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On clusters of high extremes of Gaussian stationary processes with ε-separation

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Abstract

The clustering of extremes values of a stationary Gaussian process $X(t), t \in [0, T]$ is considered, where at least two time points of extreme values above a high threshold are separated by at least a small positive value ε. Under certain assumptions on the correlation function of the process, the asymptotic behavior of the probability of such a pattern of clusters of exceedances is derived exactly where the level to be exceeded by the extreme values, tends to ∞. The excursion behaviour of the paths in such an event is almost deterministic and does not depend on the high level u. We discuss the pattern and the asymptotic probabilities of such clusters of exceedances.

Key words: Gaussian process, extreme values, clusters, separated clusters, asymptotic behavior, correlation function.

AMS 2000 Subject Classification: Primary 60G70; 60G15; 60G10.

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1 Introduction and main results

Let $X(t)$, $t \in \mathbb{R}$, be a zero mean stationary, a.s. continuous Gaussian process with unit variance and covariance function $r(t)$. We study probabilities of high extremes of the process. It is known that given a high extreme occurs in a bounded interval $[0, T]$, say, then the excursion set

$$E(u, T) := \{t \in [0, T] : X(t) > u\}$$

is non-empty, but typically very short. To prove this, one has to investigate mainly the conditional expectation of $X(t)$ given $X(0) = u_1$, where u_1 is close to u, i.e. $E(X(t)|X(0) = u_1) = u_1 r(t)$ and to notice that the conditional covariance function does not depend on u_1. It is necessary to assume that $r(t)$ is sufficiently regular at zero and $r(t) < r(0)$ for all $t > 0$. Applying then usual Gaussian asymptotical techniques, one can determine the corresponding asymptotically exact results. See for details, Berman [3], Piterbarg [8]. Notice also that high values of a Gaussian process with excursions above a high level occur rarely, and for non differentiable paths there are infinitely many crossings of the high level in a short interval, which tends to 0 as the level $u \to \infty$. Hence, they are not separated by a fixed ε, so that to use Gaussian processes modeling for "physically significant" extremes one should consider larger excursions. In other words, considering a lower high level u, one may observe longer excursions. To gain more insight in the extremal behavior of Gaussian processes, a natural step in studying high excursions is the consideration of the excursion sets, containing two points separated by some fixed $\varepsilon > 0$. Thus let us define the set $E_\varepsilon(u, T)$ by

$$E_\varepsilon(u, T) := \{\exists s, t \in [0, T] : X(s) > u,\ t \geq s + \varepsilon\,,\ X(t) > u\}.$$

We show here that for particular correlation functions $r(t)$, the trajectories spend a non-vanishing time above u given the two separated excursions $X(s) > u$ and $X(t) > u$, as $u \to \infty$.

In order to study a limit structure of such excursion sets, we introduce the collection of events S

$$\mathscr{S} := \left\{ \{\inf_{v \in A} X(v) \geq u, \sup_{v \in B} X(v) \leq u\}, A, B \in \mathscr{C} \right\}$$

(A stands for *above,* B for *below*), where $\mathscr{C}$ denotes the collection of all closed subsets of $\mathbb{R}$. Denote by $\{T_s,\ s \in \mathbb{R}\}$ the group of shifts along trajectories of the process $X(t)$. The family of probabilities

$$P_{\varepsilon,u,T}(S) := P(\exists s, t \in [0, T],\ t \geq s + \varepsilon : X(s) > u, X(t) > u, T_s S),\ S \in \mathscr{S},$$

describes the structure of the extremes containing two excursions points separated by at least ε. We study the asymptotic behavior of this probability when $u \to \infty$, which depends on the particular behavior of $r(t)$.

We describe the possible sets A with excursions above u given two exceedances which are at least ε separated. Furthermore, we can also describe in this case the sets B on which the trajectories are typically below u. Thus we study here the asymptotic behavior of the probability of "physical extremes", that is, the probability of existence of excursions above a high level with physically significant duration.

Related problems were considered in Piterbarg and Stamatovic [9], where the asymptotic behaviour of the logarithm of the probability was derived for general Gaussian processes, where the sets A and B are not used because they have no impact. Ladneva and Piterbarg [4] and Anshin [2] considered the probability of joint high values of two Gaussian processes. Clustering of extremes in time series data is a subject of modeling, e.g. in mathematical finances, meteorological studies, or reliability theory. The paper by Leadbetter et al. [5] presents some theoretical background for studying clusters of time series.

Our results depend on the behavior of the correlation function $r(t)$ of the Gaussian process $X(t)$. We introduce the following assumptions.

C1 For some $\alpha \in (0,2)$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \ \text{ as } \ t \to 0,$$
$$r(t) < 1 \ \text{ for all } \ t > 0.$$

The behavior of the clustering depends on the maximal value of $r(t)$ with $t \in [\varepsilon, T]$. Thus we restrict $r(t)$ in $[\varepsilon, T]$ by the following conditions.

C2 In the interval $[\varepsilon, T]$ there exists only one point t_m of maximum $r(t)$ being an interior point of the interval: $t_m = \arg\max_{[\varepsilon,T]} r(t) \in (\varepsilon, T)$, where $r(t)$ is twice continuously differentiable in a neighborhood of t_m with $r''(t_m) < 0$.

The following condition deals with the case $t_m = \varepsilon$, which seems somewhat more common since $r(t)$ decreases in a right neighborhood of zero. Unfortunately considerations in this case are more complicated.

C3 Assume that $r(t)$ is continuously differentiable in a neighborhood of the point $\varepsilon < T$, with $r'(\varepsilon) < 0$, and $r(\varepsilon) > r(t)$ for all $t \in (\varepsilon, T]$, hence $t_m = \varepsilon$.

Denote by $B_\alpha(t)$, $t \in \mathbb{R}$, the fractional Brownian motion with the Hurst parameter $\alpha/2 \in (0,1)$, that is a Gaussian process with a.s. continuous trajectories, and with $B_\alpha(0) = 0$ a.s., $EB_\alpha(t) \equiv 0$, and $E(B_\alpha(t) - B_\alpha(s))^2 = |t-s|^\alpha$. For any set $I \subset \mathbb{R}$ and a number $c \geq 0$, we denote

$$H_{\alpha,c}(I) = E \exp\left(\sup_{t \in I} \sqrt{2} B_\alpha(t) - |t|^\alpha - ct \right).$$

It is known, from Pickands [7] and Piterbarg [8], that there exist positive and finite limits

$$H_\alpha := \lim_{\lambda \to \infty} \frac{1}{\lambda} H_{\alpha,0}([0,\lambda]) \quad \text{(Pickands' constant)} \tag{1}$$

$$H_{\alpha,c} := \lim_{\lambda \to \infty} H_{\alpha,c}([0,\lambda]), \quad \text{for } c > 0. \tag{2}$$

Now consider the asymptotic expression for the joint exceedances of the level u by the two r.v.'s $X(0)$ and $X(t)$, i.e. for any $t > 0$,

$$P(X(0) > u, X(t) > u) = \Psi_2(u, r(t))(1 + o(1))$$

as $u \to \infty$, where

$$\Psi_2(u,r) = \frac{(1+r)^{3/2}}{2\pi u^2 \sqrt{1-r}} \exp\left(-\frac{u^2}{1+r} \right).$$

The shape of excursion sets depends on the behavior of the conditional mean $m(v)$: $m(v) = E(X(v) \mid X(0)=X(t_m)=1)$ which is

$$m(v) = \frac{r(v)+r(t_m-v)}{1+r(t_m)}\,.$$

Let

$$A_0 := \{v : m(v) > 1\} \quad \text{and} \quad B_0 := \{v : m(v) < 1\}.$$

We split the collection of events $\mathscr{S}$ into two sub-collections $\mathscr{S}_0$ and $\mathscr{S}_1$. The first sub-collection $\mathscr{S}_0$ consists of the events generated by all closed subsets $A \subset A_0$, and all closed subsets $B \subset B_0$, and the second sub-collection $\mathscr{S}_1$ is its complement, $\mathscr{S}_1 = \mathscr{S} \setminus \mathscr{S}_0$, generated by all closed A, B, having non-empty intersections with B_0 or A_0, respectively, $A \cap B_0 \neq \emptyset$ or $B \cap A_0 \neq \emptyset$. Let us single out an event $S \in \mathscr{S}$ with $A = B = \emptyset$, it equals Ω with probability one, since trajectories of X are continuous and we can simply write in this case $S = \Omega$. Clearly, we have $\Omega \in \mathscr{S}_0$.

The probability

$$P(u;\varepsilon,T) := P_{u;\varepsilon,T}(\Omega) = P(\exists s,t \in [0,T] :\ t \geq s+\varepsilon,\, X(s) \geq u,\, X(t) \geq u)$$

plays the crucial role in the study of asymptotic behavior of the set of exceedances. It turns out that the events S from $\mathscr{S}_0$ give no contribution in the asymptotic behavior of the probability $P_{u;\varepsilon,T}(S)$. Conversely, considering $S \in \mathscr{S}_1$ makes the probability exponentially smaller. Our main results show the equivalence

$$P_{\varepsilon,u,T}(S) \sim P(u;\varepsilon,T),\ \ S \in \mathscr{S}_0, \tag{3}$$

moreover, we give asymptotic expressions for $P(u;\varepsilon,T)$ and exponential bounds for $P_{\varepsilon,u,T}(S)$, $S \in \mathscr{S}_1$. Note, this means that for any $A \subset A_0$ we have $P_{\varepsilon,u,T}(A) = P\{\exists t : \min_{s \in A+t} X(s) > u\} \sim P(u;\varepsilon,T)$ as $u \to \infty$.

Theorem 1. *Let $X(t)$, $t \in \mathbb{R}$, be a Gaussian centered stationary process with a.s. continuous trajectories. Assume that the correlation function $r(t)$ satisfies **C1** and **C2**. Then we have the following.*

(i) For any $S \in \mathscr{S}_0$,

$$P_{\varepsilon,u,T}(S) = \frac{(T-t_m)\sqrt{2\pi}H_\alpha^2\, u^{-1+4/\alpha}}{\sqrt{-r''(t_m)}(1+r(t_m))^{-1+4/\alpha}}\Psi_2(u,r(t_m))(1+o(1))$$

as $u \to \infty$.

(ii) For any $S \in \mathscr{S}_1$ there exists a $\delta > 0$ with

$$P_{\varepsilon,u,T}(S) = o\left(e^{-\delta u^2}\Psi_2(u,r(t_m))\right)$$

as $u \to \infty$.

For the next results we need the following constant h. It is defined as

$$h = \liminf_{\lambda\to\infty}\liminf_{\mu\to\infty}\frac{h_1(\lambda,\mu)}{\mu} = \limsup_{\lambda\to\infty}\limsup_{\mu\to\infty}\frac{h_1(\lambda,\mu)}{\mu} \in (0,\infty)$$

with $h_1(\lambda,\mu) =$

$$= \int\int_{\mathbb{R}^2} e^{x+y} P\big(\bigcup_D \{\sqrt{2}B_1(s) - (1+r'(\varepsilon))s > x, \sqrt{2}\widetilde{B}_1(t) - (1-r'(\varepsilon))t > y\}\big)\,dx\,dy < \infty,$$

where $D = \{(s,t) : 0 \le s \le \mu/(1+r(\varepsilon))^2, 0 \le t-s \le \lambda/(1+r(\varepsilon))^2\}$ and $B_1(t)$ and $\widetilde{B}_1(t)$ denote independent copies of the standard Brownian motion.

Theorem 2. *Let $X(t)$, $t \in \mathbb{R}$, be a Gaussian centered stationary process with a.s. continuous trajectories. Assume that the correlation function $r(t)$ satisfies **C1** and **C3**. Then the following assertions take place.*

1. *If $S \in \mathscr{S}_0$, then we have:*

(i) *for $\alpha > 1$,*

$$P_{u;\varepsilon,T}(S) = \frac{(T-\varepsilon)|r'(\varepsilon)|}{(1+r(\varepsilon))^2} u^2 \Psi_2(u, r(\varepsilon))(1+o(1)).$$

(ii) *For $\alpha = 1$,*

$$P_{u;\varepsilon,T}(S) = (T-\varepsilon) h u^2 \Psi_2(u, r(\varepsilon))(1+o(1)).$$

(iii) *For $\alpha < 1$,*

$$P_{u;\varepsilon,T}(S) = \frac{(T-\varepsilon) H_\alpha^2 \, u^{-2+4/\alpha}}{|r'(\varepsilon)|(1+r(\varepsilon))^{-2+4/\alpha}} \Psi_2(u, r(\varepsilon))(1+o(1)).$$

2. *If $S \in \mathscr{S}_1$, then there exists $\delta > 0$ such that $P_{u;\varepsilon,T}(S) = o\left(e^{-\delta u^2} \Psi_2(u, r(\varepsilon))\right)$ as $u \to \infty$.*

Remark 1: Notice that the relation (3) follows by letting in both Theorems $S = \Omega \in \mathscr{S}_0$.

Remark 2: We do not consider the case of differentiable processes, $\alpha = 2$, because such considerations require quite different arguments. This case will be considered in a separate publication. In addition, we do not care about the points t such that $r(t) = -1$, because they can be deleted in the derivations, as can be noted in the proofs.

The necessary lemmas for the proof of the two results are treated in Section 3, in Section 4 follows the proof of the main results. In the next section we first discuss some examples to indicate the pattern of exceedances depending on the given correlation function.

2 Examples

A general property in case of **C3**: If $r(v)$ is above the straight line traced between $(0,1)$ and $(\varepsilon, r(\varepsilon))$, then $m(v) > 1$ for all $v \in (0,\varepsilon)$. Indeed, in this case $r(v) > 1 - (1 - r(\varepsilon))v/\varepsilon$ and $r(\varepsilon - v) > 1 - (1 - r(\varepsilon))(\varepsilon - v)/\varepsilon$. Summing we get $r(v) + r(\varepsilon - v) > 1 + r(\varepsilon)$. In particular, this holds if $r(t)$ is strictly concave on $[0,\varepsilon]$. It means that in this case A_0 contains $(0,\varepsilon)$.

Example 1: Consider the correlation function

$$r(t) = \exp(-|t/6|^{1.9})(2+\cos(3t))/3,$$

being the product of two correlation functions. It has countable many local maxima with decreasing heights. The first three local maxima after 0 are $t_m^{(1)} \approx 2.055$, $t_m^{(2)} \approx 4.115$, $t_m^{(3)} \approx 6.175$. For $k \geq 1$, denote by s_k, the maximal root of the equation $r(s) = r(t_m^{(k)})$ with $s < t_m^{(k)}$, $s_1 \approx 0.294$, $s_2 \approx 2.544$, $s_3 \approx 4.734$, (see Figure 1). Let T be larger than the considered $t_m^{(k)}$, with k fixed.

If ε has been chosen between s_1 and $t_m^{(1)}$, then $t_m = t_m^{(1)}$ and $A_0 = \emptyset$. It means that a typical trajectory with two such separated exceedances crosses (perhaps infinitely many times) a high level u, but only in two very short (vanishing as $u \to \infty$) intervals concentrated around two points separated by t_m, approximately.

If ε is larger, $\varepsilon \in (s_2, t_m^{(2)})$, then $t_m = t_m^{(2)}$, A_0 is non-empty, $A_0 \approx (1.82, 2.29)$ (see Fig. 2). That is, given two exceedances of a high level u separated by at least such an ε, say, $X(t_1) > u$ and $X(t_2) > u$, $t_2 - t_1 \geq \varepsilon$, one observes between the exceedances an interval (not tending to 0) on which the trajectory is above u. This interval is approximately congruent to A_0. Note that t_2 is rather close to $t_1 + t_m^{(2)}$ for large u.

Furthermore, if $\varepsilon \in (s_3, t_m^{(3)})$, then $t_m = t_m^{(3)}$ and $A_0 \approx (1.80, 2.31) \cup (3.86, 4.37)$ (see Fig. 2), implying in the case of two exceedances separated by at least ε that one observes two intervals on which the corresponding trajectory is entirely above u.

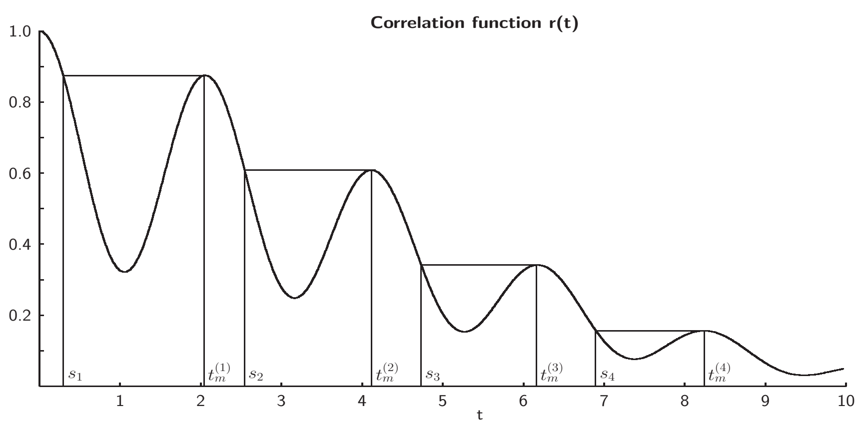

Figure 1: Correlation function $r(t)$ of the example with the local maxima $t_m^{(k)}$ and the corresponding values s_k.

Theorem 2 can be used for the other cases of ε. The correlation function $r(t)$ is strictly concave on $(0, s_1)$. For any positive $\varepsilon \in (0, s_1]$ we have also $r(t) < r(\varepsilon)$ for all $t > \varepsilon$. Thus, for such ε, Condition **C3** holds and thus Theorem 2 can be applied with $A_0 = (0, \varepsilon)$. It is easy to verify that $m(v) < 1$ outside of $[0, \varepsilon]$.

If $\varepsilon \in (t_m^{(1)}, s_2)$, one can derive that A_0 consists of two separated intervals $(0, \kappa) \cup (\varepsilon - \kappa, \varepsilon)$. For example, for $\varepsilon = 2.3$, we get $\kappa \approx 0.22$. The Conditions **C1** and **C3** are fulfilled, so the assertion (i) of Theorem 2 holds, but not the assertion of Corollary 3.

The following two examples consider the cases with ε being the maximal point of $r(t)$ in $[\varepsilon, T]$. We can show that (3) holds and want to describe the typical base of the excursions above the level

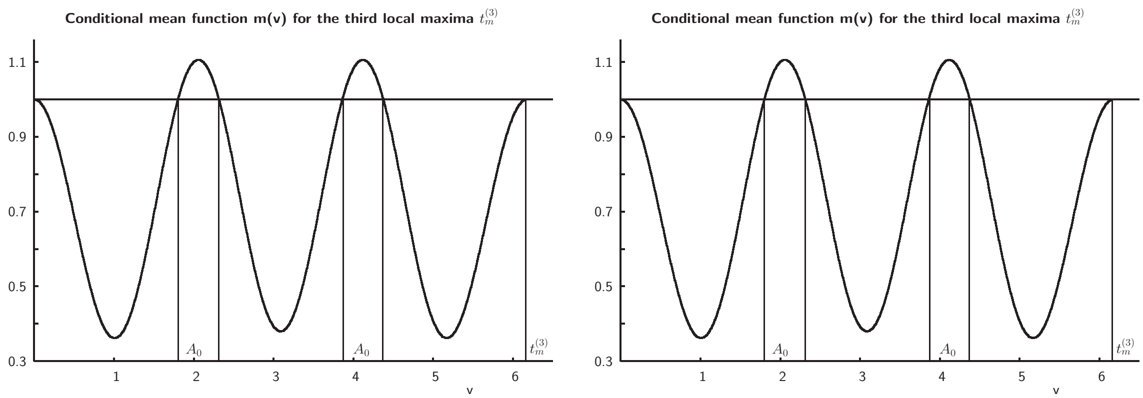

Figure 2: Conditional mean function $m(v)$ for the second and third local maxima $t_m^{(i)}, i = 2, 3$, with the corresponding sets A_0

u with ε-separation.

Example 2: Consider the correlation function $r(t) = \exp(-|t|^\alpha)$, $\alpha \in (0, 2)$. For any $\varepsilon > 0$, the point ε is the maximal point of $r(t)$ on $[\varepsilon, \infty)$. This is the situation of **C3**.
If $\alpha \leq 1$, then $r(t)$ is convex and it can be determined that $m(v) < 1$ for all $v \notin \{0, \varepsilon\}$. We have $m(v) = 1$ for $v \in \{0, \varepsilon\}$. Thus A_0 is empty.
If $\alpha > 1$, $r(t)$ is concave in a neighborhood of zero, as long as $0 < t < [(\alpha - 1)/\alpha]^{1/\alpha}$, so that for small enough ε we have $A_0 = (0, \varepsilon)$. In fact, $m(v) > 1$, for $v \in (0, \varepsilon)$, even when ε does not belong to the domain of concavity of $r(t)$. By symmetry this holds if $m(\varepsilon/2) > 1$, which means if

$$2\exp(-(\varepsilon/2)^\alpha) > 1 + \exp(-\varepsilon^\alpha).$$

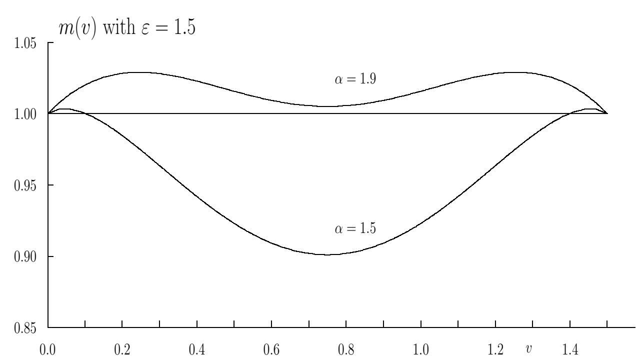

Figure 3: Conditional mean function $m(v)$ for example 2 with $\alpha = 1.5$ and 1.9

For bigger values of ε, A_0 consists of two intervals after 0 and before ε, like in Example 1 (see Fig. 2). Theorem 2 implies that the trajectories which have two points of exceedances separated by ε, spend typically some time above the high level u after the first intersection and before the second one separated by ε. If ε is small enough, these intervals overlap.

Example 3: Consider the correlation function $r(t) = (1 + |t|^\alpha)^{-1}$, $\alpha \in (0, 2]$, given e.g. in [6]. Let

$\varepsilon = 1$. For $\alpha \leq 1$, the set A_0 is empty. For $\alpha = \alpha_0 = \log 3/\log 2 \approx 1.585$, $A_0 = (0, 0.5) \cup (0.5, 1)$. For $1 < \alpha < \alpha_0$, A_0 consists of two smaller separated intervals. For $\alpha > \alpha_0$, we have $A_0 = (0, 1)$ (see Fig. 4). Again Theorem 2 can be applied with the same behavior of the trajectories as in Example 2.

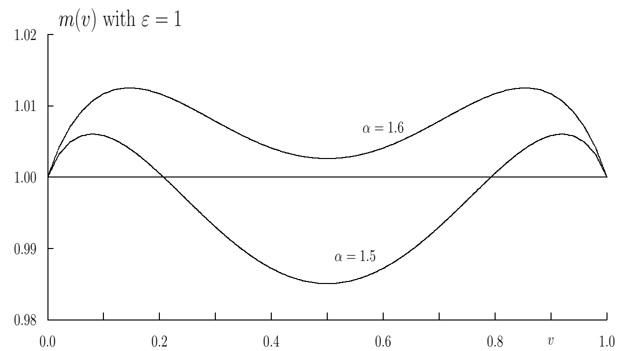

Figure 4: Conditional mean function $m(v)$ for example 3 with $\alpha = 1.5$ and 1.6, with $\varepsilon = 1$.

3 Overview on the proofs

The proof of the two results is rather lengthy and technical. Therefore, we indicate the basic ideas of the proof first without much technicalities. The applied ideas of the two proofs are the same. For the double cluster events with the path behavior given by $m(v)$, one can consider the Gaussian process $(X(s), X(t))$ on $[0, T]^2$. The events which contribute mostly to the asymptotic probability, are those with time points $(s, t) \in D = \{(s, t) : |t - s - t_m| \leq \delta\}$ for some positive δ. This domain is then split into smaller two-dimensional 'intervals' $\Delta_k \times \Delta_l$ of suitable length $\lambda u^{-2/\alpha}$ (for some $\lambda > 0$) in case of Theorem 1, and another length in Theorem 2. The probability of such double exceedance clusters and exceedance behavior in the small 'intervals' are derived asymptotically exact for the two cases assuming **C2** or **C3**. These results are given in Lemma 1 and Lemma 2. Their proofs are combined because a good part follow the same steps where we condition on the event $\{X(s) > u, X(t) > u\}$ for s, t in the subinterval separated by τ which is near t_m. Here we have to consider the conditional process converging to the limit process which defines also the Pickands type conditions. The limit is holding using a domination argument.

The Pickands type constants are considered in Lemma 4 and 5 where neighboring and separated intervals are considered. Further properties for these constants are investigated in Lemma 7 and 8.

Finally the probabilities of the events on subintervals are combined by Bonferroni inequalities, applying the double sum method for the lower bound. For the double sum we need bounds for double exceedance clusters. One such bound is derived in Lemma 3. Lemma 8 considers the bound for the probability of four clusters of exceedances, needed for the double sums in the proofs of the Theorems.

The proof of Theorem 1 is given in Section 4 which follows the ideas mentioned above, dealing with the double exceedance clusters in D and outside D, showing that a double exceedance cluster occurs with much smaller probability than within D, which gives the presented result. For the domain D with the subintervals we apply Lemma 1. The lower bound needs again Lemma 1, but also the results in Lemma 8. The proof of the second statement of Theorem 1 is much simpler.

Similar ideas are applied in the proof of Theorem 2 based on different intervals. We have to consider the three cases $\alpha > 1, = 1$ and < 1 separately since the path behavior of the conditioned Gaussian process plays a role. This is similar (but technically more complicated) to Theorem D.3 in [8], when different relations between smoothness of trajectories and smoothness of variance in its maximum point lead to quite different type of considerations.

We note that limiting conditioned processes are fractional Brownian motions with trend, where the Brownian motions have positive dependent increments if $\alpha > 1$, independent increments if $\alpha = 1$, and negative correlated increments if $\alpha < 1$. The major contribution to the asymptotic probability comes in all three cases from events where $X(s) > u, X(t) > u$ with s, t separated by not more than $\varepsilon + o(1)$ (with $o(1) \to 0$ as $u \to \infty$). Again we apply subintervals and the Bonferroni inequality, with the double sum method for the lower bounds where the subintervals are adapted to the three different cases of α. In all four cases considered by Theorems 1 and 2, one has to choose the lengths of the two-dimensional small intervals carefully in Lemma 1 and 2, to hold the double sum infinitely smaller than the sum of probabilities in the Bonferroni inequality. The cases of Theorem 1 and Theorem 2 (iii) are similar because the smoothness of the variance exceeds the smoothness of the trajectories. Therefore, we choose the same two-dimensional 'subintervals' and prove these cases in the same way.

The second part of Theorem 2 is as for the second statement of Theorem 1, and is not repeated.

4 Lemmas

We write $a\Lambda = \{ax : x \in \Lambda\}$ and $(a_1, a_2) + \Lambda = \{(a_1, a_2) + x : x \in \Lambda\}$, for any real numbers a, a_1, a_2 and set $\Lambda \subset \mathbb{R}^2$. Let A be a set in $\mathbb{R}$, and $A_\delta := \{t : \inf_{s \in A} |t - s| \le \delta\}$ its δ-extension, with $\delta > 0$. We denote the covariance matrix of two centered random vectors $\mathbf{U}, \mathbf{V}$ by

$$\operatorname{cov}(\mathbf{U}, \mathbf{V}) = E\left(\mathbf{U}\mathbf{V}^{\mathbf{T}}\right)$$

and

$$\operatorname{cov}(\mathbf{U}) = E(\mathbf{U}\mathbf{U}^{\mathbf{T}}).$$

In the following, we let τ be a point in $[0, T]$ which may depend on u and lies in the neighborhood of t_m where $r(\tau)$ is either twice continuously differentiable (in case of Condition **C2**) or continuously differentiable (in case of Condition **C3** with $t_m = \varepsilon$).

Lemma 1 and 2 deal with the events of interest on small intervals assuming the condition **C2** and **C3**, respectively. Here the limiting conditioned process enters with the Pickands type conditions. For $S \in \mathscr{S}$ and $\Lambda \subset \mathbb{R}^2$, denote

$$p(u; S, \Lambda) := P\left(\bigcup_{(s,t) \in (0,\tau) + u^{-2/\alpha}\Lambda} \{X(s) > u, X(t) > u, T_s S\}\right)$$

Lemma 1. *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions* ***C1*** *and* ***C2***. *Let Λ be a closed subset of $\mathbb{R}^2$.*

(i) Then for any $\tau = \tau(u)$ with $|\tau - t_m| = O(u^{-1}\sqrt{\log u})$ as $u \to \infty$, and any $S \in \mathscr{S}_0$,

$$p(u;S,\Lambda) \sim h_\alpha\left(\frac{\Lambda}{(1+r(t_m))^{2/\alpha}}\right)\Psi_2(u,r(\tau)), \tag{4}$$

as $u \to \infty$, where

$$h_\alpha(\Lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y} P\left(\bigcup_{(s,t)\in\Lambda}(\sqrt{2}B_\alpha(s) - |s|^\alpha > x, \sqrt{2}\tilde{B}_\alpha(t) - |t|^\alpha > y)\right) dx\,dy,$$

with $B_\alpha, \tilde{B}_\alpha$ are independent copies of the fractional Brownian motion with the Hurst parameter $\alpha/2$. In particular, for Λ_1 and Λ_2, closed subsets of $\mathbb{R}$,

$$p(u;S,\Lambda_1\times\Lambda_2) \sim H_{\alpha,0}\left(\frac{\Lambda_1}{(1+r(t_m))^{2/\alpha}}\right)H_{\alpha,0}\left(\frac{\Lambda_2}{(1+r(t_m))^{2/\alpha}}\right)\Psi_2(u,r(\tau)) \tag{5}$$

as $u \to \infty$.

(ii) Further, for any $S \in \mathscr{S}_1$ there exist $C > 0$, $\delta > 0$ such that

$$p(u;S,\Lambda) \le Ce^{-\delta u^2}\Psi_2(u,r(\tau)). \tag{6}$$

Remark 3: Note that if $|\tau - t_m| = o(u^{-1})$, then $\Psi_2(u,r(\tau)) \sim \Psi_2(u,r(t_m))$ as $u \to \infty$.

Lemma 2. *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions* ***C1*** *and* ***C3*** *with $\alpha \le 1$. Let Λ be a closed subset of $\mathbb{R}^2$.*
(i) Let $\tau = \tau(u)$, be such that $|\tau - \varepsilon| = O(u^{-2}\log u)$ as $u \to \infty$. Then for any $S \in \mathscr{S}_0$ and $\alpha < 1$,

$$p(u;S,\Lambda) \sim h_\alpha\left(\frac{\Lambda}{(1+r(\varepsilon))^{2/\alpha}}\right)\Psi_2(u,r(\tau)) \tag{7}$$

as $u \to \infty$. If $\alpha = 1$, (7) holds with h_α replaced by

$$\tilde{h}_1(\Lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y}\mathbf{P}\Big\{\bigcup_{(s,t)\in\Lambda}(\sqrt{2}B_1(s) - |s| - r'(\varepsilon)s > x,$$
$$\sqrt{2}\tilde{B}_1(t) - |t| + r'(\varepsilon)t > y)\Big\}\,dx\,dy$$

(ii) Statement (ii) of Lemma 1 holds also in this case.

Proof of Lemma 1 and 2: The proofs of both lemmas can be derived partially together with the same steps, where it does not matter whether t_m is an inner point or the boundary point ε. Some deviations are induced by this difference of t_m, hence with different smoothness conditions around t_m. Therefore, we give both proofs simultaneously, as much as possible, and some parts we have to separate for the cases $t_m > \varepsilon$ and $t_m = \varepsilon$, using condition **C2** or **C3**. This we indicate by paragraphs denoted by 'Part for Lemma 1' and 'Part for Lemma 2'. If both cases can be dealt with together, we denote the paragraph as 'Common part'.

Statement (i): **Common part:** Let $S \in \mathscr{S}_0$ which means that there are closed sets $A \subset A_0$ and $B \subset B_0$. Obviously, $r(t) > -1$ in a neighborhood of t_m. We have for any $u > 0$, denoting for short, $K = (0, \tau) + u^{-2/\alpha}\Lambda$ and

$$U(K,S) = \bigcup\nolimits_{(s,t)\in K} \{X(s) > u,\ X(t) > u, T_s S\},$$

$$p(u;S,\Lambda) = u^{-2} \int\int P\left(U(K,S) |\ X(0) = u - \frac{x}{u},\ X(\tau) = u - \frac{y}{u} \right) \times f_{X(0),X(\tau)}(u - \frac{x}{u}, u - \frac{y}{u})\, dx\, dy. \tag{8}$$

Consider first the conditional probability in (8). Denote by $P_{x,y}$ the family of conditional probabilities given $X(0) = u - \frac{x}{u}$, $X(\tau) = u - \frac{y}{u}$. Let $\kappa > 0$ be small such that the κ-extensions of A and B are still subsets of A_0 and B_0, respectively, $A_\kappa \subset A_0$, $B_\kappa \subset B_0$, then the corresponding event $S_\kappa \in \mathscr{S}_0$, and for all sufficiently large u and all $(s,t) \in K$, $S_\kappa \subset T_s S$. Note that S_κ is independent of s, if $(s,t) \in K$. Hence

$$U(K,S) \supseteq S_\kappa \cap \bigcup\nolimits_{(s,t)\in K} \{X(s) > u,\ X(t) > u\} = S_\kappa \cap U(K,\Omega).$$

Now we prove that $P_{x,y}(S_\kappa \cap U(K,\Omega)) \sim P_{x,y}(U(K,\Omega))$ as $u \to \infty$. For the conditional mean of $X(v)$, using inequality $(r(s) - r(t))^2 \le 2(1 - r(t-s))$ and the conditions of the two lemmas, we have by simple algebra,

$$\begin{aligned} M_{xy}(v,u) &:= E\left(X(v)\ \middle|\ X(0) = u - \frac{x}{u},\ X(\tau) = u - \frac{y}{u}\ . \right) \\ &= \frac{(u - x/u)(r(v) - r(\tau - v)r(\tau)) + (u - y/u)(r(\tau - v) - r(v)r(\tau))}{1 - r^2(\tau)} \\ &= u\frac{r(v) + r(\tau - v)}{1 + r(\tau)} + \frac{1}{u}(g_1(v,\tau)x + g_2(v,\tau)y) \\ &= um(v)\left(1 + O\left(u^{-\alpha}(\log u)^{\alpha/2}\right)\right) + O(u^{-1})(g_1(v,t_m)x + g_2(v,t_m)y), \end{aligned}$$

where g_1 and g_2 are continuous bounded functions. The conditional variance can be estimated as follows,

$$V_{x,y}(v) := \mathrm{var}(X(v)|X(0),X(\tau)) = \frac{\det\mathrm{cov}(X(0),X(\tau),X(v))}{1 - r^2(\tau)} \le 1. \tag{9}$$

We have by the construction of $\mathscr{S}_\kappa$, $\inf_{v\in A_\kappa} m(v) > 1$ and $\sup_{v\in B_\kappa} m(v) < 1$. Similarly as (9), we get that

$$V_{x,y}(v,v') := \mathrm{var}(X(v) - X(v')|X(0),X(\tau)) \le \mathrm{var}(X(v) - X(v')) \le C|v - v'|^\alpha.$$

Hence there exists an a.s. continuous zero mean Gaussian process $Y(v)$ with variance $V(v)$ and variance of increments $V(v,v')$. Using Fernique's inequality and (9), for any positive $\delta_1 < \min(\min_{v\in A_\kappa} m(v) - 1, 1 - \max_{v\in B_\kappa} m(v))$, we derive for all sufficiently large u,

$$\begin{aligned} &P_{x,y}\left(U(K,\Omega)\backslash S_\kappa\right) \le P_{x,y}\left(\Omega\backslash S_\kappa\right) \\ &\quad \le \min(P(\inf_{v\in A_\kappa} Y(v) + M_{xy}(v,u) < u), P(\sup_{v\in B_\kappa} Y(v) + M_{xy}(v,u) > u) \\ &\quad \le C\exp(-\delta_1^2 u^2/2), \end{aligned}$$

which gives the desired result

$$\begin{aligned} P_{x,y}\left(U(K,S)\right) &\geq P_{x,y}\left(S_\kappa \cap U(K,\Omega)\right) \\ &\geq P_{x,y}\left(U(K,\Omega)\right) - C\exp(-\delta_1^2 u^2/2). \end{aligned} \tag{10}$$

Notice that also

$$P_{x,y}\left(U(K,S)\right) \leq P_{x,y}\left(U(K,\Omega)\right). \tag{11}$$

Now we study the integrand in (8) replacing $P_{x,y}(U(K,S))$ by $P_{x,y}(U(K,\Omega))$. To this end we consider the limit behavior of the conditional distributions of the vector process $(\xi_u(t),\ \eta_u(t))$, where

$$\xi_u(t) = u(X(u^{-2/\alpha}t) - u) + x,\ \eta_u(t) = u(X(\tau + u^{-2/\alpha}t) - u) + y,$$

given $(\xi_u(0), \eta_u(0)) = (0,0)$ (that is $X(0) = u - x/u$, $X(\tau) = u - y/u$). These Gaussian processes describe the cluster behavior which are separated by at least ε. We need to know the mean and the covariance structure of $\xi_u(s)$ and $\eta_u(s)$ with the limiting expressions for the corresponding limiting processes $\xi(s)$ and $\eta(s)$. We have,

$$E\left(\begin{array}{c|c} \xi_u(t) & \xi_u(0) \\ \eta_u(t) & \eta_u(0) \end{array}\right) = E\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} + R_t R_0^{-1}\begin{pmatrix} \xi_u(0) - E\xi_u(0) \\ \eta_u(0) - E\eta_u(0) \end{pmatrix}, \tag{12}$$

where

$$R_t := E\left(\begin{pmatrix} \xi_u(t) - E\xi_u(t) \\ \eta_u(t) - E\eta_u(t) \end{pmatrix}\begin{pmatrix} \xi_u(0) - E\xi_u(0) \\ \eta_u(0) - E\eta_u(0) \end{pmatrix}^\top\right).$$

Further,

$$E\xi_u(0) = E\xi_u(t) = x - u^2,\ E\eta_u(0) = E\eta_u(t) = y - u^2, \tag{13}$$

$$\begin{aligned} \operatorname{var}\xi_u(0) = \operatorname{var}\eta_u(0) = u^2,\ \operatorname{cov}(\xi_u(0), \eta_u(0)) = u^2 r(\tau), \\ \operatorname{cov}(\xi_u(0), \xi_u(t)) = \operatorname{cov}(\eta_u(0), \eta_u(t)) = u^2 r(u^{-2/\alpha}t), \\ \operatorname{cov}(\xi_u(0), \eta_u(t)) = u^2 r(\tau + u^{-2/\alpha}t),\ \operatorname{cov}(\xi_u(t), \eta_u(0)) = u^2 r(\tau - u^{-2/\alpha}t). \end{aligned} \tag{14}$$

We write

$$\begin{aligned} r(u^{-2/\alpha}t) &= 1 - u^{-2}|t|^\alpha + o(u^{-2}), \\ r(\tau \pm u^{-2/\alpha}t) &= r(\tau) \pm u^{-2/\alpha} t r'(\tau + \theta_\pm u^{-2/\alpha}t), \end{aligned}$$

where $|\theta_\pm| \leq 1$. Obviously, if $\alpha < 1$, it follows for both lemmas, that

$$r(\tau \pm u^{-2/\alpha}t) = r(\tau) + o(u^{-2}). \tag{15}$$

Part for Lemma 1: For this lemma the last relation (15) also holds for $\alpha \in [1,2)$ by using $|\tau - t_m| = O(u^{-1}\sqrt{\log u})$. Indeed, we get $|r'(\tau + \theta_\pm u^{-2/\alpha}t) - r'(t_m)| = O(u^{-1}\sqrt{\log u})$ and again $r(\tau \pm u^{-2/\alpha}t) = r(\tau) + o(u^{-2})$. This implies that with the notation $r = r(\tau)$ and $r' = r'(\tau)$

$$R_t = u^2\begin{pmatrix} 1 - u^{-2}|t|^\alpha + o(u^{-2}) & r + o(u^{-2}) \\ r + o(u^{-2}) & 1 - u^{-2}|t|^\alpha + o(u^{-2}) \end{pmatrix} = R_0 - |t|^\alpha I + o(1),$$

where I denotes the identity matrix. Note that

$$R_0 = u^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \quad \text{and} \quad R_0^{-1} = \frac{1}{u^2(1-r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}$$

Multiplying the matrices gives

$$R_t R_0^{-1} = I + \frac{u^{-2}|t|^\alpha}{1-r^2} \begin{pmatrix} -1 & r \\ r & -1 \end{pmatrix} + o(u^{-2}),$$

as $u \to \infty$. From (12) and (13) we immediately get that

$$E\left(\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = -\frac{|t|^\alpha}{1+r} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + o(1) \tag{16}$$

as $u \to \infty$.

Part for Lemma 2: Let $\alpha = 1$. We have

$$R_t = R_0 + \begin{pmatrix} -|t| & -r't \\ r't & -|t| \end{pmatrix} + o(1).$$

Multiplying by R_0^{-1}, we get

$$\begin{aligned} R_t R_0^{-1} &= I + \frac{u^{-2}}{1-r^2} \begin{pmatrix} -|t| & -r't \\ r't & -|t| \end{pmatrix} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} + o(u^{-2}) \\ &= I + \frac{u^{-2}}{1-r^2} \begin{pmatrix} -|t| + rr't & r|t| - r't \\ r|t| + r't & -|t| - rr't \end{pmatrix} + o(u^{-2}). \end{aligned}$$

For $\alpha < 1$, we have $R_t R_0^{-1} = I - |t|^\alpha R_0^{-1} + o(u^{-2})$, as $u \to \infty$. By (12) and (13), for $\alpha \le 1$,

$$\begin{aligned} E\left(\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) &= -\frac{|t|^\alpha}{1+r} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{tr'}{1+\tilde{r}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} 1(\alpha = 1) + o(1) \\ &= -\frac{|t|^\alpha}{1+r} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{tr'}{1+r} \begin{pmatrix} -1 \\ 1 \end{pmatrix} 1(\alpha = 1) + o(1) \end{aligned} \tag{17}$$

as $u \to \infty$.

Common part: Since the conditional expectation is linear, the $o(1)$ terms in (16), (17) have the structure $(|x| + |y|)o_u$, with $o_u \to 0$ as $u \to \infty$ uniformly in $x, y \in \mathbb{R}$. Now we compute the conditional covariance matrix of the vector $(\xi_u(t) - \xi_u(s), \eta_u(t_1) - \eta_u(s_1))^\top$ given $\xi_u(0), \eta_u(0)$. We have

$$\operatorname{cov}\left(\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t_1) - \eta_u(s_1) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} \right) = \operatorname{cov}\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t_1) - \eta_u(s_1) \end{pmatrix} - C \operatorname{cov}\begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix}^{-1} C^\top, \tag{18}$$

where

$$C = \operatorname{cov}\left(\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t_1) - \eta_u(s_1) \end{pmatrix}, \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} \right)$$

is the matrix of covariances of the two random vectors. Then, as $u \to \infty$,

$$\operatorname{var}(\xi_u(t)-\xi_u(s)) = \operatorname{var}(\eta_u(t)-\eta_u(s)) = 2u^2(1-r(u^{-2/\alpha}(t-s))) \sim 2|t-s|^\alpha \tag{19}$$

Part for Lemma 1: Using the Taylor expansion, we get by **C2** as $u \to \infty$

$$\begin{aligned}
&\operatorname{cov}(\xi_u(t)-\xi_u(s), \eta_u(t_1)-\eta_u(s_1)) \\
&= u^2(r(\tau+u^{-2/\alpha}(t_1-t)) + r(\tau+u^{-2/\alpha}(s_1-s)) - r(\tau+u^{-2/\alpha}(t_1-s)) \\
&\qquad - r(\tau+u^{-2/\alpha}(s_1-t))) \\
&= u^2(u^{-2/\alpha}r'(\tau)(t_1-t+s_1-s-t_1+s-s_1+t+O(u^{-4/\alpha})) \\
&= O(u^{2-4/\alpha}) = o(1)
\end{aligned} \tag{20}$$

Part for Lemma 2: In this case the second derivative is not used. Since $\alpha \le 1$, the statement holds in the same way by **C3**.

$$\begin{aligned}
&\operatorname{cov}(\xi_u(t)-\xi_u(s), \eta_u(t_1)-\eta_u(s_1)) \\
&= u^2(u^{-2/\alpha}r'(\tau)(t_1-t+s_1-s-t_1+s-s_1+t) + o(u^{-2})) = o(1).
\end{aligned} \tag{21}$$

Common Part: Further we have for both lemmas,

$$\begin{aligned}
\operatorname{cov}(\xi_u(t)-\xi_u(s), \xi_u(0)) &= \operatorname{cov}(\eta_u(t)-\eta_u(s), \eta_u(0)) = u^2(r(tu^{-2/\alpha}) - r(su^{-2/\alpha})) \\
&= O(1), \\
\operatorname{cov}(\xi_u(t)-\xi_u(s), \eta_u(0)) &= u^2(r(\tau-u^{-2/\alpha}t) - r(\tau-u^{-2/\alpha}s)) = O(u^{2-2/\alpha}), \\
\operatorname{cov}(\eta_u(t_1)-\eta_u(s_1), \xi_u(0)) &= O(u^{2-2/\alpha}),
\end{aligned}$$

so each element of the matrix

$$C \operatorname{cov}\begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix}^{-1} C^\top$$

is bounded by

$$\frac{O(u^{4-4/\alpha})}{u^2} = O(u^{2-4/\alpha}) = o(1) \tag{22}$$

as $u \to \infty$. This implies together that (18) can be written as

$$\operatorname{cov}\left(\begin{array}{c|c} \xi_u(t)-\xi_u(s) & \xi_u(0) \\ \eta_u(t_1)-\eta_u(s_1) & \eta_u(0) \end{array} \right) = \begin{pmatrix} 2|t-s|^\alpha & 0 \\ 0 & 2|t_1-s_1|^\alpha \end{pmatrix}(1+o(1))$$

as $u \to \infty$. Since the conditional variance is bounded by the unconditional one, we get that

$$\operatorname{var}(\xi_u(t)-\xi_u(s) \mid \xi_u(0), \eta_u(0)) \le C|t-s|^\alpha, \tag{23}$$

$$\operatorname{var}(\eta_u(t)-\eta_u(s) \mid \xi_u(0), \eta_u(0)) \le C|t-s|^\alpha, \tag{24}$$

for all $t, s \in [0, \infty)$. Thus we proved that for any $T > 0$, the distribution of the Gaussian vector process $(\xi_u(t), \eta_u(t))$ conditioned on $\xi_u(0) = \eta_u(0) = 0$ converges weakly in $C[-T, T]$ to the distribution of the Gaussian vector process $(\xi(t), \eta(t))$, $t \in [-T, T]$. This implies that

$$\lim_{u\to\infty} P_{x,y}(U(K,\Omega)) = P\left(\bigcup_{(s,t)\in\Lambda} \{\, \xi(s) > x,\ \eta(t) > y \} \right).$$

Furthermore, we have for ξ and η the following representations:

Part for Lemma 1: The limit process are

$$\xi(t)=\sqrt{2}B_\alpha(t)-\frac{|t|^\alpha}{1+r(t_m)} \quad \text{and} \quad \eta(t)=\sqrt{2}\widetilde{B}_\alpha(t)-\frac{|t|^\alpha}{1+r(t_m)}.$$

Part for Lemma 2: The limit processes are

$$\xi(t)=\sqrt{2}B_\alpha(t)-\frac{|t|^\alpha+r't1(\alpha=1)}{1+r} \quad \text{and} \quad \eta(t)=\sqrt{2}\widetilde{B}_\alpha(t)-\frac{|t|^\alpha-r't1(\alpha=1)}{1+r}.$$

Common Part: Domination: We want to apply the dominated convergence theorem for the integral in (8) divided by $\Psi_2(u,r)$, hence to

$$(1+r)^{-2}\int\int P_{x,y}(U(K,\Omega))\, f_u dx\, dy, \text{ where } f_u=\exp\left(\frac{x+y}{1+r}-\frac{x^2-2rxy+y^2}{2u^2(1-r^2)}\right). \tag{25}$$

We construct an integrable dominating function with separate representations in the four quadrants as follows. Use (11) and bound the probability $P_{x,y}(U(K,\Omega))$. Let $T>0$ be such that $\Lambda\subset[-T,T]\times[-T,T]$.

1. For the quadrant $(x<0,y<0)$, we bound the probability P_u by 1, and the function f_u by $\exp(\frac{x+y}{1+r})$, using the relations $|r(t)|\le 1$ and $x^2+y^2\ge 2xy$.

2. Within the quadrant $(x>0,y<0)$, we bound the probability P_u by

$$p_u(x)=P\left(\max_{t\in[-T,T]}\xi_u(t)>x \,\middle|\, \xi_u(0)=0,\eta_u(0)=0\right),$$

and the function f_u by

$$\exp\left(\frac{y}{1+r}+\frac{x}{0.9+r}\right),$$

for sufficiently large u, using arguments similar to **1.** The function $p_u(x)$ can be bounded by $C\exp(-bx^2)$, $b>0$, using the Borel inequality with relations (16) - (24). Similar arguments were applied in Ladneva and Piterbarg [4].

3. The consideration in the quadrant $(x<0,y>0)$ is similar to 2. with obvious changes. Thus the dominating function is

$$C\exp(-by^2)\exp\left(\frac{x}{1+r}+\frac{y}{0.9+r}\right).$$

4. In the quadrant $(x>0,y>0)$ we bound f_u by

$$\exp\left(\frac{x}{0.9+r}+\frac{y}{0.9+r}\right)$$

and the probability by

$$P\left(\max_{(s,t)\in[-T,T]^2}\xi_u(s)+\eta_u(t)>x+y \,\middle|\, \xi_u(0)=0,\eta_u(0)=0\right).$$

Again, in the same way we apply the Borel inequality for the probability, to get the bound $C\exp(-b(x+y)^2)$, with a positive b.

The four bounds give together the dominating function for the integrand in (25).

Asymptotic probability: Finally we transform the limit of (25) using the self-similarity of the fractional Brownian motion. We give the transformation for Lemma 1 with $\tilde r = r(t_m)$. The corresponding transformation for Lemma 2 with $\alpha < 1$ and $\tilde r = r(\varepsilon)$ is the same and for $\alpha = 1$ it is similar. Let

$$\begin{aligned}
&\iint \exp\left(\frac{x+y}{1+\tilde r}\right) P\left(\bigcup_{\Lambda}\{\,\xi(s)>x,\ \eta(t)>y\}\right)\frac{dx\,dy}{(1+\tilde r)^2}\\
&=\iint e^{x+y} P\left(\bigcup_{\Lambda}\left\{\sqrt{2}B_\alpha(s)-\frac{|s|^\alpha}{1+\tilde r}>(1+\tilde r)x,\ \sqrt{2}\widetilde B_\alpha(t)-\frac{|t|^\alpha}{1+\tilde r}>(1+\tilde r)y\right\}\right)dx\,dy\\
&=\iint e^{x+y} P\left(\bigcup_{\Lambda}\left\{\frac{\sqrt{2}B_\alpha(s)}{1+\tilde r}-\frac{|s|^\alpha}{(1+\tilde r)^2}>x,\ \frac{\sqrt{2}\widetilde B_\alpha(t)}{1+\tilde r}-\frac{|t|^\alpha}{(1+\tilde r)^2}>y\right\}\right)dx\,dy\\
&=\iint e^{x+y} P\left(\bigcup_{\Lambda/(1+\tilde r)^{2/\alpha}}\left\{\sqrt{2}B_\alpha(s)-|s|^\alpha>x,\ \sqrt{2}\widetilde B_\alpha(t)-|t|^\alpha>y\right\}\right)dx\,dy. \qquad (26)
\end{aligned}$$

This shows first statements of the two lemmas.

Statement (ii): It remains to prove the statements (ii) of both lemmas, it means the bound (6). Since $S \in \mathscr{S}_1$, the set A contains an inner point $v \in B_0$ or B contains an inner point $w \in A_0$. In the first case, for all sufficiently large u, we have $v \in \cap_{(s,t)\in u^{-2/\alpha}\Lambda}\{s+A\}\cap\{s+B_0\}$ and $m(v)<1$. In the second case, for all sufficiently large u, $w \in \cap_{(s,t)\in u^{-2/\alpha}\Lambda}\{s+B\}\cap\{s+A_0\}$ and $m(w)>1$. Define the Gaussian field

$$X(s,t)=\frac{X(s)+X(t)}{\sqrt{2(1+r(t-s))}},$$

with

$$b(v)=\sqrt{\frac{2}{1+r(v)}},\quad \text{and}\quad m_{s,t}(v)=\frac{r(v-s)+r(v-t)}{1+r(t-s)}.$$

Note that the Gaussian field $X(s,t)$ has variance 1.

The event we consider implies that there are at least three exceedances of which two are separated by at least ε, and at least one additional exceedance occurs at v with $m(v)<1$, or there are at least two exceedances which are separated by at least ε, and in addition at least a non-exceedance occurs at some point w with $m(w)>1$.

We have for all sufficiently large u that

$$
\begin{aligned}
P(u,S,\Lambda) &\leq P\left(X(v)\geq u, \bigcup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}\{X(s)>u,X(t)>u\}\right)\\
&\quad + P\left(X(w)\leq u, \bigcup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}\{X(s)>u,X(t)>u\}\right)\\
&\leq P\left(X(v)\geq u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}(X(s)+X(t))>2u\right)\\
&\quad + P\left(X(w)\leq u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}(X(s)+X(t))>2u\right)\\
&\leq P\left(X(v)\geq u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}\frac{X(s,t)}{b(t-s)}>u\right)\\
&\quad + P\left(X(w)\leq u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}\frac{X(s,t)}{b(t-s)}>u\right) \qquad (27)
\end{aligned}
$$

Let us consider the first term in (27) which is for any $\tilde{\alpha}\in(0,1)$ at most

$$
\begin{aligned}
&P\left(\tilde{\alpha}X(v)\geq\tilde{\alpha}u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}(1-\tilde{\alpha})\frac{X(s,t)}{b(t-s)}>(1-\tilde{\alpha})u\right)\\
&\leq P\left(\sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda}\tilde{\alpha}X(v)+(1-\tilde{\alpha})\frac{X(s,t)}{b(t-s)}>u\right).
\end{aligned}
$$

We estimate the variance of the Gaussian field $Y(s,t)=\tilde{\alpha}X(v)+(1-\tilde{\alpha})\frac{X(s,t)}{b(t-s)}$ in this probability and minimize it in $\tilde{\alpha}$. The variance equals ($b=b(t-s)$, $m=m_{s,t}(v)$)

$$
\begin{aligned}
&\tilde{\alpha}^2+b^{-2}(1-\tilde{\alpha})^2+2\tilde{\alpha}(1-\tilde{\alpha})b^{-2}m\\
&=b^{-2}((b^2+1-2m)\tilde{\alpha}^2-2(1-m)\tilde{\alpha}+1),
\end{aligned}
$$

with its minimal point at

$$\tilde{\alpha}=\frac{1-m}{1-m+b^2-m}$$

and with minimal value

$$D^2_{s,t}(v):=\frac{1}{b^2(t-s)}\left(1-\frac{(1-m_{s,t}(v))^2}{1+b^2(t-s)-2m_{s,t}(v)}\right).$$

In the domain of interest, $r(t-s)\to r(t_m)$, $m_{s,t}(v)\to m(v)$, as $u\to\infty$, moreover, $m(v)<1$ and $b(t_m)>1$, so that for any $\epsilon>0$ and all sufficiently large u, we have $\tilde{\alpha}\in(0,1)$ and

$$D^2_{s,t}(v)\leq\frac{1+r(t_m)}{2}\left(1-\frac{(1-m(v))^2}{1+b^2(t_m)-2m(v)}+\varepsilon\right).$$

Choose ϵ small such that

$$\delta_1 = \frac{(1 - m(v))^2}{1 + b^2(t_m) - 2m(v)} - \epsilon > 0.$$

Since the Gaussian field $Y(s,t)$ satisfies the variance condition of Theorem 8.1 of [8], we can use this inequality result to get that

$$P_{u,S} \leq C u^c \exp\left(-\frac{u^2}{(1 + r(t_m))(1 - \delta_1)}\right),$$

for some positive constants C and c. This implies that the statements of the lemmas hold for the first probability of (27) for any positive $\delta < \delta_1/(1 + r(t_m))$.

Now we turn to the second probability term in (27) with w such that $m(w) > 1$. We estimate the probability in the same way from above again with any $\tilde{\alpha} > 0$.

$$\begin{aligned}
P(u,S,\Lambda) &\leq P\left(X(w) \leq u, \bigcup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda} \{X(s) > u, X(t) > u\}\right) \\
&\leq P\left(X(w) \leq u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda} (X(s) + X(t)) > 2u\right) \\
& P\left(-\tilde{\alpha}X(w) \geq -\tilde{\alpha}u, \sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda} (1+\tilde{\alpha})b^{-1}X(s,t) > (1+\tilde{\alpha})u\right) \\
&\leq P\left(\sup_{(s,t)\in(0,\tau)+u^{-2/\alpha}\Lambda} -\tilde{\alpha}X(w) + (1+\tilde{\alpha})\frac{X(s,t)}{b(t-s)} > u\right).
\end{aligned}$$

The variance of the field $Y(s,t) = -\tilde{\alpha}X(w) + (1+\tilde{\alpha})\frac{X(s,t)}{b(t-s)}$ equals

$$\begin{aligned}
&\tilde{\alpha}^2 + b^{-2}(1+\tilde{\alpha})^2 - 2\tilde{\alpha}(1+\tilde{\alpha})b^{-2}m \\
&= b^{-2}((b^2 + 1 - 2m)\tilde{\alpha}^2 - 2(m-1)\tilde{\alpha} + 1).
\end{aligned}$$

Notice that for any s,t,w, we have $b^2 + 1 - 2m > 0$, otherwise we would have a negative variance. The minimum of the parabola is at

$$\tilde{\alpha} = \frac{m-1}{b^2 + 1 - 2m} > 0$$

for all sufficiently large u,, with value

$$D_{s,t}^2(v) := \frac{1}{b^2(t-s)}\left(1 - \frac{(m_{s,t}(w) - 1)^2}{1 + b^2(t-s) - 2m_{s,t}(w)}\right).$$

By the same steps as above, the stated bound holds again for the second probability of (27), which show the second statements of both lemmas. □

From this proof the following result can be derived also which we use in the proof of Theorem 2 (i). Let us denote by $\tilde{E}$ the conditional expectation: $\tilde{E}(\cdot) = E(\cdot|\xi_u(0) = \eta_u(0) = 0)$.

Corollary 3. *For any $\alpha \in (0,2)$, any $\Lambda > 0$ and τ as in Lemma 1, the sequence of Gaussian processes $\xi_u(s) - \tilde{E}\xi_u(s)$ and $\eta_u(t) - \tilde{E}\eta_u(t)$ conditioned on $\xi_u(0) = \eta_u(0) = 0$ converges weakly in $C[-\Lambda, \Lambda]$ to the Gaussian processes $\sqrt{2}B_\alpha(s)$ and $\sqrt{2}\widetilde{B}_\alpha(t)$), respectively, as $u \to \infty$.*

Indeed, to prove this we need only relations (18, 19, 22) with (23, 24), which are valid by the assumptions of the corollary.

The following lemma is proved in Piterbarg [8], Lemma 6.3, for the multidimensional time case. We formulate it here for the one-dimensional time where we consider the event of a double exceedance separated by $C(u^{-2/\alpha})$ for some constant $C > 0$.

Lemma 3. *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption **C1**. Let ϵ with be such that $\frac{1}{2} > \epsilon > 0$ and*

$$1 - \frac{1}{2}|t|^\alpha \geq r(t) \geq 1 - 2|t|^\alpha$$

for all $t \in [0, \varepsilon]$. Then there exists a positive constant F such that the inequality

$$P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]} X(t) > u, \max_{t\in[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t) > u\right) \leq F\lambda^2 u^{-1} e^{-\frac{1}{2}u^2 - \frac{1}{8}(\lambda_0-\lambda)^\alpha}$$

holds for any $\lambda < \lambda_0$, and for any $u \geq (4(\lambda + \lambda_0)/\epsilon)^{\alpha/2}$.

The following two lemmas are straightforward consequences of Lemma 6.1, Piterbarg [8] giving the accurate approximations for probabilities of exceedances in neighboring intervals or of a double exceedance in neighboring intervals.

Lemma 4. *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption **C1**. Then for any $\lambda, \lambda_0 > 0$,*

$$P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]\cup[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t) > u\right) \sim H_\alpha([0,\lambda]\cup[\lambda_0, \lambda_0+\lambda])\frac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2}$$

as $u \to \infty$, where

$$H_\alpha([0,\lambda]\cup[\lambda_0,\lambda_0+\lambda]) = \mathbf{E}\exp\left(\max_{t\in[0,\lambda]\cup[\lambda_0,\lambda_0+\lambda]}(B_\alpha(t) - |t|^\alpha)\right).$$

Lemma 5. *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption **C1**. Then for any $\lambda, \lambda_0 > 0$,*

$$\begin{aligned} &P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]} X(t) > u, \max_{t\in[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t) > u\right) \\ &= H_\alpha([0,\lambda],[\lambda_0,\lambda_0+\lambda])\frac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2}(1+o(1)) \end{aligned}$$

as $u \to \infty$, where

$$H_\alpha([0,\lambda],[\lambda_0,\lambda_0+\lambda]) = \int_{-\infty}^{\infty} e^x \mathbf{P}\left(\max_{t\in[0,\lambda]} B_\alpha(t) - |t|^\alpha > x, \max_{t\in[\lambda_0,\lambda_0+\lambda]} B_\alpha(t) - |t|^\alpha > x\right) dx.$$

Proof. Write

$$
\begin{aligned}
&P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]} X(t)>u,\ \max_{t\in[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t)>u\right)\\
&=P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]} X(t)>u\right)+P\left(\max_{t\in[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t)>u\right)\\
&\quad -P\left(\max_{t\in[0,\lambda u^{-2/\alpha}]\cup[\lambda_0 u^{-2/\alpha},(\lambda_0+\lambda)u^{-2/\alpha}]} X(t)>u\right)
\end{aligned}
$$

and apply Lemma 6.1, Piterbarg [8] and Lemma 4. □

From Lemmas 5 and 3 we get a bound for $H_\alpha([0,\lambda],[\lambda_0,\lambda_0+\lambda])$, the Pickands type constant, which depends on the separation $\lambda_0-\lambda$.

Lemma 6. *For any $\lambda_0>\lambda$,*

$$H_\alpha([0,\lambda],[\lambda_0,\lambda_0+\lambda])\le F\sqrt{2\pi}\lambda^2 e^{-\frac{1}{8}(\lambda_0-\lambda)^\alpha}.$$

When $\lambda_0=\lambda$ the bound is trivial. A non-trivial bound for $H_\alpha([0,\lambda],[\lambda,2\lambda])$ is derived from the proof of Lemma 7.1, Piterbarg [8], see page 107, inequality (7.5). This inequalitiy, Lemma 6.8, Piterbarg [8] and Lemma 3 give the following bound.

Lemma 7. *There exists a constant F_1 such that for all $\lambda\ge 1$,*

$$H_\alpha([0,\lambda],[\lambda,2\lambda])\le F_1\left(\sqrt{\lambda}+\lambda^2 e^{-\frac{1}{8}\lambda^\alpha}\right).$$

Applying the conditioning approach of Lemmas 1 and 2 to the following event of four exceedances, we can derive the last preliminary result by using $H_\alpha(\cdot)$ of Lemma 5.

Lemma 8. *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ with $\alpha<1$, satisfying assumptions **C1** and either **C2** or **C3**. Let $\tau=\tau(u)$ satisfies either the assumptions of Lemma 1 or the assumptions of Lemma 2. Then for all $\lambda>0$, $\lambda_1\ge\lambda$, $\lambda_2\ge\lambda$*

$$
\begin{aligned}
&P\left(\max_{t\in[0,u^{-2/\alpha}\lambda]} X(t)>u,\ \max_{t\in[u^{-2/\alpha}\lambda_1,u^{-2/\alpha}(\lambda_1+\lambda)]} X(t)>u,\right.\\
&\qquad\left.\max_{t\in[\tau,\tau+u^{-2/\alpha}\lambda]} X(t)>u,\ \max_{t\in[\tau+u^{-2/\alpha}\lambda_2,\tau+u^{-2/\alpha}(\lambda_2+\lambda)]} X(t)>u\right)\\
&=\prod_{i=1,2} H_\alpha\left([0,\kappa\lambda],\,[\kappa\lambda_i,\kappa(\lambda_i+\lambda)]\right)\Psi_2(u,r(\tau))(1+o(1)),
\end{aligned}
$$

as $u\to\infty$, where $\kappa=(1+r(t_m))^{-2/\alpha}$.

5 Proof of Theorem 1

The proof follows the ideas mentioned in the overview in Section 3. We begin with the first part. The second part of the theorem is much easier to show.

First part:
Denote $\Pi = \{(s,t) : s,t \in [0,T] \text{ and } \ t-s \geq \varepsilon\}$, $\delta = \delta(u) = C\sqrt{\log u}/u$, where the positive constant C is specified later, and $D = \{(s,t) \in \Pi : \ |t-s-t_m| \leq \delta\}$.

a) We want to show that we have to deal mainly with the domain D, since events occuring outside of D occurs with asymptotically smaller probability. We have for any $S \in \mathscr{S}_0$,

$$P_{\varepsilon,u,T}(S) \leq P\left(\bigcup_{(s,t)\in D} \{X(s)>u\} \cap \{X(t)>u\} \cap T_s S\right)$$
$$+ P\left(\bigcup_{(s,t)\in \Pi\setminus D} \{X(s)>u\} \cap \{X(t)>u\}\right) \tag{28}$$

and on the other hand, for all sufficiently large u,

$$P_{\varepsilon,u,T}(S) \geq P\left(\bigcup_{(s,t)\in D} \{X(s)>u\} \cap \{X(t)>u\} \cap T_s S\}\right). \tag{29}$$

The second term of the right-hand side of (28) is bounded by

$$P\left(\bigcup_{(s,t)\in \Pi\setminus D} \{X(s)>u\} \cap \{X(t)>u\}\right) \leq P\left(\max_{(s,t)\in\Pi\setminus D} (X(s)+X(t)) > 2u\right).$$

Making use of Theorem 8.1, Piterbarg [8], we can bound the last probability by

$$\text{const} \cdot u^{-1+2/\alpha} \exp\left(-\frac{u^2}{1+\max_{(t,s)\in\Pi\setminus D} r(t-s)}\right).$$

For all sufficiently large u, the maximal correlation on $\Pi \setminus D$ is bounded by

$$\max_{(s,t)\in\Pi\setminus D} r(t-s) \leq r(t_m) - 0.4|r''(t_m)|\delta^2(u) = r(t_m) - 0.4C^2|r''(t_m)|u^{-2}\log u.$$

Hence, the second term is of smaller order than the leading term, since

$$P\left(\bigcup_{(s,t)\in \Pi\setminus D} \{X(s)>u\} \cap \{X(t)>u\}\right) \leq \text{const} \cdot u^{-1+2/\alpha} \exp\left(-\frac{u^2}{1+r(t_m)}\right) u^{-G}, \tag{30}$$

where

$$G = \frac{0.4C^2|r''(t_m)|}{(1+r(t_m))^2}.$$

b) Now we deal with the first probability in the right-hand side of (28), with events occuring in D. We bound the probability from above and from below such that the bounds are asymptotically equivalent. Denote $\Delta = \lambda u^{-2/\alpha}$, for some $\lambda > 0$, and define the intervals

$$\Delta_k = [k\Delta, (k+1)\Delta],\ 0 \leq k \leq N,\ N = [T/\Delta],$$

where $[\cdot]$ denotes the integer part. We will apply Lemma 1 for sets $\Delta_k \times \Delta_l = (0,(l-k)\Delta) + \Delta_k \times \Delta_k$, where $\tau = (l-k)\Delta$ and $\Lambda = [k\lambda,(k+1)\lambda] \times [k\lambda,(k+1)\lambda]$ in this lemma. For $S \in \mathscr{S}_0$, in virtue of (5) of Lemma 1, we have

$$
\begin{aligned}
&P\left(\bigcup_{(s,t)\in D}\{X(s)>u\}\cap\{X(t)>u\}\cap T_s S\right) \\
&\leq \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} P\big(\exists (s,t)\in(0,\tau)+\Delta_k\times\Delta_k : X(s)>u, X(t)>u, T_s S\big) \qquad (31)\\
&\quad\leq \frac{(1+\gamma(u))(1+r(t_m))^{3/2}}{2\pi u^2\sqrt{1-r(t_m)}} H^2_{\alpha,0}\left(\frac{\lambda}{(1+r(t_m))^{2/\alpha}}\right) \\
&\qquad\qquad \times \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right), \qquad (32)
\end{aligned}
$$

where $\gamma(u)\downarrow 0$ as $u\to\infty$. For the last sum, denoted by Σ_1, we get

$$
\begin{aligned}
\Sigma_1 &= \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right) \\
&= \exp\left(-\frac{u^2}{1+r(t_m)}\right) \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} \exp\left(-u^2\frac{r(t_m)-r((l-k)\Delta)}{(1+r((l-k)\Delta))(1+r(t_m))}\right).
\end{aligned}
$$

Further, for any exponent with $n = l-k$

$$
\begin{aligned}
\frac{r(t_m)-r(n\Delta)}{(1+r(n\Delta))(1+r(t_m))} &\lesseqgtr \frac{\frac{1}{2}|r''(t_m)|(n\Delta-t_m)^2}{(1+r(t_m))^2}(1\pm\gamma_1(u)) \\
&= C_r(n\Delta-t_m)^2(1\pm\gamma_1(u)),
\end{aligned}
$$

where $\gamma_1(u)\downarrow 0$, as $u\to\infty$, and

$$
C_r = \frac{\frac{1}{2}|r''(t_m)|}{(1+r(t_m))^2}.
$$

The index k of the sum varies between 0 and $(T-t_m)/\Delta+\theta$, as $u\to\infty$, where $|\theta|\leq 1$, depending of u. The index n varies between $(t_m-\delta)/\Delta+\theta_1$ and $(t_m+\delta)/\Delta+\theta_2$ as $u\to\infty$, where θ_1,θ_2 have the same properties as θ. Denote $x_n = (n\Delta - t_m)u$ and note that $x_{n+1}-x_n = u\Delta\to 0$ as $u\to\infty$. Using this, we can approximate the sum Σ_1 by an integral

$$
\begin{aligned}
\Sigma_1 &= (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{T-t_m}{u\Delta^2}\sum_{x_n=-u\delta+o(1)}^{u\delta+o(1)} e^{-C_r x_n^2}(x_{n+1}-x_n) \\
&= (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{T-t_m}{u\Delta^2}\int_{-\infty}^{\infty} e^{-C_r x^2}\,dx \qquad (33)
\end{aligned}
$$

since $u\delta = C\sqrt{\log u}\to\infty$. The integral equals $\sqrt{\pi/C_r}$, hence we get for the right-hand side of (32) using the definition of $\Psi(u,r)$

$$P\left(\bigcup_{(s,t)\in D}\{\inf_{v\in s+A}X(v)\geq u\}\cap\{X(s)>u\}\cap\{X(t)>u\}\right)$$
$$\leq \frac{(1+\gamma_2(u))(T-t_m)\sqrt{\pi}u^{-1+4/\alpha}}{\lambda^2\sqrt{C_r}}H^2_{\alpha,0}\left(\frac{\lambda}{(1+r(t_m))^{2/\alpha}}\right)\Psi_2(u,r(t_m)), \tag{34}$$

where $\gamma_2(u)\downarrow 0$ as $u\to\infty$. Letting $\lambda\to\infty$, using that by (2)

$$H_{\alpha,0}\left(\frac{\lambda}{(1+r(t_m))^{2/\alpha}}\right)\sim\frac{\lambda}{(1+r(t_m))^{2/\alpha}}H_{\alpha,0}$$

this bound gives asymptotically the asymptotic term of Theorem 1.

We choose the value of C. Let C be so large that $G>2-2/\alpha$, which implies that the left-hand side of (30) is infinitely smaller than the left-hand side of (34), as $u\to\infty$.

c) We bound now the probability in the right-hand side of (29) from below. By the Bonferroni inequality we get

$$P\left(\bigcup_{(s,t)\in D}\{X(s)>u\}\cap\{X(t)>u\}\cap T_sS\right)$$
$$\geq\sum_{(k,l):\,\Delta_k\times\Delta_l\subset D}P\left(\exists(s,t)\in(0,\tau)+\Delta_k\times\Delta_l: X(s)>u, X(t)>u,\ T_sS,\right)$$
$$-\sum\sum P\left(\max_{s\in\Delta_l}X(s)>u,\ \max_{t\in\Delta_k}X(t)>u,\ \max_{s'\in\Delta_{l'}}X(s')>u,\ \max_{t'\in\Delta_{k'}}X(t')>u\right), \tag{35}$$

where second term, the double sum, has been increased by omitting the events T_sS. This double-sum in (35) is taken over the set

$$K=\{(k,l,k',l'):\ (k',l')\neq(k,l),\ \Delta_k\times\Delta_l\subset D,\ \Delta_{k'}\times\Delta_{l'}\subset D\}.$$

The first sum in the right-hand side of (35) can be bounded from below in the same way as the previous sum in (32), therefore it is at least the right-hand side of (34) with a change of $(1+\gamma_2(u))$ by $(1-\gamma_2(u))$.

Consider the double-sum Σ_2, say, in the right-hand side of (35). For simplicity we denote

$$H(m)=H_{\alpha,0}\left(\left[0,\frac{\lambda}{(1+r(t_m))^{2/\alpha}}\right),\left[\frac{m\lambda}{(1+r(t_m))^{2/\alpha}},\frac{(m+1)\lambda}{(1+r(t_m))^{2/\alpha}}\right]\right)$$

and note that

$$H(0)=H_{\alpha,0}\left(\left[0,\frac{\lambda}{(1+r(t_m))^{2/\alpha}}\right]\right).$$

With Lemma 8 we derive for the probability

$$P\left(\max_{s\in\Delta_l}X(s)>u,\ \max_{s'\in\Delta_{l'}}X(s')>u,\ \max_{t\in\Delta_k}X(t)>u,\ \max_{t'\in\Delta_{k'}}X(t')>u\right)$$

with $k \le k'$, $l \le l'$, taking $\lambda_1 = (k'-k)\lambda$, $\lambda_2 = (l'-l)\lambda$, and $\tau = (l-k)\Delta$. Note that $\tau \to t_m$ uniformly for the possible k and l. Since the process X is stationary, we have for the double-sum Σ_2 in (35) (which includes only different pairs (k,l) and (k',l')) with $L = \{k \le k', l \le l', (k,l) \ne (k',l')\}$

$$\begin{aligned}\Sigma_2 &\le 4\sum_{L\cap K} P\left(\max_{s\in\Delta_l} X(s) > u,\ \max_{s'\in\Delta_{l'}} X(s') > u,\ \max_{t\in\Delta_k} X(t) > u,\ \max_{t'\in\Delta_{k'}} X(t') > u\right)\\
&\le \frac{4(1+\Gamma(u))(1+r(t_m))^2}{2\pi u^2\sqrt{1-r^2(t_m)}}\sum_{L\cap K} H(k'-k)H(l'-l)\exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right)\\
&\le \frac{4(1+\Gamma(u))(1+r(t_m))^2}{2\pi u^2\sqrt{1-r^2(t_m)}}\sum_{n=1}^{\infty} H(n)\left(H(0)+\sum_{n=0}^{\infty} H(n)\right)\\
&\times \sum_{(k,l):\,\Delta_k\times\Delta_l\cap D\ne\emptyset} \exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right),\end{aligned}$$

where $\Gamma(u) \downarrow 0$ as $u \to \infty$. Note that for the last step we fix k and l, and consider the summation on k' and l'. Since $(k,l) \ne (k',l')$, either $k'-k > 0$ or $l'-l > 0$. This explains the sums on $H(n)$ which do not depend on k and l. The last sum was considered already in (33).

d) Hence, it remains to bound the sum of $H(n)$. By (5), (6) and (7) and Lemma 6.8 of Piterbarg [8], we get that $H(0) \le \text{const}\cdot\lambda$, $H(1) \le \text{const}\cdot\sqrt{\lambda}$, and

$$H(n) \le \text{const}\cdot\lambda^2\exp(-\frac{1}{8}(n-1)^\alpha\lambda^\alpha) \quad \text{for } n > 1,$$

hence

$$\sum_{n=1}^{\infty} H(n)\left(H(0)+\sum_{n=1}^{\infty} H(n)\right) \le const\cdot\lambda^{3/2}.$$

Thus

$$\Sigma_2 \le const\cdot\lambda^{-1/2}u^{-1+4/\alpha}\Psi_2(u, r(t_m))$$

which shows that the double-sum Σ_2 is infinitely smaller than (34) or the asymptotic term of the statement, letting $\lambda \to \infty$. Thus the first assertion of Theorem 1 follows.

Second part:
Finally we turn to the second assertion of Theorem 1, where $S \in \mathscr{S}_1$. By Lemma 1, each term in the sum (31) can be bounded uniformly by

$$\exp\{-\delta u^2\}\Psi_2(u, r(t_m)),$$

for all sufficiently large u, where $\delta > 0$. Since the number of summands in (31) is at most a power of u, the statement of Theorem 1 follows. □

6 Proof of Theorem 2.

We begin with the proof of the first part of Theorem, assuming $S \in \mathscr{S}_0$, dealing separately with the three cases $\alpha > 1, = 1$ and < 1 since the subintervals have to selected differently in the three cases.

6.1 Proof of Theorem 2(i).

In this case $\alpha > 1$. Let $b \geq a$, introduce the event

$$U_{u;a,b,T}(S) := \{\exists (s,t):\ s,t \in [0,T],\ b \geq t-s \geq a,\ X(t) \geq u,\ X(s) \geq u, T_s S\},$$

with $U_{u;a,b,T} := U_{u;a,b,T}(\Omega)$. In particular, for any S, $P(U_{u;\varepsilon,b,T}(S)) = P_{u;\varepsilon,T}(S)$ for all $b \geq T$, and

$$P(U_{u;\varepsilon,\varepsilon,T}(S)) = P(\exists t \in [\varepsilon, T] : X(t-\varepsilon) \wedge X(t) > u, T_{t-\varepsilon}S).$$

Let $\delta(u) = cu^{-2}\log u$, $c > 0$, and $\gamma > 0$ a small positive number. Then for all u such that $\delta(u) \geq \gamma u^{-2}$, (which holds for u such that $\log u \geq \gamma/c$),

$$\begin{aligned} &U_{u;\varepsilon,T,T}(S) \subseteq U_{u;\varepsilon,T,T} \\ &\qquad \subseteq U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T} \cup (U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \setminus U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) \cup U_{u;\varepsilon+\delta(u),T,T} \end{aligned} \tag{36}$$

and

$$U_{u;\varepsilon,T,T}(S) \supseteq U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}(S) \supseteq U_{u;\varepsilon,\varepsilon,T}(S). \tag{37}$$

We estimate from above the probabilities of the third and second event in the right-hand side of (36). Then we derive the asymptotic behavior of the probability of the first event in the right-hand side with small γ and show that it dominates the two other ones. Finally, we need to estimate the probability of the event in the right hand part of (37) from below, and show that it is equivalent to the upper bound.

6.1.1 Large separation of the clusters

We estimate from above the probability of the third event in the right-hand side of (36). Using the inequality of Theorem 8.1, [8], with $r(t) < r(\varepsilon + \delta(u))$ for $t > \varepsilon + \delta(u)$ by **C3**, we get with $r = r(\varepsilon), r' = r'(\varepsilon)$

$$\begin{aligned} &P(U_{u;\varepsilon+\delta(u),T,T}) \\ &\leq P(\exists (s,t):\ s,t \in [0,T],\ t \geq s+\varepsilon+\delta(u),\ X(t)+X(s) > 2u) \\ &= P\left(\max_{s,t\in[0,T],\ t\geq s+\varepsilon+\delta(u)} (X(t)+X(s)) > 2u\right) \leq Cu^{4/\alpha-1}\exp\left(-\frac{u^2}{1+r(\varepsilon+\delta(u))}\right) \\ &\leq Cu^{4/\alpha-1}\exp\left(-\frac{u^2}{1+r} + \frac{u^2\delta(u)r'(1+o(1))}{(1+r)^2}\right) \leq Cu^{-R}\exp\left(-\frac{u^2}{1+r}\right), \end{aligned} \tag{38}$$

for any $R > 0$ by choosing c in $\delta(u)$ sufficiently large, with some constant $C > 0$, since $r' < 0$. This estimate holds for any $\alpha \leq 2$.

6.1.2 Intermediate separation of clusters

Now we estimate the probability of the second event in the right-hand side of (36). We have

$$P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \setminus U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) \leq P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \cap U^c_{u;\varepsilon,\varepsilon,T}).$$

For any positive g with $g < \gamma$, we use time points on the grid kgu^{-2} to estimate

$$P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \cap U^c_{u;\varepsilon,\varepsilon,T}) \le P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \cap U^c_g),$$

where

$$U^c_g = \left\{ \max_{k:kgu^{-2}\in[\varepsilon,T]} X(kgu^{-2}-\varepsilon) \wedge X(kgu^{-2}) \le u \right\}.$$

For any pair (s,t) with $t-s \in [\varepsilon+\gamma u^{-2}, \varepsilon+\delta(u)]$ set $k=[tu^2/g]$. For a fixed k and large u, we have $t \in [kgu^{-2}, kgu^{-2}+\delta(u)]$ and $s \in [kgu^{-2}-\varepsilon-\delta(u), kgu^{-2}-\varepsilon)$. Therefore the probability $P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \cap U_g)$ is at most

$$\sum_{k:kgu^{-2}\in[\varepsilon,T]} P(\{X(kgu^{-2}-\varepsilon) \wedge X(kgu^{-2}) \le u\} \cap B_k \cap B'_k),$$

where

$$B_k = \{ \max_{s\in[kgu^{-2}-\varepsilon-\delta(u),kgu^{-2}-\varepsilon)} X(s) > u \}$$

and

$$B'_k = \{ \max_{t\in[kgu^{-2},kgu^{-2}+\delta(u)]} X(t) > u \}.$$

Since X is stationary the latter sum is equal to

$$\left[\frac{T-\varepsilon}{gu^{-2}}\right] P\left(X(0)\wedge X(\varepsilon) \le u, \max_{[-\delta(u),0]} X(s) > u, \max_{[\varepsilon,\varepsilon+\delta(u)]} X(t) > u \right)$$
$$\le \left[\frac{T-\varepsilon}{gu^{-2}}\right] (p_1+p_2),$$

where

$$p_1 = P\left(X(0) \le u, \max_{[-\delta(u),0]} X(s) > u, \max_{[\varepsilon,\varepsilon+\delta(u)]} X(t) > u \right)$$

and

$$p_2 = P\left(X(\varepsilon) \le u, \max_{[-\delta(u),0]} X(s) > u, \max_{[\varepsilon,\varepsilon+\delta(u)]} X(t) > u \right).$$

For any sufficiently small $\kappa > 0$ and all sufficiently large u, such that $\delta(u) \le \kappa u^{-2/\alpha}$ (since $\alpha > 1$), we have

$$p_1 \le P\left(X(0) \le u, \max_{[-\kappa u^{-2/\alpha},0]} X(s) > u, \max_{[\varepsilon,\varepsilon+\kappa u^{-2/\alpha}]} X(t) > u \right).$$

Apply similar steps as for the approximation of (8) with $\tau = \varepsilon$ and take into account that $X(0) \le u$ to derive

$$p_1 \le \frac{1}{2\pi u^2\sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r}\right) \int_0^\infty \int_{-\infty}^\infty \exp\left(\frac{x+y}{1+r} - \frac{x^2-2rxy+y^2}{2u^2(1-r^2)}\right) \widetilde{P}_u dx dy,$$

where

$$\widetilde{P}_u = P\left(\max_{[-\kappa,0]} \xi_u(s) > x, \max_{[0,\kappa]} \eta_u(t) > y \,\middle|\, \xi_u(0) = \eta_u(0) = 0 \right)$$

with $\xi_u(s) = u(X(u^{-2/\alpha}s) - u) + x$ and $\eta_u(t) = u(X(\tau + u^{-2/\alpha}t) - u) + y$. Now we estimate the conditional expectations of $\xi_u(t)$ and $\eta_u(t)$, as in the proof of Lemma 1 and 2,

$$R_t = R_0 + u^{2-2/\alpha} \begin{pmatrix} 0 & -r't \\ r't & 0 \end{pmatrix} + o(u^{2-2/\alpha}),$$

and

$$R_t R_0^{-1} = I + \frac{u^{-2/\alpha}}{1-r^2} \begin{pmatrix} rr'\,t & -r't \\ r't & -rr't \end{pmatrix} + o(u^{-2/\alpha})$$

as $u \to \infty$. Thus, by (12) and (13)

$$E\left(\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \frac{u^{2-2/\alpha} r't}{1-r^2} \begin{pmatrix} r-1 \\ 1-r \end{pmatrix} + o(u^{2-2/\alpha}) \tag{39}$$

as $u \to \infty$. The process $\xi_u(s)$ is considered for $s \le 0$, whereas $\eta_u(t)$ is considered for $t \ge 0$. So

$$\tilde{E}(\xi_u(s)) = E(\xi_u(s) \mid \xi_u(0) = \eta_u(0) = 0) \le (1-a)\frac{u^{2-2/\alpha}|r'|s}{1+r},\ s \le 0, \tag{40}$$

$$\tilde{E}(\eta_u(t)) = E(\eta_u(t) \mid \xi_u(0) = \eta_u(0) = 0) \le -(1-a)\frac{u^{2-2/\alpha}|r'|t}{1+r},\ t \ge 0, \tag{41}$$

for any $a \in (0,1)$ and all sufficiently large u. Similar to the derivation in Lemma 2, the conditional variance of the increments is bounded by the unconditional variance of the increments and the process converges as $u \to \infty$, see Corollary 3. We have therefore,

$$\begin{aligned} \widetilde{P}_u &\le P\left(\max_{[-\kappa,0]} \xi_u(s) > x \middle| \xi_u(0) = \eta_u(0) = 0 \right) \\ &\le P\left(\exists s \in [-\kappa, 0] : (\xi_u(s) - \tilde{E}\xi_u(s)) > x - \tilde{E}\xi_u(s) \middle| \xi_u(0) = \eta_u(0) = 0 \right). \end{aligned}$$

Since for any negative s, $\tilde{E}\xi_u(s) \to -\infty$ as $u \to \infty$, we have $\widetilde{P}_u \to 0$ for any small κ and any $x > 0$, $y \in R$. By the domination in the proof of Lemma 1 and 2, we have

$$\widetilde{P}_u\, dx dy = 0.$$

By symmetry, the same is valid for p_2 by using (41), thus we have shown that

$$\lim_{u\to\infty} \exp\left(\frac{u^2}{1+r(\varepsilon)} \right) P(U_{u;\varepsilon+\gamma u^{-2},\varepsilon+\delta(u),T} \setminus U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) = 0. \tag{42}$$

□

6.1.3 Asymptotic behavior of the probability $P(U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}(S))$.

First we study the conditional means and covariance functions of the process

$$\xi_u(t) = u(X(u^{-2}t) - u) + x,\ \eta_u(t) = u(X(\varepsilon + u^{-2}t) - u) + y, \tag{43}$$

given $(\xi_u(0), \eta_u(0)) = (0,0)$. Note that another time scaling is used, u^{-2}, instead of $u^{-2/\alpha}$ as in section 3.1.2. We have with $r = r(\varepsilon)$ and $r' = r'(\varepsilon)$,

$$R_t = R_0 + \begin{pmatrix} 0 & -r't \\ r't & 0 \end{pmatrix} + o(1)$$

and so

$$R_t R_0^{-1} = I + \frac{u^{-2} r' t}{1-r^2} \begin{pmatrix} r & -1 \\ 1 & -r \end{pmatrix} + o(u^{-2})$$

as $u \to \infty$. Thus, by (12) and (13),

$$E\left(\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \frac{r't}{1-r^2} \begin{pmatrix} r-1 \\ 1-r \end{pmatrix} + o(1)$$

as $u \to \infty$. For the conditional covariance matrix Σ of the vector $(\xi_u(t) - \xi_u(s), \eta_u(t_1) - \eta_u(s_1))^\top$, we have

$$\operatorname{var}(\xi_u(t) - \xi_u(s)) = \operatorname{var}(\eta_u(t) - \eta_u(s)) = 2u^2(1 - r(u^{-2}(t-s))) = O(u^{2-2\alpha}) = o(1)$$

as $u \to \infty$. Using Taylor expansion for $r(t)$ at $t = \varepsilon$, one easily gets that $\operatorname{cov}(\xi_u(t) - \xi_u(s), \eta_u(t_1) - \eta_u(s_1)) = o(1)$ as $u \to \infty$. By the same argument,

$$\operatorname{cov}(\xi_u(t) - \xi_u(s), \eta_u(0)) = -(t-s)r' + o(1),$$
$$\operatorname{cov}(\eta_u(t_1) - \eta_u(s_1), \xi_u(0)) = (t_1 - s_1)r' + o(1),$$

and by **C1,**

$$\operatorname{cov}(\xi_u(t) - \xi_u(s), \xi_u(0)) = O(u^{2-2\alpha}),\ \operatorname{cov}(\eta_u(t_1) - \eta_u(s_1), \eta_u(0)) = O(u^{2-2\alpha}),$$

thus the elements of Σ are bounded, which implies

$$\Sigma \operatorname{cov}\begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix}^{-1} \Sigma^\top = \frac{O(1)}{u^2(1-r^2)} = O(u^{-2}).$$

Hence,

$$\operatorname{cov}\left(\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t_1) - \eta_u(s_1) \end{pmatrix} \middle| \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} \right) = o(1)$$

as $u \to \infty$. Hence, the vector process $(\xi_u(t), \eta_u(t))$ in (43) conditioned on $(\xi_u(0), \eta_u(0)) = (0,0)$ converges weakly to the deterministic process $(-r't/(1+r), r't/(1+r))$. For a large positive L and any non-negative integer m we have

$$\begin{aligned} P_m &:= \\ &= P(\exists (s,t) \ni t - s \in [\varepsilon, \varepsilon + \gamma u^{-2}], s \in [mLu^{-2}, (m+1)Lu^{-2}] : X(s) > u, X(t) > u) \\ &= \frac{(1+o(1))}{2\pi u^2 \sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r}\right) \\ &\qquad \times \int\int \exp\left(\frac{x+y}{1+r}\right) \mathbf{I}_{\{\exists (s,t): t-s \in [0,\gamma],\, s \in [0,L]: -r's/(1+r) > x, r't/(1+r) > y\}}\, dx\, dy = P_0, \end{aligned}$$

where we use the stationarity and the domination, as in the proof of Lemma 2. The probability has been changed to the indicator function in view of the deterministic limiting behavior. Denoting $\delta_1 = \gamma|r'|/(1+r)^2$, $L_1 = L|r'|/(1+r)^2$ and changing variables $\tilde{x} = x/(1+r), \tilde{y} = y/(1+r), \tilde{t} = t|r'|/(1+r)^2$ and $\tilde{s} = s|r'|/(1+r)^2$, we get

$$\begin{aligned}
&\iint\limits_{\{\exists(s,t):t-s\in[0,\gamma],\, s\in[0,L], -r's/(1+r)>x, r't/(1+r)>y\}} \exp\left(\frac{x+y}{1+r}\right) dx dy \\
&= (1+r)^2 \iint\limits_{\{\exists(\tilde{s},\tilde{t}):\tilde{t}-\tilde{s}\in[0,\delta_1],\, \tilde{s}\in[0,L_1],\, \tilde{s}>\tilde{x}, -\tilde{t}>\tilde{y}\}, \tilde{x}\le 0} \exp(\tilde{x}+\tilde{y})\, d\tilde{x} d\tilde{y} \\
&\quad + (1+r)^2 \iint\limits_{\{\exists(\tilde{s},\tilde{t}):\tilde{t}-\tilde{s}\in[0,\delta_1],\, \tilde{s}\in[0,L_1],\, \tilde{s}>\tilde{x}, -\tilde{t}>\tilde{y}\}, \tilde{x}>0} \exp(\tilde{x}+\tilde{y})\, d\tilde{x} d\tilde{y} \\
&\le (1+r)^2 \left(\int_{-\infty}^{0} \int_{-\infty}^{0} \exp(\tilde{x}+\tilde{y}) d\tilde{x} d\tilde{y} + \int_{0}^{L_1} \int_{-\infty}^{-\tilde{x}} \exp(\tilde{x}+\tilde{y})\, d\tilde{x} d\tilde{y} \right) \\
&= (1+r)^2(1+L_1) = (1+r)^2 + L|r'|. \qquad (44)
\end{aligned}$$

since $\tilde{t} \ge \tilde{s} \ge \tilde{x}$. Similar to Section 3.1.2 we get

$$\begin{aligned}
P(U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) &\le \frac{T-\varepsilon+o(1)}{Lu^{-2}} P_0 \\
&\sim \frac{T-\varepsilon}{Lu^{-2}} \frac{\exp\left(-u^2/(1+r)\right)}{2\pi u^2 \sqrt{1-r^2}} \left((1+r)^2 + L|r'|\right),
\end{aligned}$$

and

$$\limsup_{L\to\infty} \limsup_{u\to\infty} P(U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) \exp\left(u^2/(1+r)\right) \le \frac{(T-\varepsilon)|r'|}{2\pi\sqrt{1-r^2}}. \qquad (45)$$

It remains to derive a lower bound of the probability of right hand part of (37). Using stationarity we write

$$\begin{aligned}
P(U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}(S)) &\ge P(U_{u;\varepsilon,\varepsilon,T}(S)) \\
&\ge \frac{T-\varepsilon}{Lu^{-2}} \left(p_0(S) - \sum_{m=1}^{[(T-\varepsilon)u^2/L]+1} p_{0,m} \right), \qquad (46)
\end{aligned}$$

where

$$\begin{aligned}
p_0(S) &= P(\exists t : t\in[0, Lu^{-2}] : X(t)\ge u, X(t+\varepsilon)\ge u, T_t S), \\
p_{0,m} &= P(\exists (s,t):\ s\in[0,Lu^{-2}],\ t\in[mLu^{-2},(m+1)Lu^{-2}]: \\
&\qquad X(s)\ge u, X(s+\varepsilon)\ge u, X(t)\ge u, X(t+\varepsilon)\ge u).
\end{aligned}$$

Further, with some $A \subset A_0$ and $B \subset B_0$

$$p_0(S) = p_0(\Omega) \tag{47}$$
$$-P(\exists t : t \in [0, Lu^{-2}] : X(t) \geq u, X(t+\varepsilon) \geq u, T_t\{\inf_{v\in A} X(v) < u\} \cup \{\sup_{v\in B} X(v) > u\})$$

Similarly to (44), as $u \to \infty$,

$$\begin{aligned} p_0(\Omega) &\geq \frac{(1+o(1))}{2\pi u^2\sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r}\right) \int\int_{[\tilde{x},-\tilde{y}]\cap[0,L_1]\neq\emptyset} \exp\left(\tilde{x}+\tilde{y}\right) d\tilde{x}d\tilde{y} \\ &= \frac{(1+o(1))}{2\pi u^2\sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r}\right)(1+r)^2(1+L_1), \end{aligned} \tag{48}$$

since

$$\int\int_{[\tilde{x},-\tilde{y}]\cap[0,L_1]\neq\emptyset} = \int_{-\infty}^{0}\int_{-\infty}^{0} + \int_{0}^{L_1}\int_{-\infty}^{-x}.$$

Now we estimate the difference between $p_0(\Omega)$ and $p_0(S)$, the second term in the right hand side of (47). We have by **C1**, for all sufficiently large u,

$$\sup_{t\in[0,Lu^{-2}]} \operatorname{var}(X(t) - X(0)) \leq 3|Lu^{-2}|^{\alpha},$$

therefore for some $C > 0$ and any $d > 0$

$$P(\sup_{t\in[0,Lu^{-2}]} |X(t) - X(0)| > d) \leq Cu^{2/\alpha-1} \exp\left(-\frac{d^2u^{2\alpha}}{6L^{\alpha}}\right).$$

One may select $d = \tilde{c}u^{1-\alpha}$ since $\alpha > 1$, with sufficiently large $\tilde{c}$ such that the above probability is smaller than $\exp\left(-\frac{2u^2}{1+r(\varepsilon)}\right)$, for all sufficiently large u. The same is valid for the difference $X(t+\varepsilon) - X(\varepsilon)$. Therefore, for all sufficiently large u,

$$\begin{aligned} &P(\exists t : t \in [0, Lu^{-2}] : X(t) \geq u, X(t+\varepsilon) \geq u, \inf_{v\in t+A} X(v) < u, \sup_{v\in t+B} X(v) > u) \\ &\leq 2\exp\left(-\frac{2u^2}{1+r(\varepsilon)}\right) + P(X(0) \geq u-d, X(\varepsilon) \geq u-d, \inf_{s\in A_{lu^{-2}}} X(s) < u) \\ &\quad + P(X(0) > u-d, X(\varepsilon) > u-d, \sup_{v\in t+B_{lu^{-2}}} X(v) > u) \end{aligned}$$

with the extended sets $A_{lu^{-2}}$ and $B_{lu^{-2}}$ of A and B, respectively. The estimation of the last two probabilities is based on the same methods as for the proof of Lemma 2. The first probability equals

$$\frac{1}{\tilde{u}^2}\int_0^{\infty}\int_0^{\infty} f(\tilde{u}+\frac{x}{\tilde{u}}, \tilde{u}+\frac{y}{\tilde{u}})P(\inf_{s\in A_{lu^{-2}}} X(s) < u \mid X(0) = \tilde{u}+\frac{x}{\tilde{u}}, X(\varepsilon) = \tilde{u}+\frac{y}{\tilde{u}})dxdy,$$

where $\tilde{u} = u - d$ and f is the Gaussian two-dimensional density with means zero, variances one and covariance $r(\varepsilon)$. The density contributes $\exp\left(-\frac{\tilde{u}^2}{1+r(\varepsilon)}\right)$. Since the conditional mean in the above conditional probability is uniformly greater than one and $d \to 0$, the probability contributes $\exp\left(-a\tilde{u}^2\right)$ with some $a > 0$. Thus the probability has order $\exp\left(-\frac{bu^2}{1+r(\varepsilon)}\right)$ with some $b > 1$. Similarly for the second probability, since the conditional mean on $B_{lu^{-2}}$ is uniformly less than one. Together, the second term of the right hand side of (47) is at most of the same order, and of smaller order than (48).

Now we derive a bound for the sum in (46). First note that for any $\tau > \varepsilon$,

$$\exp\left(-\frac{u^2}{1+r(\tau)}\right) \le \exp\left(-\frac{u^2}{1+r(\varepsilon)}\right)\exp\left(-\frac{u^2\kappa(\tau-\varepsilon)}{(1+r(\varepsilon))^2}\right), \tag{49}$$

where

$$\kappa = \inf_{t\in[0,T-\varepsilon]} \frac{r(\varepsilon)-r(\varepsilon+t)}{t} > 0.$$

We have

$$p_{0,m} \le P(\exists(s,t) : s,t \in [0, Lu^{-2}],\ X(s) > u, X(t+\varepsilon+mLu^{-2}) > u).$$

The above right-hand side can be bounded similarly to the derived bound of P_m (see (44) and (45)), but by using $\tau = \varepsilon + mLu^{-2}$ instead of ε. We have,

$$p_{0,m} \le \frac{(1+o(1))}{2\pi u^2\sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r(\tau)}\right) \int\int I_m \exp\left(\frac{x+y}{1+r(\tau)}\right) dx\,dy \tag{50}$$

with the indicator function

$$I_m = I_{\{\exists(s,t):s\in[0,L]:t\in\varepsilon+[mL,(m+1)L],-r'(\tau)s/(1+r(\tau))>x,r'(\tau)t/(1+r(\tau))>y\}}$$

Since $I_m \le I_{\{x\le -r'(\tau)L/(1+r(\tau)),\, y\le r'(\tau)(\varepsilon+mL)/(1+r(\tau)\}}$, the double integral in (50) is bounded for all large u by $(1+r(\tau))^2 \le (1+r)^2$. Thus for sufficiently large u, using (49), we have

$$\begin{aligned}
\sum_{m\ge1} p_{0,m} &\le \sum_{m\ge1} P(\exists(s,t) : s,t \in [0, Lu^{-2}],\ X(s) > u, X(t+\varepsilon+mLu^{-2}) > u)\\
&\le \frac{(1+r)^2(1+o(1))}{2\pi u^2\sqrt{1-r(\varepsilon)^2}} \exp\left(-\frac{u^2}{1+r(\varepsilon)}\right) \sum_{m\ge1} \exp\left(-\frac{\kappa mL}{1+r(\varepsilon)}\right)\\
&\le o\left(u^{-2}\exp\left(-\frac{u^2}{1+r(\varepsilon)}\right)\right) = o(p_0(S))
\end{aligned} \tag{51}$$

as $L \to \infty$. Combining the bounds (46), (48) and (51), we get

$$\liminf_{L\to\infty} \liminf_{u\to\infty} P(U_{u;\varepsilon,\varepsilon+\gamma u^{-2},T}) \exp\left(u^2/(1+r)\right) \ge \frac{(T-\varepsilon)|r'|}{2\pi\sqrt{1-r^2}}.$$

Hence the statement of Theorem 1, part 1(i) follows.

6.2 Proof of Theorem 2(ii).

In this case $\alpha = 1$. By the same arguments as in (36) and (37) we have

$$P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S)) \le P_{u;\varepsilon,T}(S) \le P(U_{u;\varepsilon,\varepsilon+\delta(u),T}) + P_{u;\varepsilon+\delta(u),T}\,, \tag{52}$$

where $\delta(u) = cu^{-2}\log u$, $c > 0$. The second probability in the right hand side of (52) is bounded by (38), for any sufficiently large c.

UPPER BOUND

We derive now an upper bound of $P(U_{u;\varepsilon,\varepsilon+\delta(u),T})$. We split the set $\{(s,t) : s,t \in [0,T],\ \varepsilon \le t-s \le \varepsilon+\delta(u)\}$ into small parallelograms $K_{k,m} = K_{k,m}(\lambda,\mu)$, $\lambda,\mu > 0$, defined as

$$K_{k,m} := \left\{(s,t) : k\mu u^{-2} \le s \le (k+1)\mu u^{-2}, \varepsilon + m\lambda u^{-2} \le t-s \le \varepsilon + (m+1)\lambda u^{-2}\right\},$$

$$m = 0,1,...,M_u = \left[\frac{\delta(u)}{\lambda u^{-2}}\right],\ \ k = 0,1,...,N_{u,m} = \left[\frac{T-(\varepsilon+m\lambda u^{-2})}{\mu u^{-2}}\right].$$

We have

$$P(U_{u;\varepsilon,\varepsilon+\delta(u),T}) \le \sum_{m=0}^{M_u+1}\sum_{k=0}^{N_{u,m}+1} P(\exists (s,t)\in K_{k,m} :\ X(s)>u,\ X(t)>u). \tag{53}$$

As in (49) with $\tau = \varepsilon + m\lambda u^{-2}$ and $r = r(\varepsilon)$, $r' = r'(\varepsilon)$, $\rho = (1+r)^{-2}$, we get

$$\exp\left(-\frac{u^2}{1+r(\varepsilon+m\lambda u^{-2})}\right) \le \exp\left(-\frac{u^2}{1+r}\right)\exp\left(-m\lambda\rho|r'|(1+o(1))\right) \tag{54}$$

as $u \to \infty$, where $o(1)$ in the right-hand side is uniform in m because $m\lambda u^{-2} \le cu^{-2}\log u$. By Lemma 2, with $\alpha = 1$ and $\tau = \varepsilon + m\lambda u^{-2}$, using (54) and the stationarity of X, we have

$$\begin{aligned} p_{k,m} &:= P\left(\bigcup\nolimits_{(s,t)\in K_{k,m}}\{X(s)>u,\ X(t)>u\}\right) \equiv p_{0,m} \\ &\le h_\alpha(\lambda,\mu)e^{-m\lambda\rho|r'|(1+o(1))}\Psi_2(u,r)(1+o(1)) \end{aligned}$$

as $u \to \infty$, where

$$h_\alpha(\lambda,\mu) = \int\!\!\int e^{x+y}P\Big(\bigcup_{(s,t)\in\tilde K}\{\sqrt{2}B_1(s)-(1+r')s > x,\ \sqrt{2}\widetilde{B}_1(t)-(1-r')t > y\}\Big)dx\,dy,$$

with $\tilde K = \{(s,t) : 0 \le t-s \le \rho\lambda,\ 0 \le s \le \rho\mu\}$. For some γ_u, with $\gamma_u \downarrow 0$ as $u \uparrow \infty$, we have

$$\begin{aligned} \frac{\mu u^{-2}P(U_{u;\varepsilon,\varepsilon+\delta(u),T})}{h_\alpha(\lambda,\mu)\Psi_2(u,r)(T-\varepsilon)} &\le (1+\gamma_u)\left(1+\sum_{m=1}^{M_u+1}\frac{T-\varepsilon-m\lambda u^{-2}}{T-\varepsilon}e^{-m\lambda\rho|r'|}\right) + O(u^{-1}) \\ &\le (1+\gamma_u)\left(1-e^{-\lambda\rho|r'|}\right)^{-1} + O(u^{-1}). \end{aligned} \tag{55}$$

which tends to $\left(1-e^{-\lambda\rho|r'|}\right)^{-1}$ as $u\to\infty$, and then to 1 as $\lambda\to\infty$.

Lower Bound

We consider a lower bound of $P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S))$. We need to introduce another set of parallelograms,

$$K_l = K_l(\lambda,\mu) := \left\{(s,t) : l\mu u^{-2} \le s \le ((l+1)\mu - \sqrt{\mu})u^{-2}, \varepsilon \le t-s \le \varepsilon + \lambda u^{-2}\right\},$$

$l = 0,1,2,...,L_u = \left[(T-(\varepsilon+\lambda u^{-2}))/\mu u^{-2}\right]$, where again μ and λ are large with $\mu > \lambda > 1$. Observe that there is a gap of width $\sqrt{\mu}$ between any neighboring parallelograms K_l and K_{l+1}.

Denote for $S \in \mathscr{S}$,

$$q_0 := P\left(\bigcup_{(s,t)\in K_0} \{X(s) \ge u,\ X(t) \ge u, T_s S\}\right).$$

and

$$q_l := P\left(\bigcup_{(s,t)\in K_0} \{X(s) \ge u,\ X(t) \ge u\} \cap \bigcup_{(s,t)\in K_l} \{X(s) \ge u,\ X(t) \ge u\}\right),\quad l \ge 1.$$

We have by the stationarity of X,

$$\begin{aligned} P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S)) &\ge P\left(\bigcup_{l=0}^{L_u}\bigcup_{(s,t)\in K_l} \{X(s) \ge u,\ X(t) \ge u, T_s S\}\right) \\ &\ge L_u q_0 - \sum_{l=1}^{L_u+1}(L_u+1-l)q_l. \end{aligned} \tag{56}$$

By Lemma 2,

$$q_0 \ge (1-\gamma_u)h_\alpha(\lambda,\mu-\sqrt{\mu})\Psi_2(u,r),$$

where $\gamma_u \downarrow 0$ as $u \uparrow \infty$ and $h_\alpha(\lambda,\mu-\sqrt{\mu})$ denotes the factor $h_\alpha(\Lambda/(1+r)^2)$ of the Lemma with $\Lambda = \{(s,t) : 0 \le s \le \mu-\sqrt{\mu}, 0 \le t-s \le \lambda\}$. Thus

$$L_u q_0 \ge (1-\gamma_u)u^2\frac{(T-\varepsilon)-\lambda u^{-2}}{\mu}h_\alpha(\lambda,\mu-\sqrt{\mu})\Psi_2(u,r). \tag{57}$$

We show at the end of this proof that the right-hand side is asymptotically equal to the upper bound (55), as u tends to infinity and also $\mu \to \infty$ and $\lambda \to \infty$. So it remains to show that the second term in the right-hand side of (56) is of smaller order. Let $\tilde{\delta} > 0$ be such that $\tilde{\delta} < T-\varepsilon-\lambda\mu^{-2}$ and $r'(\varepsilon+t) < 0$ for all $t \in [0,\tilde{\delta}]$. Set $l_0 = [\tilde{\delta}/\mu u^{-2}]+1$. We derive upper bounds for q_l separately for $l \le l_0$ and for $l > l_0$.

We have $\max\{(r(t), t \ge \varepsilon+\tilde{\delta}\} < r-\vartheta$ for some $\vartheta > 0$. Therefore by Fernique's inequality we get for $l > l_0$

$$\begin{aligned} q_l &\le P\left(\max_{(s,t)\in[0,\mu u^{-2}]\times[\varepsilon+l\mu u^{-2},\varepsilon+(\lambda+(l+1)\mu)u^{-2}]}(X(s)+X(t)) \ge 2u\right) \\ &\le C\exp\left(-\frac{u^2}{1+r-\vartheta}\right) \end{aligned}$$

for some positive constant C. For any $R > 0$, it follows

$$\sum_{l>l_0}^{L_u+1} L_u q_l \le C L_u^2 \exp\left(-\frac{u^2}{1+r(\varepsilon)-\vartheta}\right) = o\left(u^{-R}\right)\Psi_2(u, r(\varepsilon)) \tag{58}$$

as $u \to \infty$.

For $l \le l_0$, we write

$$q_l \le P\left(\bigcup_{s\in[0,(\mu-\sqrt{\mu})u^{-2}]} \bigcup_{t\in[\varepsilon+l\mu u^{-2},\varepsilon+(\lambda+(l+1)\mu-\sqrt{\mu})u^{-2}]} \{X(s) > u,\, X(t) > u\}\right).$$

Applying again Lemma 2 for $\tau = \varepsilon + (l-1)\mu u^{-2} \in [\varepsilon, \varepsilon + \tilde{\delta}] \subset C_\varepsilon$, we get using for short $r = r(\tau), r' = r'(\tau)$, for sufficiently large u,

$$q_l \le 2b(\lambda,\mu)\Psi_2(u, r(\tau))$$

with

$$b(\lambda,\mu) = \int\int e^{x+y} P\left(\max_{0\le s\le\tilde{\rho}(\mu-\sqrt{\mu})}\{\sqrt{2}B_1(s) - (1+r')s\} > x,\right.$$
$$\left.\max_{\tilde{\rho}\mu\le t\le\tilde{\rho}(\lambda+2\mu-\sqrt{\mu})}\{\sqrt{2}\widetilde{B}_1(t) - (1-r')t\} > y\right) dx dy,$$

$\tilde{\rho} := 1/(1+r^2)$). Note that $b(\lambda,\mu)$ depends on l through τ. Since $r' < 0$, the probability under the integral is at most

$$P\left(\max_{0\le s\le\tilde{\rho}(\mu-\sqrt{\mu})}\{\sqrt{2}B_1(s) - s\} > x - |r'|\rho(\mu-\sqrt{\mu}),\right.$$
$$\left.\max_{\tilde{\rho}\mu\le t\le\tilde{\rho}(\lambda+2\mu-\sqrt{\mu})}\{\sqrt{2}\widetilde{B}_1(t) - t\} > y + |r'|\tilde{\rho}\mu\right).$$

Changing x by $x - |r'|\tilde{\rho}(\mu - \sqrt{\mu})$ and y by $y + |r'|\tilde{\rho}\mu$ and using the independence of the Brownian motions B and $\widetilde{B}$, we get that

$$\begin{aligned} b(\lambda,\mu) &\le e^{-|r'|\tilde{\rho}\sqrt{\mu}} \int e^x P\left(\max_{0\le s\le\tilde{\rho}(\mu-\sqrt{\mu})}\{\sqrt{2}B_1(s) - s\} > x\right) dx \\ &\quad \times \int e^y P\left(\max_{\mu\le t\le\tilde{\rho}(\lambda+2\mu-\sqrt{\mu})}\{\sqrt{2}\tilde{B}_1(t) - t\} > y\right) dy \\ &\le e^{-|r'|\tilde{\rho}\sqrt{\mu}} H_1(\tilde{\rho}(\mu-\sqrt{\mu})) H_1(\tilde{\rho}(\lambda+2\mu-\sqrt{\mu})) \\ &\le C\mu(\lambda+2\mu)e^{-|r'|\tilde{\rho}\sqrt{\mu}} \le C\mu(\lambda+2\mu)e^{-|r'|\sqrt{\mu}/(1+r)^2}. \end{aligned}$$

where we applied the inequality $H_1(T) \le CT$, see (1). Furthermore, using (49), we derive

$$\Psi_2(u, r(\tau)) = \Psi_2(u, r(\varepsilon + (l-1)\mu u^{-2})) \le C\Psi_2(u, r)\exp\left(-\frac{\kappa(l-1)\mu}{(1+r)^2}\right),$$

for some C, where κ is a positive number with $\kappa < \min_{t\in[\varepsilon,\varepsilon+\tilde{\delta}]} |r'(t)|$. Thus

$$q_l \le C\mu(\lambda+2\mu)\Psi_2(u,r)\exp\left(-\frac{\kappa(l-1)\mu}{(1+r)^2} - \frac{\kappa\sqrt{\mu}}{(1+r)^2}\right)$$

and therefore

$$\begin{aligned}\sum_{l=1}^{l_0}(L_u+1-l)q_l &\le CL_u\mu(\lambda+2\mu)\Psi_2(u,r)\sum_{l=1}^{l_0}\exp\left(-\frac{\kappa(l-1)\mu}{(1+r)^2} - \frac{\kappa\sqrt{\mu}}{(1+r)^2}\right)\\ &\le C\frac{T-\varepsilon}{\mu u^{-2}}\mu^2\Psi_2(u,r)\exp\left(-\kappa\sqrt{\mu}/(1+r)^2\right) \qquad (59)\end{aligned}$$

since $\lambda < \mu$. Combining the bounds (56), (57), (58) and (59), it follows that

$$\begin{aligned}&\liminf_{u\to\infty}\frac{P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S)}{(T-\varepsilon)u^2\Psi_2(u,r)} \ge \frac{h_\alpha(\lambda,\mu-\sqrt{\mu})}{\mu} - C\mu\exp\left(-\kappa\sqrt{\mu}/(1+r)^2\right)\\ &= \frac{h_\alpha(\lambda,\mu-\sqrt{\mu})}{\mu}(1-\tilde{\gamma}_\mu), \qquad (60)\end{aligned}$$

with $0 < \tilde{\gamma}_\mu \to 0$ as $\mu \to \infty$.

Asymptotic Equivalence

Finally, with the upper and the lower bounds we derive the asymptotic behavior of $P_{u;\varepsilon,T}(S)$. From (55) we have for large λ and μ

$$\limsup_{u\to\infty}\frac{P_{u;\varepsilon,T}(S)}{(T-\varepsilon)u^2\Psi_2(u,r)} \le \frac{h_\alpha(\lambda,\mu)}{\mu}(1-e^{-\lambda\rho|r'|})^{-1} \le \frac{h_\alpha(\lambda,\mu)}{\mu}(1+e^{-\delta\lambda}),$$

for some positive δ. From (60) we get for any sufficiently large μ_1 and λ_1 with $\mu_1 > 1$ and $\lambda_1 < \mu_1$

$$\liminf_{u\to\infty}\frac{P_{u;\varepsilon,T}(S)}{(T-\varepsilon)u^2\Psi_2(u,r)} \ge \frac{h_\alpha(\lambda_1,\mu_1-\sqrt{\mu_1})}{\mu_1}(1-\tilde{\gamma}_{\mu_1}).$$

It remains to show that the two bounds are asymptotically equivalent. We have for any λ,μ and λ_1,μ_1, with $\mu_1 > 1$

$$\frac{h_\alpha(\lambda,\mu)}{\mu}(1+e^{-\delta\lambda}) \ge \frac{h_\alpha(\lambda_1,\mu_1-\sqrt{\mu_1})}{\mu_1}(1-\tilde{\gamma}_{\mu_1}). \qquad (61)$$

Since $h_\alpha(\lambda_1,\mu_1-\sqrt{\mu_1}) \ge h_\alpha(0,0) = 1$, for any $\lambda_1 > 0$, one can find some $\mu_1 > 1$ such that the right-hand side of (61) is positive. Fixing these λ_1, μ_1, we get for any λ

$$\liminf_{\mu\to\infty}\frac{h_\alpha(\lambda,\mu)}{\mu} > 0.$$

Fixing now λ and μ we get by (61)

$$\limsup_{\mu_1\to\infty}\frac{h_\alpha(\lambda_1,\mu_1-\sqrt{\mu_1})}{\mu_1} = \limsup_{\mu_1\to\infty}\frac{h_\alpha(\lambda,\mu_1)}{\mu_1} < \infty,$$

and thus

$$\infty > \liminf_{\mu\to\infty} \frac{h_\alpha(\lambda,\mu)}{\mu}(1+e^{-\delta\lambda}) \geq \limsup_{\mu_1\to\infty} \frac{h_\alpha(\lambda_1,\mu_1)}{\mu_1} > 0.$$

Consequently,

$$\infty > \liminf_{\lambda\to\infty} \liminf_{\mu\to\infty} \frac{h_\alpha(\lambda,\mu)}{\mu} \geq \limsup_{\lambda_1\to\infty} \limsup_{\mu_1\to\infty} \frac{h_\alpha(\lambda_1,\mu_1)}{\mu_1} > 0,$$

so that

$$h = \liminf_{\lambda\to\infty} \liminf_{\mu\to\infty} \frac{h_\alpha(\lambda,\mu)}{\mu} = \limsup_{\lambda\to\infty} \limsup_{\mu\to\infty} \frac{h_\alpha(\lambda,\mu)}{\mu} \in (0,\infty), \tag{62}$$

and Theorem 2(ii) follows.

6.3 Proof of Theorem 2(iii).

Now $\alpha < 1$. This proof is similar to the proof of Theorem 1. The relations (52) and (38) are still valid with the same $\delta = \delta(u) = cu^{-2}\log u$, (with $c > 0$), and we immediately turn to the probability $P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S))$. Let $\Delta = \lambda u^{-2/\alpha}$, $\lambda > 0$, and define the intervals

$$\Delta_k = [k\Delta, (k+1)\Delta],\ 0 \leq k \leq N,\ N = [T/\Delta].$$

We cover the set $D = \{s,t \in [0,T] :\ 0 \leq t-s-\varepsilon \leq \delta\}$ with squares $\Delta_k \times \Delta_l = (0,(l-k)\Delta)+\Delta_k\times\Delta_k$. We apply Lemma 2 with $\tau = (l-k)\Delta$ and $\Lambda = [k\lambda,(k+1)\lambda] \times [k\lambda,(k+1)\lambda]$. Since these sets are squares and the Brownian motions are independent, the double integral in the definition of $h_\alpha(\Lambda)$ is factorized into two integrals which are both equal to $H_{\alpha,0}(\lambda/(1+r(\tau))^{2/\alpha})$. Since $\tau \to \varepsilon$ uniformly for all k,l such that $\Delta_k \times \Delta_l \cap D \neq \emptyset$, we have, writing again $r = r(\varepsilon)$, $r' = r'(\varepsilon)$,

$$\begin{aligned}
P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S)) &\leq \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} P\left(\max_{s\in\Delta_k} X(s) \geq u,\ \max_{t\in\Delta_l} X(t) \geq u\right) \\
&\leq \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} \frac{(1+\gamma(u))(1+r(\tau))^{3/2}}{2\pi u^2\sqrt{1-r(\tau)}} H^2_{\alpha,0}\left(\frac{\lambda}{(1+r)^{2/\alpha}}\right) \\
&\qquad \times \exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right) \\
&\leq \frac{(1+\gamma(u))(1+r)^{3/2}}{2\pi u^2\sqrt{1-r}} H^2_{\alpha,0}\left(\frac{\lambda}{(1+r)^{2/\alpha}}\right) \\
&\qquad \times \sum_{(k,l):\ \Delta_k\times\Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r((l-k)\Delta)}\right),
\end{aligned} \tag{63}$$

where $\gamma(u) \downarrow 0$ as $u \uparrow \infty$. Using that $-\Delta \leq (l-k)\Delta - \varepsilon \leq \delta + \Delta \to 0$, it follows

$$\begin{aligned}
\frac{1}{1+r((l-k)\Delta)} &= \frac{1}{1+r} - (1+\gamma_1(u))\frac{r((l-k)\Delta)-r}{(1+r)^2} \\
&= \frac{1}{1+r} + (1+\gamma_1(u))\frac{|r'|((l-k)\Delta-\varepsilon)}{(1+r)^2} \\
&= \frac{1}{1+r} + (1+\gamma_1(u))R((l-k)\Delta-\varepsilon),
\end{aligned}$$

where $\gamma_1(u) \downarrow 0$, as $u \uparrow \infty$, and $R = |r'|/(1+r)^2$. Therefore the sum Σ in the right-hand side of (32) equals

$$\Sigma = \exp\left(-\frac{u^2}{1+r}\right) \sum_{(k,l):\ \Delta_k \times \Delta_l \cap D \neq \emptyset} \exp\left(-Ru^2((l-k)\Delta - \varepsilon)\right).$$

In the sum, the index k varies between 0 and $(T-\varepsilon)/\Delta + \theta$, as $u \to \infty$, where $|\theta| \leq 1$, depending of u. The index $n = (l-k)$ varies between $\varepsilon/\Delta + \theta_1$ and $(\varepsilon + \delta)/\Delta + \theta_2$ as $u \to \infty$, where θ_1, θ_2 have the same properties as θ. Let $x_n = (n\Delta - \varepsilon)u^2$ and note that $x_{n+1} - x_n = u^2\Delta \to 0$ as $u \to \infty$, since $\alpha < 1$. It implies that

$$\begin{aligned}\Sigma &= (1+o(1)) \exp\left(-\frac{u^2}{1+r}\right) \frac{T-\varepsilon}{u^2\Delta^2} \sum_{x_n = o(1)}^{u^2\delta + o(1)} e^{-Rx_n}(x_{n+1} - x_n) \\ &= (1+o(1)) \exp\left(-\frac{u^2}{1+r}\right) \frac{T-\varepsilon}{u^2\Delta^2} \int_0^\infty e^{-Rx}\,dx. \end{aligned} \tag{64}$$

Since the integral equals $1/R$, it follows by (63) and the properties of Pickands' constants for large λ, that

$$\begin{aligned} &P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S)) \\ &\leq \frac{(1+\gamma_2(u))(1+r)^{3/2}}{2\pi u^2\sqrt{1-r}} H_\alpha^2\left(\frac{\lambda}{(1+r)^{2/\alpha}}\right) \exp\left(-\frac{u^2}{1+r}\right) \frac{T-\varepsilon}{u^2\Delta^2 R} \\ &= \frac{(1+\gamma_2(u))(T-\varepsilon)(1+r)^{2-4/\alpha}}{u^{2-4/\alpha}|r'|} (1+\gamma_3(\lambda)) H_\alpha^2 \Psi_2(u,r) \end{aligned} \tag{65}$$

where $\gamma_2(u) \downarrow 0$ as $u \uparrow \infty$ and $\gamma_3(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$. Letting u and then $\lambda \to \infty$, this bound is asymptotically equal to the term of the statement.

The estimation of the probability $P(U_{u;\varepsilon,\varepsilon+\delta(u),T}(S))$ from below repeats word-by-word the corresponding estimation from the proof of Theorem 1.

6.4 Proof of the second part of Theorem 2

This proof is the same as the proof of the second part of Theorem 1. □

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