}(t_0,\omega)\right| &\le \left|R^{\alpha}(t_0,\omega) - R^{\alpha}(\widetilde{k}_J(t_0)2^{-J},\omega)\right| \\
&+ \quad \left|R^{\alpha}(\widetilde{k}_J(t_0)2^{-J},\omega) - R_J^{\alpha}(\widetilde{k}_J(t_0)2^{-J},\omega)\right| \\
&+ \quad \left|R_J^{\alpha}(2^{-J}\widetilde{k}_J(t_0),\omega) - R_J^{\alpha}(t_0,\omega)\right| \\
&\le \quad C_2(\omega)\,2^{-J(\alpha-1/2)}\sqrt{\log(2+2^J)} \\
&+ \quad \sup_{0\le K<2^J, K\in\mathbb{N}_0}\left|R^{\alpha}(K2^{-J},\omega) - R_J^{\alpha}(K2^{-J},\omega)\right| + C_{10}(\omega)\,2^{-J(\alpha-1/2)}\sqrt{J+1}
\end{aligned}$$

and thus one gets the lemma. □

Proof. (of Theorem 5.5) Putting together Lemma 5.9, Lemma 3.2, the fact that $\widetilde{\sigma}_J^2 \geq \sup_{0 \leq K < 2^J, K \in \mathbb{N}_0} \widetilde{\sigma}_J^2(2^{-J}K)$ and Proposition 5.6 one obtains the theorem. □

5.4 The case $\alpha = 3/2$

Recall (cf. [9]) that $l_n(R_{3/2}) \approx n^{-1}\sqrt{\log n}$; this clearly implies that

$$l_n^{\mathbf{H}}(R_{3/2}) \geq c n^{-1} \sqrt{\log n}.$$

The goal of this subsection is to show that a slightly stronger result holds, namely the following theorem.

Theorem 5.10. *There exists a constant $c_{14} > 0$ such that, for any $n \in \mathbb{N}$ and any set $N \subseteq \{(j,k) \in \mathbb{N}^2 : 0 \leq k \leq 2^j - 1\}$ satisfying $\#N < n$ one has*

$$\Big(2^{-J_n} \sum_{l=0}^{2^{J_n}-1} \sum_{(j,k)\notin N} \Big|(R_{3/2}h_{j,k})(l/2^{J_n})\Big|^2\Big)^{1/2} \geq c_{14} n^{-1} \sqrt{\log n},$$

where $J_n \geq 2$ is the unique integer such that $2^{J_n-2} \leq n < 2^{J_n-1}$.

Remark 5.2. A straightforward consequence of (5.4), (5.5) and Theorem 5.10 is that for all $J \in \mathbb{N}_0$ one has

$$\widetilde{\sigma}_J \geq c_{14} 2^{-J} \sqrt{J+1}.$$

In fact by using the same technics as before one can prove that $2^{-J}\sqrt{J+1}$ is the right order of $\widetilde{\sigma}_J$ i.e. one has for some constant $c_{15} > 0$ and all $J \in \mathbb{N}_0$, $\widetilde{\sigma}_J \leq c_{15} 2^{-J}\sqrt{J+1}$. But observe that $\widetilde{\sigma}_J \approx 2^{-J}\sqrt{J+1}$ does, unfortunately, not answer the question whether or not representation (5.1) is optimal in the case $\alpha = 3/2$.

Proof. (of Theorem 5.10) Let us set

$$M := \Big\{k \in \mathbb{N} : 0 \leq k \leq 2^{J_n-1} - 1 \text{ and } (J_n, k) \notin N\Big\}.$$

Clearly,

$$\#M \geq 2^{J_n} - \#N > 2^{J_n-1} \tag{5.24}$$

and

$$2^{-J_n} \sum_{l=0}^{2^{J_n}-1} \sum_{(j,k)\notin N} \Big|(R_{3/2}h_{j,k})(l/2^{J_n})\Big|^2 \geq 2^{-J_n} \sum_{l=0}^{2^{J_n}-1} \sum_{k \in M} \Big|(R_{3/2}h_{J_n,k})(l/2^{J_n})\Big|^2. \tag{5.25}$$

Putting together (5.11) in which we replace α by 3/2, (5.14) in which we replace γ by 3/2, the fact that $\widetilde{k}_{J_n}(l/2^{J_n}) = l$ for any integer l satisfying $0 \le l \le 2^{J_n} - 1$ and (5.24), it follows that

$$\begin{aligned}
& 2^{-J_n} \sum_{l=0}^{2^{J_n}-1} \sum_{k \in M} \left| (R_{3/2} h_{J_n,k})(l/2^{J_n}) \right|^2 \\
\ge \; & c_5^2 \, \Gamma(\alpha+1)^{-2} 2^{-3J_n} \sum_{k \in M} \sum_{l=k+2}^{2^{J_n}-1} \Big(l - k + 1 \Big)^{-1} \\
\ge \; & c_5^2 \, \Gamma(\alpha+1)^{-2} 2^{-3J_n} \sum_{k=2^{J_n-1}-1}^{2^{J_n}-3} \; \sum_{p=1}^{2^{J_n}-k-2} (p+2)^{-1} \\
\ge \; & c_5^2 \, \Gamma(\alpha+1)^{-2} 2^{-3J_n} \sum_{n=1}^{2^{J_n-2}} \Big(2^{J_n-1} - n \Big) (n+2)^{-1} \\
\ge \; & c_5^2 \Gamma(\alpha+1)^{-2} 2^{-2J_n-2} \sum_{n=1}^{2^{J_n-2}} (n+2)^{-1} \\
\ge \; & c_5^2 \, \Gamma(\alpha+1)^{-2} 2^{-2J_n-2} \int_1^{2^{J_n-2}+1} (x+2)^{-1} \, dx \\
\ge \; & c_{14}^2 n^{-2} \log n, \qquad (5.26)
\end{aligned}$$

with the convention that $\sum_{l=k+2}^{2^{J_n}-1} \cdots = \sum_{p=1}^{2^{J_n}-k-2} = \cdots = 0$ whenever we have $2^{J_n} - 2 \le k \le 2^{J_n} - 1$. Finally combining (5.25) with (5.26) we obtain the theorem □

5.5 Non–optimality of the Haar basis for $\alpha > 3/2$

The goal of this subsection is to prove the following theorem.

Theorem 5.11. *If $\alpha > 3/2$, then we have:*

(i) For any $t_0 \in (0,1]$ there exists a constant $c_{16} > 0$ such that, for each $n \in \mathbb{N}$ and each set $N \subseteq \{(j,k) \in \mathbb{N}^2 : 0 \le k \le 2^j - 1\}$, satisfying $\#N < n$ one has

$$\Big(\sum_{(j,k) \notin N} \left| (R_\alpha h_{j,k})(t_0) \right|^2 \Big)^{1/2} \ge c_{16} n^{-1}. \qquad (5.27)$$

(ii) There exists a constant $c_{17} > 0$ such that for each $J \in \mathbb{N}$ one has

$$\mathbb{E} \, \| R^\alpha - R_J^\alpha \|_\infty \le c_{17} 2^{-J} \sqrt{J+1}.$$

A straightforward consequence of Theorem 5.11 is that the Haar basis $\mathbf{H}$ is rearrangement non–optimal for R_α when $\alpha > 3/2$. More precisely:

Corollary 5.12. *If $\alpha > 3/2$, then there are two constant $0 < c \le c'$, only depending on α, such that for each $n \ge 2$ one has*

$$c \, n^{-1} \le l_n^{\mathbf{H}}(R_\alpha) \le c' \, n^{-1} \sqrt{\log n} \, .$$

Remark 5.3. We conjecture that

$$l_n^{\mathbf{H}}(R_\alpha) \approx n^{-1}\sqrt{\log n}$$

for all $\alpha > 3/2$.

Proof. (of Theorem 5.11 part (i)) Let $J_n \geq 4$ be the unique integer with

$$2^{J_n-4}t_0 \leq n < 2^{J_n-3}t_0 \tag{5.28}$$

and set

$$L := \Big\{k \in \mathbb{N} : 0 \leq k \leq \widetilde{k}_{J_n}(t_0) - 2 \text{ and } (J_n, k) \notin N\Big\}. \tag{5.29}$$

It is clear that

$$\sum_{(j,k)\notin N} \Big|(R_\alpha h_{j,k})(t_0)\Big|^2 \geq \sum_{k \in L} \Big|(R_\alpha h_{J,k})(t_0)\Big|^2. \tag{5.30}$$

Moreover, it follows from (5.29), (5.28) and (5.12) that

$$\#L \geq \widetilde{k}_{J_n}(t_0) - 1 - \#N \geq \widetilde{k}_{J_n}(t_0) - 2^{J_n-3}t_0 > 3 \cdot 2^{J_n-2}t_0. \tag{5.31}$$

On the other hand (5.11) and (5.14) imply that

$$\sum_{k\in L} \Big|(R_\alpha h_{J,k})(t_0)\Big|^2 \geq c_5^2 2^{-(2\alpha-1)J_n} \sum_{k\in L} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)}. \tag{5.32}$$

Let us now assume that $3/2 < \alpha < 2$; then (5.31), the fact that

$$x \mapsto \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)}$$

is an increasing function on $[0, \widetilde{k}_{J_n}(t_0)]$ and (5.12) imply that

$$\begin{aligned}
\sum_{k\in L} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)} &\geq \sum_{k=0}^{[3\cdot 2^{J_n-2}t_0]} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)} \\
&\geq \int_1^{[3\cdot 2^{J_n-2}t_0]} \Big(1 + \widetilde{k}_{J_n}(t_0) - x\Big)^{2(\alpha-2)} dx \\
&\geq c_{18} 2^{J_n(2\alpha-3)},
\end{aligned} \tag{5.33}$$

where $c_{18} > 0$ is a constant only depending on t_0 and α. Next, let us assume that $\alpha \geq 2$; then (5.31), the fact that $x \mapsto \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)}$ is an nonincreasing function on $[0, \widetilde{k}_{J_n}(t_0)]$ and (5.12) entail that

$$\begin{aligned}
&\sum_{k\in L} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)} \\
&\geq \sum_{k=\widetilde{k}_{J_n}(t_0)-2-[3\cdot 2^{J_n-2}t_0]}^{\widetilde{k}_{J_n}(t_0)-2} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)} \\
&\geq \int_{\widetilde{k}_{J_n}(t_0)-2-[3\cdot 2^{J_n-2}t_0]}^{\widetilde{k}_{J_n}(t_0)-3} \Big(1 + \widetilde{k}_{J_n}(t_0) - k\Big)^{2(\alpha-2)} dx \\
&\geq c_{19} 2^{J_n(2\alpha-3)},
\end{aligned} \tag{5.34}$$

where $c_{19} > 0$ is a constant only depending on t_0 and α. Finally, putting together, (5.30), (5.32), (5.33), (5.34) and (5.28) one obtains (5.27). □

In order to be able to prove part (ii) of Theorem 5.11 we need some preliminary results.

Proposition 5.13. *If $\alpha > 3/2$, there exists a constant $c_{17} > 0$ such that one has for any $J \in \mathbb{N}$,*

$$\widetilde{\sigma}_J^2 \le c_{17}^2 2^{-2J}.$$

Proof. (of Proposition 5.13) It follows from (5.4), (5.11) and parts (i) and (ii) of Lemma 5.3, that

$$\widetilde{\sigma}_J^2(t) \le c_4^2 \Gamma(\alpha+1)^{-2} \sum_{j=J}^{\infty} 2^{-j(2\alpha-1)} \sum_{k=0}^{\widetilde{k}_j(t)} \Big(1+\widetilde{k}_j(t)-k\Big)^{-2(2-\alpha)} \tag{5.35}$$

Let us assume that $3/2 < \alpha < 2$; then the fact that

$$x \mapsto \Big(1+\widetilde{k}_{J_n}(t_0)-k\Big)^{2(\alpha-2)}$$

is an increasing function on $[0, \widetilde{k}_{J_n}(t_0)+1)$ implies

$$\begin{aligned} \sum_{k=0}^{\widetilde{k}_j(t)} \Big(1+\widetilde{k}_j(t)-k\Big)^{-2(2-\alpha)} &\le \int_0^{\widetilde{k}_j(t)+1} \Big(1+\widetilde{k}_j(t)-x\Big)^{-2(2-\alpha)} dx \\ &\le c_{20} 2^{2\alpha-3}, \end{aligned} \tag{5.36}$$

where $c_{20} > 0$ is a constant only depending on α. Next let us assume that $\alpha \ge 2$; then the fact that $x \mapsto \Big(1+\widetilde{k}_{J_n}(t_0)-k\Big)^{2(\alpha-2)}$ is an nonincreasing function on $[-1, \widetilde{k}_{J_n}(t_0)]$ entails that

$$\begin{aligned} \sum_{k=0}^{\widetilde{k}_j(t)} \Big(1+\widetilde{k}_j(t)-k\Big)^{-2(2-\alpha)} &\le \int_{-1}^{\widetilde{k}_j(t)} \Big(1+\widetilde{k}_j(t)-x\Big)^{-2(2-\alpha)} dx \\ &\le c_{21} 2^{2\alpha-3}, \end{aligned} \tag{5.37}$$

where $c_{21} > 0$ is a constant only depending on α. Finally putting together (5.35), (5.36) and (5.37) one obtains the proposition. □

Lemma 5.14. *For any $\alpha > 3/2$, there exists a random variable $C_{22} > 0$ of finite moment of any order such that one has almost surely for any real $t \in [0,1]$ and any integer $J \in \mathbb{N}$,*

$$\Big| R_J^\alpha(t) - R_J^\alpha\Big(\widetilde{k}_J(t) 2^{-J}\Big)\Big| \le C_{22} 2^{-J} \sqrt{J+1}.$$

We refer to (5.12) for the definition of the integer $\widetilde{k}_J(t)$.

Proof. (of Lemma 5.14) By using Lemma 5.1 and the fact that

$$0 \le t^\alpha - \Big(\widetilde{k}_J(t) 2^{-J}\Big)^\alpha \le \alpha 2^{-J},$$

one gets that

$$\left|R_J^\alpha(t) - R_J^\alpha\left(2^{-J}\widetilde{k}_J(t)\right)\right| \le |\epsilon_{-1}| \frac{\alpha 2^{-J}}{\Gamma(\alpha+1)}$$

$$+ \quad C_1\sqrt{J+1}\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left|(R_\alpha h_{j,k})(t) - (R_\alpha h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right|. \tag{5.38}$$

On the other hand, it follows from Lemma 5.8 that

$$\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left|(R_\alpha h_{j,k})(t) - (R_\alpha h_{j,k})\left(\widetilde{k}_J(t)2^{-J}\right)\right|$$

$$\le \quad c_{11}\sum_{j=0}^{J-1}2^{(3/2-\alpha)j-J}\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3}. \tag{5.39}$$

Let us assume that $3/2 < \alpha < 2$; then one has

$$\sum_{j=0}^{J-1}2^{(3/2-\alpha)j-J}\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3} \le c_{23}2^{-J}, \tag{5.40}$$

where $c_{23} = c_{11}\left(1-2^{3/2-\alpha}\right)^{-1}\sum_{l=1}^{\infty} l^{\alpha-3} < \infty$. Next let us assume that $\alpha \ge 2$. By using the same technics as in the proof of Proposition 5.13 one can show that there is a constant $c_{24} > 0$, only depending on α, such that for each $j \in \mathbb{N}$,

$$\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3} \le c_{24}\left(2^{j(\alpha-2)}+j+1\right).$$

One has therefore,

$$\sum_{j=0}^{J-1}2^{(3/2-\alpha)j-J}\sum_{k=0}^{\widetilde{k}_j(t)}\left(1+\widetilde{k}_j(t)-k\right)^{\alpha-3} \le c_{25}2^{-J}, \tag{5.41}$$

where $c_{25} = c_{24}\sum_{j=0}^{\infty} 2^{(3/2-\alpha)j}\left(2^{j(\alpha-2)}+j+1\right) < \infty$. Finally putting together (5.38), (5.39), (5.40) and (5.41) one obtains the lemma. □

Lemma 5.15. *There is a random variable $C_{26} > 0$ of finite moments of any order such that one has almost surely for every $J \in \mathbb{N}$*

$$\sup_{t\in[0,1]}\left|R^\alpha(t) - R_J^\alpha(t)\right|$$

$$\le \quad \sup_{0\le K<2^J, K\in\mathbb{N}_0}\left|R^\alpha(K2^{-J}) - R_J^\alpha(K2^{-J})\right| + C_{26}\,2^{-J}\sqrt{J+1}.$$

Proof. (of Lemma 5.15) We use Lemma 5.14, part (ii) of Lemma 5.2 and exactly the same method as the proof of Lemma 5.9. □

We are now in position to prove part (ii) of Theorem 5.11.

Proof. (of Theorem 5.11 part (ii)) Putting together Lemma 5.15, Lemma 3.2, the fact that $\widetilde{\sigma}_J^2 \geq \sup_{0 \leq K < 2^J, K \in \mathbb{N}_0} \widetilde{\sigma}_J^2(2^{-J}K)$ and Proposition 5.13 one obtains the theorem. □

Acknowledgments

The authors are very grateful to Helga Wendel (formerly Helga Schack) for several helpful discussions about the subject of this paper.

References

[1] A. Ayache and W. Linde, *Approximation of Gaussian random fields: general results and optimal wavelet representation of the Lévy fractional motion.* J. Theoret. Probab. **21** (2008), 69–96. MR2384473

[2] A. Ayache and M. S. Taqqu, *Rate optimality of wavelet series approximations of fractional Brownian motion.* J. Fourier Anal. Appl. **9** (2003), 451–471. MR2027888

[3] E. Belinsky and W. Linde, *Small ball probabilities of fractional Brownian sheets via fractional integration operators.* J. Theoret. Probab. **15** (2002), 589–612. MR1922439

[4] B. Carl and I. Stephani, *Entropy, Compactness and Approximation of Operators.* Cambridge Univ. Press, Cambridge, 1990. MR1098497

[5] K. Dzhaparidze and H. van Zanten, *Optimality of an explicit series expansion of the fractional Brownian sheet.* Statist. Probab. Lett. **71** (2005), 295–301. MR2145497

[6] X. Fernique, *Fonctions aléatoires gaussiennes, vecteurs aléatoires gaus-siens.* Montréal: Les Publications CRM, 1997. MR1472975

[7] H. Gilsing and T. Sottinen, *Power series expansions for fractional Brownian motions.* Theory Stoch. Process. **9** (2003), 38–49. MR2306058

[8] E. Iglói, *A rate–optimal trigonometric series expansion of the fractional Brownian motion.* Electron. J. Probab. **10** (2005), 1381–1397. MR2183006

[9] Th. Kühn and W. Linde, *Optimal series representation of fractional Brownian sheets.* Bernoulli **8** (2002), 669–696. MR1935652

[10] S. Kwapień and W. A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple.* Birkhäuser, Boston, Basel, Berlin, 1992. MR1167198

[11] W. V. Li and W. Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures.* Ann. Probab. **27** (1999), 1556–1578. MR1733160

[12] W. V. Li and Q. M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications.* Stochastic Processes: Theory and Methods, Handbook of Statistics. **19** (2001), 533–597. MR1861734

[13] M. A. Lifshits, *On Haar expansion of Riemann–Liouville process in a critical case.* Available at arXiv:0910.2177 (2009).

[14] M. A. Lifshits, W. Linde and Z. Shi, *Small deviations for Riemann-Liouville processes in L_q-spaces with respect to fractal measures.* Proc. London Math. Soc. **92** (2006), 224–250. MR2192391

[15] M. Lifshits and T. Simon. *Small deviations for fractional stable processes.* Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), 725–752. MR2144231

[16] H. Luschgy and G. Pages. *High–resolution product quantization for Gaussian processes under sup–norm distortion.* Bernoulli **13** (2007), 653–671. MR2348745

[17] H. Luschgy and G. Pages. *Expansions for Gaussian processes and Parseval frames.* Electron. J. Probab. **14** (2009), 1198–1221. MR2511282

[18] A. Malyarenko, *An optimal series expansion of the multiparameter fractional Brownian motion.* J. Theoret. Probab. **21** (2008), 459–475. MR2391256

[19] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry.* Cambridge Univ. Press, Cambridge, 1989. MR1036275

[20] H. Schack, *An optimal wavelet series expansion of the Riemann–Liouville process.* To appear J. Theoret. Probab.

[21] A. Zygmund, *Trigonometric Series.* Cambridge Univ. Press, Cambridge, 1950.